Universiteit Utrecht

DIRECTIONAL-FIELD PROCESSING

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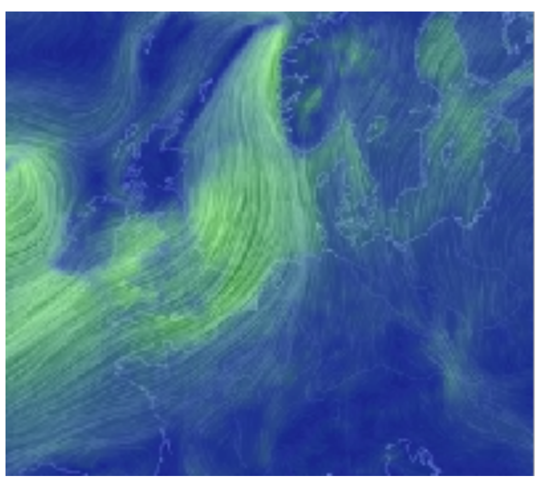
Developed with Marcel Campen, Olga Diamanti, David Bommes, Mirela Ben-Chen, Klaus Hildebrandt, and Daniele Panozzo.

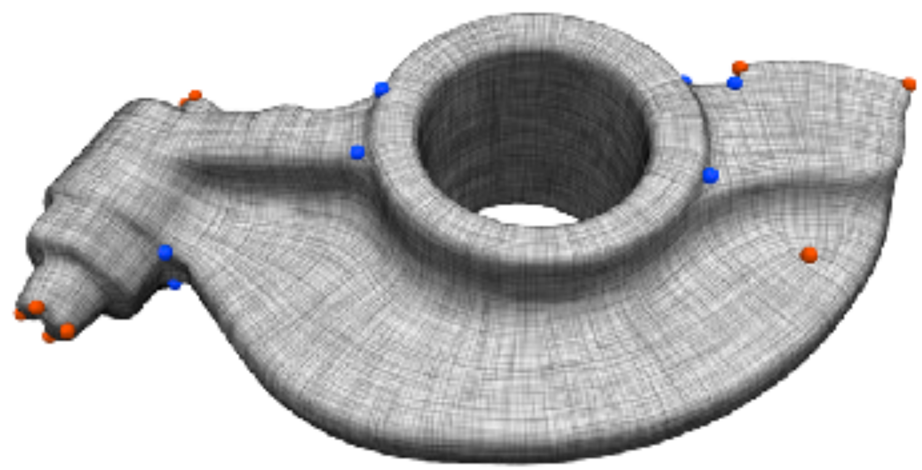
Materials: https://github.com/SGI-2021/directional-fields-tutorial

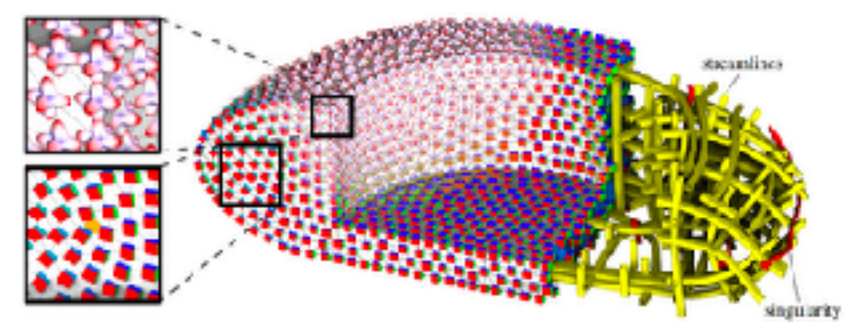
DIRECTIONAL FIELDS

N-**Directional field**: a collection of vectors $\{u_1(p), u_2(p), \cdots, u_N(p)\}$ associated with each point $p \in \Omega$ in a domain Ω.



https://earth.nullschool.net/





[Sokolov *et al.* 2016]

Domain: *2D* image*

Field: *single (1-)vector field.*

Domain: *2-manifold in 3D*

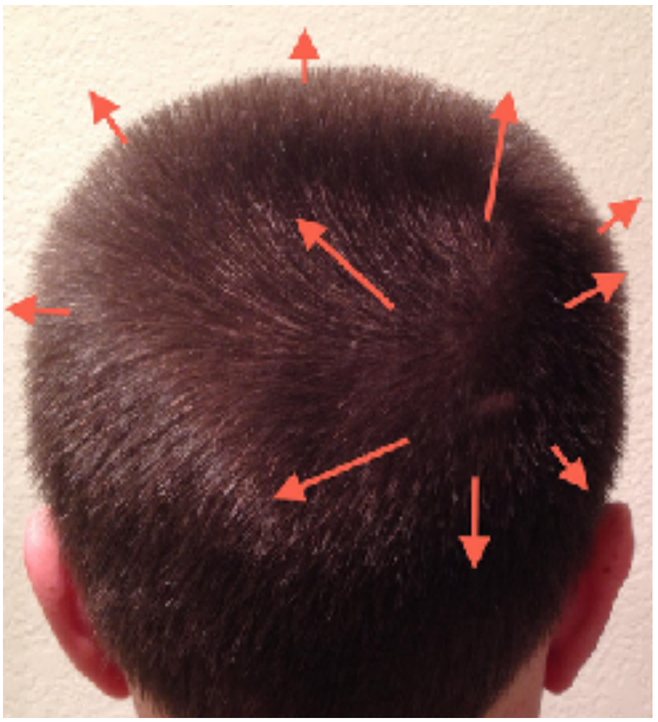
Field: *tangent 4-field ("cross-field")*

Domain: *3D bounded volume*

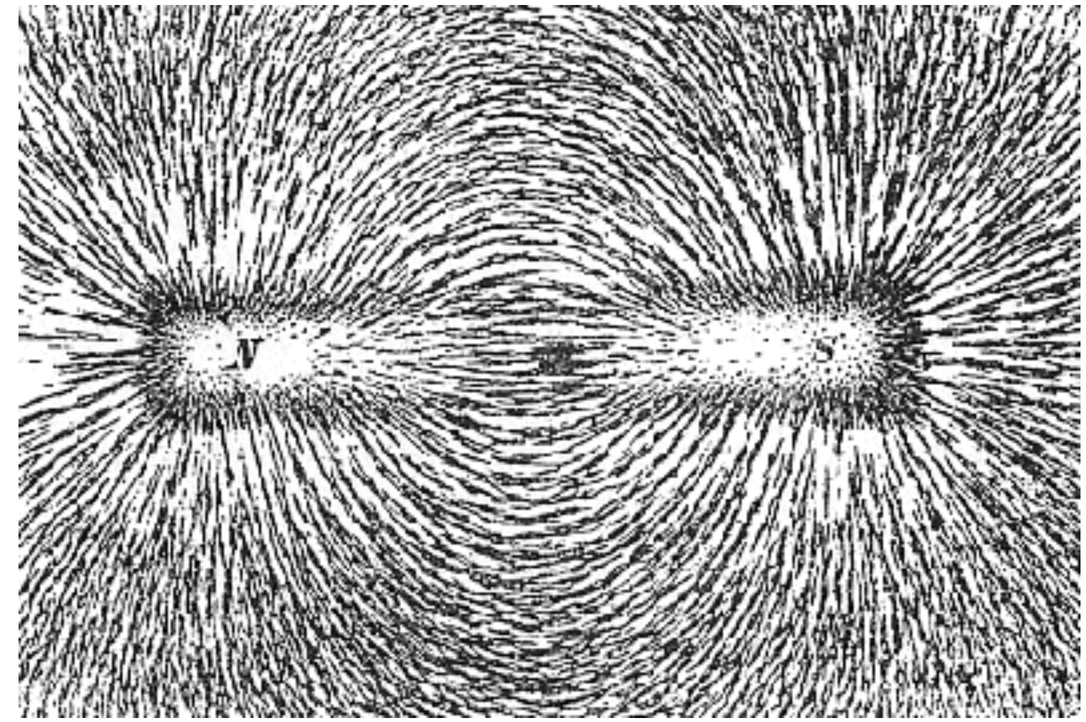
Field: *3D orthonormal frame field ("octahedral")*

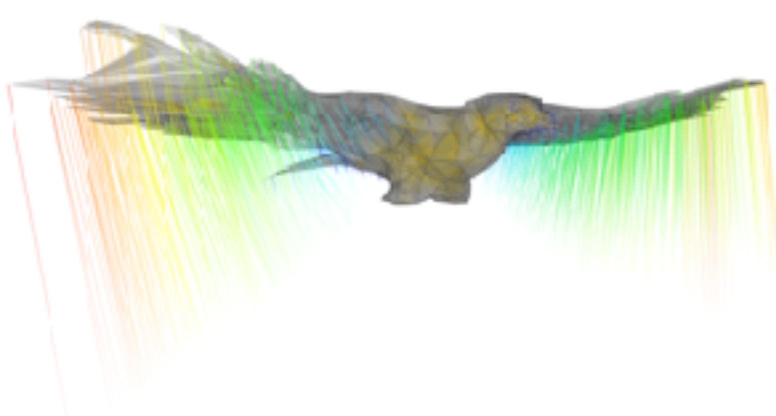
**spherical*

DIRECTIONAL FIELDS IN NATURE AND DESIGN



[Alexander *et al.* 2007]



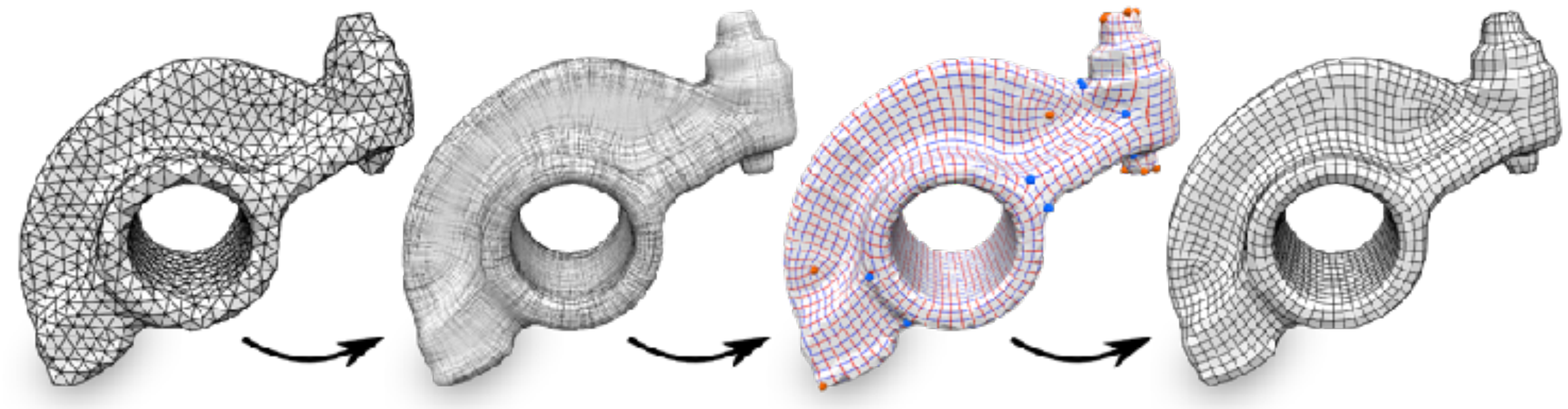


[Martinez Esturo *et al.* 2014]

[Ferreira *et al.* 2013]

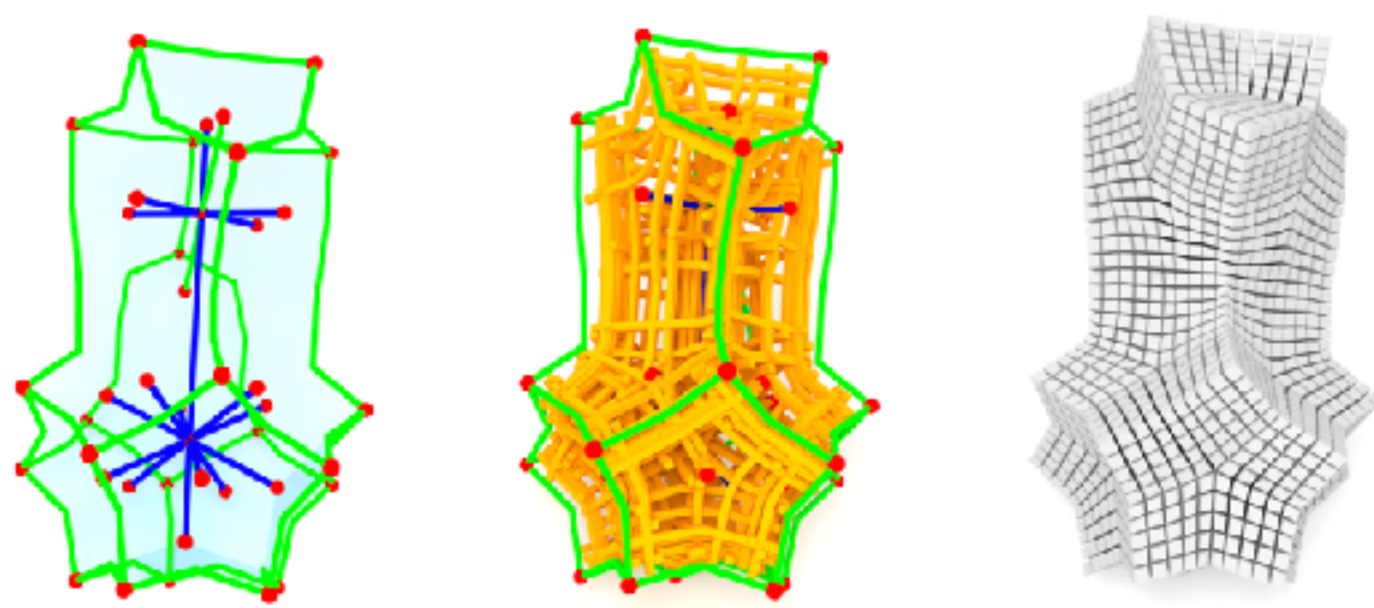
APPLICATIONS

- Meshing



N=4 [Ebke et al. 2014]

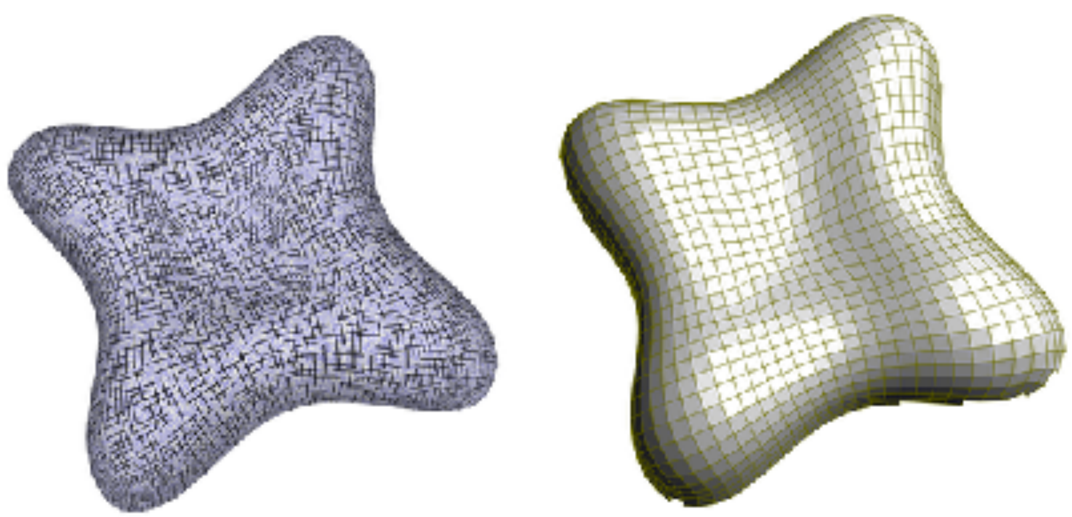
N=3,6 [Nieser et al. 2011]



Octahedral [Liu et al. 2018]

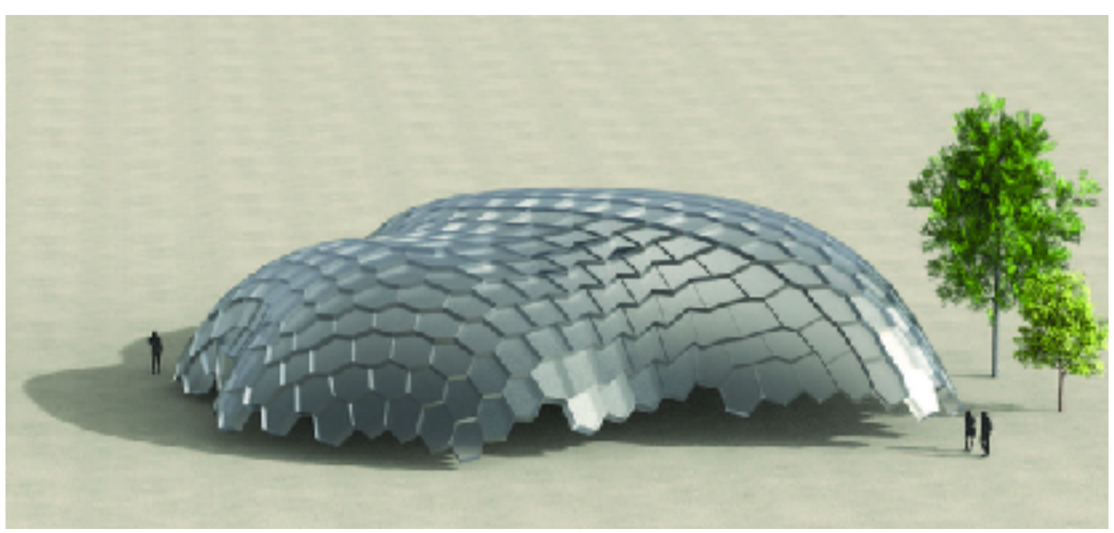
APPLICATIONS

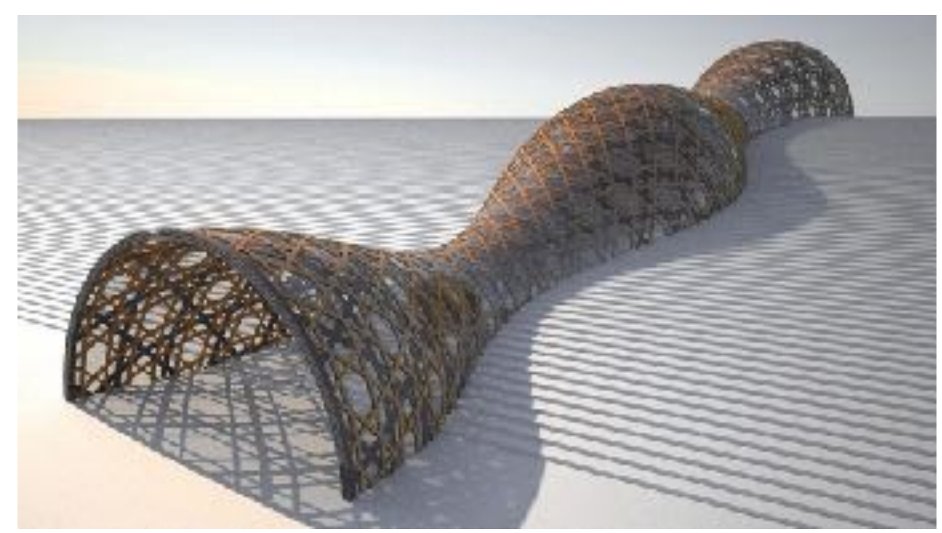
- Meshing with a special function



Conjugacy->planarity [Liu *et al.* 2011, Diamanti *et al.* 2014,2015, Pluta *et al.* 2021]

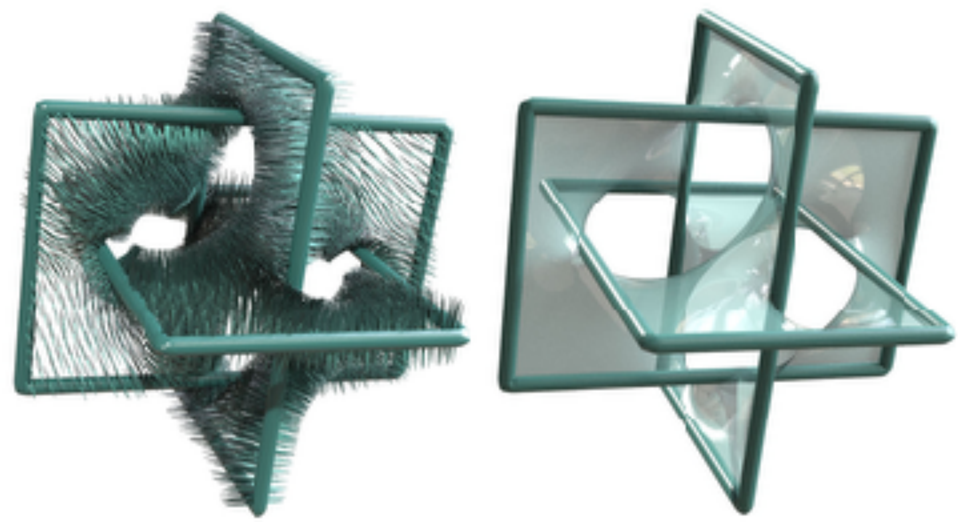
Geodesic [Pottmann et al. 2010, Vekhter *et al.* 2019]



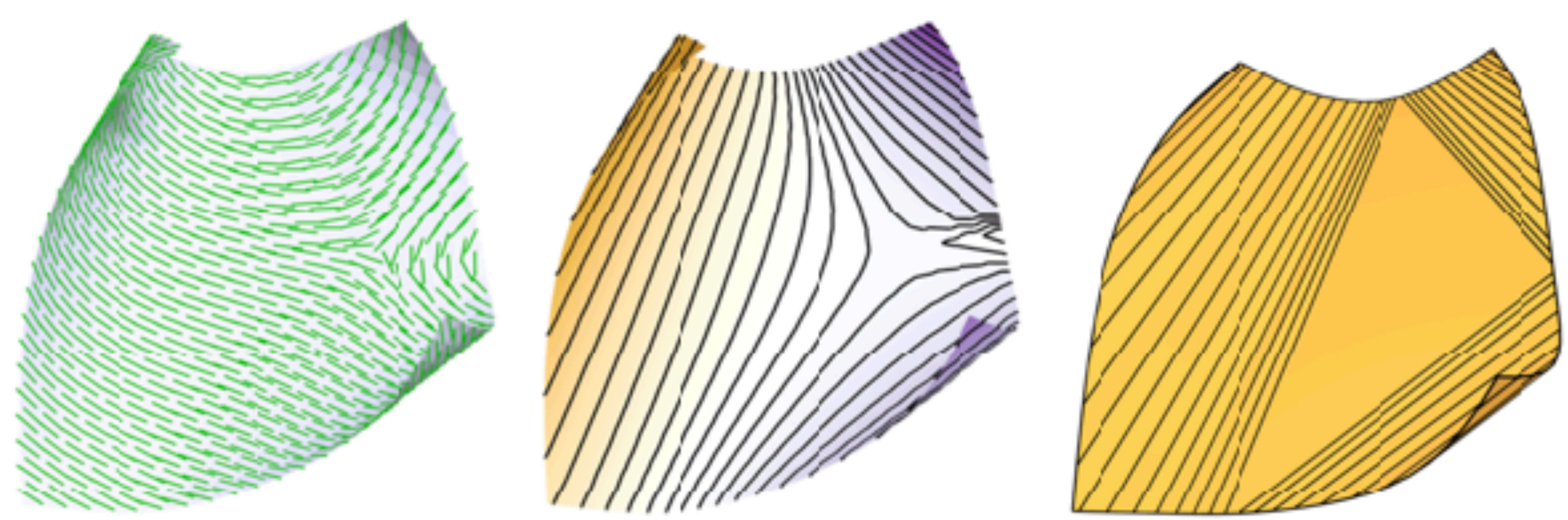


APPLICATIONS

- Meshing with a special function



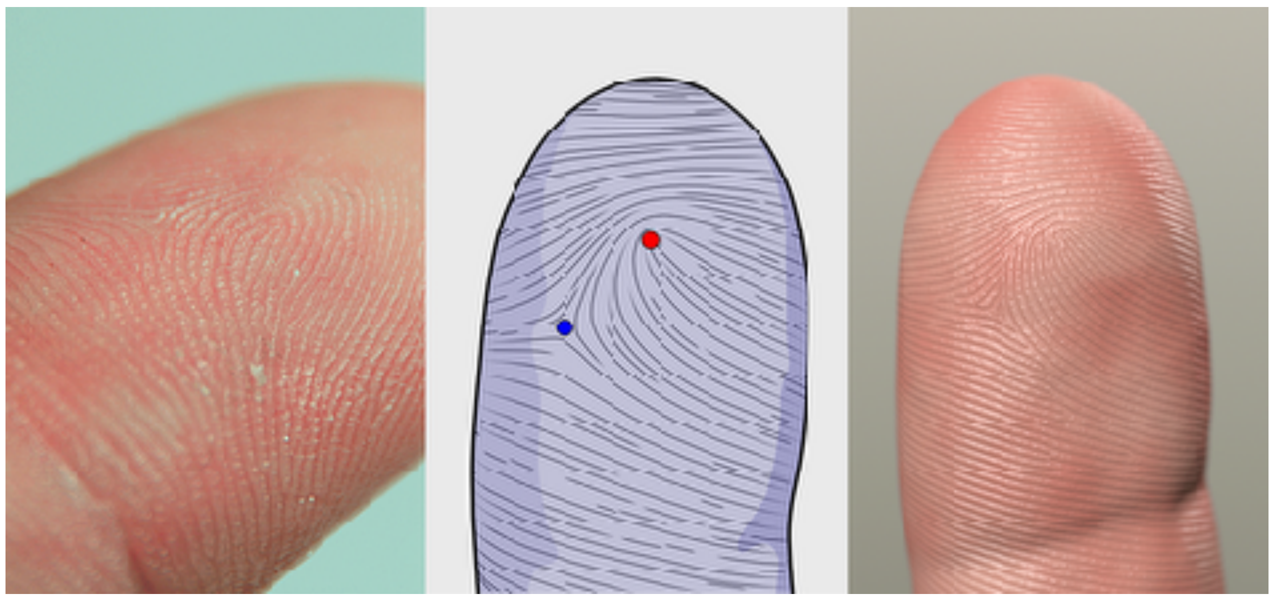
Minimal surfaces [Wang and Chern 2021]



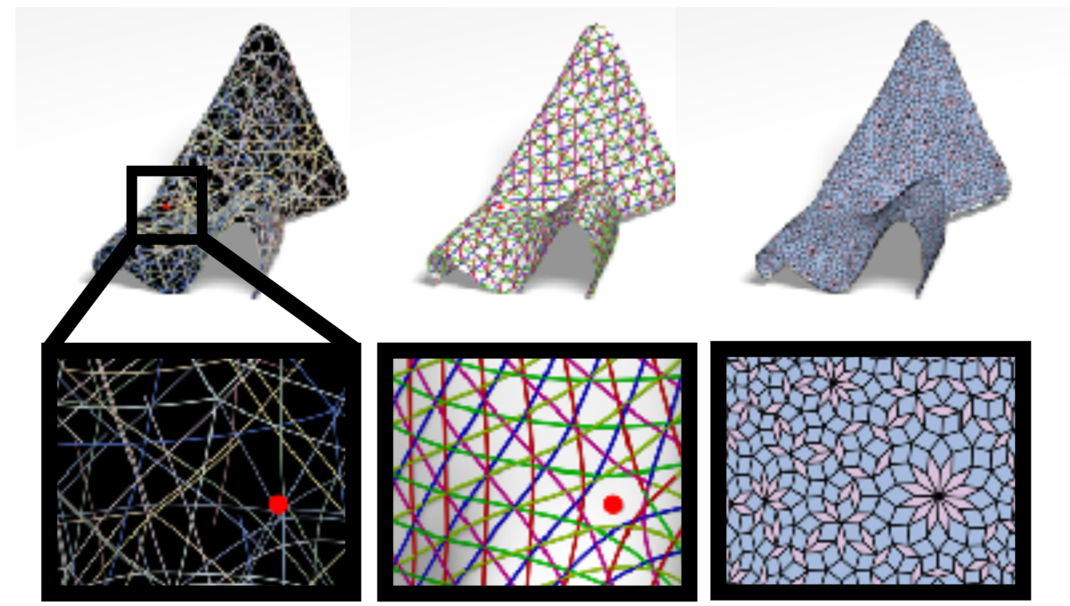
Developability [Verhoeven *et al.* 2021]

APPLICATIONS

- Alternate symmetries



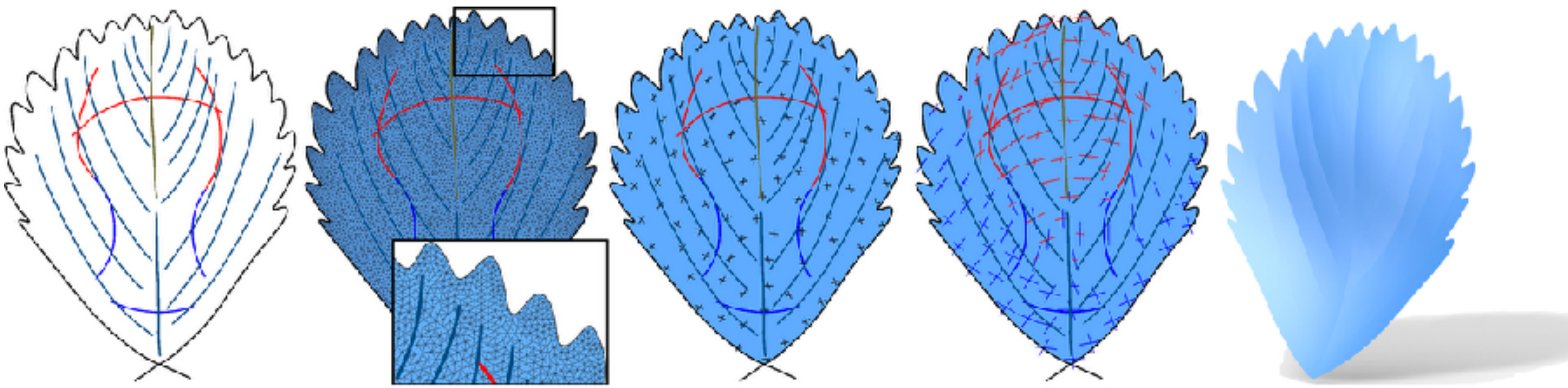
N=2 Stripe patterns [Knöppel *et al.* 2015]



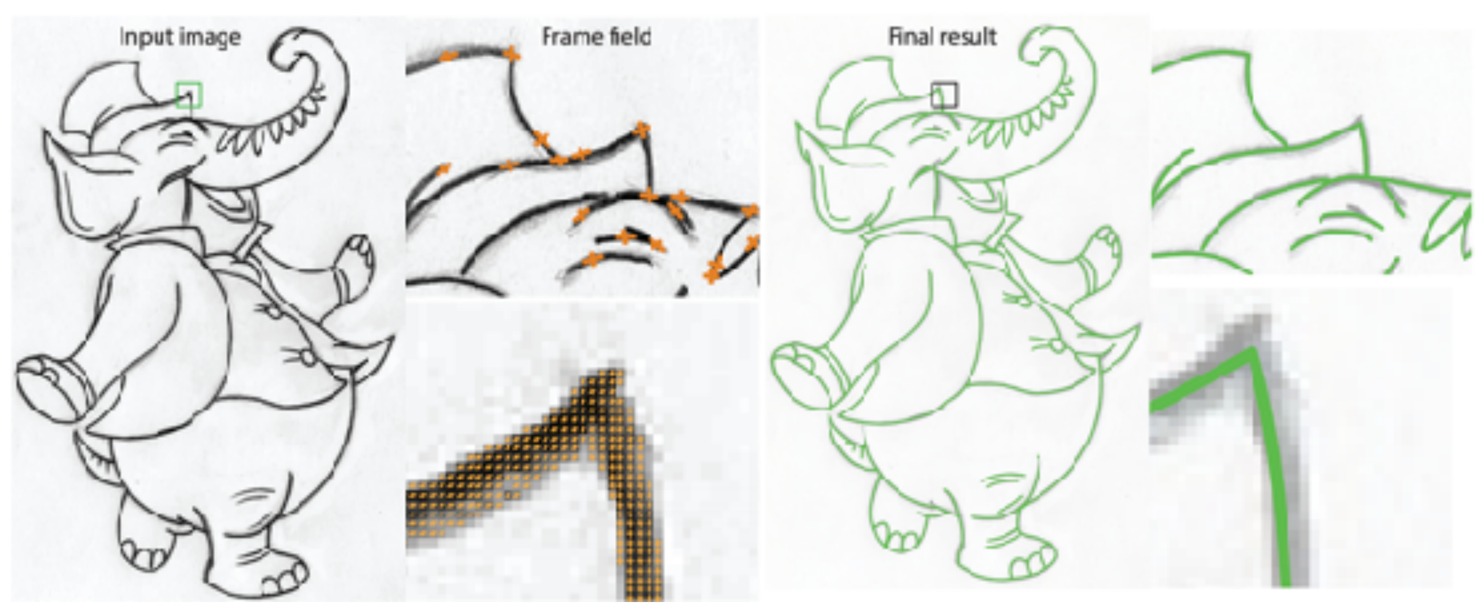
N=5,>7 [Meekes *et al.* 2021]

