SGI 2021 Friday Tutorial Exercise 2: Index Prescription

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Errata

Introduction

This exercise implements a complementary field-design approach to the power-field paradigm: here, instead of first computing a field, and then deriving its topology (matchings and singularities), we will *prescribe* vertex-based singularities and compute a face-based field that conforms to these singularities. For simplicity, we will only consider vertex-based dual cycles, and thus omit high-genus and boundary meshes from this practical (the script will warn you when you are trying to load one, by checking the Euler characteristic).

The exercise

The main practical directory comprises two subfolders: `code`, where there are the main codes files, including the ones you need to update, `data`, containing the meshes. The files contain already given envelope and setup code which is (hopefully!) well-documented enough for you to work with, and you need to complete to parts marked in "`TODO:`". In the end of the day, you will receive the reference solution, and can compare your approach and reflect on it.

The code will automatically plot the results of the steps of your solution, properly titled. Currently the plots will be of trivial results, which are mostly zeros; these are the variables you need to fill in.

1 Index Prescription

The setup The envelope script which runs the algorithm and visualizes the result is `IndexPrescription.m`. You will notice that it shares much of the code of exercise 1. Part of which should also be completed with code you already wrote, such as the Gaussian curvature computation. The envelope script proceeds as follows (see Figure 1 for how the result should look):

- Loading a triangle mesh with `readOBJ()` into variables `V` and `F`, and using `make_edge_list()` to compute edge-based quantities.
- Computing required geometric quantities: normals, face areas, face barycenter, and basis vectors B_1, and B_2. This should be the same code from the previous exercise.
- Choosing the singularities and their indices on a subset of vertices, where the rest have index 0. You are welcome to "inherit" the singularities derived from the previous exercise and try to create a new field on their basis. Otherwise, the set is randomized to automatically be consistent.
- The field is computed
- Visualizing the field on the original mesh.

1.1 Computing index-prescribed fields

Given a field order N, and prescribed indices $I(v \in V)$, that are integer multiples of $\frac{1}{N}$, we will compute a face-based N-RoSy field (like in the previous exercise), that conforms to these singularities. Consider the dual cycles around vertices that measure effort, which are the only independent cycles in simply-connected meshes. We then have the condition that:

$$I(v) = \frac{1}{2\pi N}\left((d_0)^T \Theta + NK(v)\right)$$

which can be written as a condition on the effort as follows:

$$(d_0)^T \Theta = 2\pi N I(v) - NK(v) \tag{1}$$

This condition is necessary and sufficient for the existence of an N-RoSy field which conforms to this effort. The effort is a (dual) differential quantity that lives on edges, and therefore is of dimension $|E|$. However, the Condition 1 only has $|V|$ constraints. Since we always have $|V| < |E|$ (in fact, we have $|V| = \frac{1}{3}|E| + \chi$ for closed triangle meshes), we have quite a big space of fields that conform to this effort. We are then looking (again!) for the *smoothest* field that has this property. That means we are solving for:

$$\Theta = \text{argmin} \sum_e w(e)\left|\Theta(e)\right|^2, s.t. \tag{2}$$

$$(d_0)^T \Theta = 2\pi N I(v) - NK(v).$$

We showed in class that this actually amounts to a very simple Poisson system of the form:

$$\left[(d_0)^T W d_0\right] \psi(v) = 2\pi N I(v) - NK(v). \tag{3}$$

d_0 is the same matrix you computed in the previous exercise, but now we are going to use the inverse W which has diagonal weights of $\frac{A_f+A_g}{3l_e^2}$. This is because we are converting a primal quantity ($d_0\psi$) to a dual one rather than comparing dual quantities (the power fields on neighboring faces). $\psi(v)$ is a $|V|$-dimensional vertex-based function which is essentially the "effort potential". One can then compute:

$$\Theta(e) = W d_0 \psi(v)$$

as the $|E|$-sized vector that we require.