APPLICATIONS

- Shape from illustration and sketching



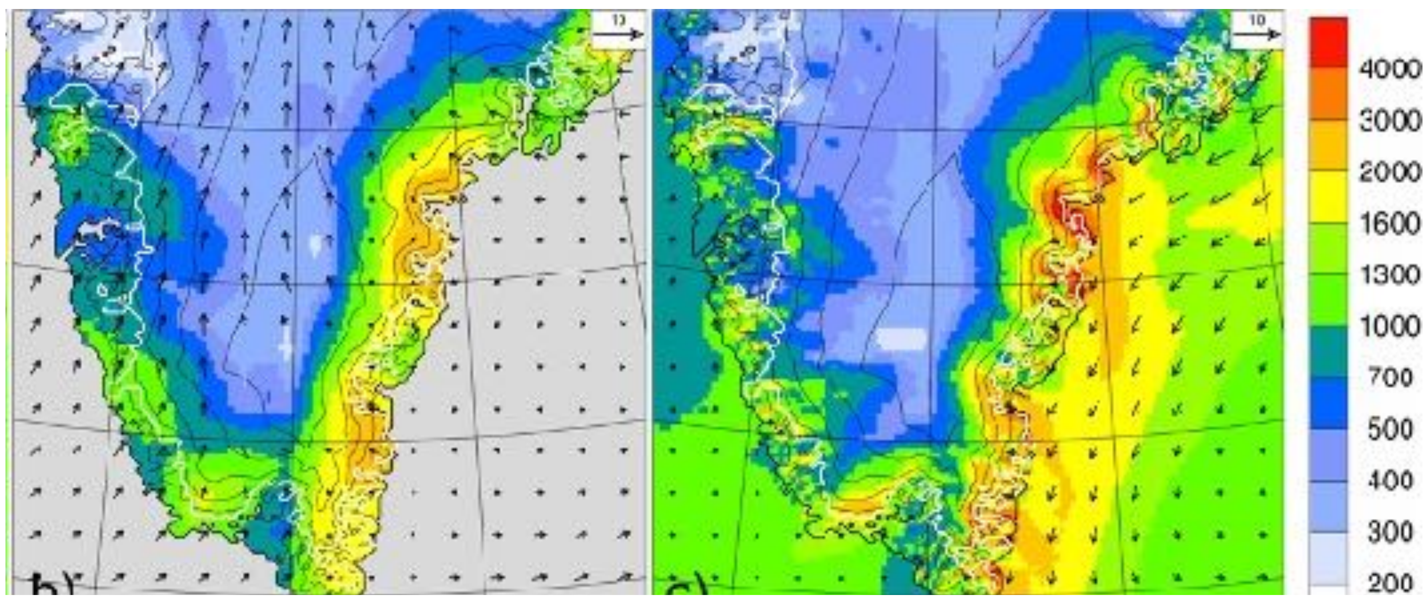
[Li *et al.* 2017]



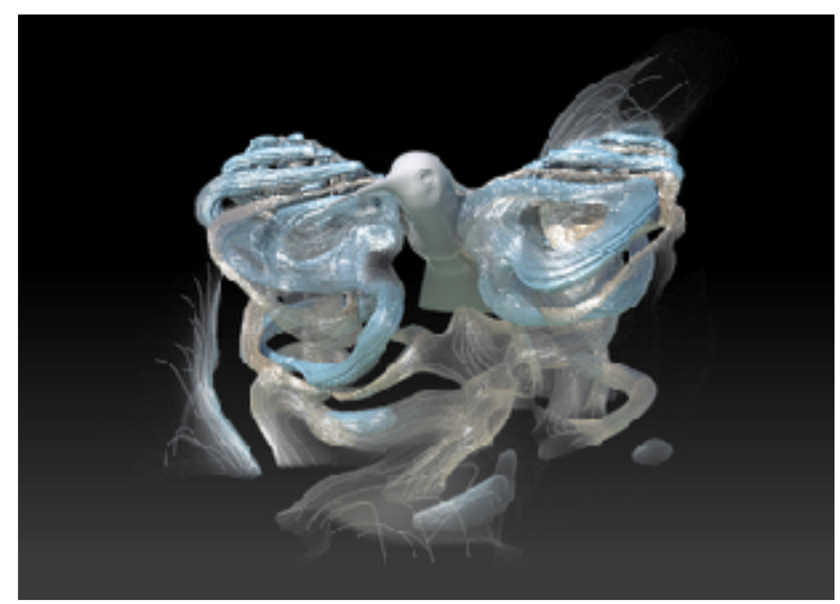
[Bessmeltsev & Solomon 2000]

APPLICATIONS

- Simulation



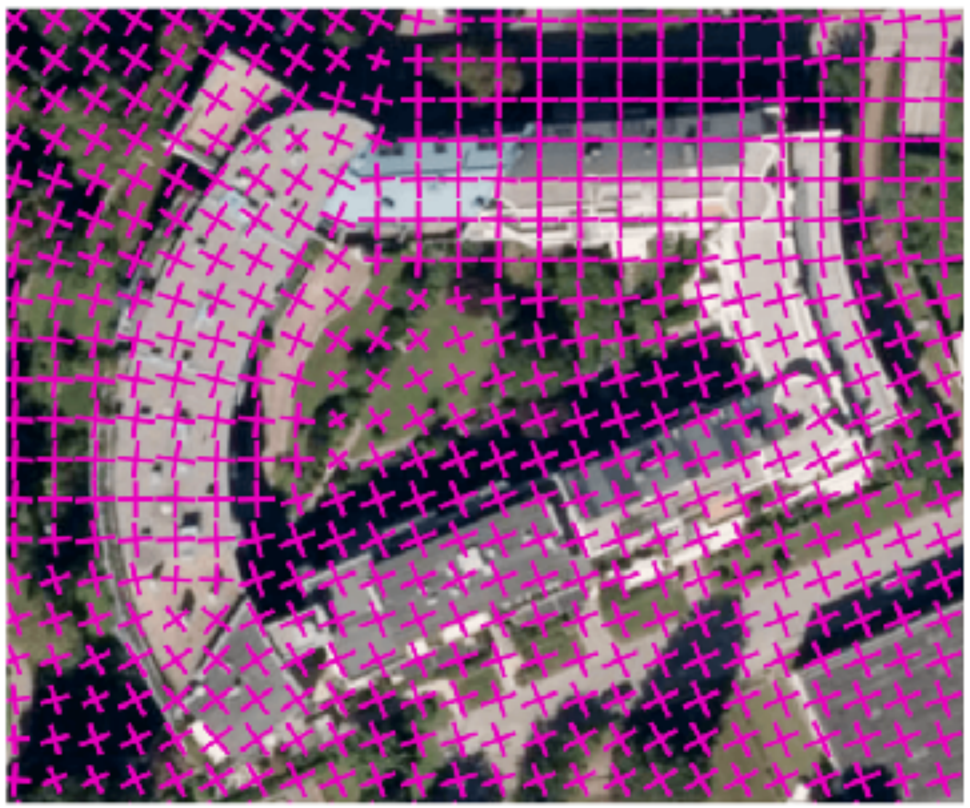
Climate [Noel *et al. 2016*]



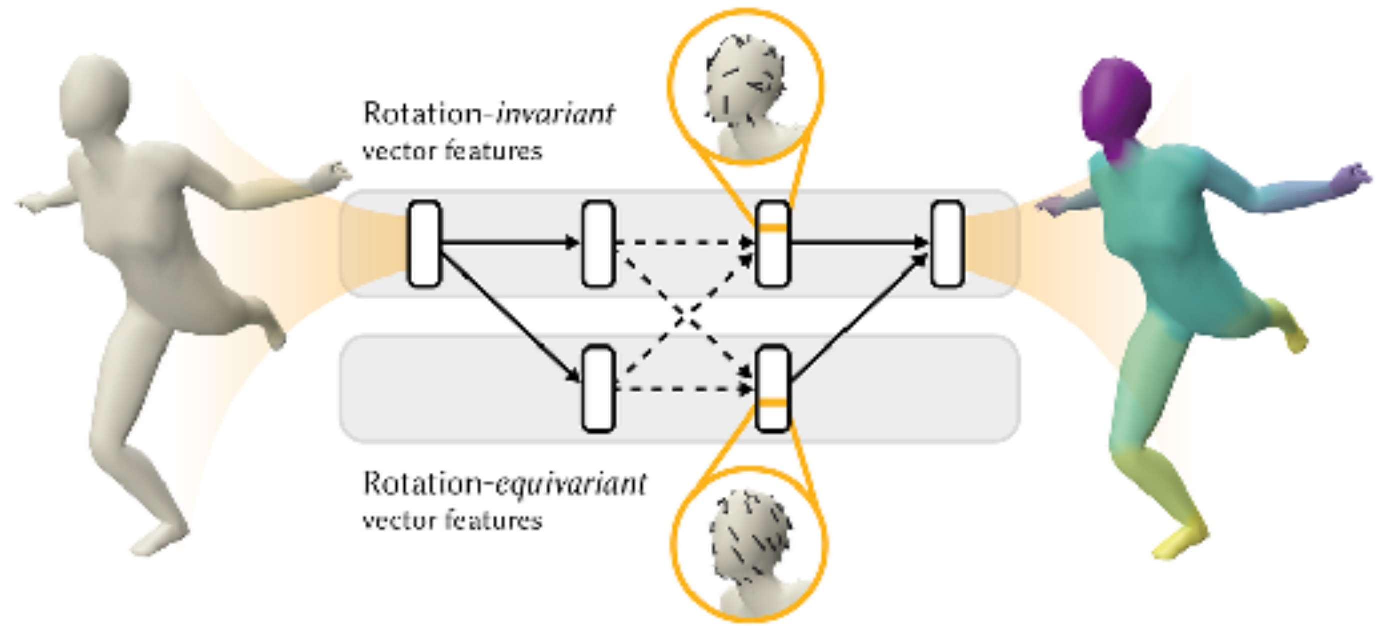
Fluid [Chern *et al.* 2017]

APPLICATIONS

- Learning



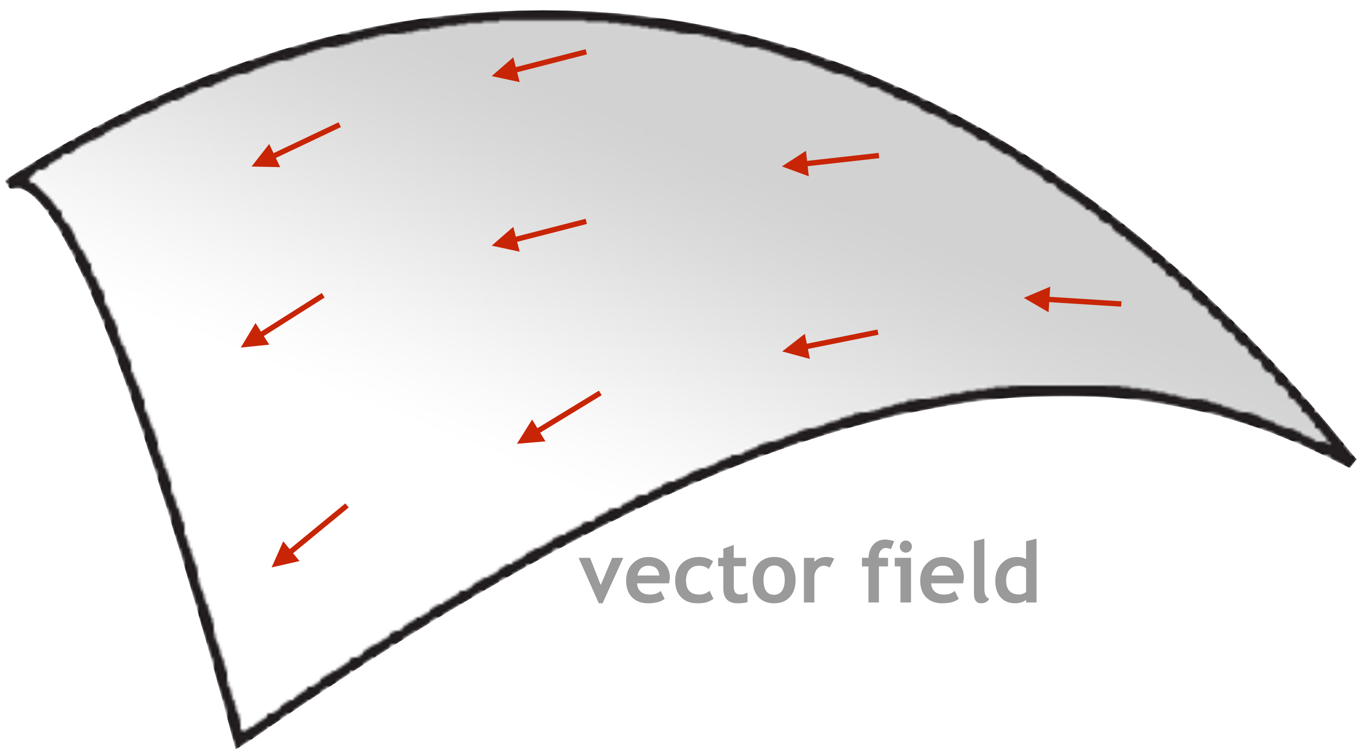
Polygonal building extraction [Girard *et al.* 2021]

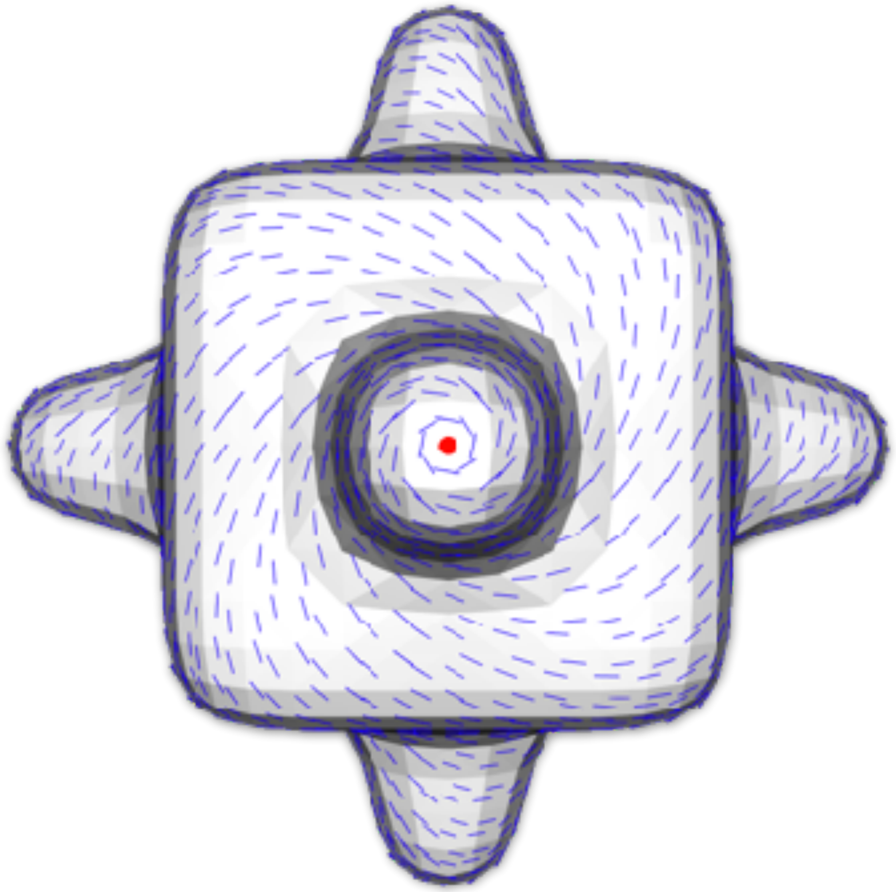


Learning surface features [Wiersma *et al.* 2020]

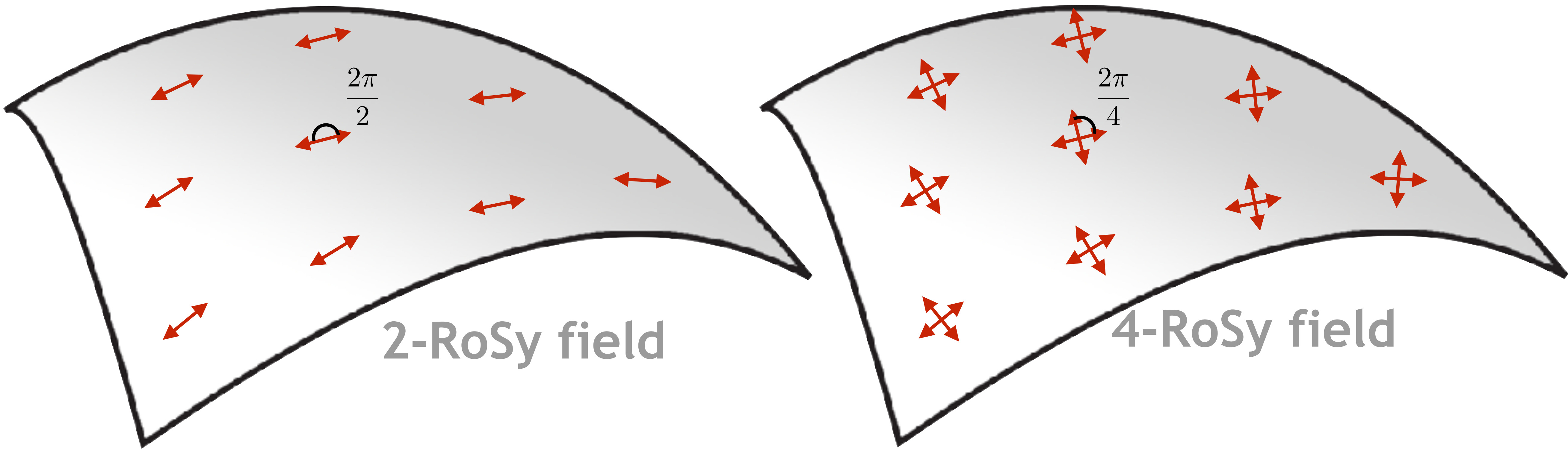
IN THE CONTINUUM

FIELDS ON SURFACES

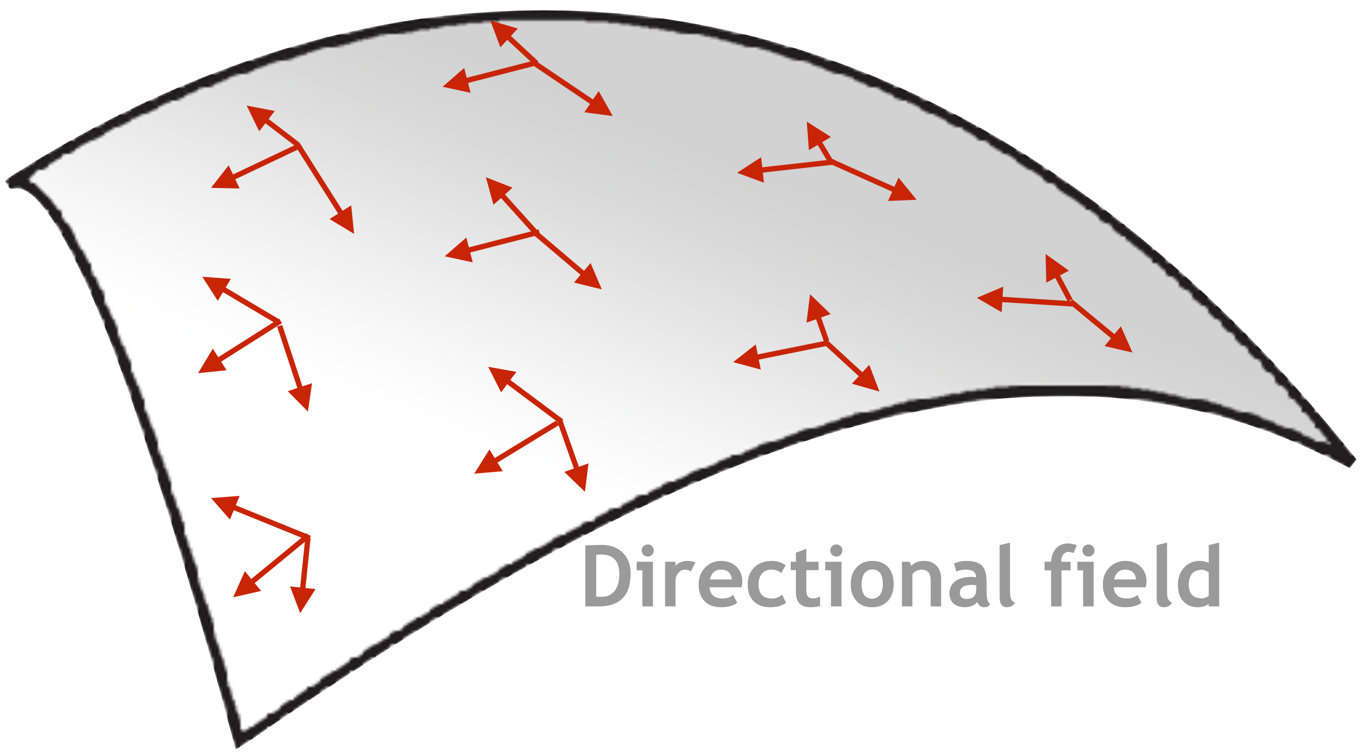


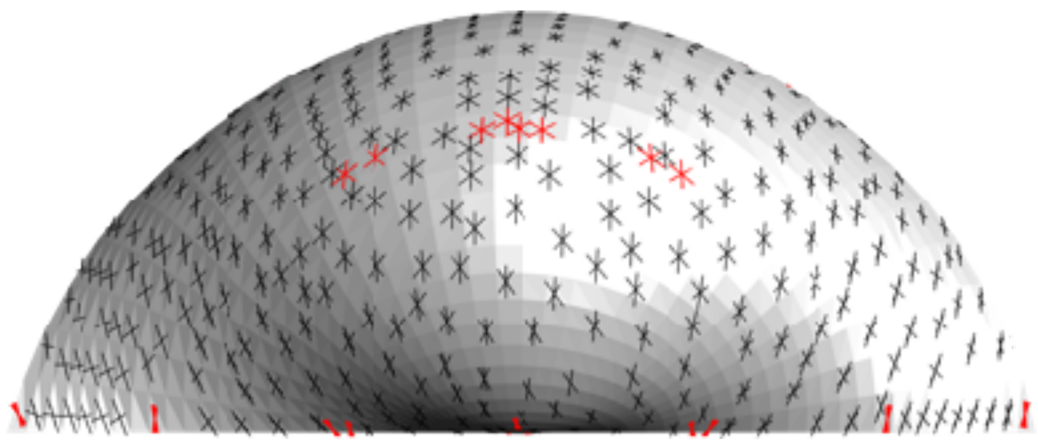


ROTATIONALLY-SYMMETRIC FIELDS



GENERAL, NON-SYMMETRIC FIELDS





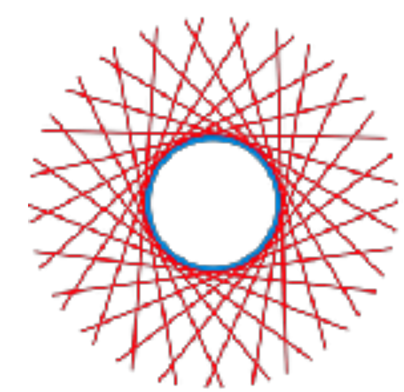
FORMALITIES

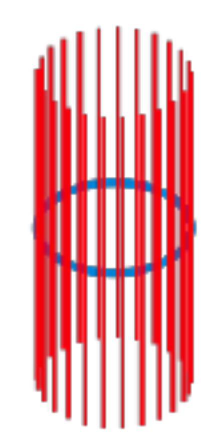
- A tangent N-directional field is a map $V : T\Omega \rightarrow \mathbb{R}^{N\times 3}$

Tangent Bundle

N vectors, each (x,y,z)

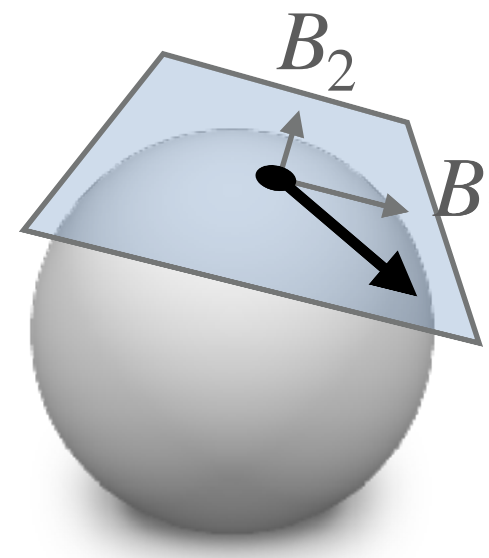
- Tangent $=>$ $N \times 2$ d.o.f. in each point, where $V(p) = \left\{ v_i(p) \,|\, \forall i,\ v_i \perp N(p) \right\}$

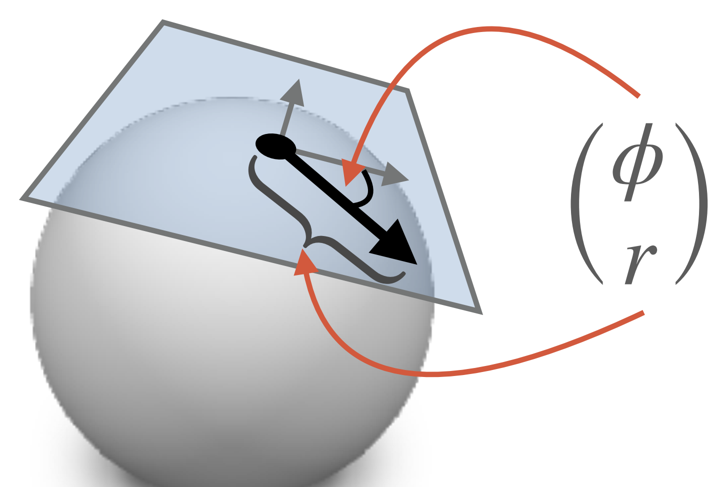




LOCAL COORDINATES

- Admit basis vectors (fields on their own!): $B_1(p), B_2(p)$.
- Then field can be represented in its true d.o.f. in *Cartesian coordinates* $v_i = \left(x, y\right)$.
- Alternatives:
  - complex representation: $z = x + iy$
  - polar representation

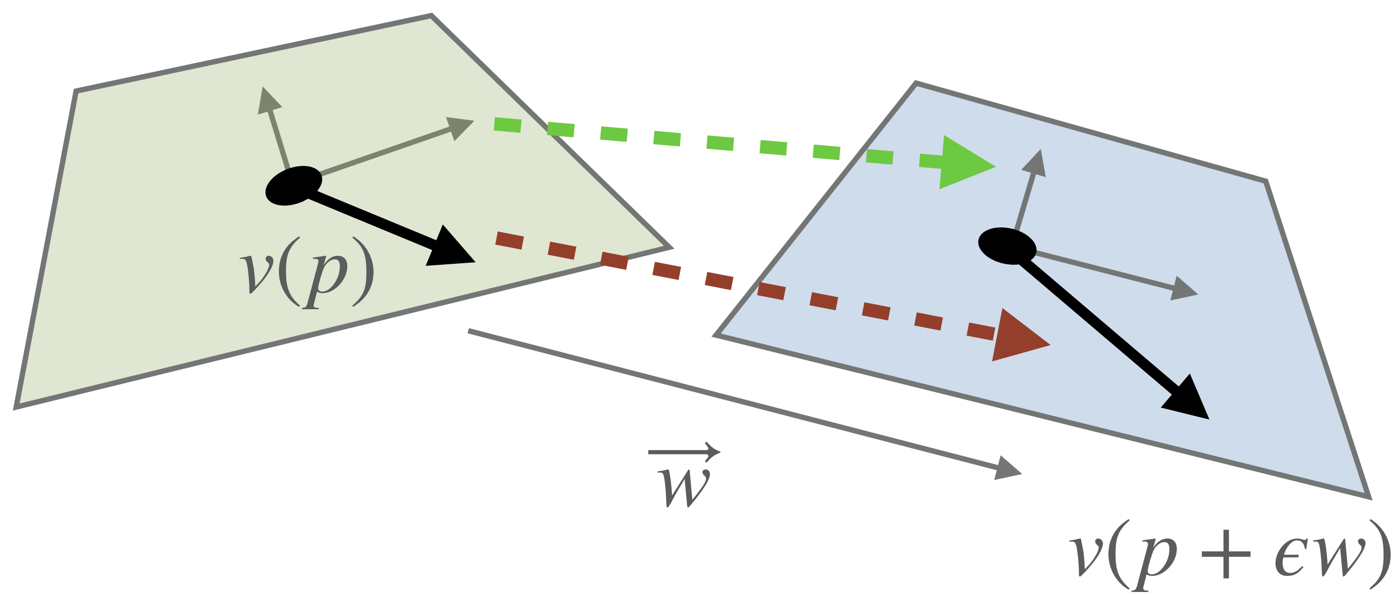




CONNECTION AND PARALLEL TRANSPORT

- How to differentiate/compare a vector in two points on a surface?
  - Take into account the change of basis
- Covariant derivative (assuming affine metric):

$$\nabla_w(v) = \langle \nabla v, w \rangle + Jv$$



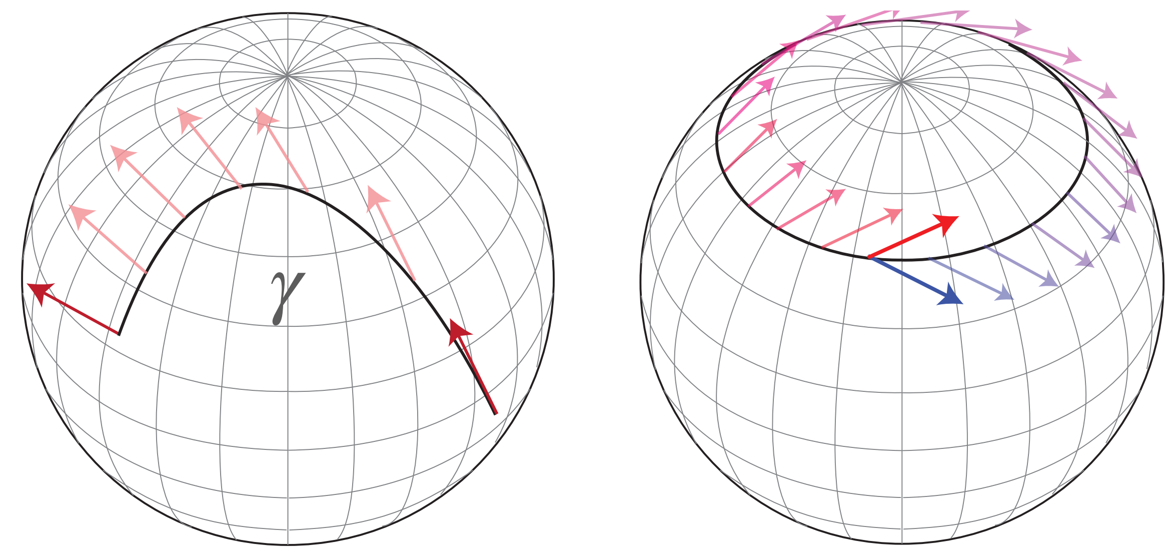
Usual derivative of coordinates

J is “basis angular velocity”

CONNECTION AND PARALLEL TRANSPORT

- Parallel transport: zero cov. derivative along a path:

$$\nabla_{\dot{\gamma}} v = 0$$

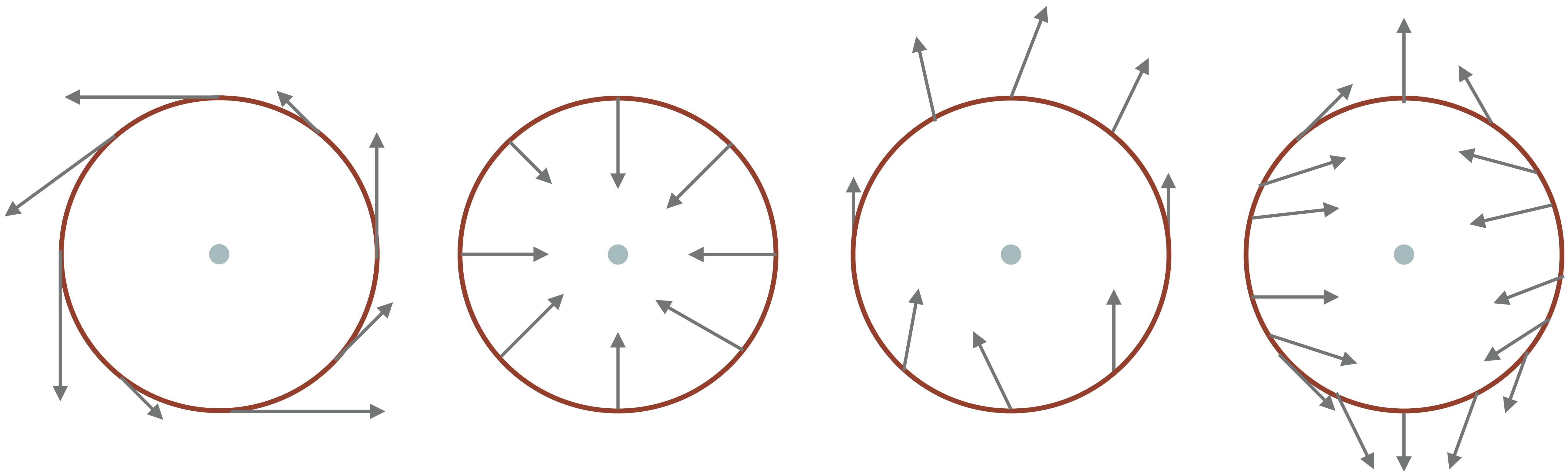


- Holonomy: transport is non-trivial with Gaussian curvature.
- If path is simply connected, then:

$$\phi_{\text{end}} - \phi_{\text{start}} = \iint_{S(\gamma)} K dS \pmod{2\pi}$$

SINGULARITIES

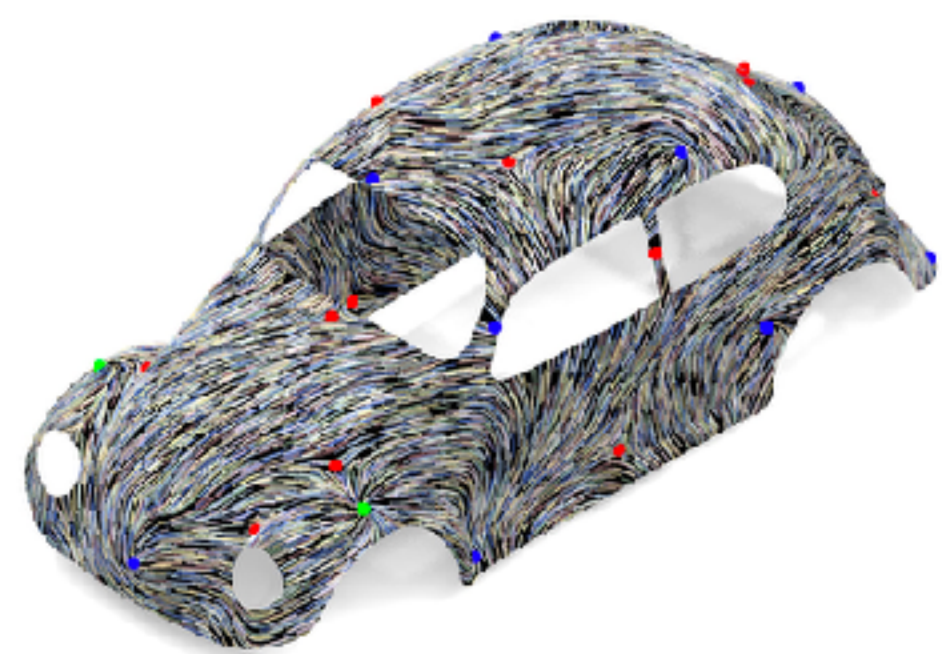
- Index I_p of field $v(p)$: #rotations of field around an infinitesimal cycle
    - Always integer!



$I_p = 1$ $\quad$ $I_p = 1$ $\quad$ $I_p = 0$ $\quad$ $I_p = -1$

SINGULARITIES

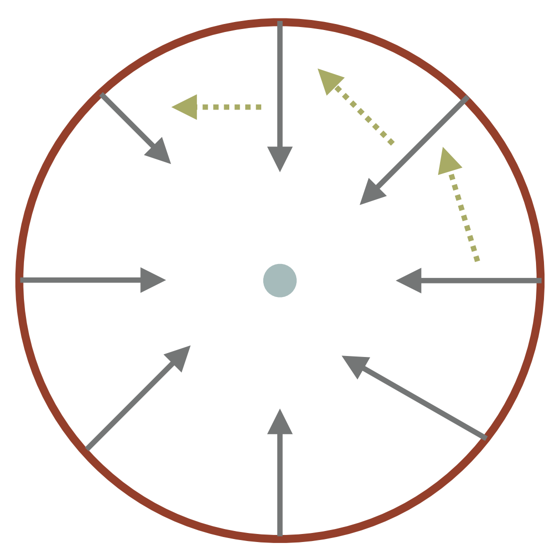
- Index I_p of field $v(p)$: #rotations of field around an infinitesimal cycle
  - Always integer!
- Does not distinguish between source, sink, or vortex.
- Poincare-Hopf Theorem: $\sum I_p = 2\pi\chi = 2\pi(2 - 2g - b)$
  - g - genus of surface
  - b - number of boundary loops



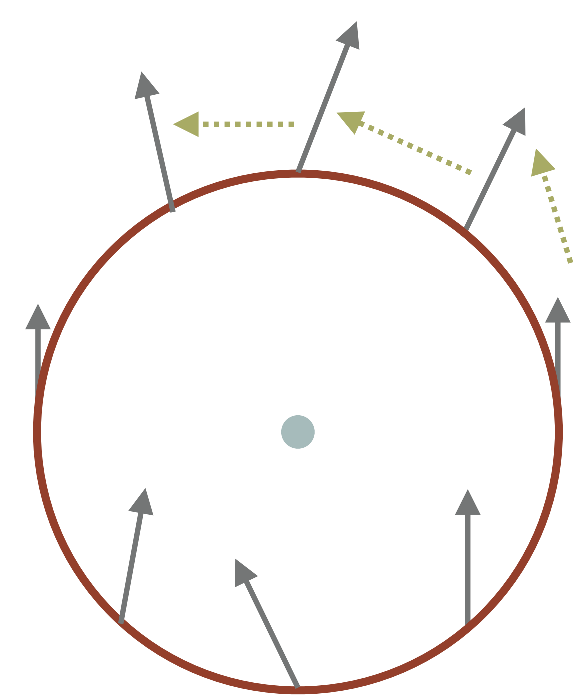
VECTOR FIELD AS A TRIVIAL CONNECTION

- The connection measures how much neighboring fields are parallel
- Reproduces Gaussian curvature around cycles
- We *define* a new “parallelity” as the already-given change induced by a field
- *Induced* curvature $= 2\pi \times$ field index
- Total curvature:

$$\int K dS = \int K_p dS = 2\pi\chi$$



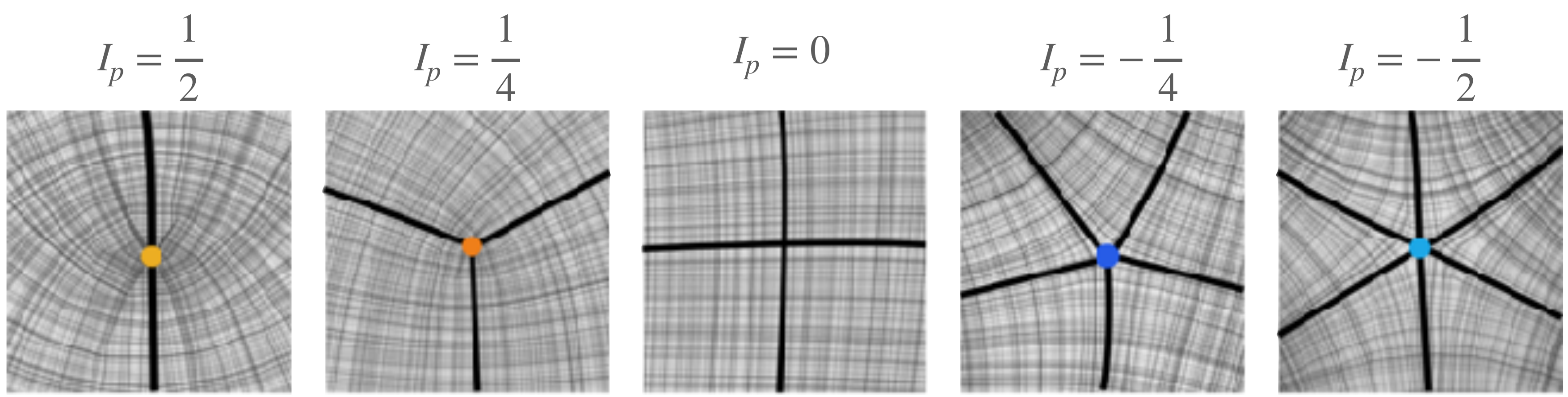
$I_p = 1,\ K_p = 2\pi$

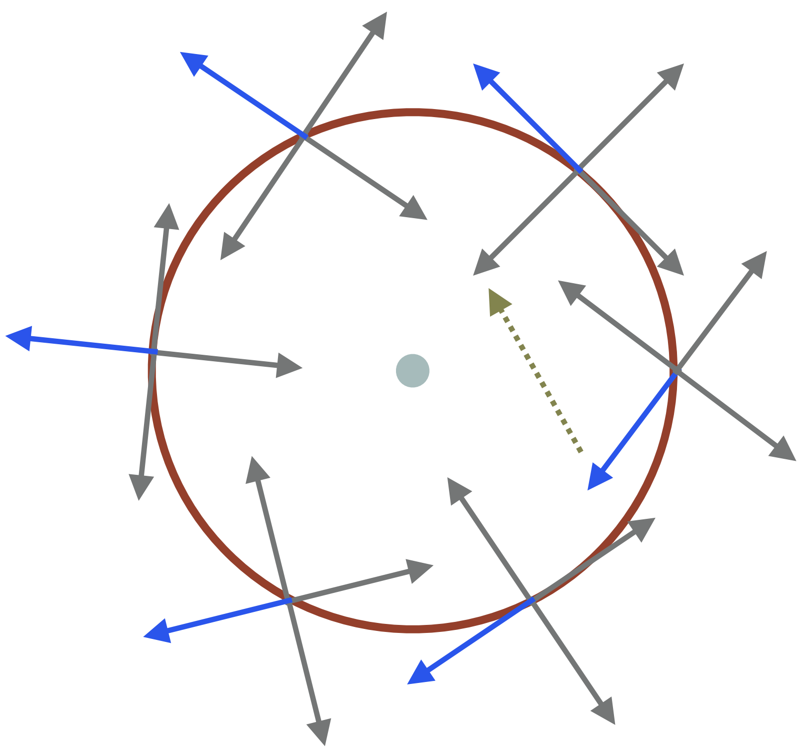


$I_p = 0,\ K_p = 0$

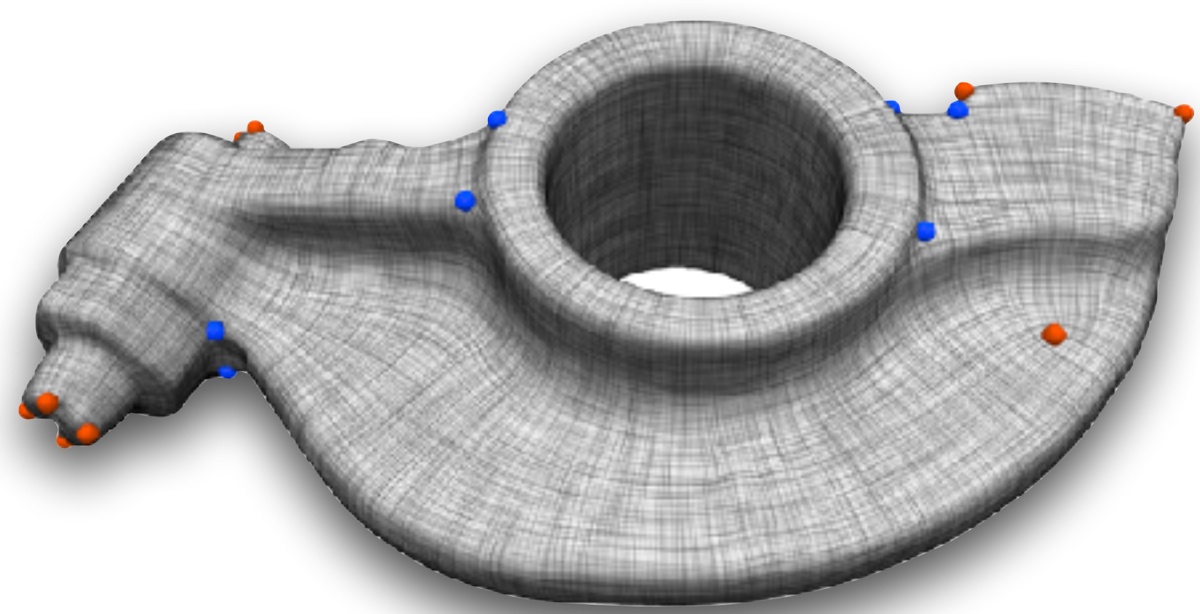
N-FIELD SINGULARITIES

- The same definition, but index can be in multiples of $\frac{1}{N}$.
- Interpretation: field “returns to itself” but with a different *matching* of vectors.





$$I_p = \frac{1}{4},\ K_p = \frac{2\pi}{4}$$

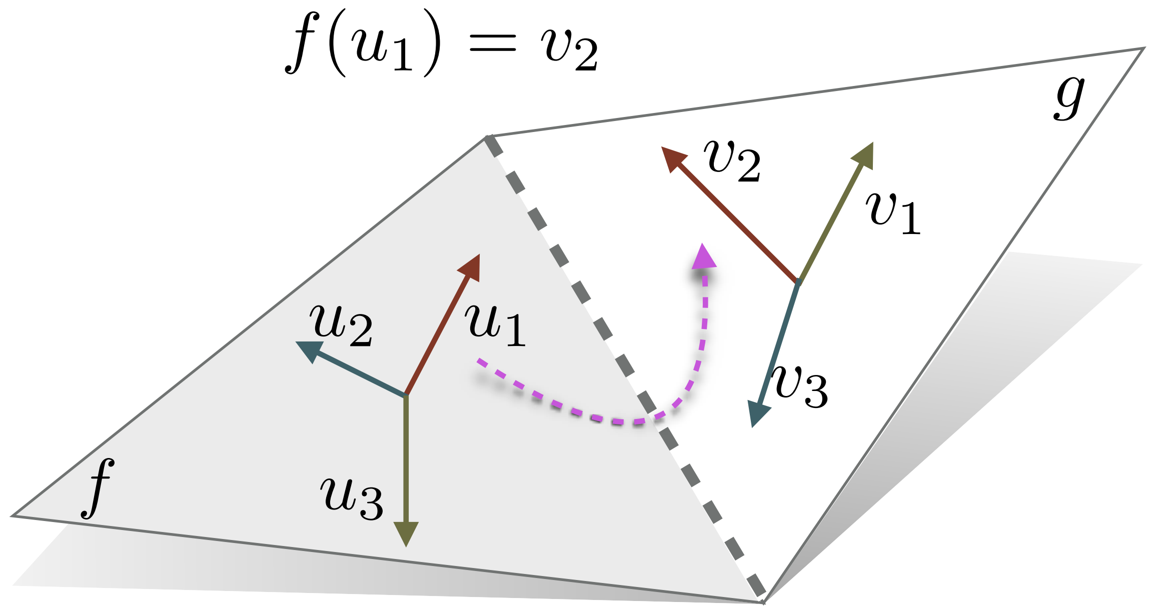


COVECTORS (1-FORMS)

- A linear functional $\eta : T\Omega \to \mathbb{R}$.
  - Taking vectors into scalar.
  - In coordinates: $\eta_x dx + \eta_y dy$
- Given a metric, we can dualize to the vector $\left(\eta_x, \eta_y\right)$ so that the map is implemented as:

$$\eta(v \in T_p\Omega) = \eta_x v_x + \eta_y v_y$$

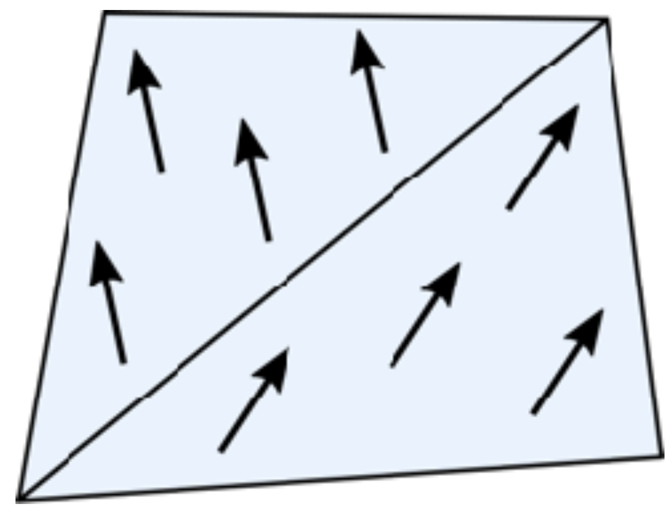
- Differential: $df(v) = \langle \nabla f, v \rangle$
  - Identifying a field with an operator on functions!



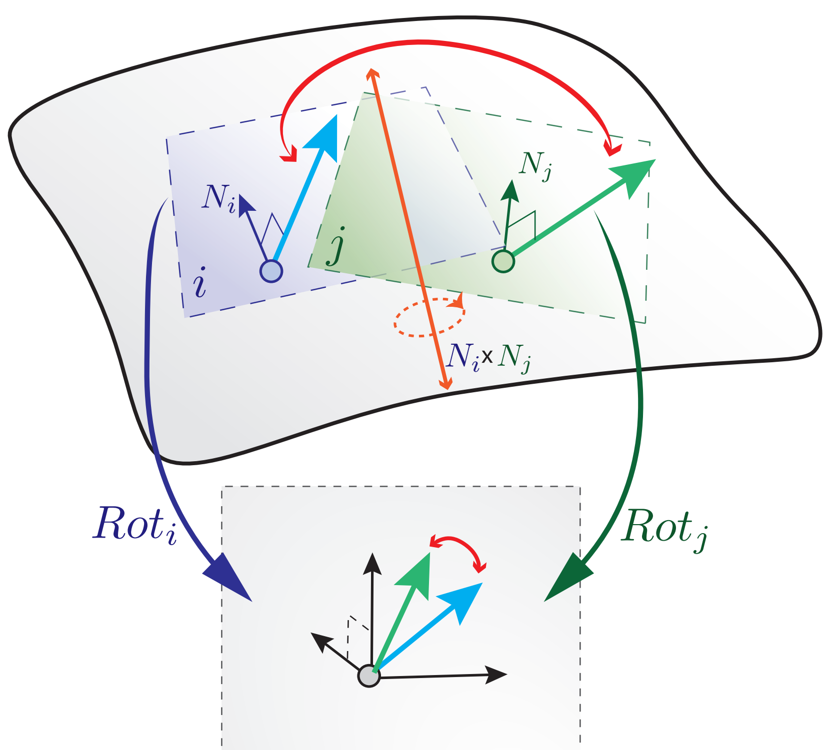
DISCRETIZATION

CHALLENGES IN THE DISCRETE SETTING

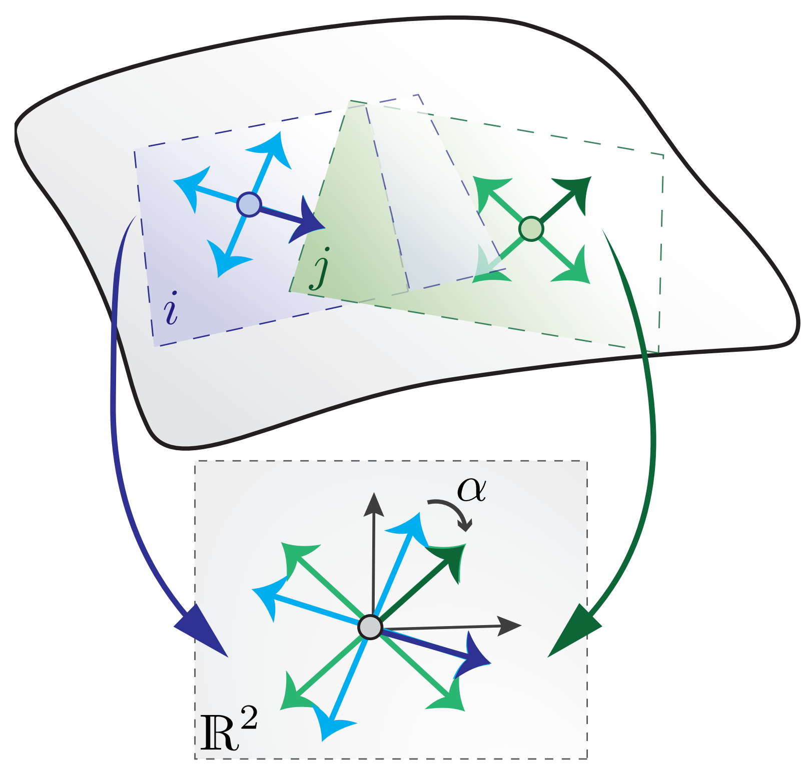
Discontinuity



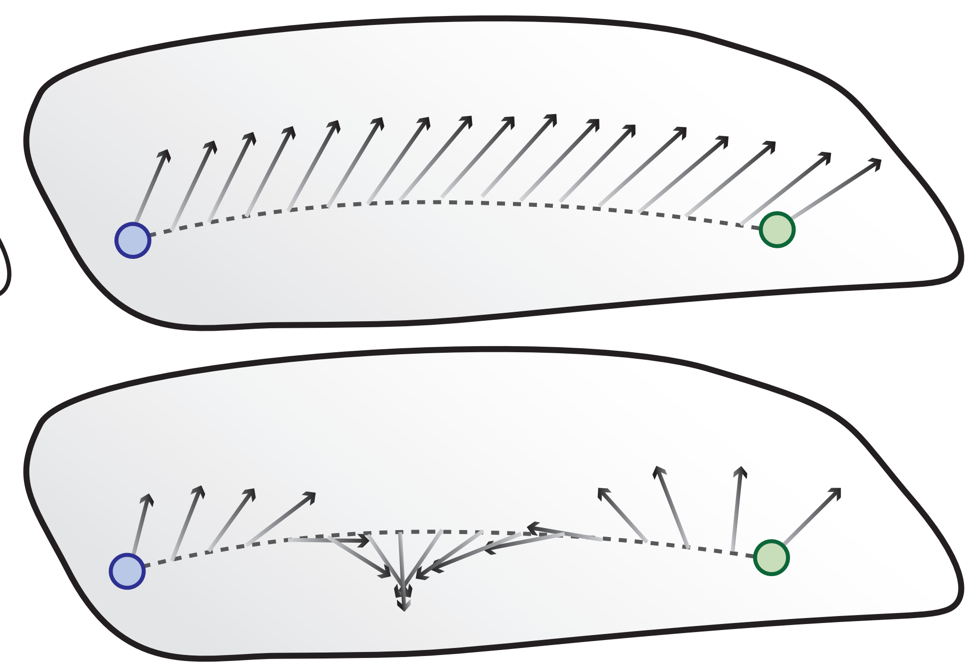
Connection



Matching

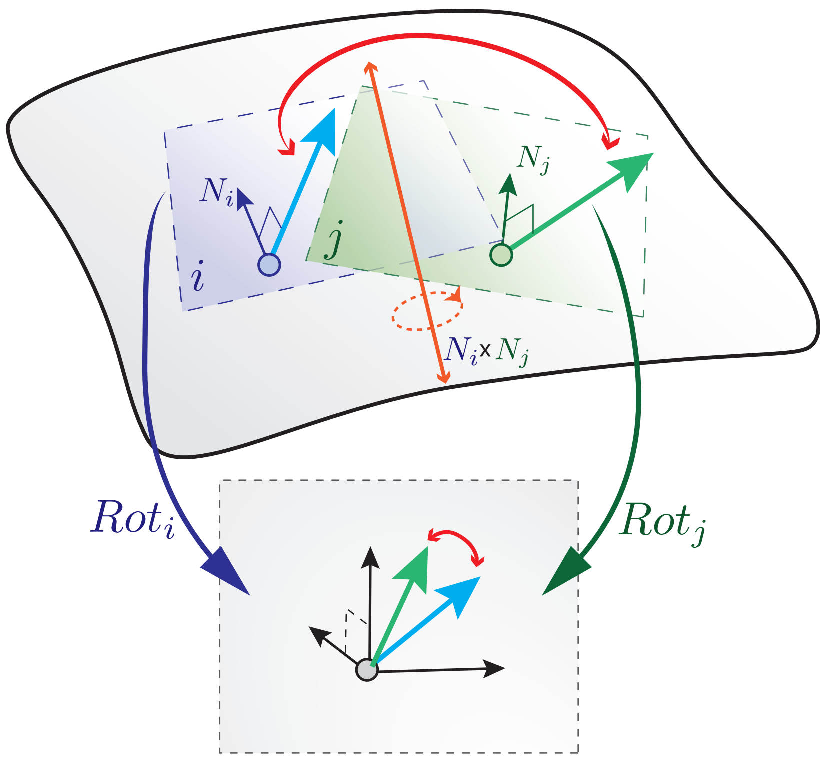


Interpolation



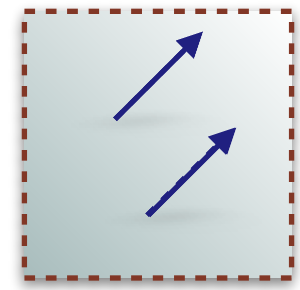
DISCRETE CONNECTION

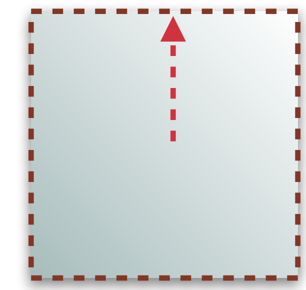
- Bijective linear map between adjacent tangent spaces.
- Popular choice: flattening + single axis system.
- Alternatively: align the representation of a mutual vector.



DISCRETE TOPOLOGY: ROTATION

- What happens in between?
- Valid rotation choices:

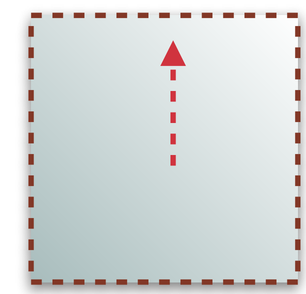




$$\theta_{ij} = \frac{\pi}{4} + 2\pi k, \; k \in \mathbb{Z}$$

Principal Rotation

Period Jump

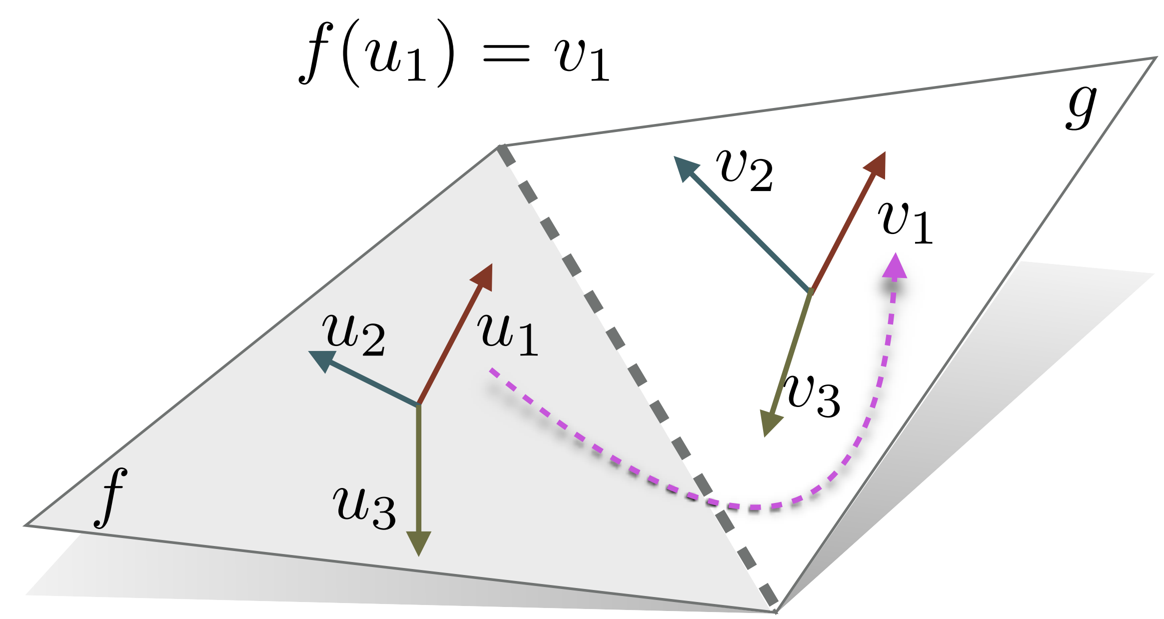


[Li *et al.* 2006]

- Implicit/Cartesian: can only assume principal.
- Explicit/polar/angle-based: period given.

MATCHING

- Which direction to which other?
- Reduction: order-preserving. [Diamanti *et al.* 2014]
- *N*-directional: *N* possible choices.
  - How best to choose?