Note that the Poisson system has a single null space (you can add any global constant to ψ to get the same Θ). MATLAB will automatically choose a solution, though it may complain about conditioning. However, this also means that the right-hand side has to sum up to 0, which only happens when

$$\sum_v I(v) = \chi.$$

The `presIndices` you choose must preserve that, and it will be checked by `indexPresError`.

2 Field Reconstruction

We still need to obtain the actual $\mathbf{v}(f \in F)$ from the effort $\Theta(e \in E)$. For this, we solve again the least-squares "as-parallel-as-possible" system from the previous exercise, just with a twist: instead of trying to have zero effort, we are trying to have exactly Θ as the effort; since Θ is integrable, the system is nominally least-squares, but the fitting is perfect, and it should have zero energy—this is checked in the confidence check of `vectorFromEffortError`. We are specifically solving for:

$$E(y) = \sum_e w_e \left| \mathbf{y}(f) \left(\overline{e_f}\right)^N \cdot e^{i\Theta(e)} - \mathbf{y}(g) \left(\overline{e_g}\right)^N \right|^2, \tag{4}$$

Where the A matrix from the previous exercise is altered so that each row has $\overline{e_f} \cdot e^{i\Theta(e)}$ instead of just $\overline{e_f}$ in the appropriate column f in each row. The rest of the reconstruction proceeds as in the previous practical.

Directional constraints: the index-prescription algorithm in this form does not support directional constraints directly. However, you need to set *a single* face $b \in B$ in order for the Equation 4 to be well-defined. That means there is a single degree of freedom for the field with a given effort, and this culminates in a *global rotation* of the entire field, which doesn't change the effort. If you set more than one constraint, the reconstruction Equation 4 will be solved in an actual least-squares fashion, and the error would not be zero (but please try! it's still going to be "as-fitting-to-the-effort-as-possible").

The singularities will be shown as spheres on the mesh with the field, as seen in Figure 1, similar to the previous exercise.

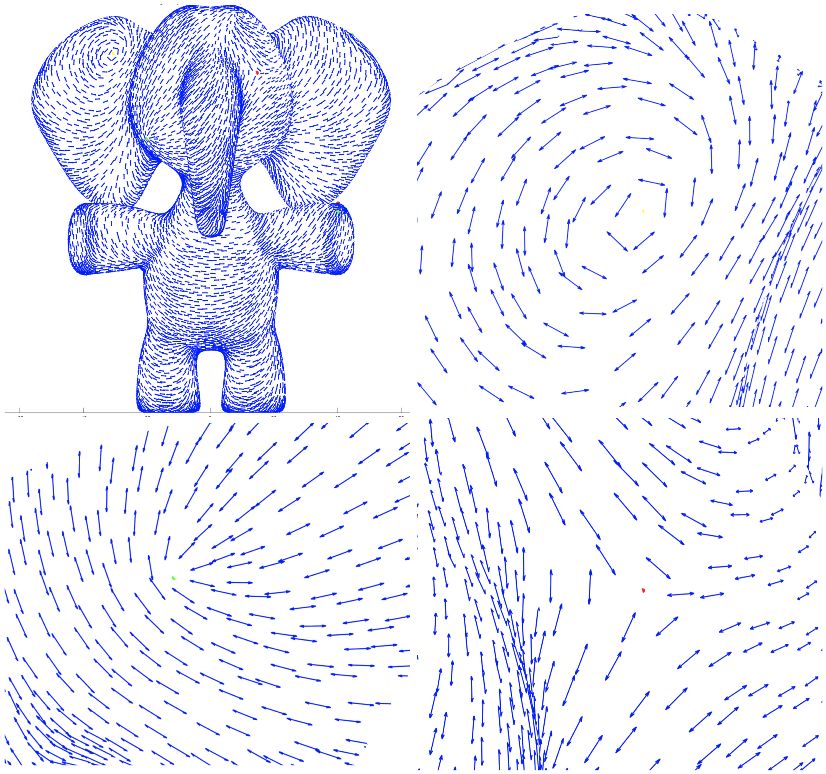

Figure 1: A 2-field computed from its prescribed singularities (top left). clockwise from top right: $I = 2, -1, 1$ prescribed singularities