MATCHING

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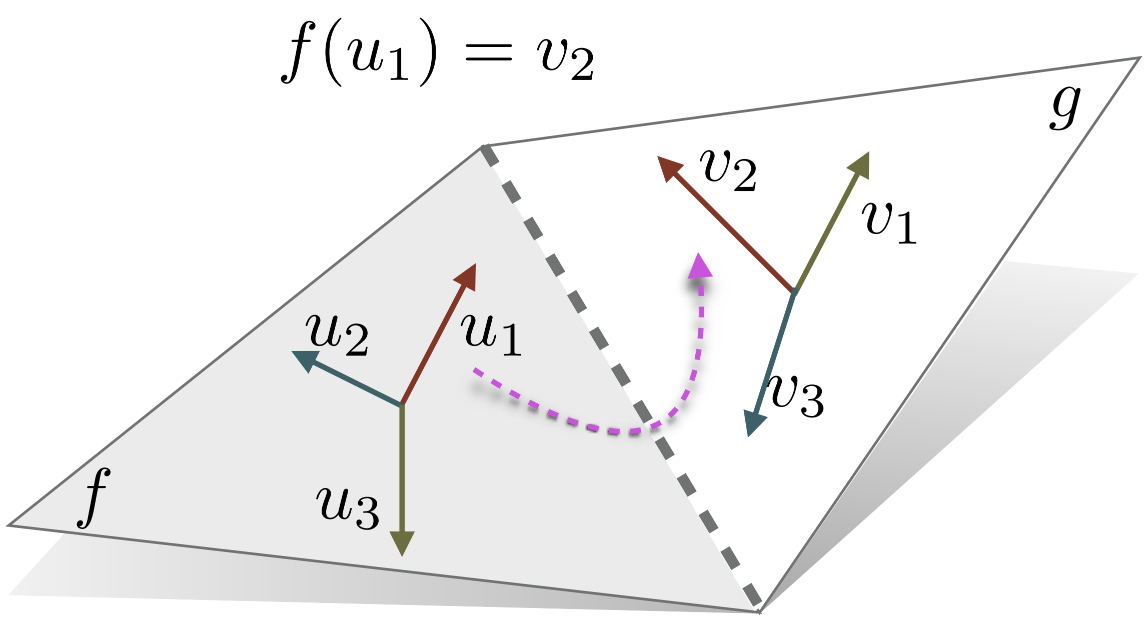
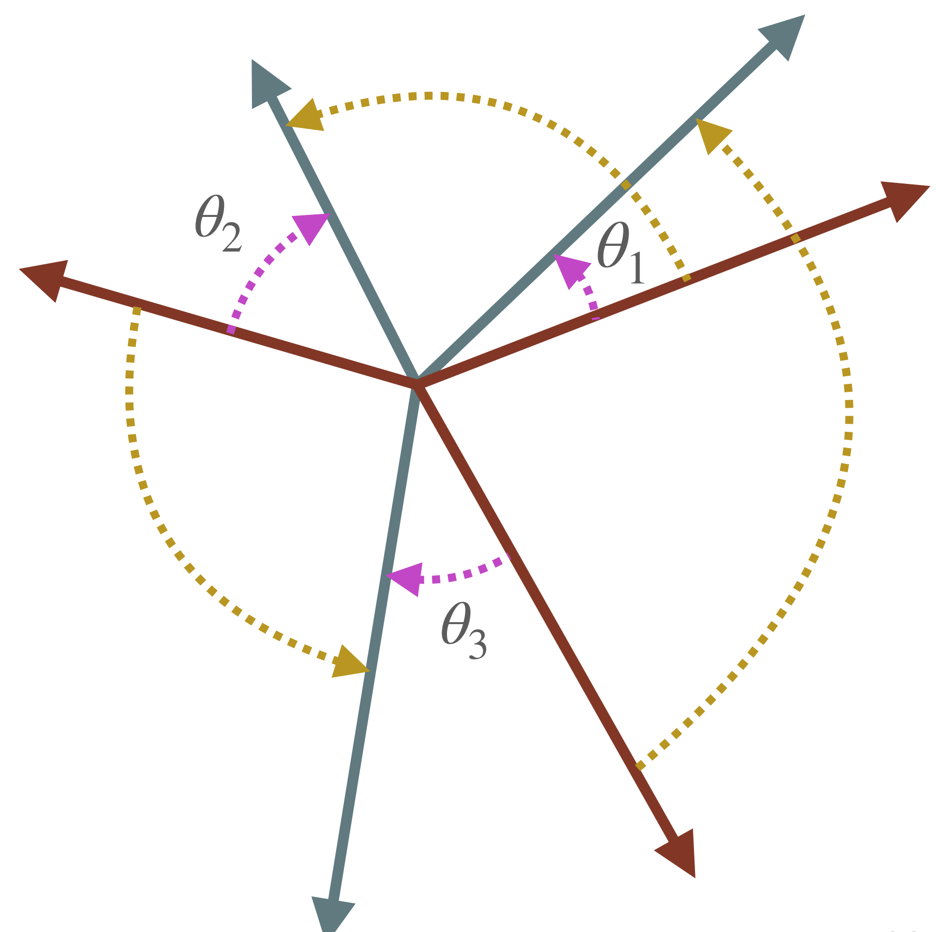


EFFORT

- Sum of matching rotations.
  - Intuition: generalize "closest angle" to "minimum effort".
- All order-preserving matchings differ by $2\pi k, k \in \mathbb{Z}$
- Principal matching: the matching with $\Theta \in [-\pi, \pi)$

$$\Theta_1 = \theta_1 + \theta_2 + \theta_3$$

$$\Theta_2 = \theta_1 + \theta_2 + \theta_3 + 2\pi$$

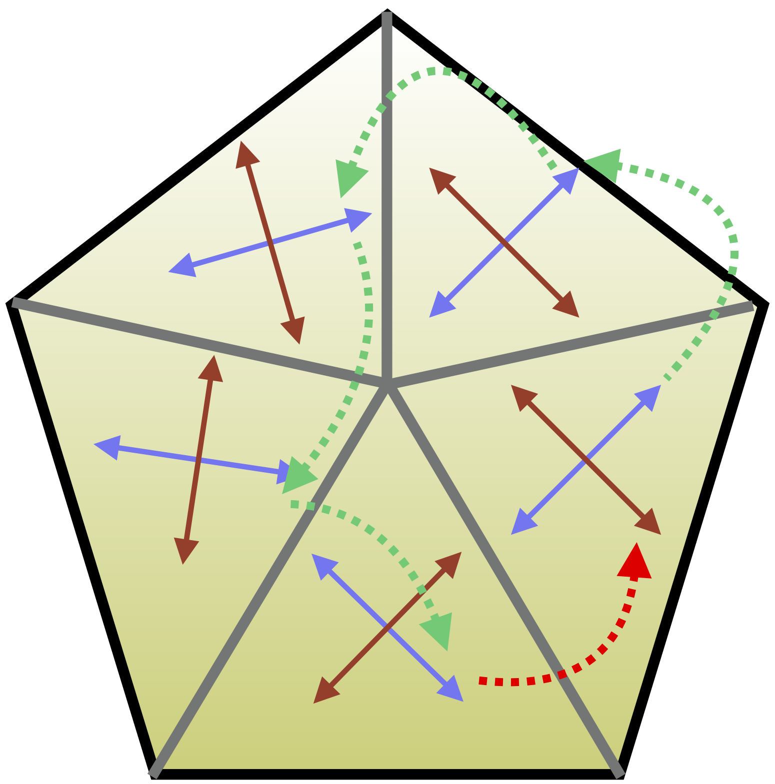


SINGULARITIES

- Around a matching cycle, a directional set returns to itself.
- Up to a different matching!

- Directional field as trivial connections. [Crane *et al.* 2010]
  - Induced Curvature: $\frac{2\pi}{k}$

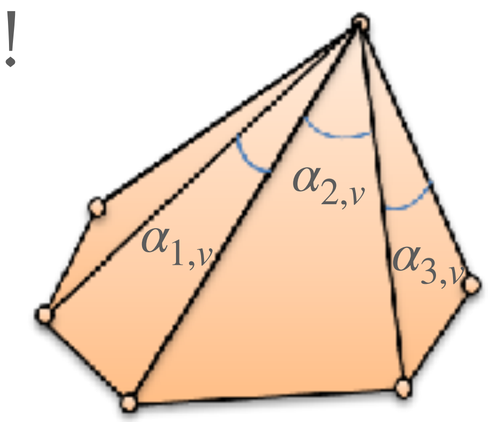
- Regular cycles: index 0
- Sum of indices: $2 - 2g$

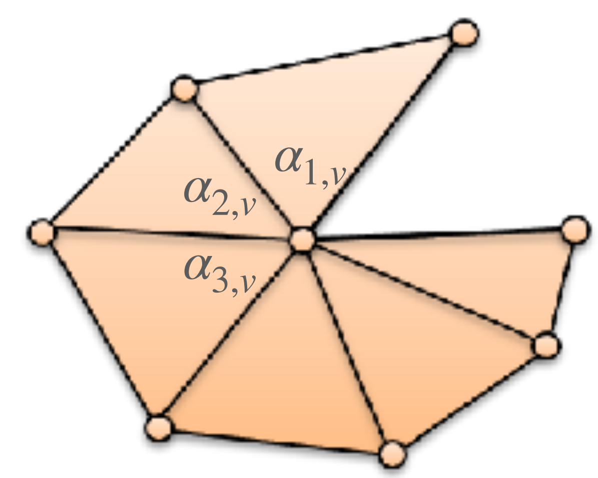


Index of singularity: $\frac{1}{k}$

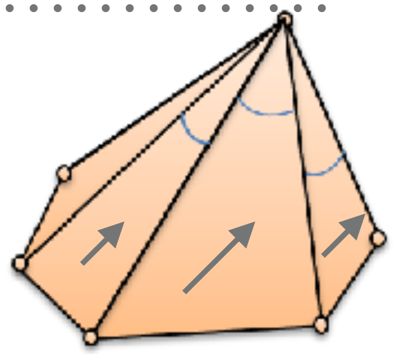
DISCRETE GAUSSIAN AND FIELD CURVATURE

- Discrete Gaussian curvature: angle defect:
  $$K(v) = 2\pi - \sum_{f \in N(v)} \alpha_{f,v}$$
- Discrete Gaussian curvature preserves total curvature!
  $$\int K dS = \sum_{v} K(v) =$$

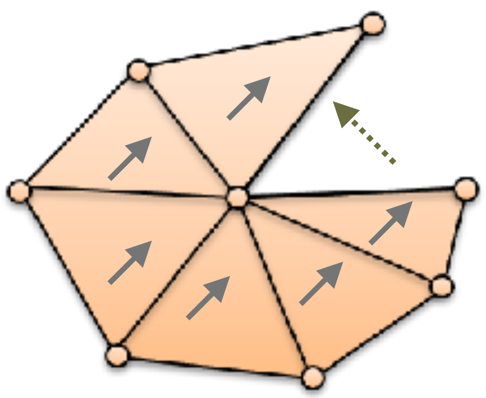




DISCRETE GAUSSIAN AND FIELD CURVATURE

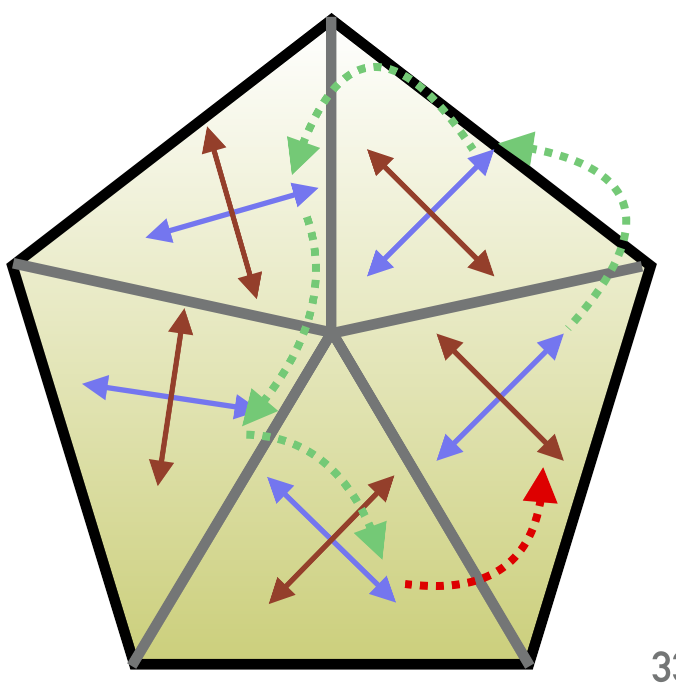


- Discrete parallel transport reproduces DGC.
    - Around a cycle, a parallel vector will have rotated exactly $K(v)$!



- For a given field, the rotation angle $\theta(e)$ cancels this effect:

$$\sum_{e \in N(v)} \theta(e) = 2\pi I(v) - K(v)$$

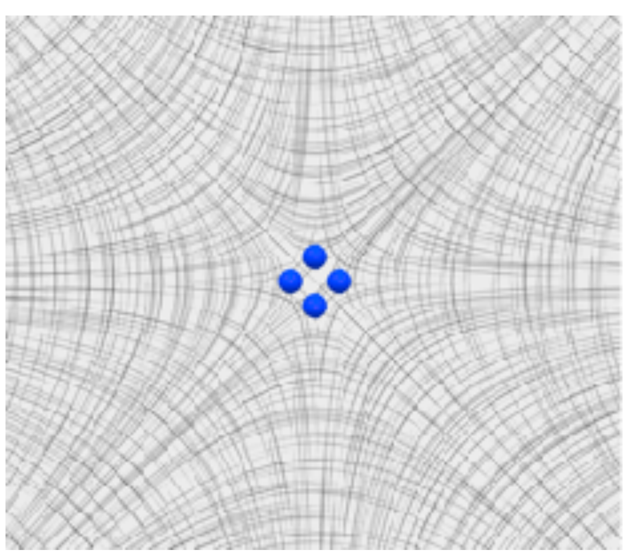


- Note: cycle is oriented wrt. vertex!
- This cancellation is the "trivial" in trivial connection.

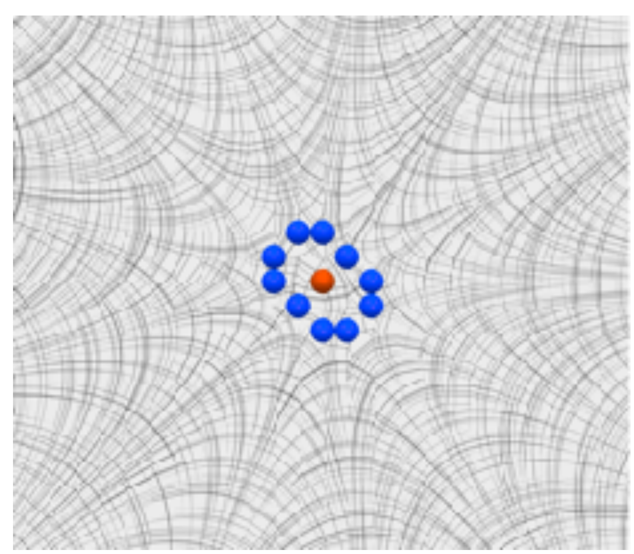
SAMPLING PROBLEM

- Implicit field: principal matching assumed.
- Low valence cycles: limited rotation sums.
- Higher order singularities cannot be represented!
- In practice: promoting low-degree singularity cycles ("singularity party").

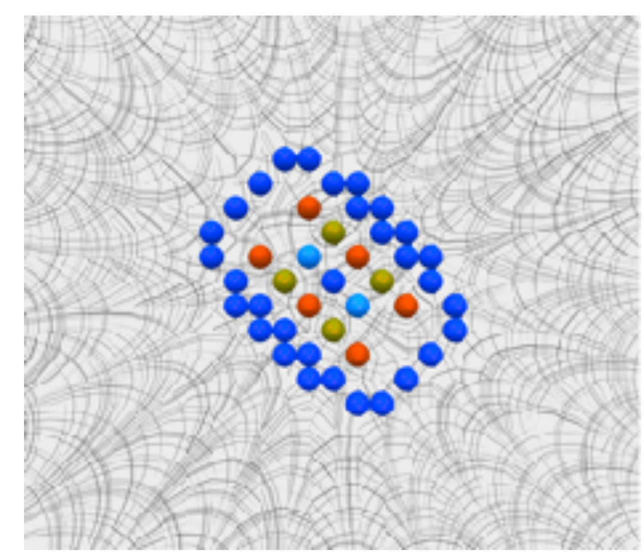
$$-\frac{4}{4}$$



$$-\frac{9}{4}$$

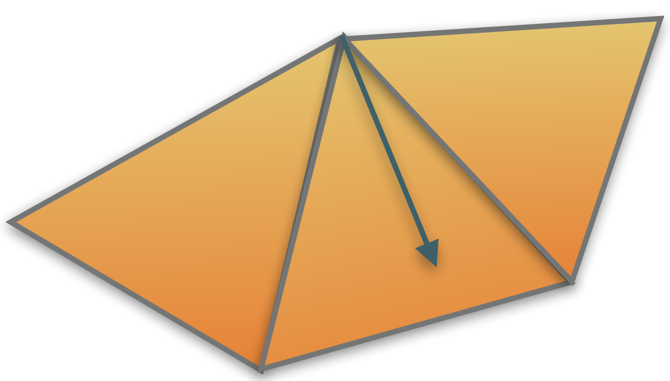


$$-\frac{21}{4}$$



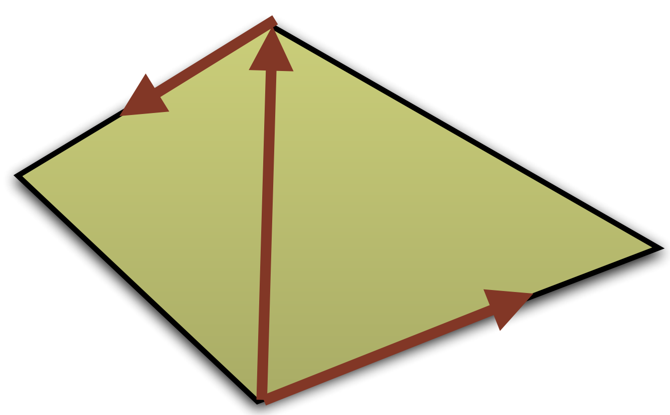
TANGENT SPACES

Vertex based



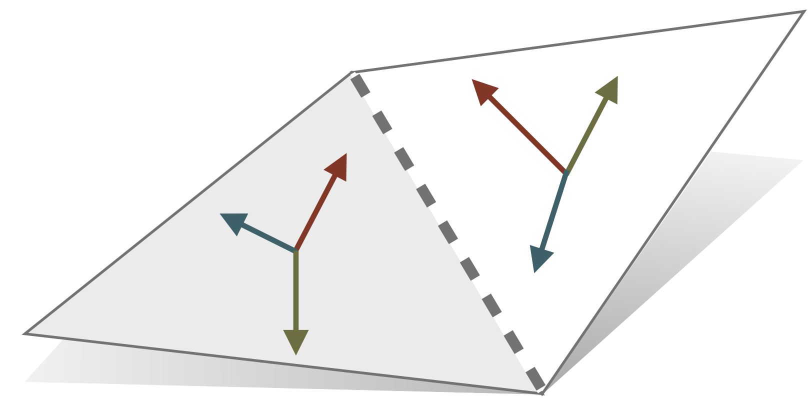
[Polthier and Schmies 98]
[Zhang *et al.* 2006]
[Knöppel *et al.* 2013]

Edge based



[Desbrun *et al.* 2005]
[Fisher *et al.* 2007]
[Ben-Chen *et al.* 2010]

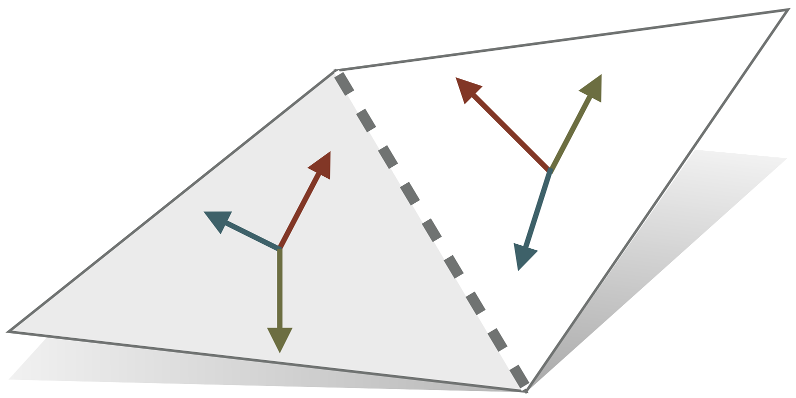
Face based



[Bommes *et al.* 2009]
[Crane *et al.* 2010]
[Diamanti *et al.* 2014]

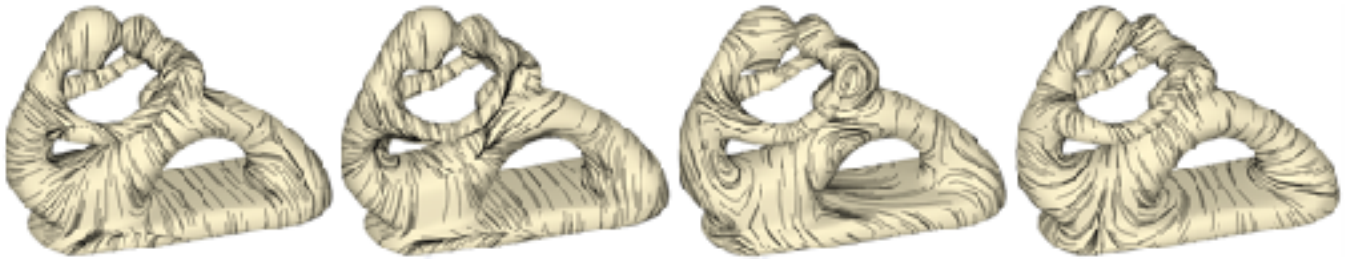
FACE-BASED FIELDS

- Natural normal => natural discrete tangent spaces
- Usually piecewise constant
- Connection: representation of mutual edge
- Discontinuous at edges and vertices
- Structure-preserving:



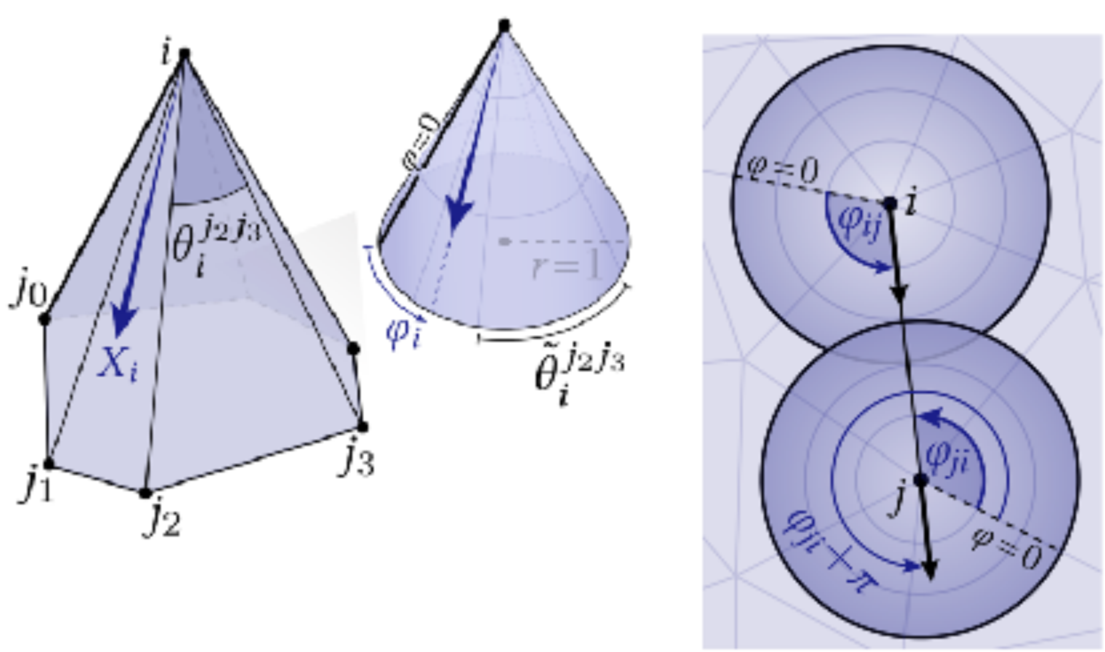
$$v = \nabla f_V + J \nabla g_E + H$$

$$2|F| \qquad |V| - 1 \qquad |E| - 1 \qquad 2g$$

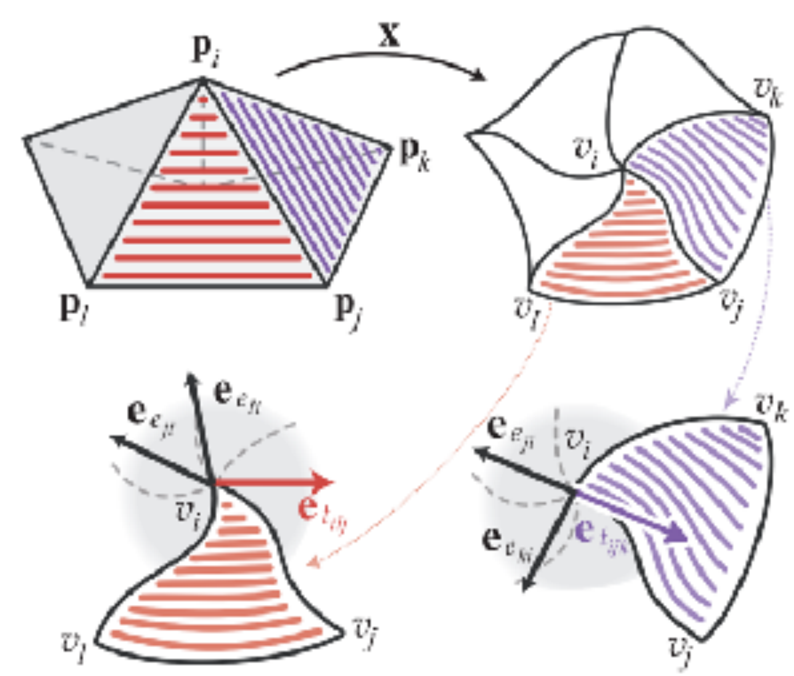


VERTEX-BASED FIELDS

- Common flavor: parameterize vertex tangent space
  - Effectively “pushing” curvature inside faces
- Connection: through the representation of a mutual edge or by using PL charts
- Not structure-preserving, but allows for 2nd-order pointwise operators



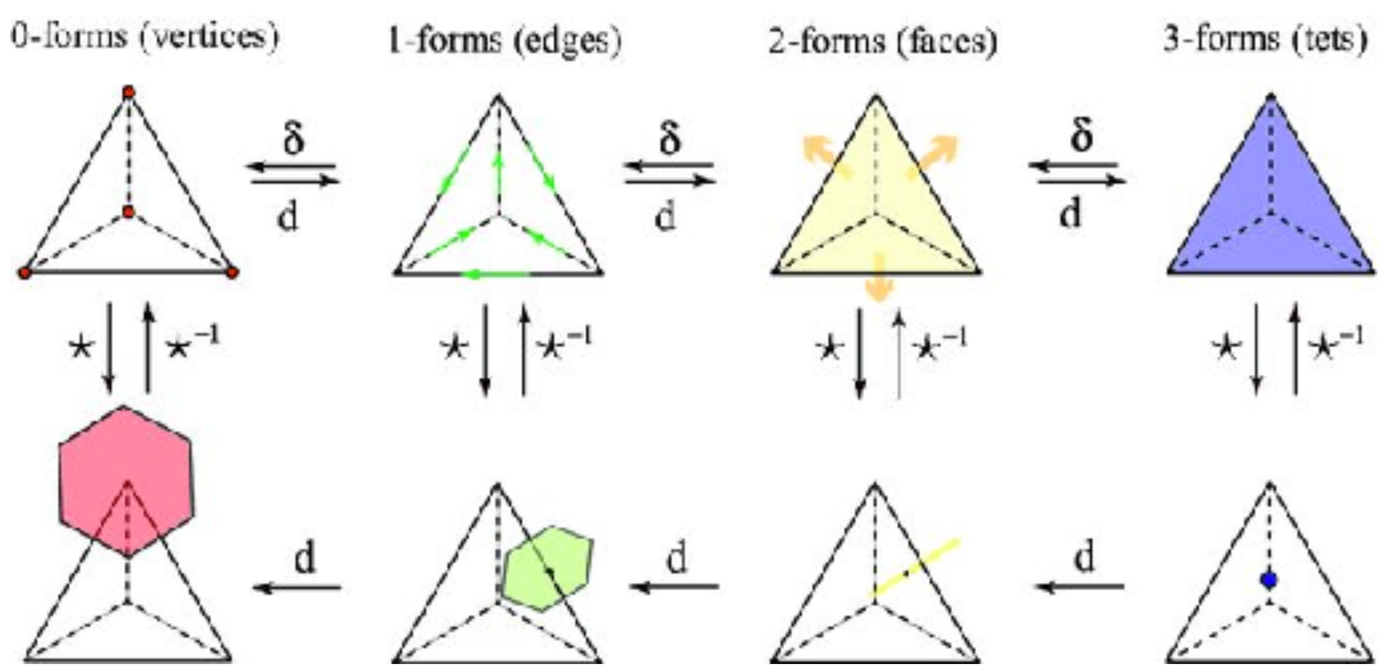
[Sharp *et al.* 2019]



[Liu *et al.* 2016]

EDGE-BASED FIELDS - DIFFERENTIAL 1-FORMS

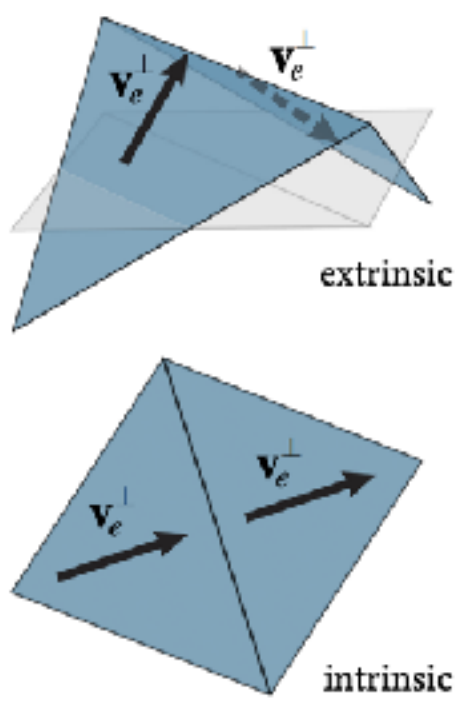
- Differential forms:
    - Encoding integration of field over edges: $\int_e \langle v, e \rangle dt$
    - A single number per oriented edge!
- Structure preserving: $d^2 = 0,\ \delta^2 = 0.$



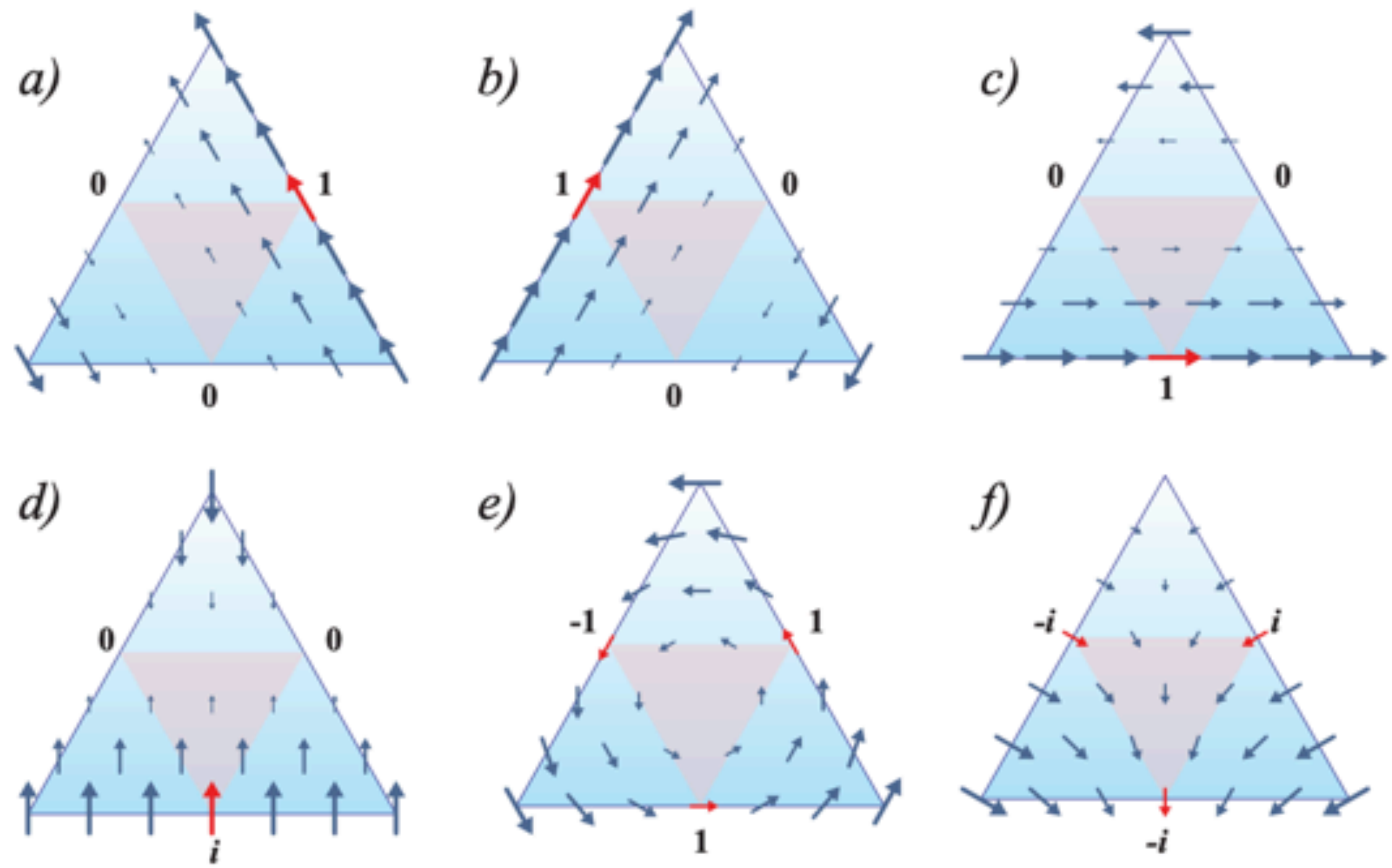
[Desbrun *et al.* 2005]

EDGE-BASED FIELDS - NON-CONFORMING SPACES

- Field is defined per-edge and linear inside faces.
- Field is encoded in parallel and orthogonal components to the mutual edge.



[Stein *et al.* 2020]

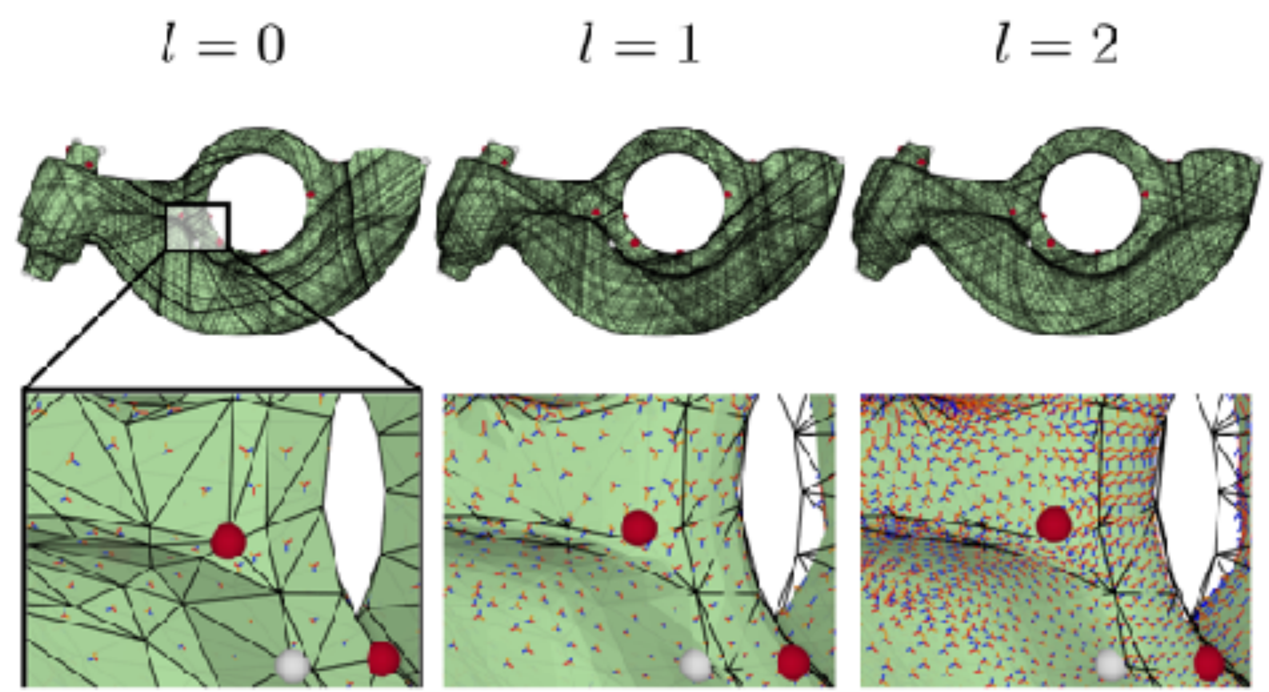


[Djerbetian and Ben-Chen 2016]

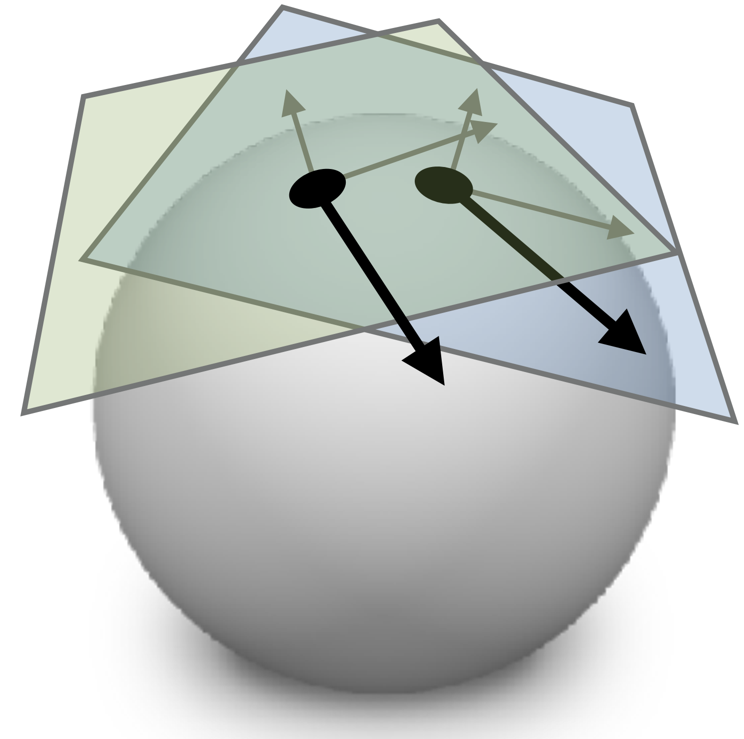
SUBDIVISION FIELDS

- Higher-order continuity in less d.o.f.
- Overcoming the sampling issue.
- Structure Preserving
    - Gradient coarse fields subdivide into gradient fine fields
    - The coarse curl subdivides into fine curl.

Differential forms [De Goes *et al. 2016*]

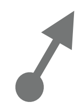


Face-based fields [Custers and Vaxman 2020]

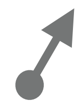


REPRESENTATION

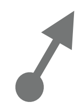
REPRESENTATION

- 1 directional 

REPRESENTATION

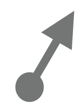
- 1 directional 
    - Cartesian

REPRESENTATION

- 1 directional 
    - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

REPRESENTATION

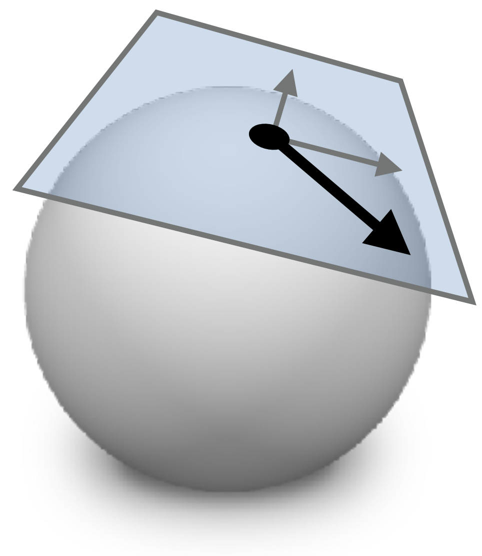
- 1 directional 
    - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

REPRESENTATION

- 1 directional
  - Cartesian

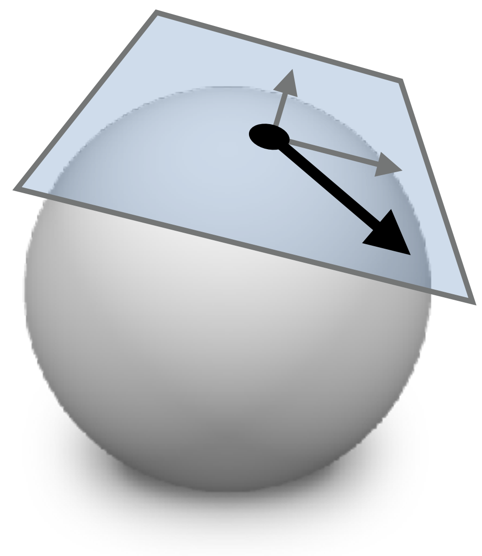
$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



REPRESENTATION

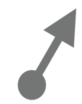
- 1 directional
  - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

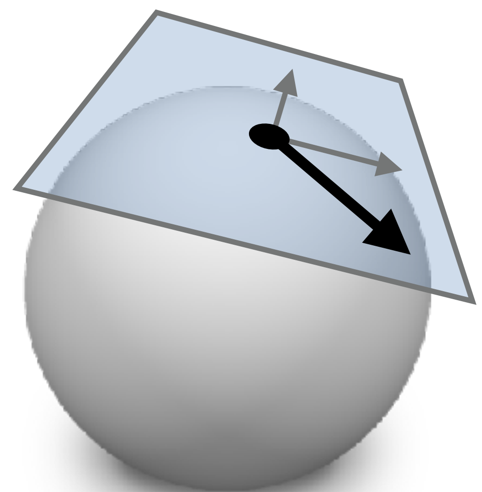


$$\begin{pmatrix} b_{1,p} & b_{2,p} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

REPRESENTATION

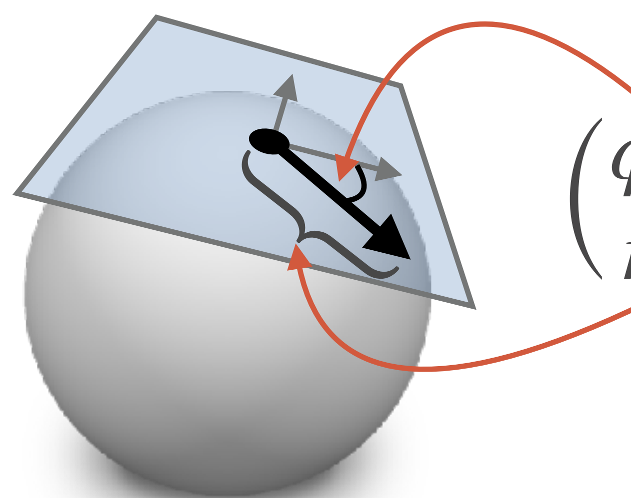
- 1 directional 
  - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$$\begin{pmatrix} b_{1,p} & b_{2,p} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  - Polar



$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

REPRESENTATION

- 1 directional

direction (unit length) field

  - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

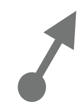
$$x^2 + y^2 = 1$$

  - Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

$$(\phi)$$

REPRESENTATION

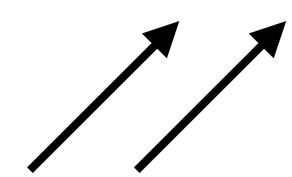
- 1 directional 

direction (unit length) field

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1 \qquad \begin{pmatrix} 0.7 \\ 0.7 \end{pmatrix} - \begin{pmatrix} 0.7 \\ 0.7 \end{pmatrix} = 0$$

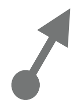


- Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

$$(\phi) \qquad 45^\circ - 405^\circ \neq 0$$

REPRESENTATION

- 1 directional 
  - Cartesian

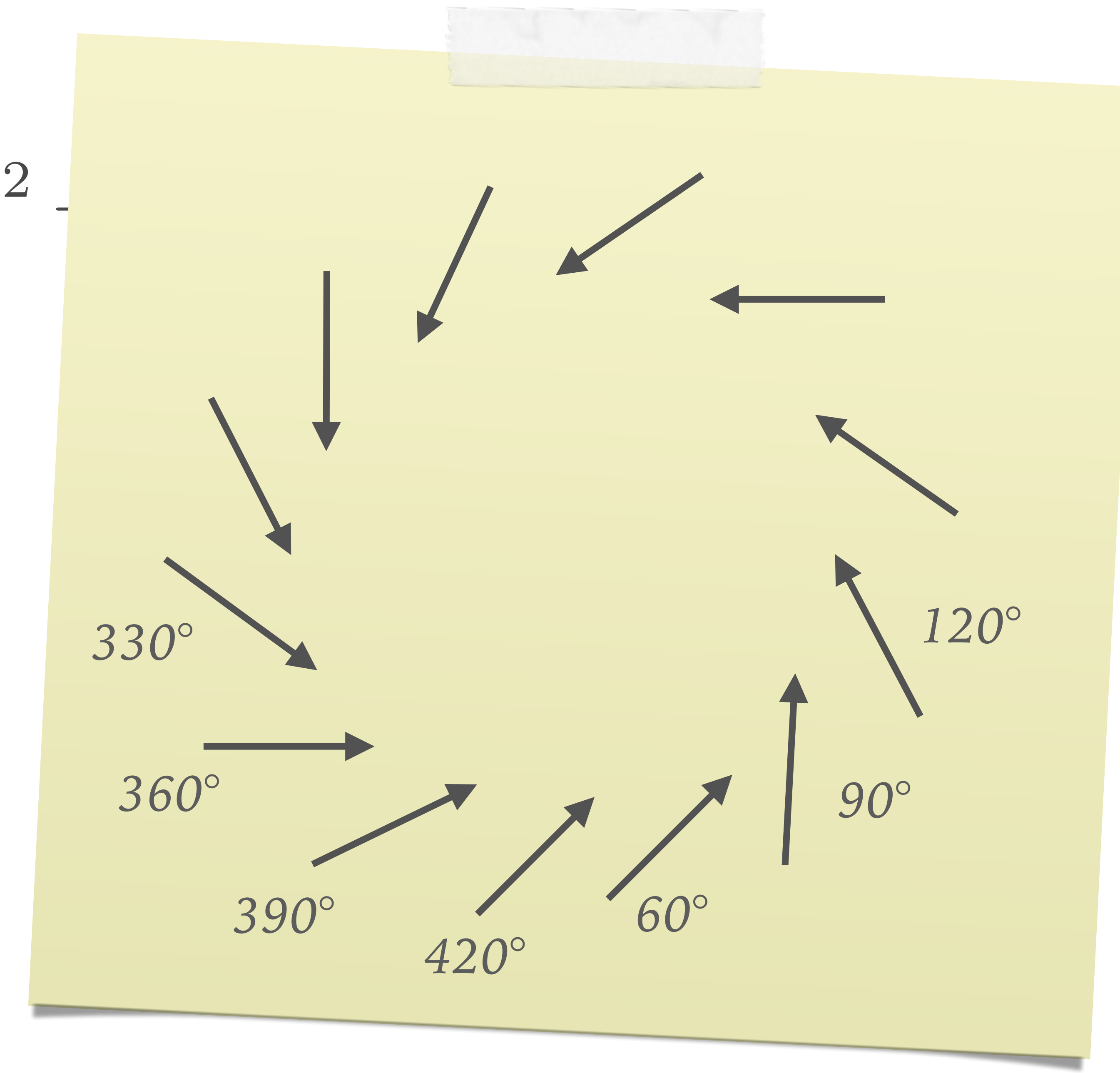
    $$\begin{pmatrix} x \\ y \end{pmatrix}$$

  - Polar

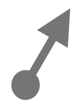
    $$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

direction field

x^2 - 0



REPRESENTATION

- 1 directional 
  - Cartesian

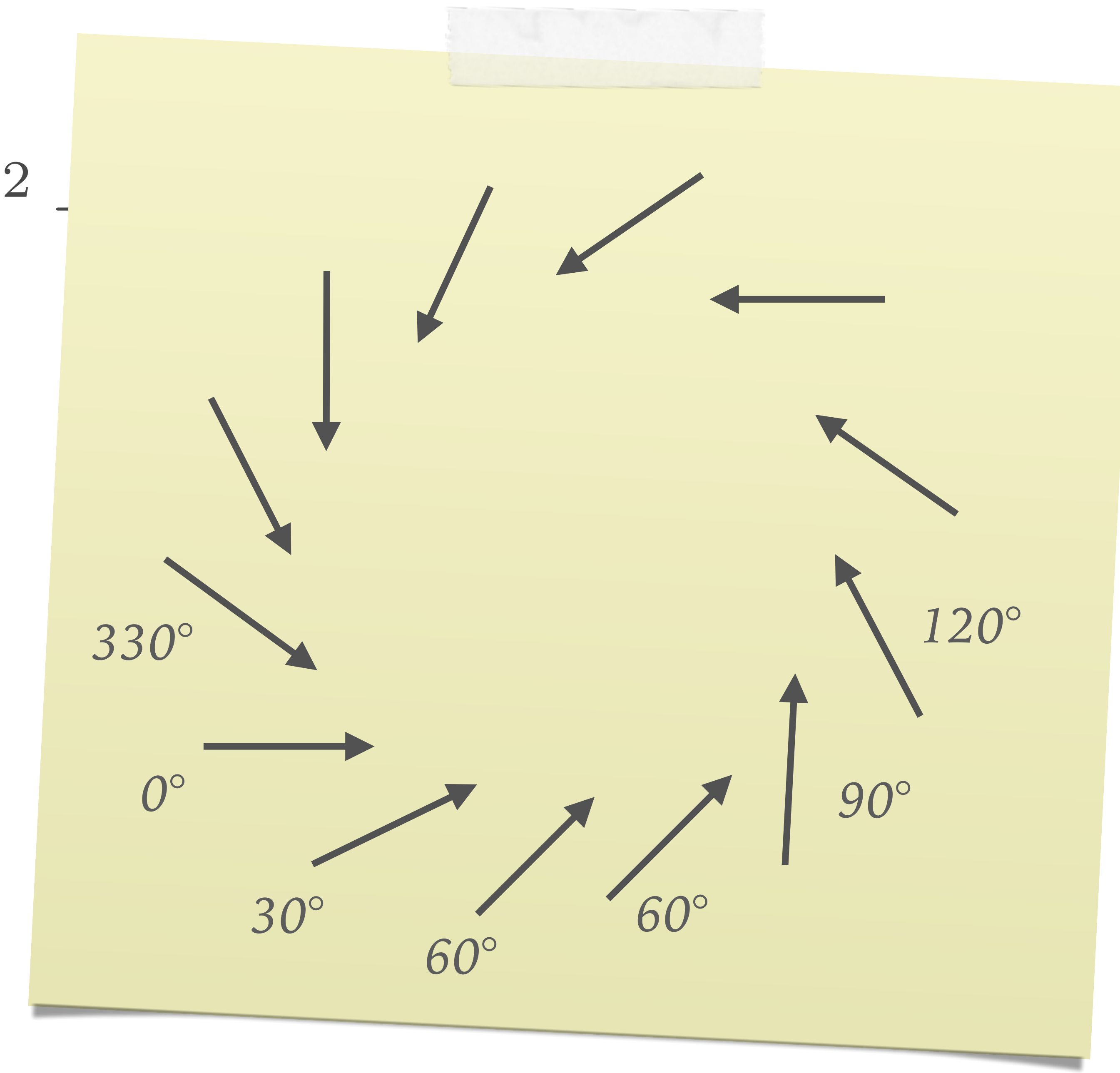
$$\begin{pmatrix} x \\ y \end{pmatrix}$$

  - Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

direction field

x^2 - 0



REPRESENTATION

- 1 directional

  direction **field** — 2π-invariance

  - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1$$

built-in

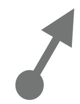
  - Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

$$(\phi)$$

$$\phi_i - \phi_j \mod 2\pi$$

REPRESENTATION

- 1 directional 

direction field | 2π-invariance

  - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad x^2 + y^2 = 1 \qquad \text{built-in}$$

  - Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix} \qquad (\phi) \qquad min_{k \in \mathbb{Z}} \left| \phi_i - \phi_j + 2\pi k \right|^2$$

REPRESENTATION

- 1 directional
  - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

direction **field**

$x^2 + y^2 = 1$

2π-invariance

built-in

  - Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

(ϕ)

$$min_{k \in \mathbb{Z}} \left| \phi_i - \phi_j + 2\pi k \right|^2$$

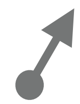
k const.

explicit choice of period

$\Rightarrow$ control over topology

[Li et al. 2006]

REPRESENTATION

- 1 directional 
  - Cartesian

    $$\begin{pmatrix} x \\ y \end{pmatrix}$$

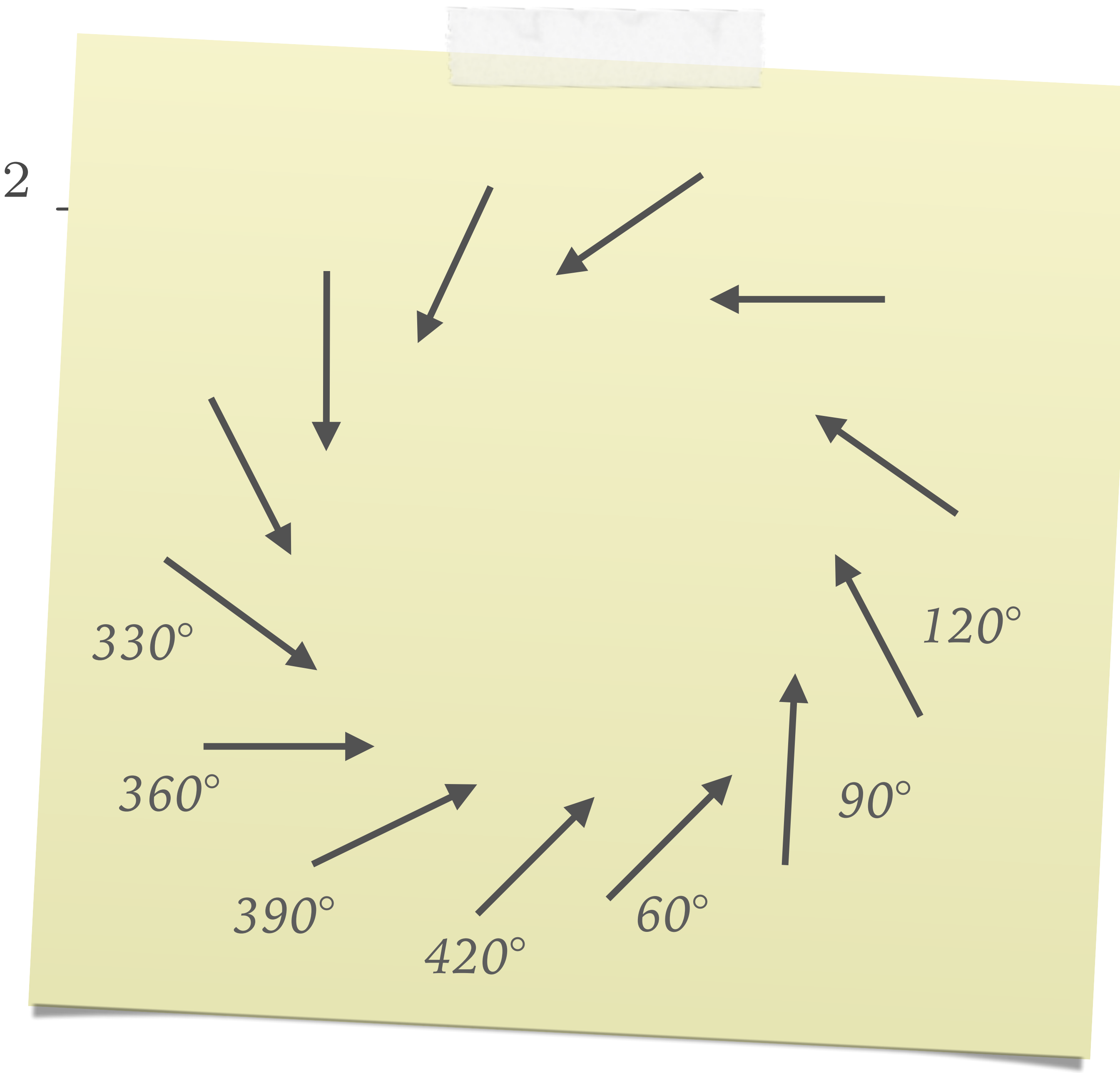
  - Polar

    $$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

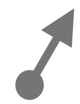
direction field

x^2 -

0



REPRESENTATION

- 1 directional 
    - Cartesian

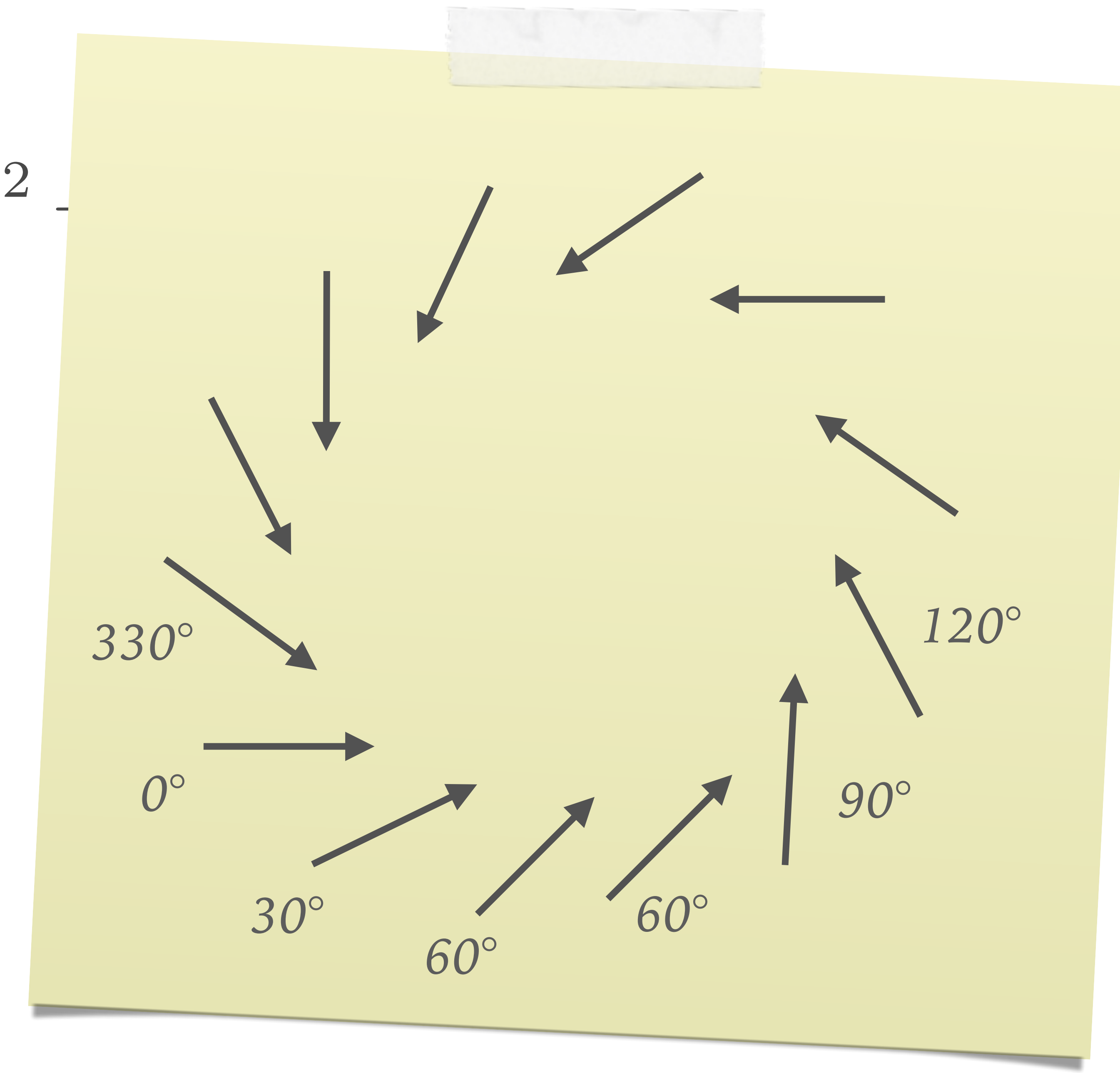
$$\begin{pmatrix} x \\ y \end{pmatrix}$$

    - Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

direction field

x^2 -



0

REPRESENTATION

- 1 directional
  - Cartesian

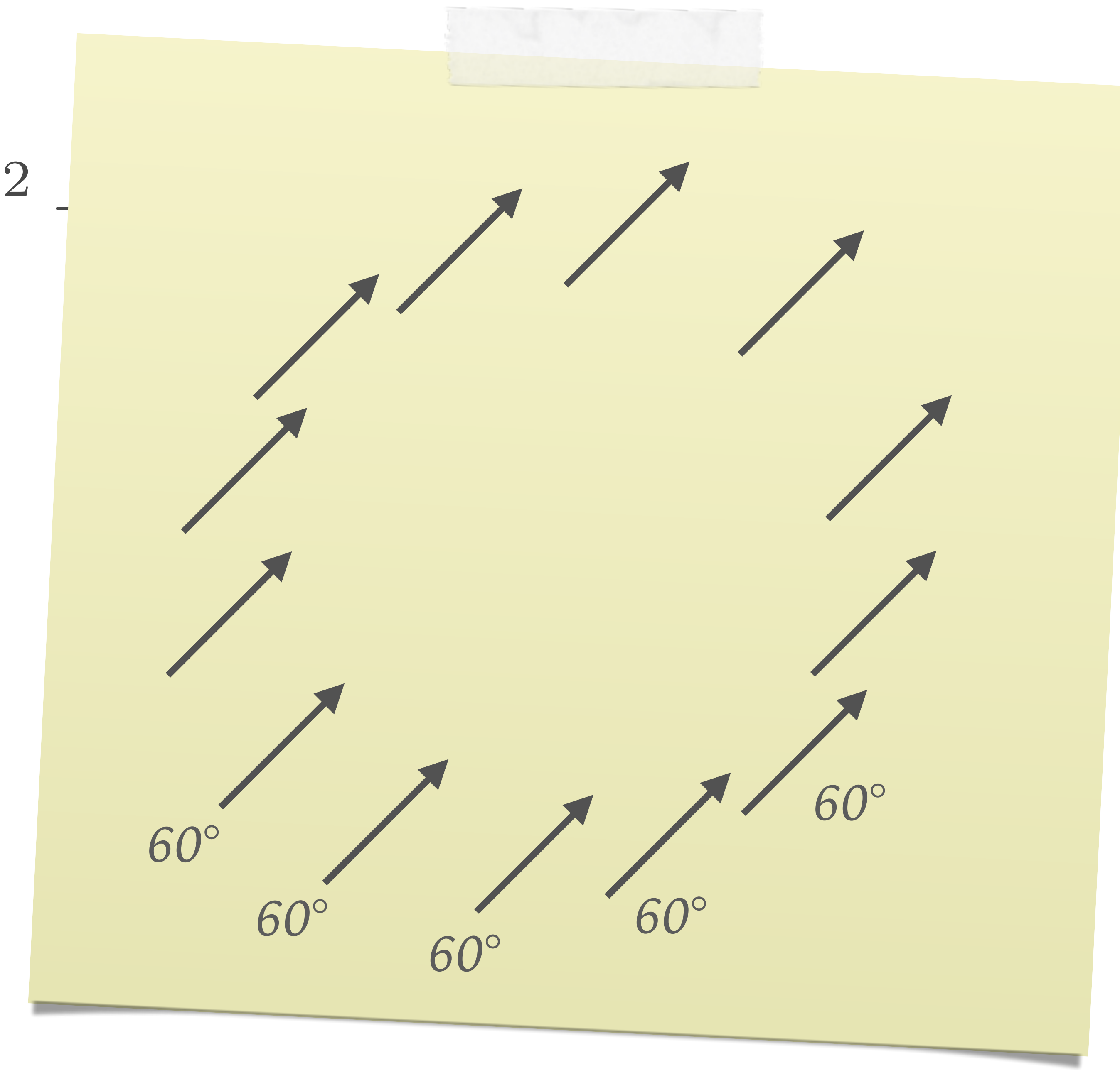
$$\begin{pmatrix} x \\ y \end{pmatrix}$$

  - Polar

$$\begin{pmatrix} \phi \\ r \end{pmatrix}$$

direction field

x^2 -



0

REPRESENTATION

- 1 directional

	direction **field**	2π-invariance
• Cartesian		
$\begin{pmatrix} x \\ y \end{pmatrix}$	$x^2 + y^2 = 1$	built-in
• Polar		
$\begin{pmatrix} \phi \\ r \end{pmatrix}$ $re^{i\phi}$	(ϕ)	$\phi_i - \phi_j + 2\pi k$

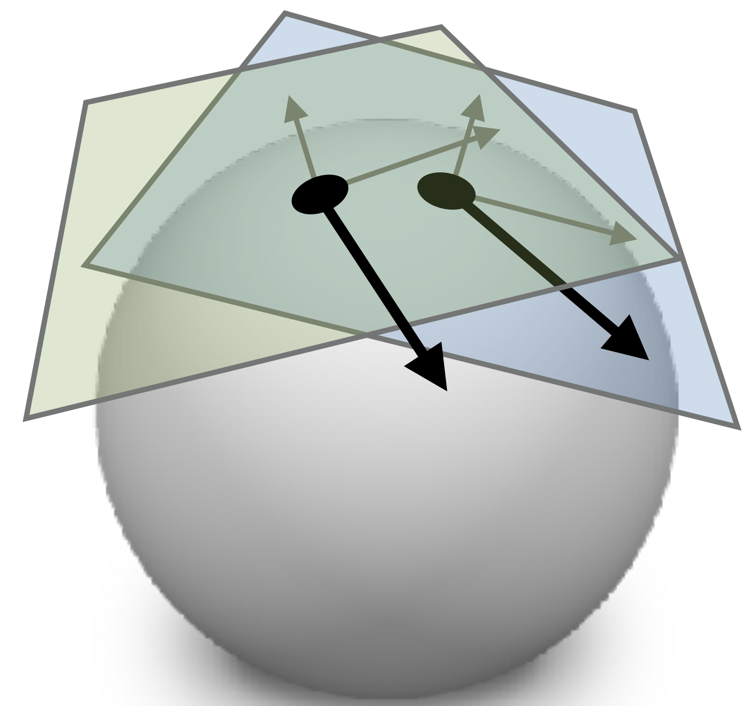
REPRESENTATION

- Differences between tangent vectors?
  - Cartesian

    $$\begin{pmatrix} x \\ y \end{pmatrix}$$

  - Angle

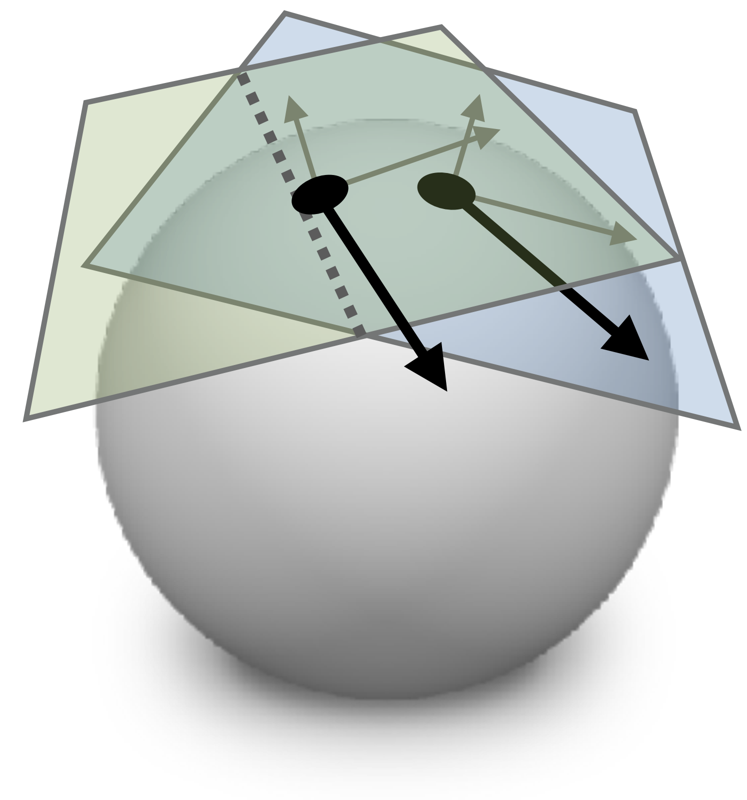
    $$\phi$$



REPRESENTATION

- Differences between tangent vectors?
  - Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}_j - \begin{pmatrix} \cos X_{ij} & -\sin X_{ij} \\ \sin X_{ij} & \cos X_{ij} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_i$$



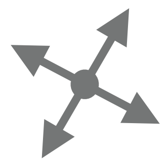
  - Angle

Period Jump

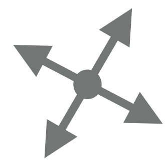
$$\phi_j - \phi_i + X_{ij} + p_{ij} 2\pi$$

Transition angle

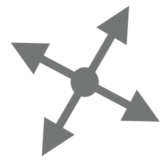
REPRESENTATION

- N directionals 

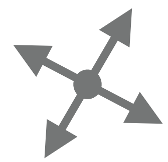
REPRESENTATION

- N directionals 
    - simply use multiple $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} \phi \\ r \end{pmatrix}$ per location?

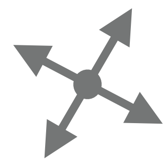
REPRESENTATION

- N directionals 
  - simply use multiple $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} \phi \\ r \end{pmatrix}$ per location?
    - perhaps okay for mere representation,
      but problematic for synthesis, optimization, …

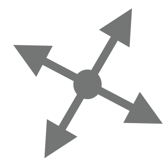
REPRESENTATION

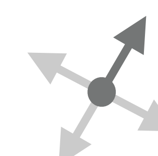
- N directionals 
  - simply use multiple $\binom{x}{y}$ or $\binom{\phi}{r}$ per location?
    - perhaps okay for mere representation,
      but problematic for synthesis, optimization, …
      - symmetries $\Rightarrow$ additional constraints

REPRESENTATION

- *N* directionals 
  - symmetric
    - just use one representative

REPRESENTATION

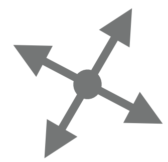
- N directionals 
    - symmetric
        - just use one representative
            - others implied by rotation by $k2\pi/N$

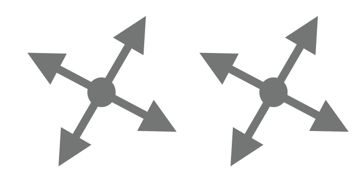


[Palacios & Zhang 2007]

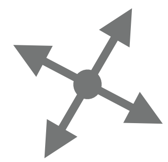
[Ray et al. 2008]

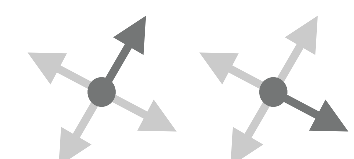
REPRESENTATION

- N directionals 
  - symmetric
    - just use one representative
      - others implied by rotation by $k2\pi/N$



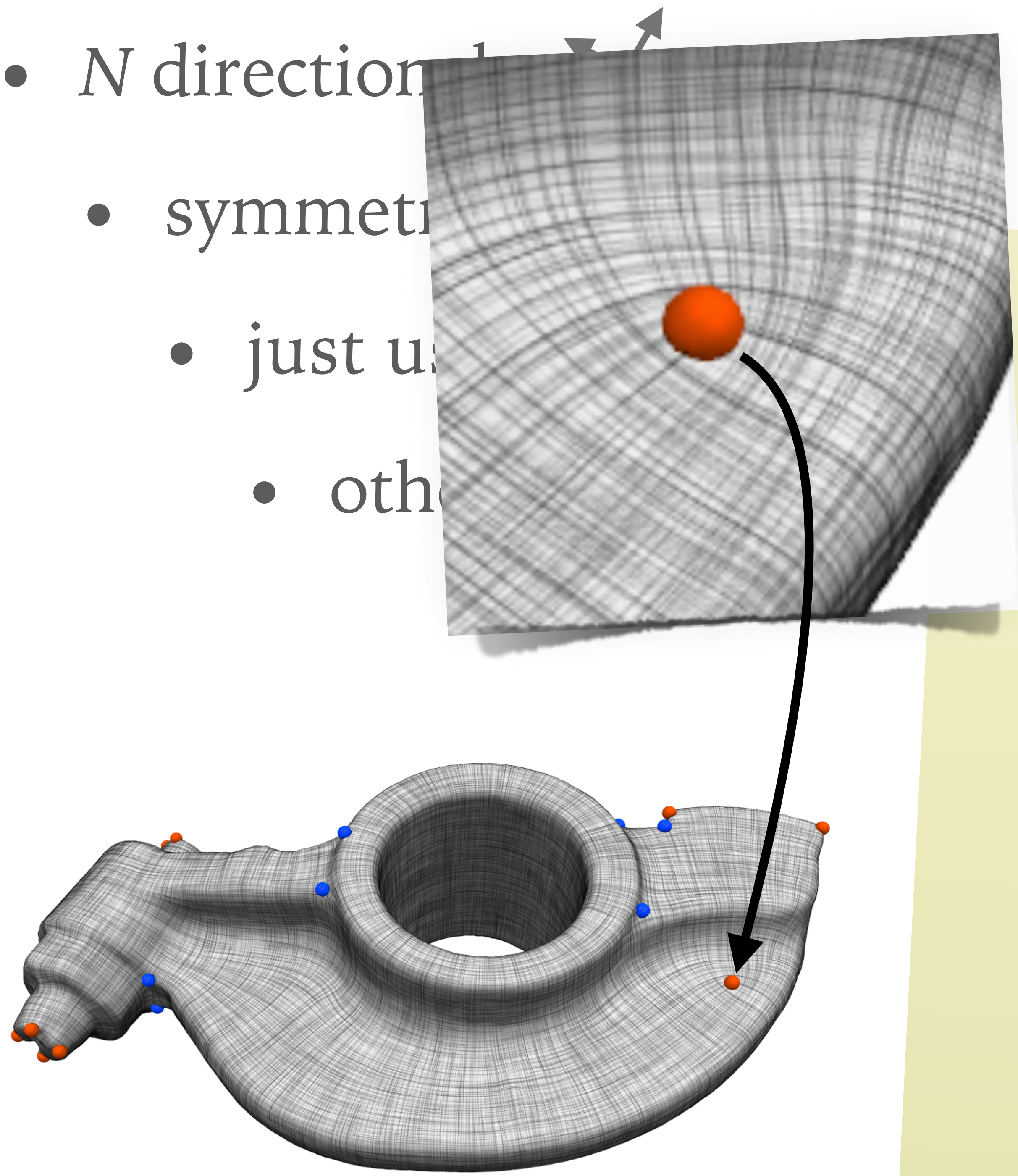
REPRESENTATION

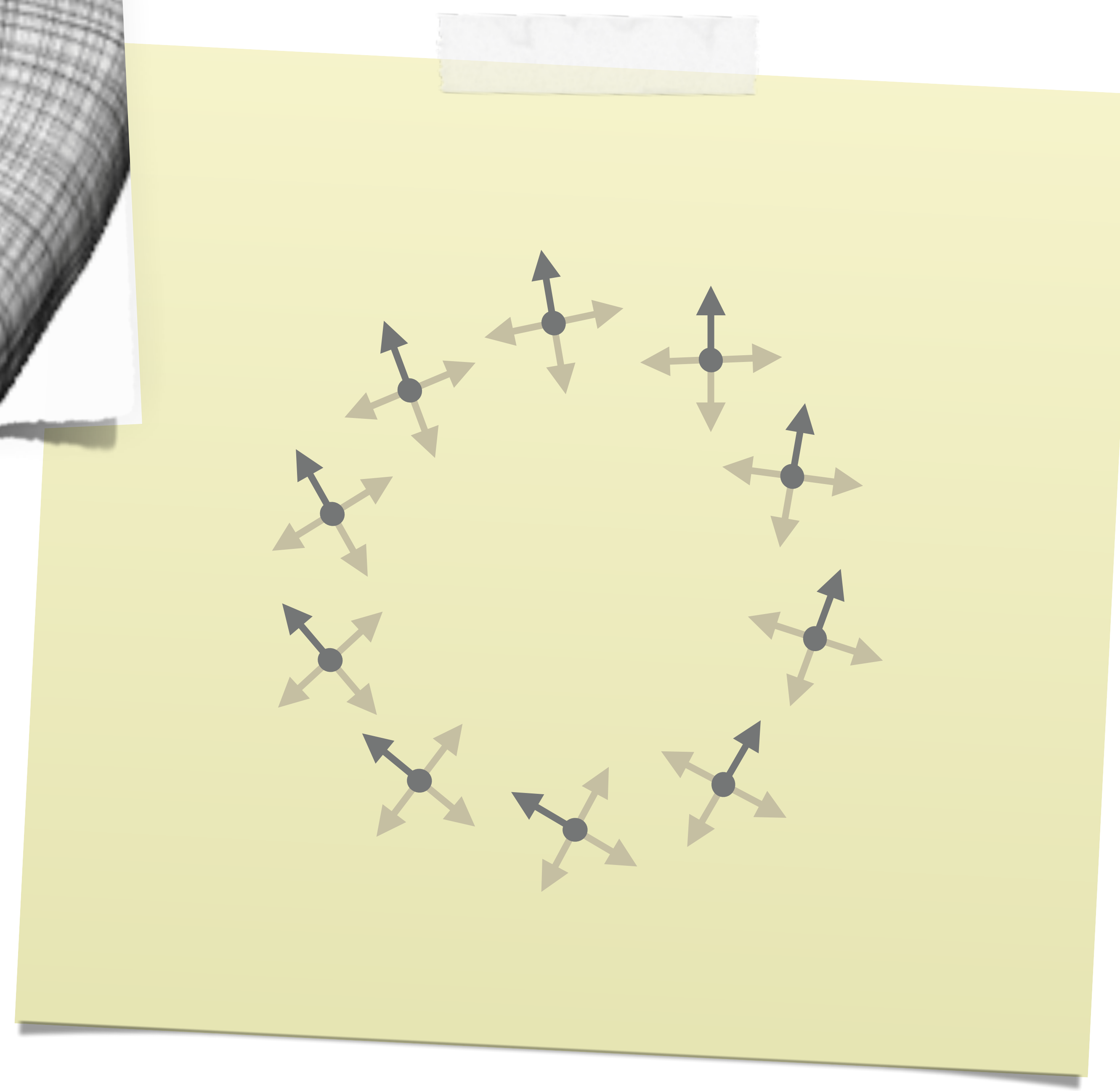
- N directionals 
    - symmetric
        - just use one representative
            - others implied by rotation by $k2\pi/N$



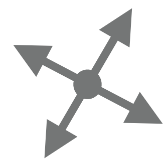
REPRESENTATION

- *N* direction
  - symmetr
    - just us
      - oth

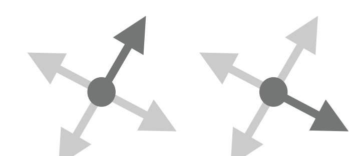




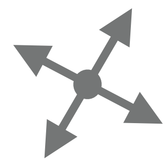
REPRESENTATION

- N directionals 
    - symmetric
        - just use one representative
            - others implied by rotation by $k2\pi/N$

$$\phi_j - \phi_i + X_{ij} + p_{ij}2\pi$$



REPRESENTATION

- N directionals 
  - symmetric
    - just use one representative
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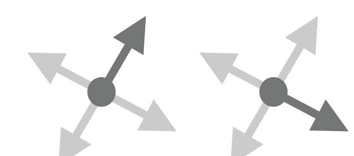
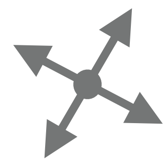
$$\phi_j - \phi_i + X_{ij} + p_{ij}2\pi$$



- Cartesian:

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

REPRESENTATION

- N directionals 
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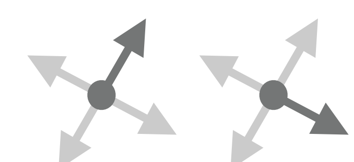
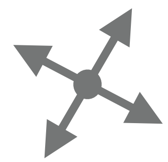
$$\phi_j - \phi_i + X_{ij} + p_{ij}2\pi$$



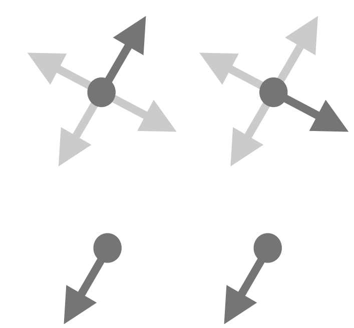
- Cartesian:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \phi = atan2\left(y, x\right)$$

REPRESENTATION

- N directionals 
  - symmetric
    - just use one representative
      - others implied by rotation by $k2\pi/N$

$$\phi_j - \phi_i + X_{ij} + p_{ij}2\pi$$



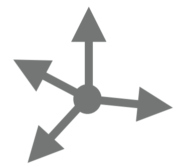
      - Cartesian:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \phi = atan2\left(y, x\right) \rightarrow \begin{pmatrix} cosN\phi \\ sinN\phi \end{pmatrix}$$

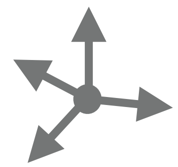
"representation/power vector"

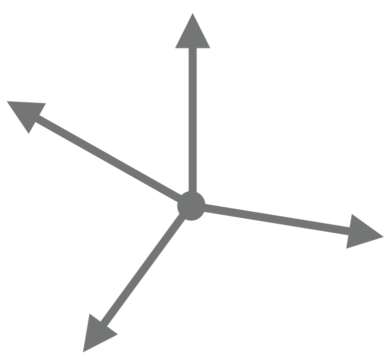
[Hertzmann & Zorin 2000]

REPRESENTATION

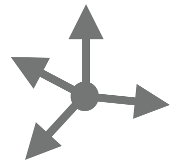
- N directionals 
  - non-symmetric

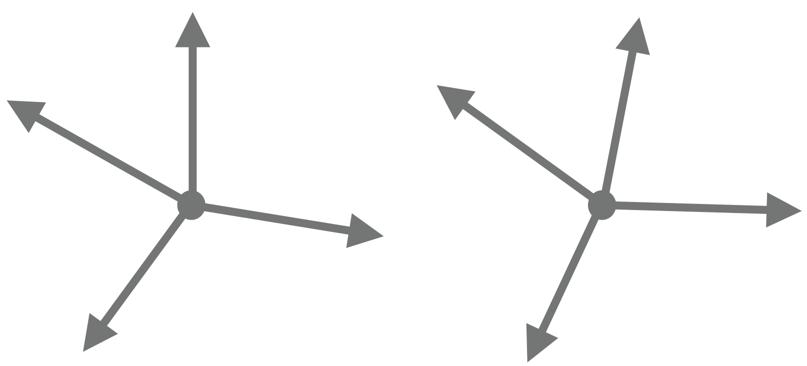
REPRESENTATION

- *N* directionals 
  - non-symmetric

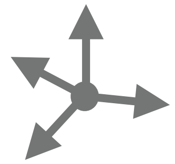


REPRESENTATION

- *N* directionals 
    - non-symmetric



REPRESENTATION

- *N* directionals 
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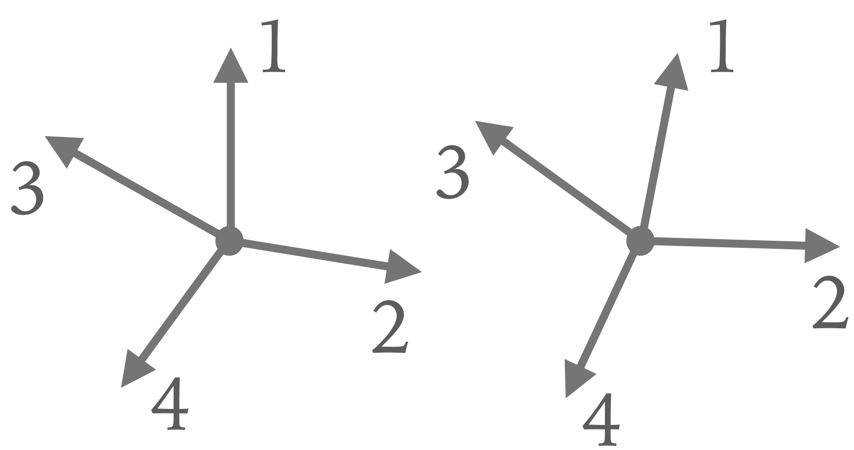
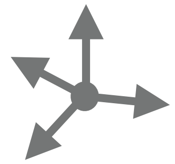
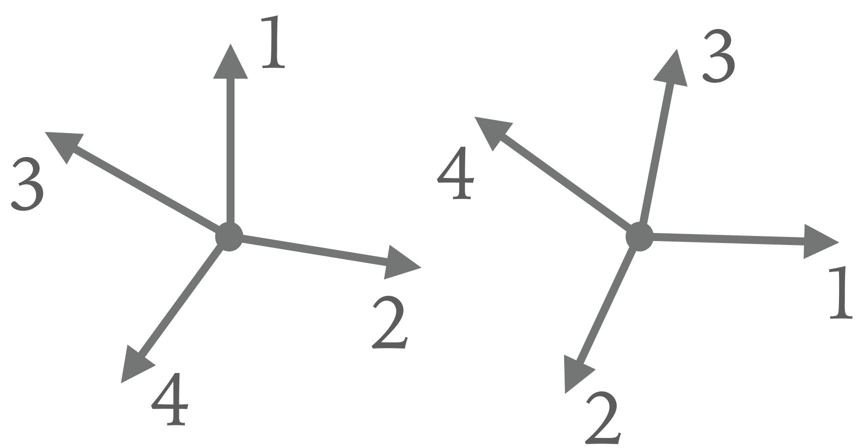
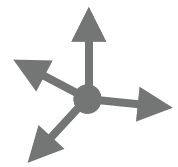


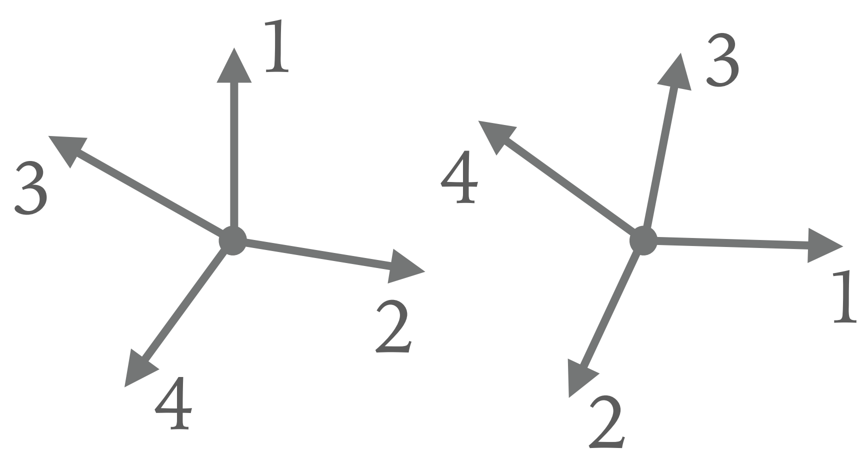
REPRESENTATION

- N directionals 
    - non-symmetric



REPRESENTATION

- N directionals 
  - non-symmetric

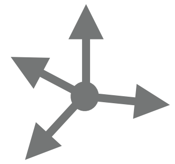


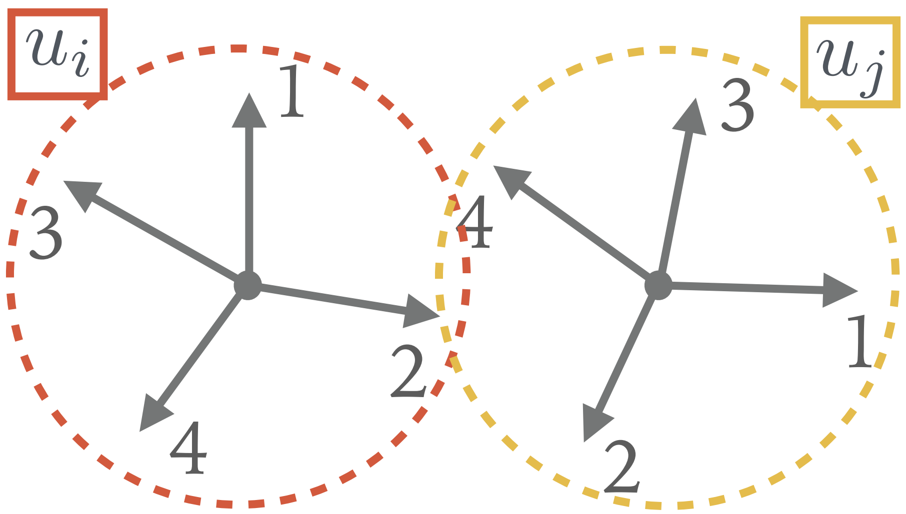
$F(v_1, v_2, v_3, v_4) = u$

F symmetric

$F(v_1, v_2, v_3, v_4) = F(v_3, v_2, v_4, v_1)$

REPRESENTATION

- N directionals 
  - non-symmetric



$F(v_1, v_2, v_3, v_4) = u$

F symmetric

F invertible (up to symmetry)

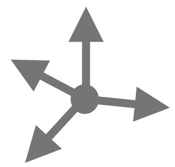
$$(x-a)(x-b)(x-c)$$
$$= x^3 + (a+b+c)x^2 + (ab+ac+bc)x + (abc)$$
$$\rightarrow (a+b+c \mid ab+ac+bc \mid abc)$$
$$(\quad u \quad , \quad v \quad , \quad w \;) =: F(a,b,c)$$

N-PolyVector

[Diamanti et al. 2014]

$$(a+b+c+d \mid ab+ac+ad+bc+bd+cd \mid abc+abd+acd+bcd \mid abcd)$$

REPRESENTATION

- N directionals 
  - use complex numbers, complex polynomials

    $$a = x + iy$$

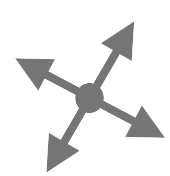
    $$a = re^{i\theta}$$

  - non-symmetric

$$(a+b+c+d \mid ab+ac+ad+bc+bd+cd \mid abc+abd+acd+bcd \mid abcd)$$

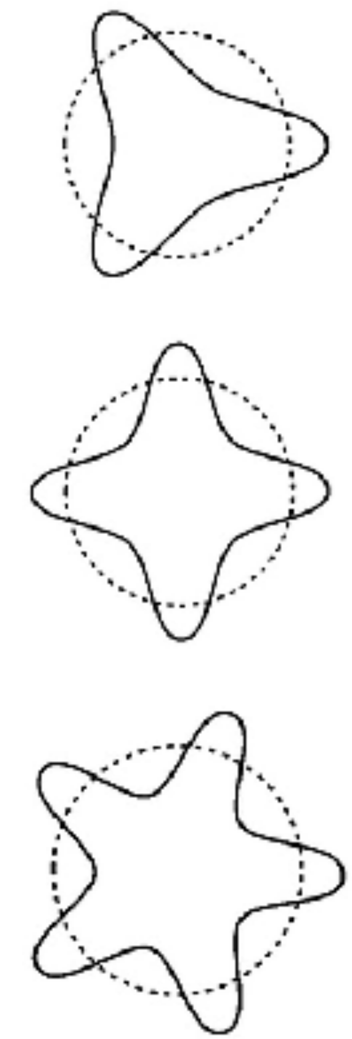
  - symmetric

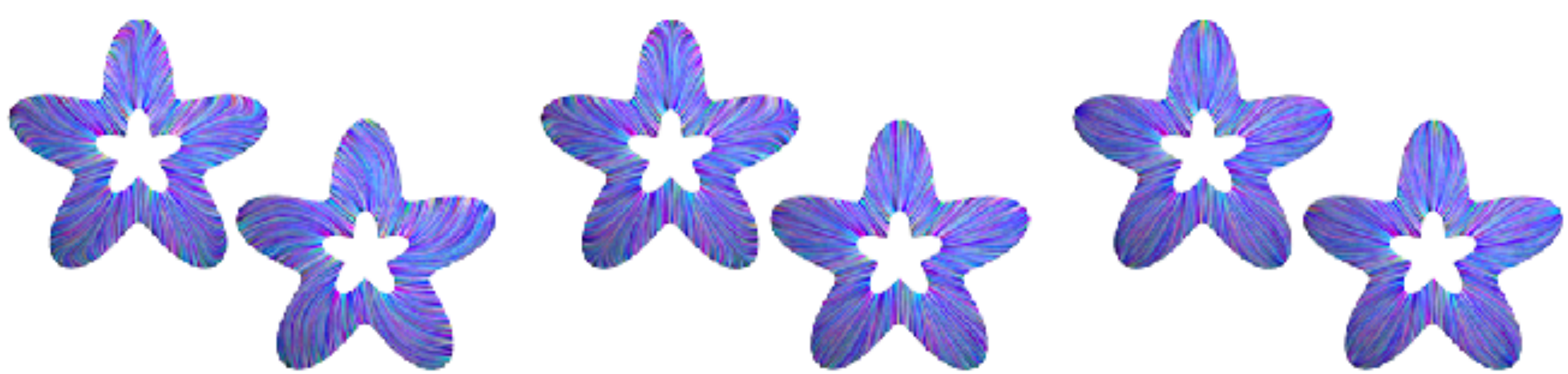
$$(\underbrace{a+b+c+d}_{0} \mid \underbrace{ab+ac+ad+bc+bd+cd}_{0} \mid \underbrace{abc+abd+acd+bcd}_{0} \mid \underbrace{abcd}_{\substack{-a^4 \\ =-re^{i4\theta}}})$$

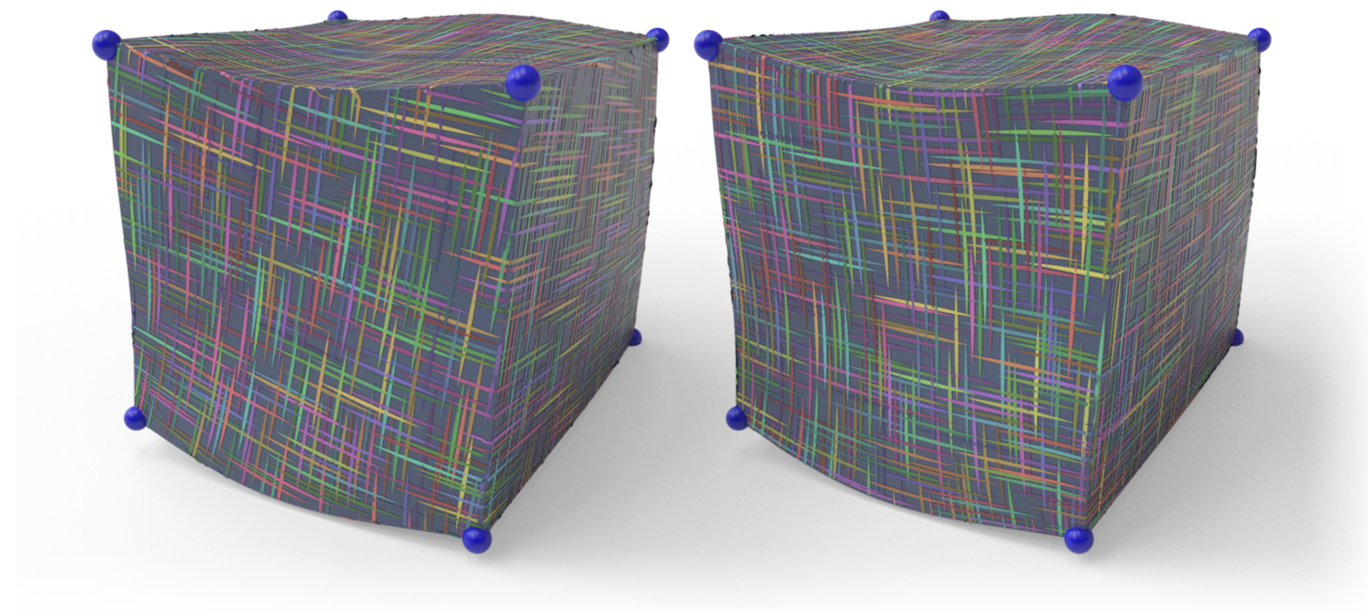
 $\rightarrow \begin{pmatrix} \sin N\theta \\ \cos N\theta \end{pmatrix}$

REPRESENTATION

- Alternatives
  - Extrema of periodic circular functions (SH)
    - Also spherical harmonics in 3D [Ray *et al.* 2016]
  - Eigenvectors of tensors (spd matrices)
  - Functional representation [Azencot *et al.* 2013]



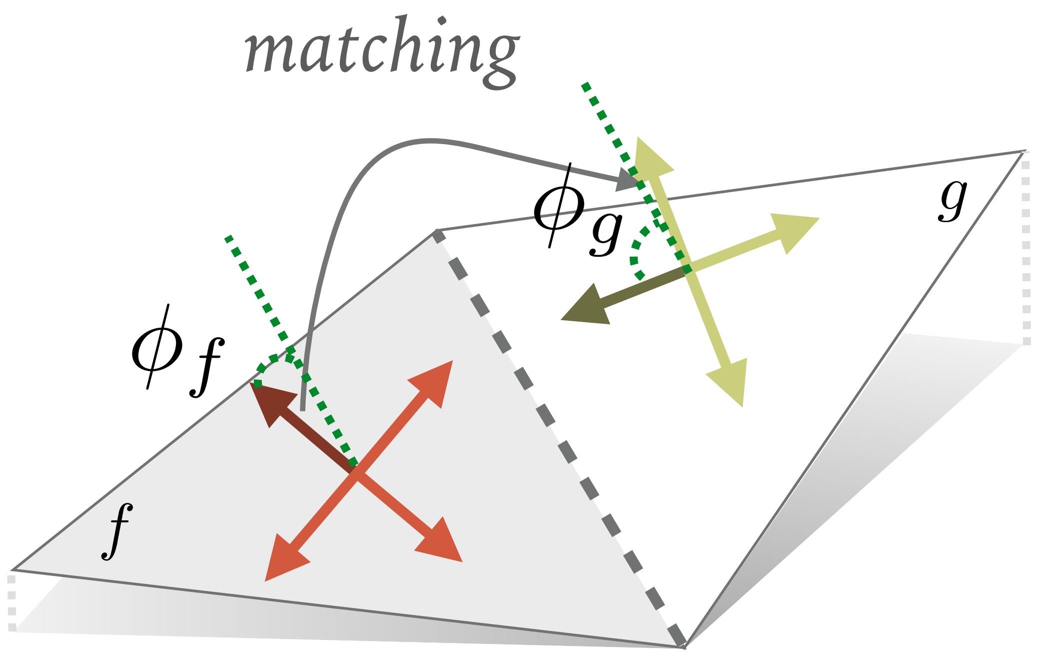


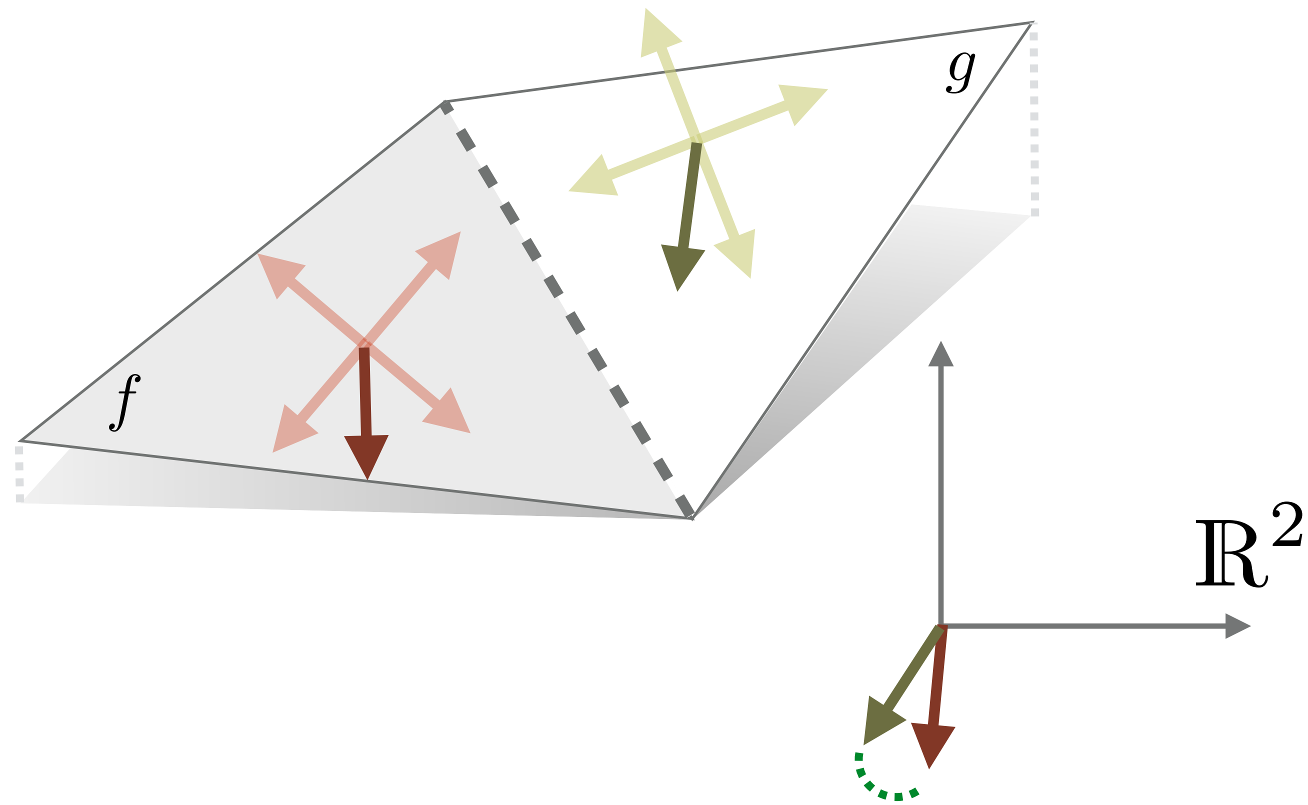


OBJECTIVES

OBJECTIVES – FIELD FAIRNESS

- “As-parallel-as-possible”





angle-based approaches

compare matching angles

[Hertzmann et al. 2000] [Crane et al. 2010]
[Ray et al. 2008] [Ray et al. 2009]
[Bommes et al. 2009] [Jakob et al. 2015]

cartesian/complex approaches

compare representative vectors

[Ray et al. 2006] [Knöppel et al. 2013]
[Palacios et al. 2007] [Diamanti et al. 2014]

DIRICHLET ENERGY

- Smoothest field - minimizing the Dirichlet energy:

$$E_D = \int_\Omega \left| \nabla v \right|^2 d\Omega = \langle \Delta v, v \rangle_{L^2}$$

- Discrete version (in Cartesian discretizations/ representations):

$$E_D = \sum_{i,j} w_{ij} \left| v_i - R_{ij} v_j \right|^2 = v^T L v$$

Metric (stiffness)

Connection matrix

Discrete (vector) Laplacian

DIRICHLET ENERGY FOR FACE-BASED FIELDS

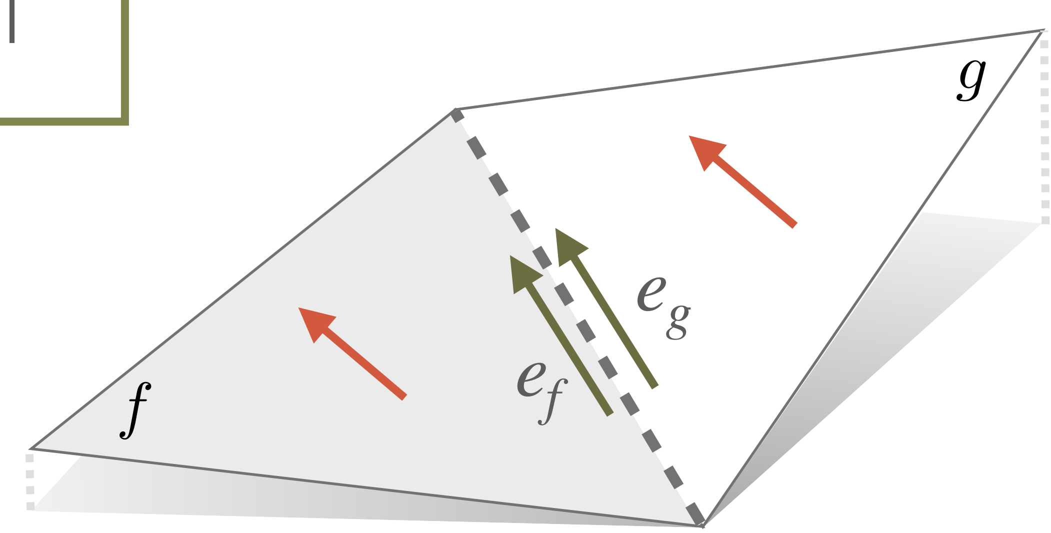
$$E_D = \sum_{i,j} w_{ij} \left| v_i - R_{ij} v_j \right|^2 = v^T L v$$

- In complex numbers we get a simple expression
- Two vectors in neighboring faces $f, g \in F$ are perfectly parallel when: $v_f \overline{e_f} = v_g \overline{e_g}$ (then $R_{fg} = \frac{\overline{e_g}}{\overline{e_f}}$)
  - Aligning both bases to agree on (unit vector) e as an x *axis*.
- As parallel-as-possible Dirichlet energy:

$$\boxed{E_D = \sum_{e=(f,g)} w_e \left| v_f \overline{e_f} - v_g \overline{e_g} \right|^2}$$

- Choice of w_e : $\dfrac{3l_e^2}{A_f + A_g}$

[Brandt *et al.* 2018]



EXAMPLE AND DEMO: POWER FIELDS

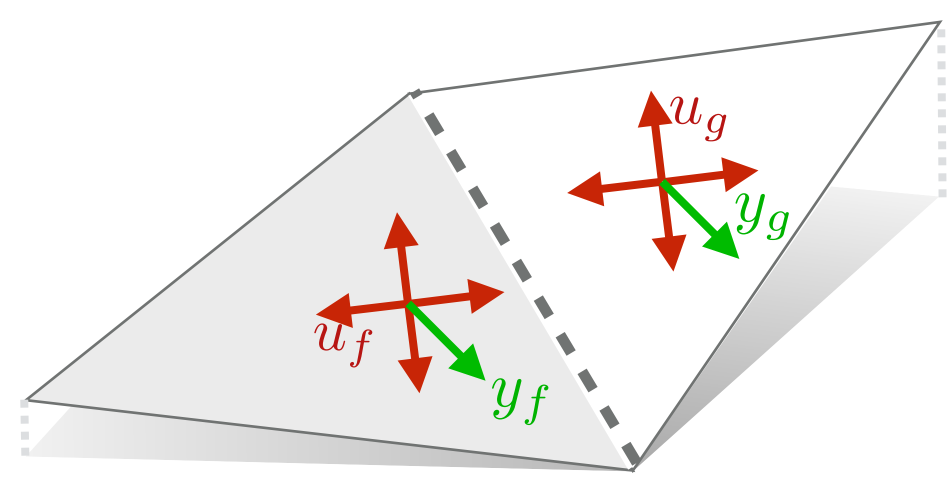
- **Input**: N-RoSy as single complex number per constrained face

$$\forall f \in B,\ y_f = \left(u_f\right)^N$$

- **Output**: interpolation to all free faces $B = F \backslash U,\ y_U = \left(u_U\right)^N$
- Minimizing Dirichlet energy:

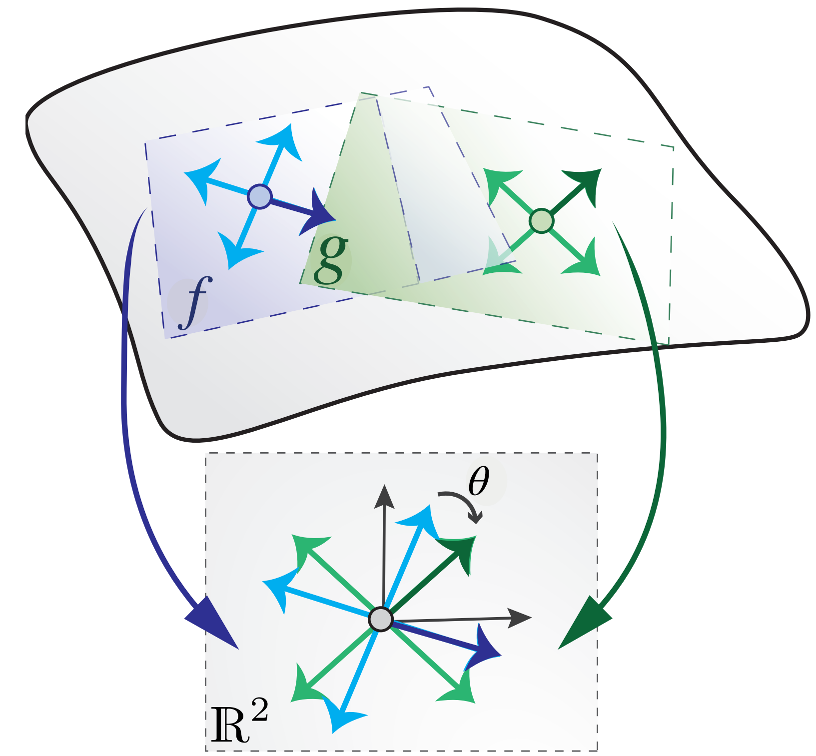
$$\sum_{e=(f,g)} w_e \left| y_f \overline{e}_f^{N} - y_g \overline{e}_g^{N} \right|^2 = y^T L_N y$$

- $\overline{e}^N$ corresponds with $\left(u_f\right)^N$
- Take roots of all y_f



POWER-FIELD PRINCIPAL MATCHING

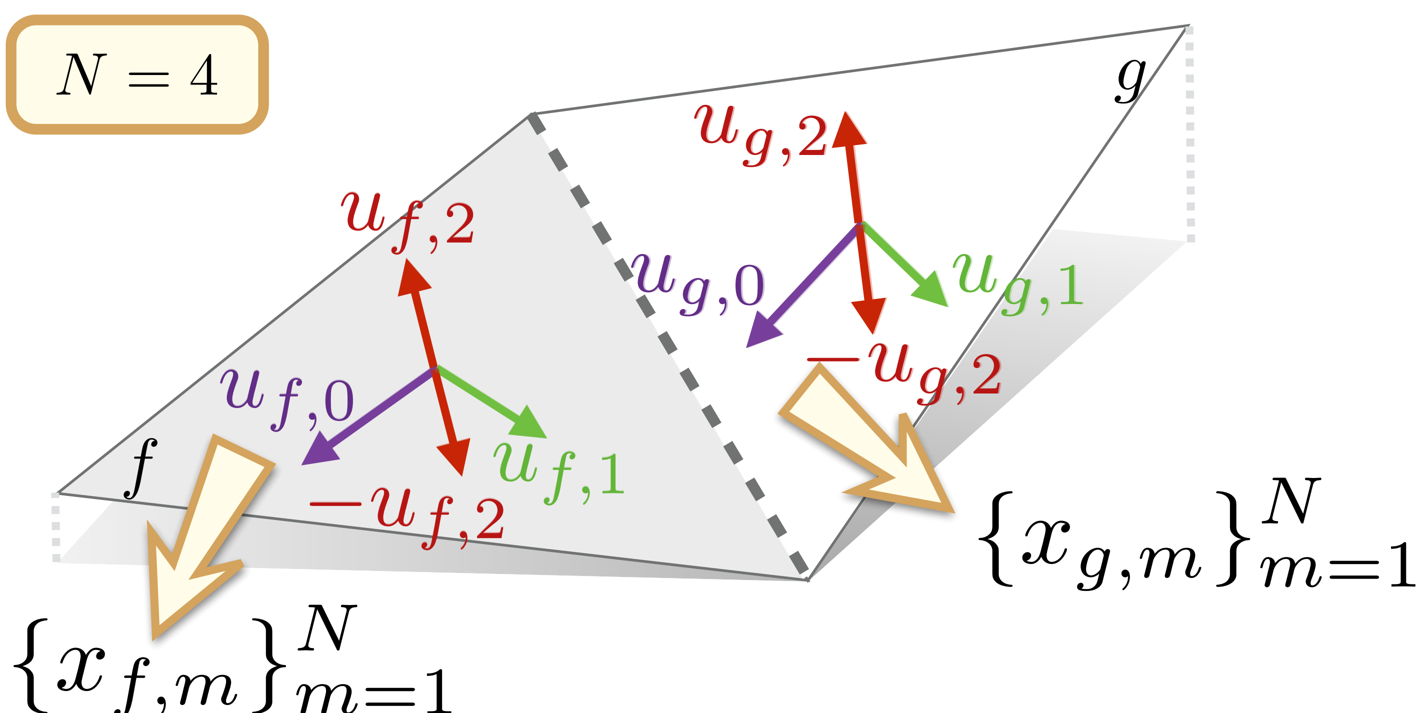
- The effort reduces to $N\theta(e)$, which is just N times the smallest-angle rotation.



- Obtained by: $\Theta(e) = \arg\left(\frac{y_g \overline{e_g}}{y_f \overline{e_f}}\right)$
- Note $\Theta(e) = 0$ for parallel transport.
- Vertex index:
$$I(v) = \frac{1}{2\pi N}\left(\left(d_0\right)^T \Theta(e) + NK(v)\right)$$
- d_0 is the *differential matrix*: $d_0(e, v) = \begin{cases} -1, & v = source(e) \\ 1, & v = target(e) \\ 0 & else \end{cases}$

EXAMPLE DEMO: POLYVECTORS

- Representing set of polynomial coefficients instead of single N-RoSy.
- Roots of polynomial = directional field in face.
- Interpolating each like in power fields.

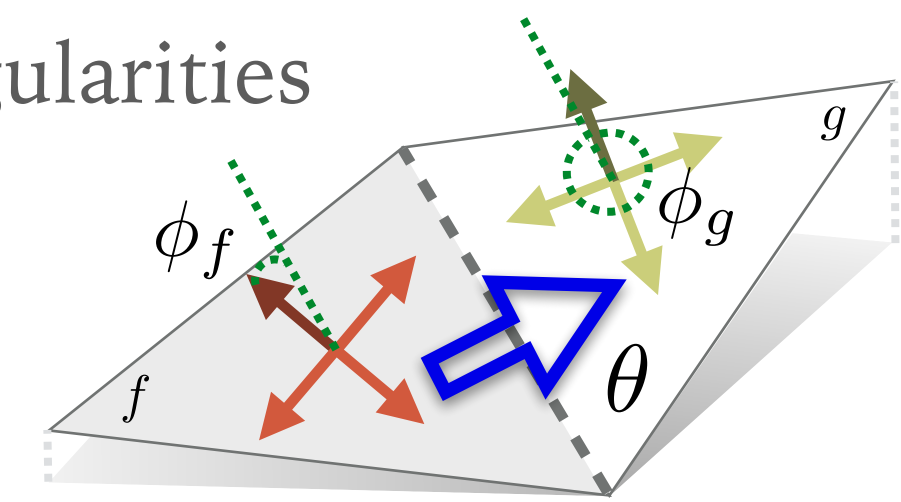


$$\forall m \in [1,N], \quad \sum_{e=(f,g)} w_e \left| y_f \overline{e_f}^m - y_g \overline{e_g}^m \right|^2 = y^T L_m y$$

DEMO: INDEX PRESCRIPTION

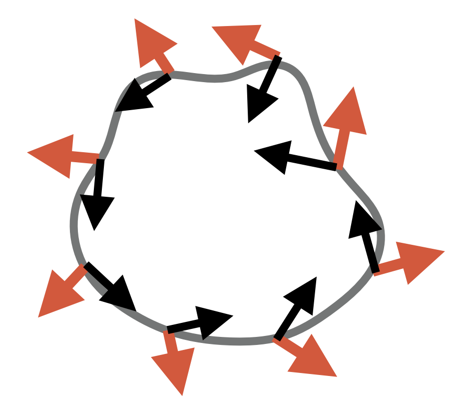
$$I(v) = \frac{1}{2\pi N}\left(\left(d_0\right)^T \Theta(e) + NK(v)\right)$$

- Input:
  - Original vertex curvatures $K(v)$.
  - Prescribed curvatures $2\pi I(v)$ (cone singularities and flat vertices).



- Polar Dirichlet energy:

$$E_D(\theta_E) = \sum_{e \in E} \left| \theta(e) \right|^2$$



- Under the linear constraint:
  $\left(d_0\right)^T \theta(E) = 2\pi I(v) - K(v)$

INDEX PRESCRIPTION

- How do we solve?

$$\theta(e) = \operatorname{argmin} \sum_{e \in E} \left| \theta(e) \right|^2 \text{ s.t. } \left(d_0 \right)^T \theta(E) = 2\pi I(v) - K(v)$$

- Quadratic energy with linear constraints.
  - But: many constraints!
  - Lagrange multipliers are cumbersome.

INDEX PRESCRIPTION – POLAR HODGE DECOMPOSITION

- Any dual-edge quantity $\theta(e)$ on a closed surface can always be decomposed as:

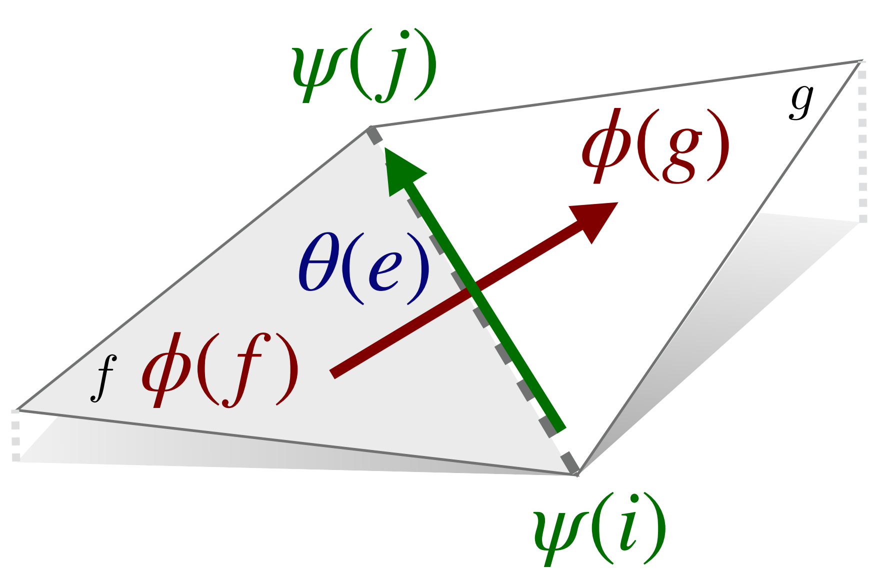
$$\theta(e) = \star\, d_0 \psi(v) + (d_1)^T \phi(f) + H(e)$$

primal-dual conversion

vertex-based function

face-based function

harmonic field



INDEX PRESCRIPTION - POLAR HODGE DECOMPOSITION

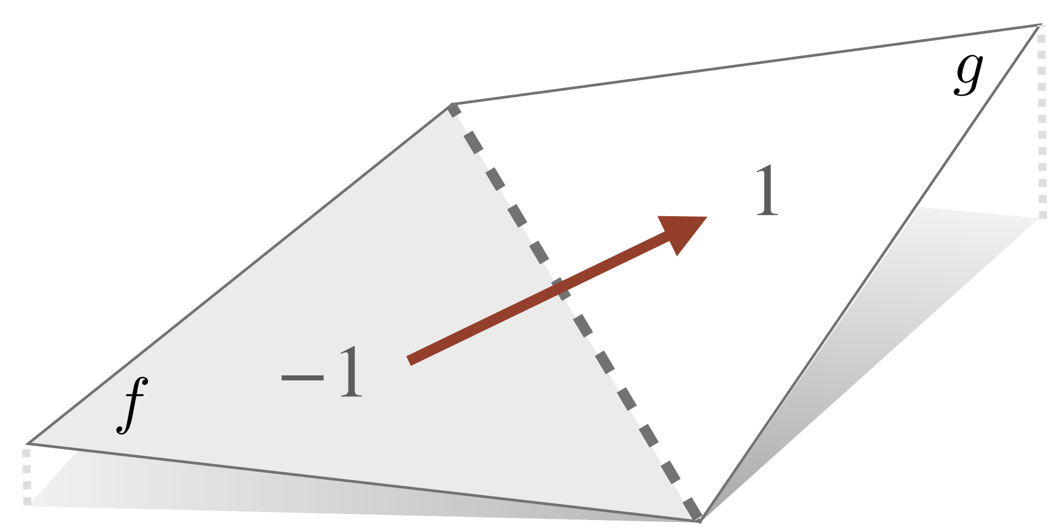
- Any dual-edge quantity $\theta(e)$ on a closed surface can always be decomposed as:

$$\theta(e) = \star\, d_0 \psi(v) + (d_1)^T \phi(f) + H(e)$$

differential matrix

edge-face differential

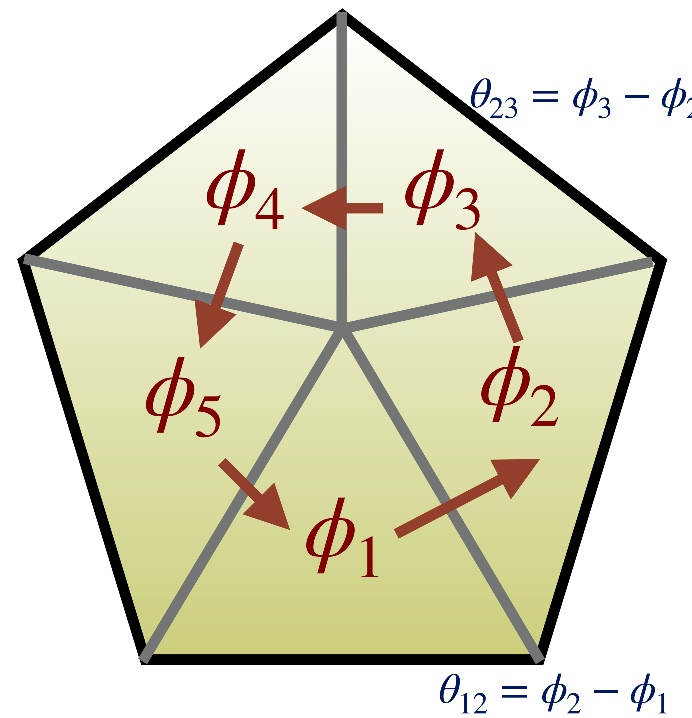
$$d_1(f,e) = \begin{cases} -1 & f = left(e) \\ 1 & g = \; right(e) \\ 0 & else \end{cases}$$



INDEX PRESCRIPTION - POLAR HODGE DECOMPOSITION

- For simply-connected meshes without boundary

$$\theta(e) = \star\, d_0 \psi(v) + (d_1)^T \phi(f)$$



- deRham exact sequence: $(d_0)^T (d_1)^T = 0$
- Then the linear condition transforms to:
$\left(d_0\right)^T \theta(E) = 2\pi I(v) - K(v) = \left[\left(d_0\right)^T \star d_0\right] \psi(v) + 0$

No dependence on $\phi(f)$!

$2\pi I(v) - K(v) = \left[(d_0)^T \star d_0 \right] \psi(v) + 0$

INDEX PRESCRIPTION - POLAR HODGE DECOMPOSITION

- For simply-connected meshes without boundary

$$\theta(e) = \star\, d_0 \psi(v) + (d_1)^T \phi(f)$$

- The decomposition is orthogonal, and then:

$$\left|\theta\right|^2 = \left| \star\, d_0 \psi \right|^2 + \left| \cancel{(d_1)^T \phi} \right|^2$$

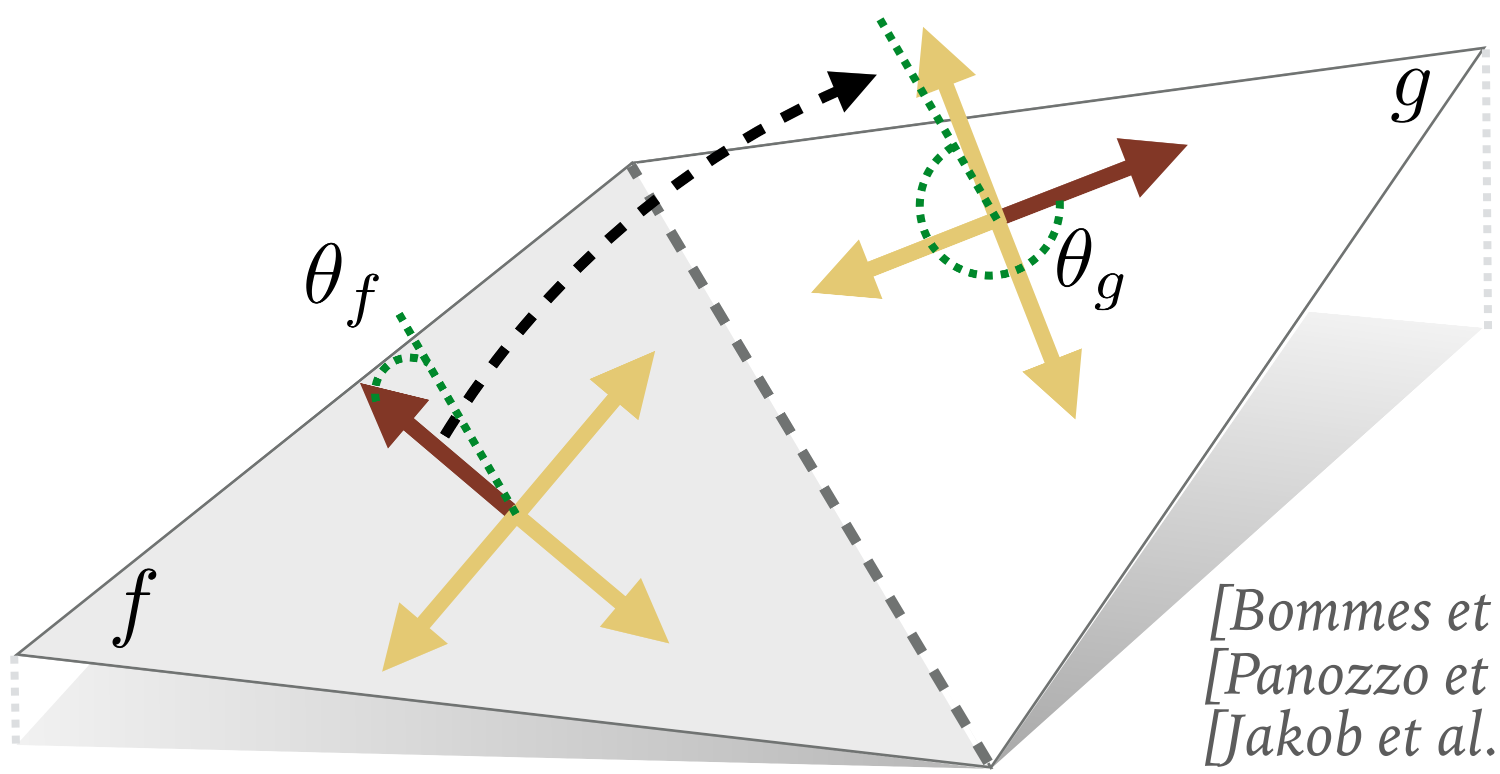
- Since the condition is invariant to $\phi(f)$, the minimum energy solution is only $\theta(e) = \star\, d_0 \psi(v)$!

INDEX PRESCRIPTION ALGORITHM

- Solve for ψ: $\left[\left(d_0\right)^T \star d_0\right] \psi(v) = 2\pi I(v) - K(v)$
- Compute: $\theta = \star\, d_0 \psi$.
- In the discrete setting:
  - $\star$ is a diagonal matrix $W(e,e) = \dfrac{A_f + A_g}{3 l_e}$
    - Inverse of the power-field weight matrix!
  - $\left[\left(d_0\right)^T \star d_0\right]$ is a *Laplacian*.

POLAR DIRICHLET: FREE TOPOLOGY

- Explicitly model topology (typically angle-based)
    - Matchings are explicitly modeled
    - Mixed Integer Optimization
    - Local minima

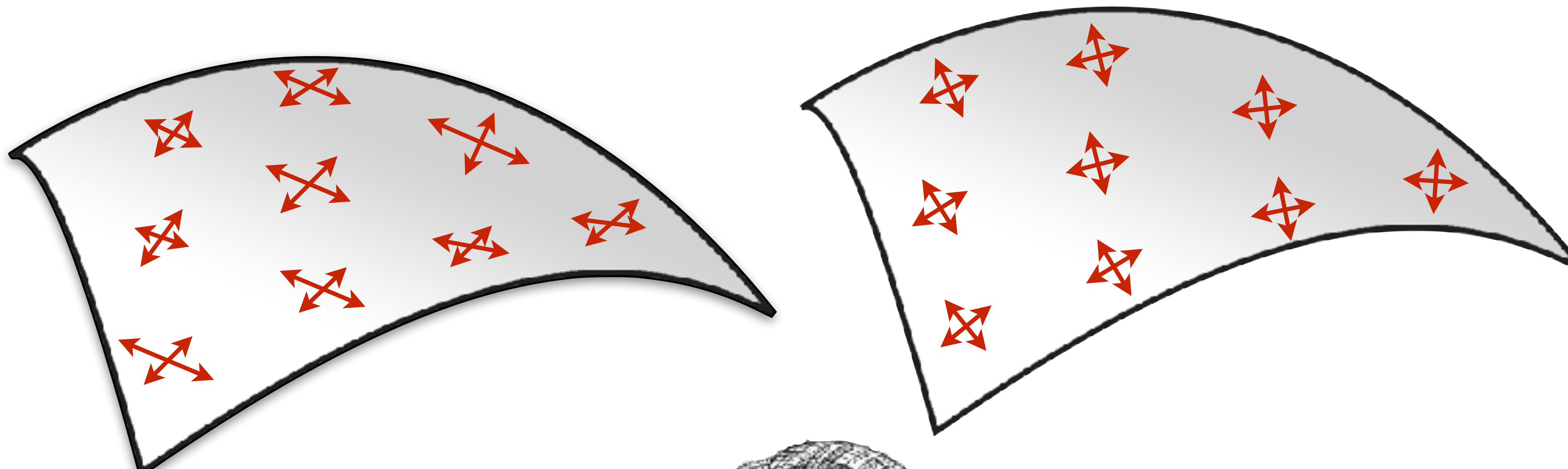


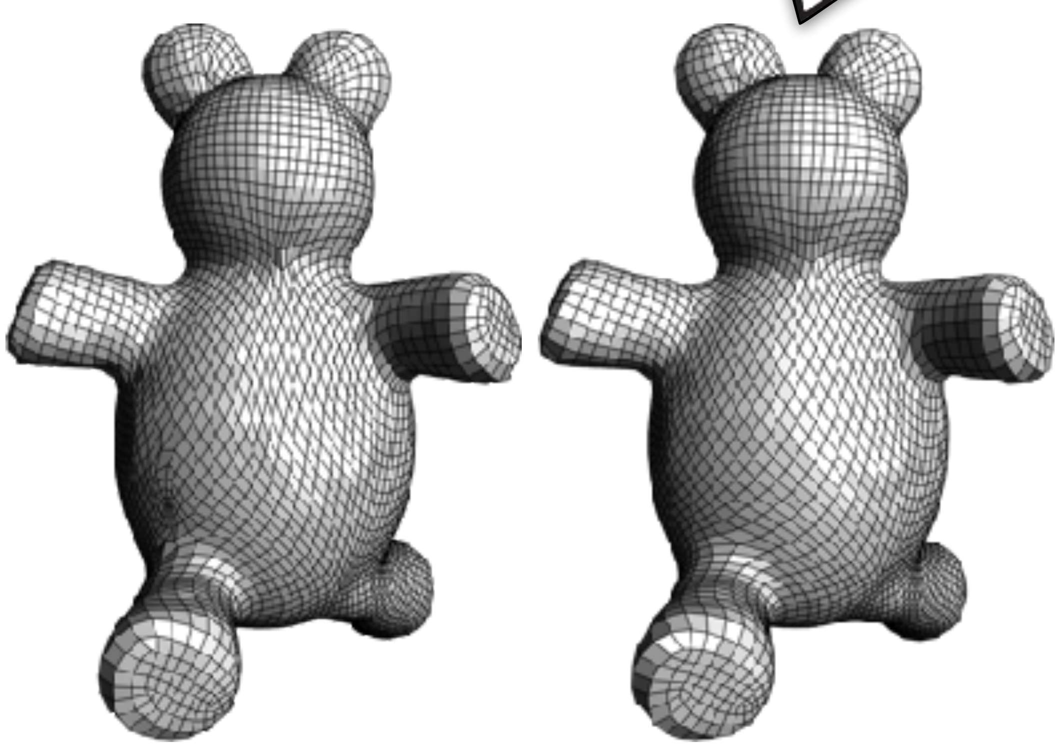
$$(\theta_f + \rho_{fg}\frac{\pi}{2} - \theta_g)^2 \quad \rho_{fg} \in \mathbb{I}$$

[Bommes et al. 2009]
[Panozzo et al. 2012]
[Jakob et al. 2015]

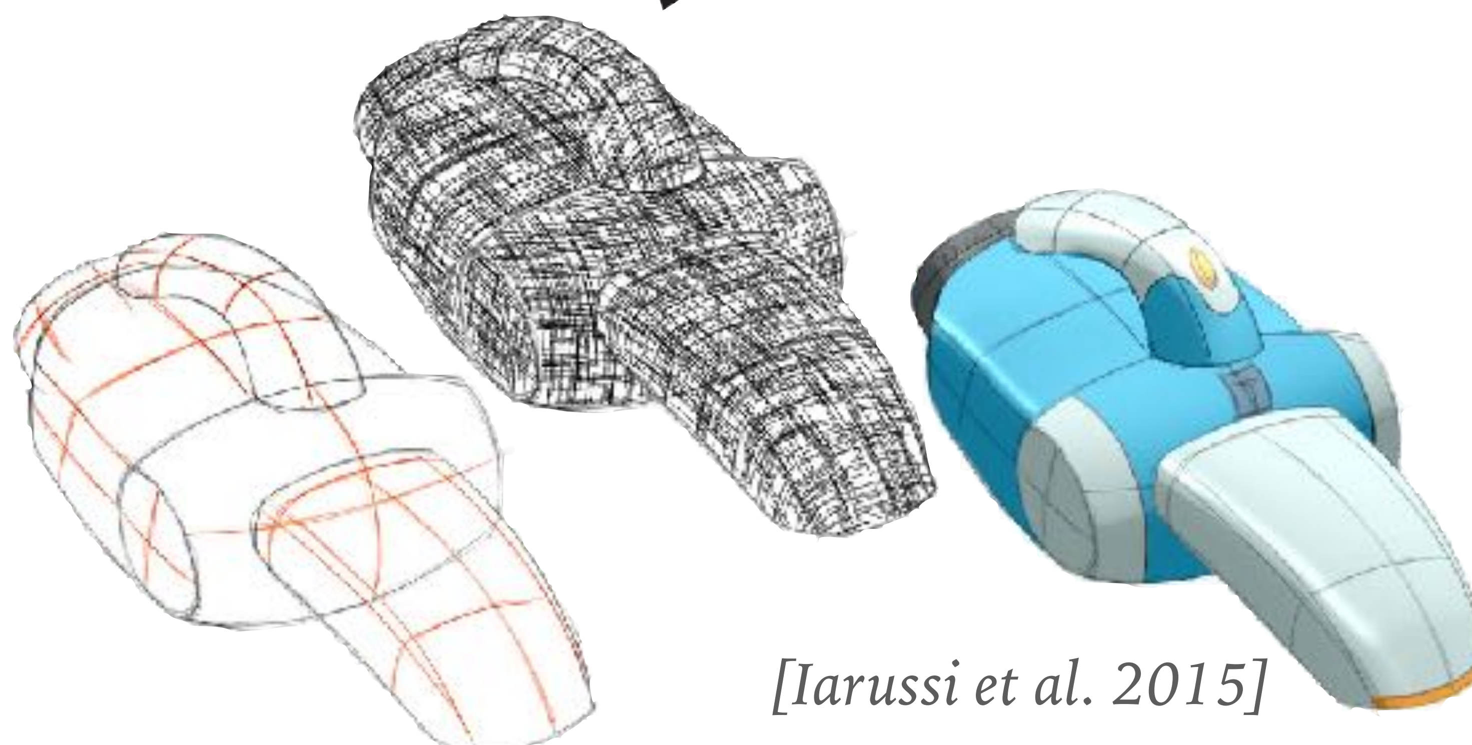
OBJECTIVES - ORTHOGONALITY

- In PolyVectors - minimize all non-free coefficients ("as-power-field-as-possible")



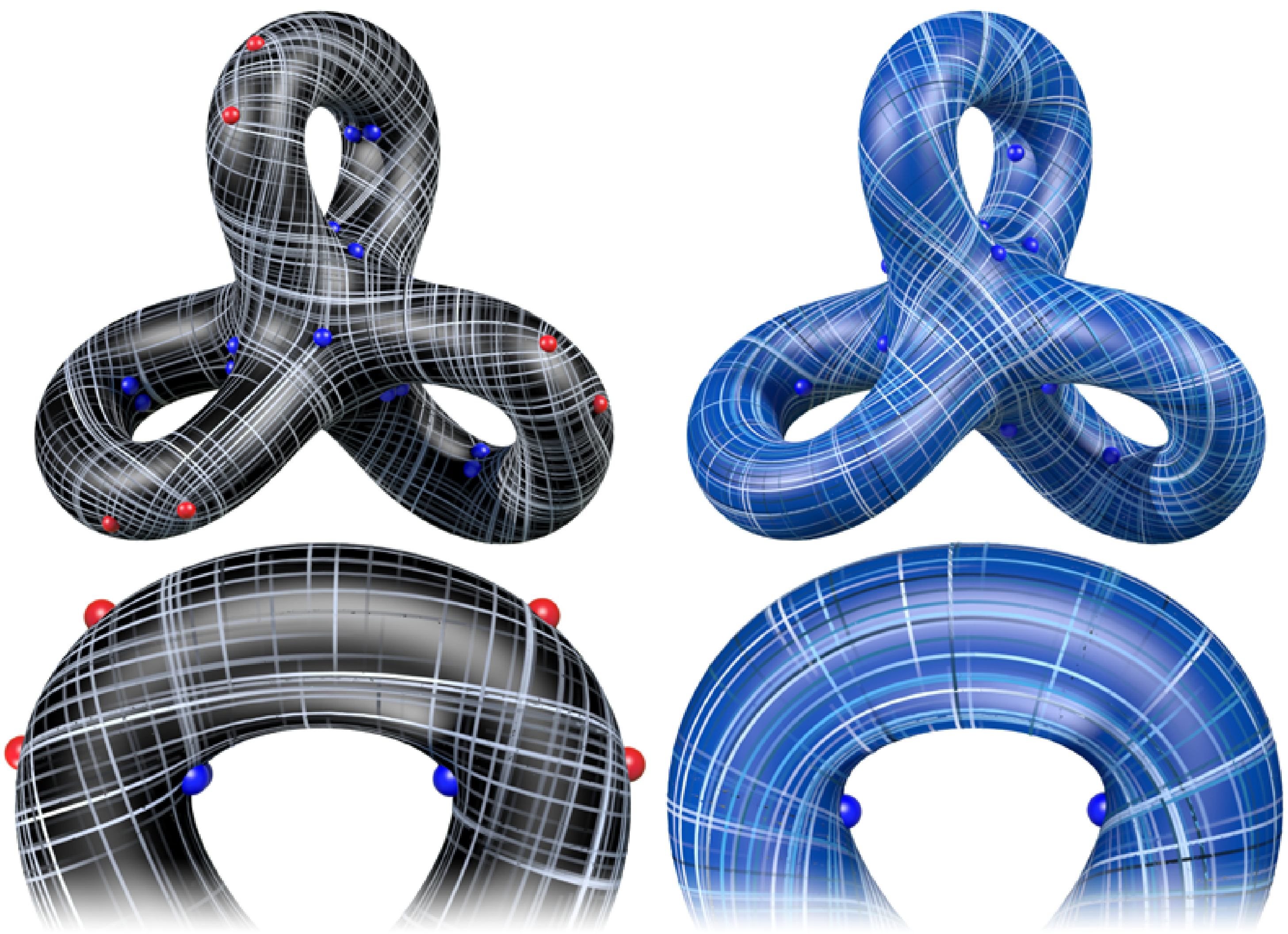


[Diamanti et al. 2014]



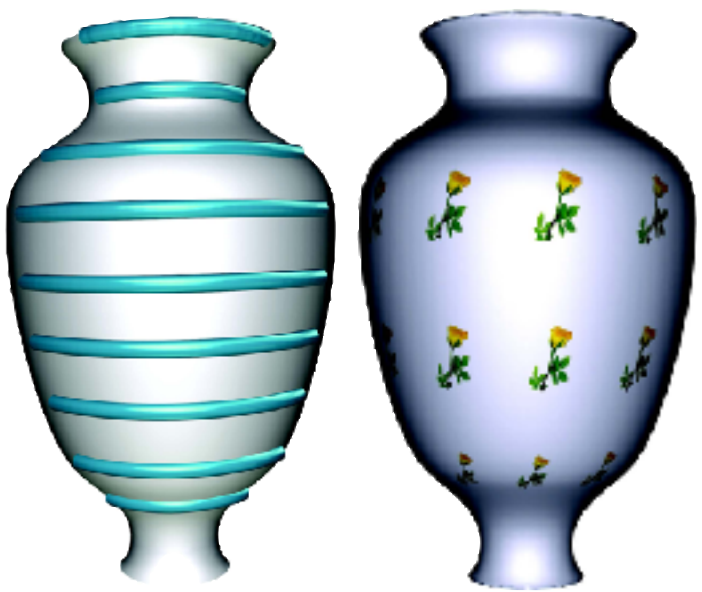
[Iarussi et al. 2015]

OBJECTIVES – SINGULARITY CONTROL



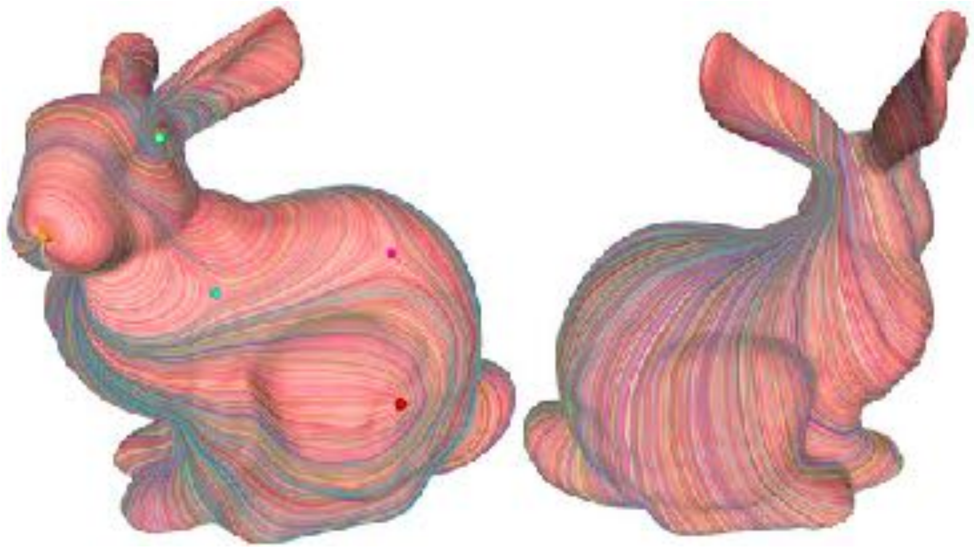
[Knöppel et al. 2013]

OBJECTIVES – ISOMETRY INDUCING



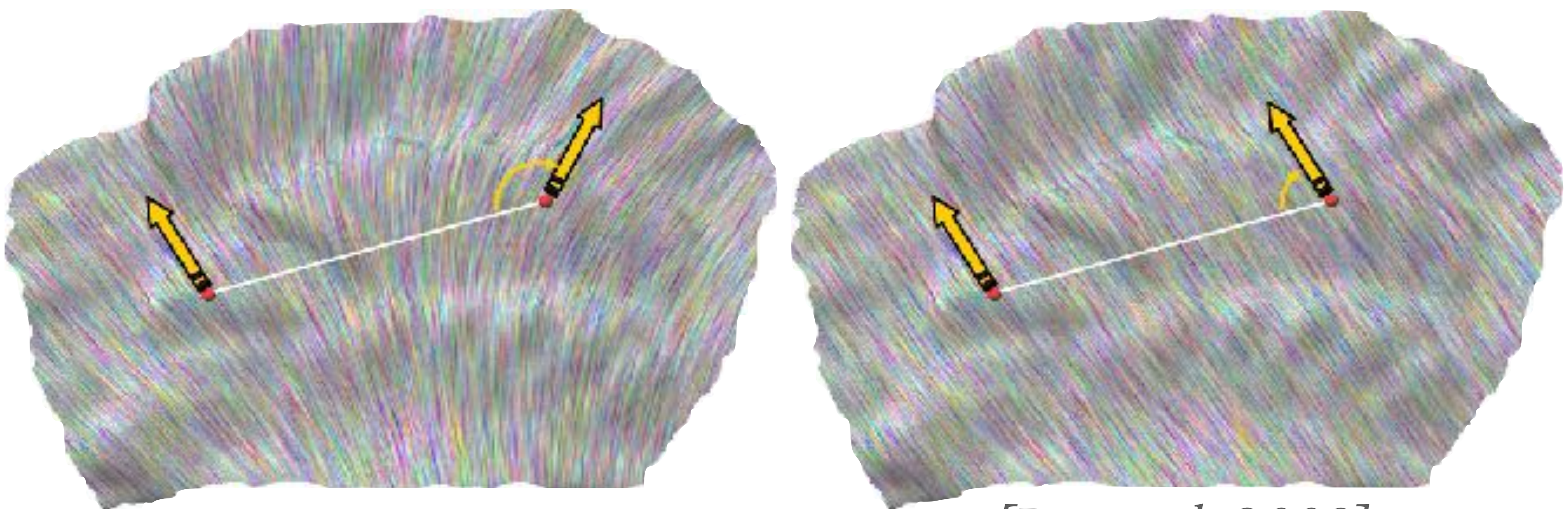
[Ben-Chen et al. 2010]

[Solomon et al. 2011]

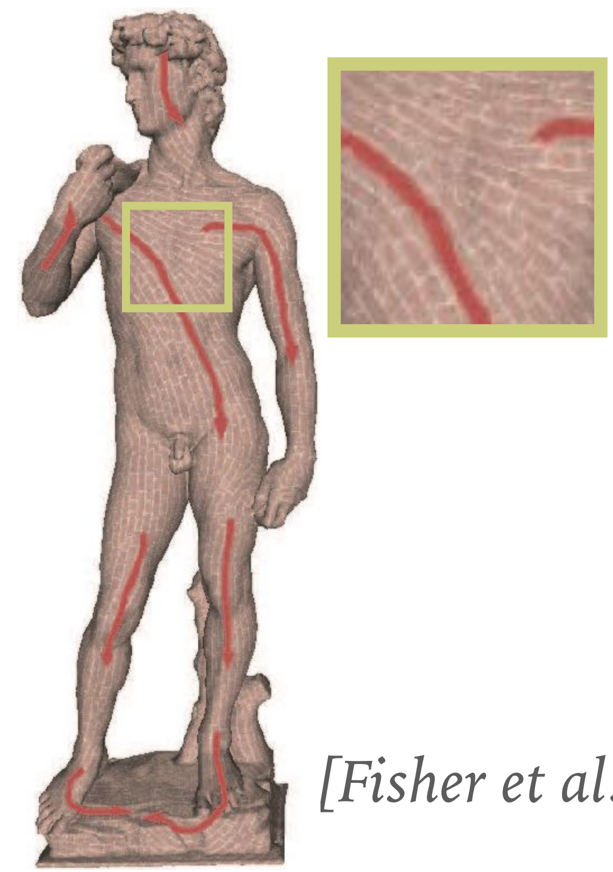


CONSTRAINTS

CONSTRAINTS - ALIGNMENT

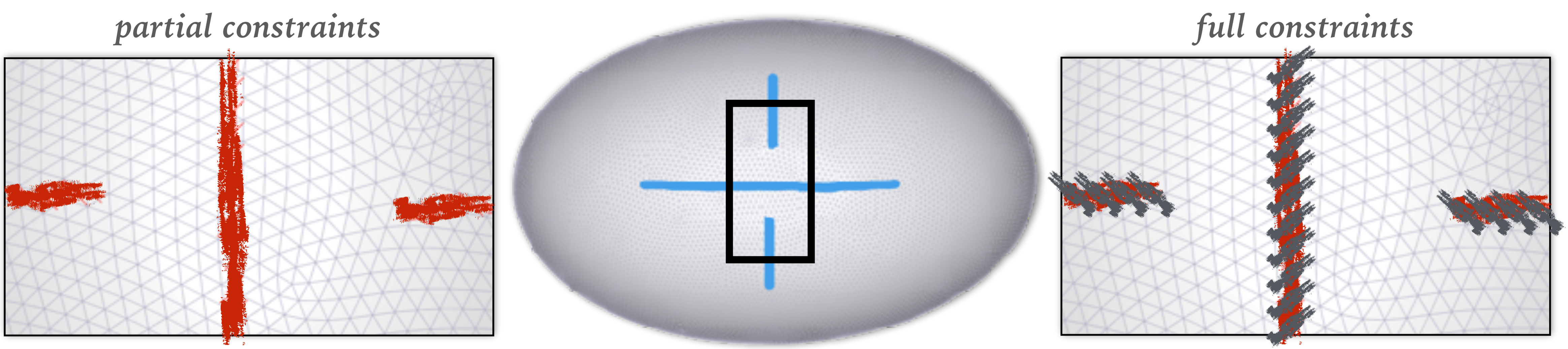


[Ray et al. 2008]

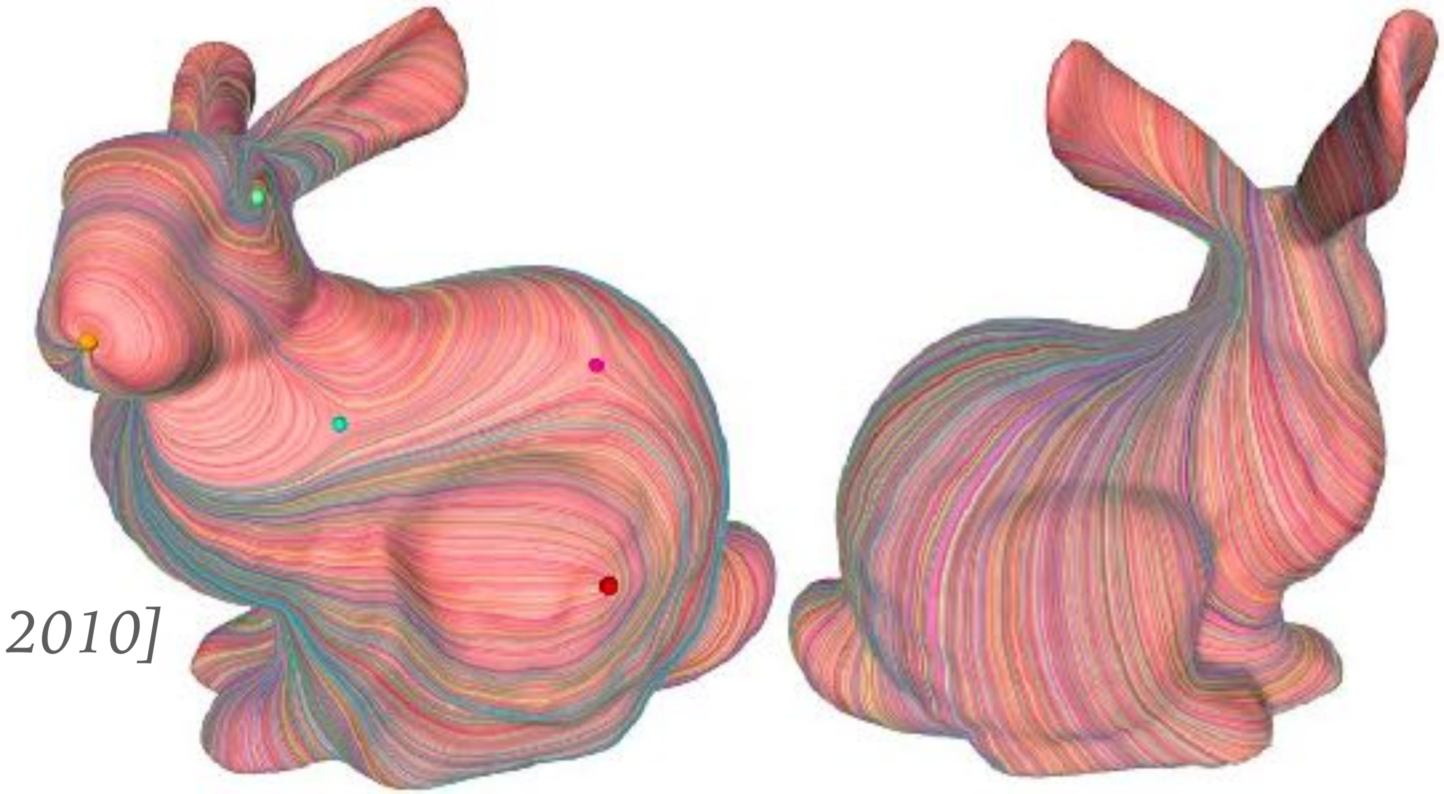


[Fisher et al. 2007]

- Complete or Partial

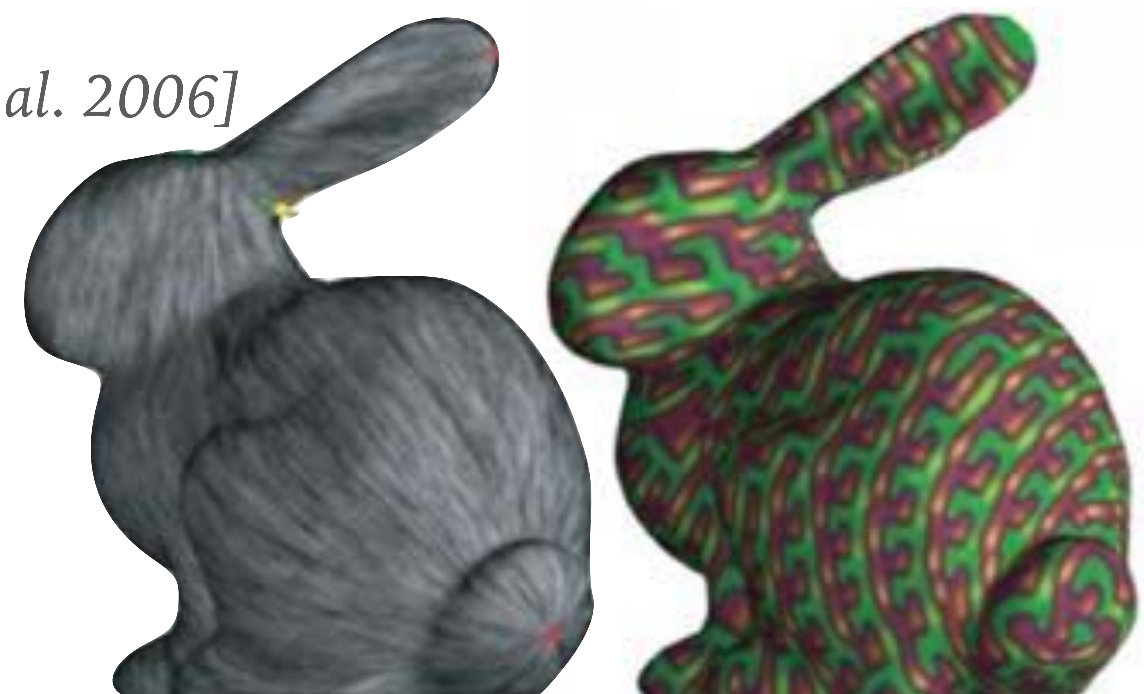


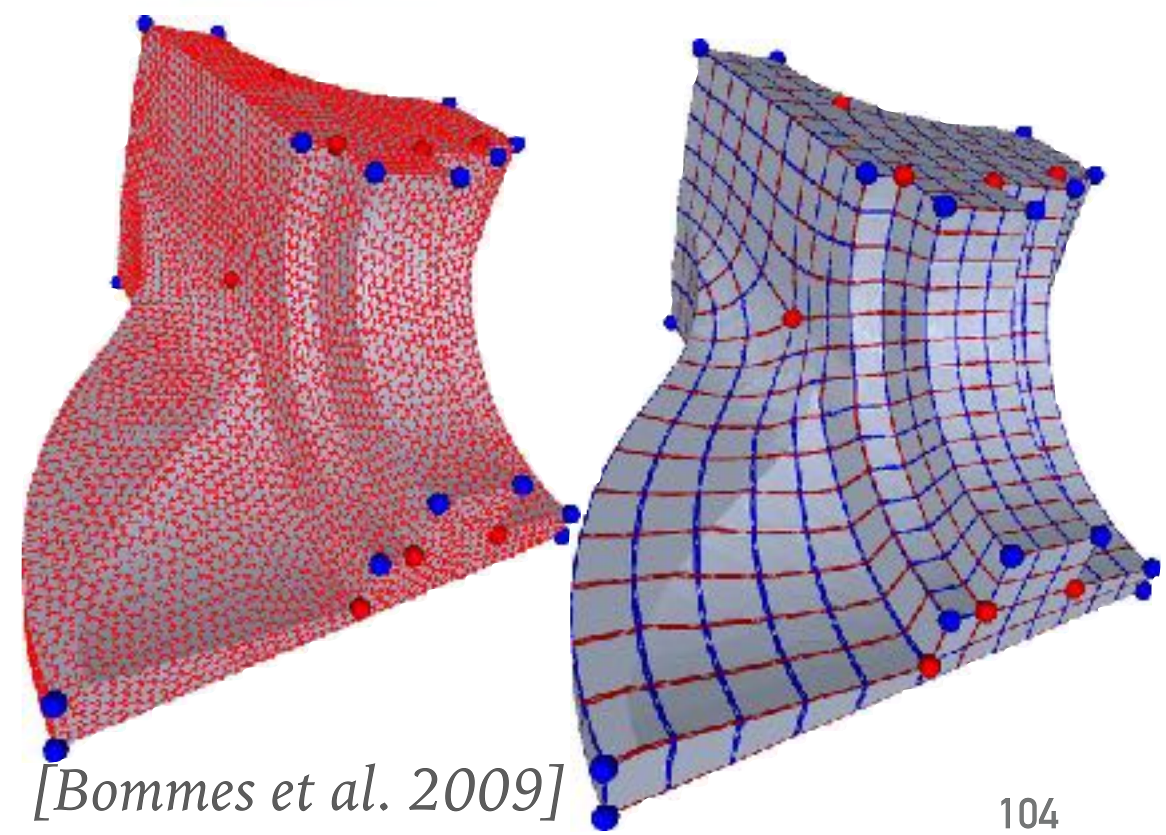
CONSTRAINTS – TOPOLOGY



[Crane et al. 2010]

[Zhang et al. 2006]

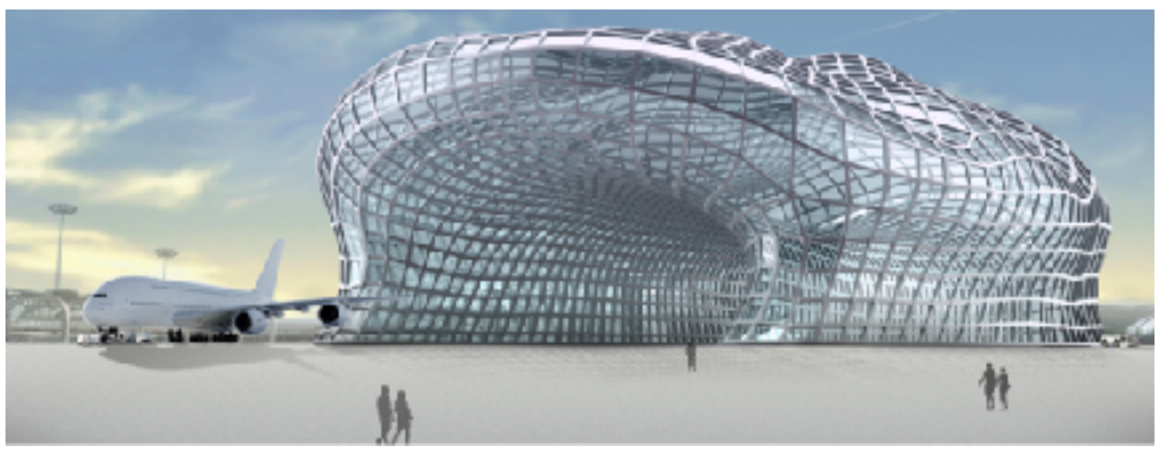




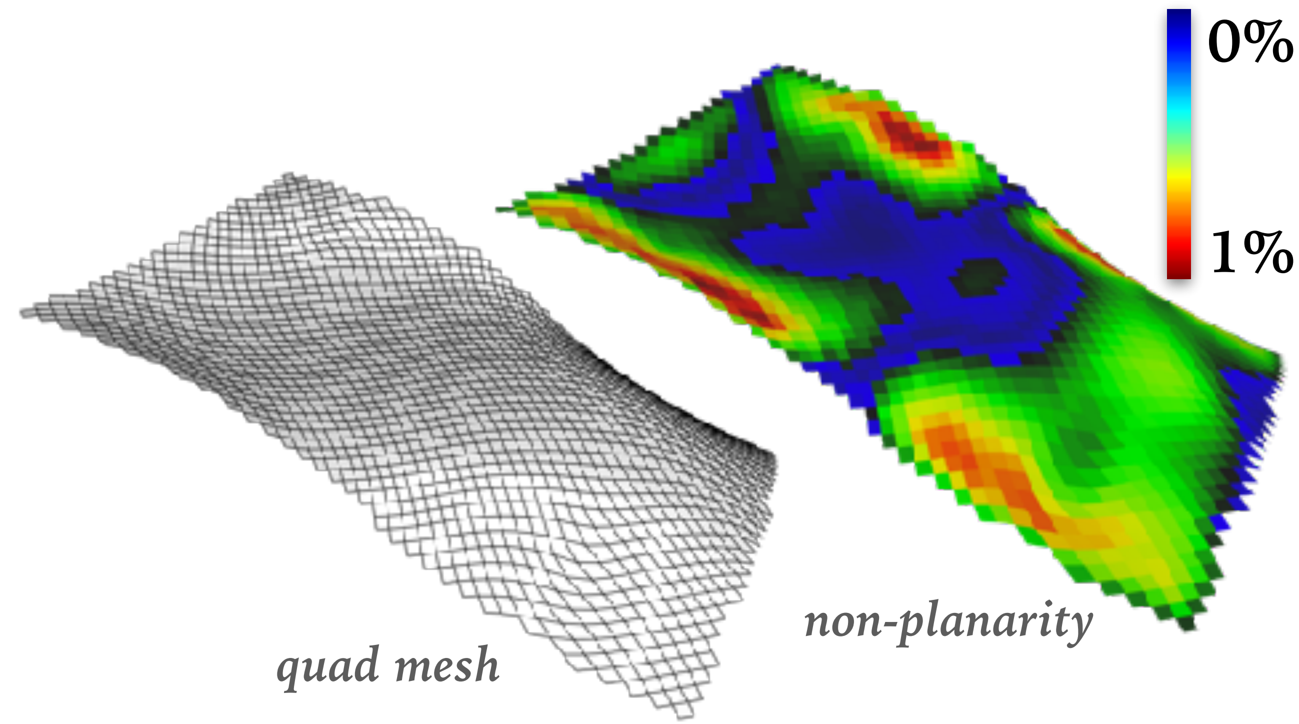
[Bommes et al. 2009]

CONSTRAINTS – CONJUGACY (PLANARITY)

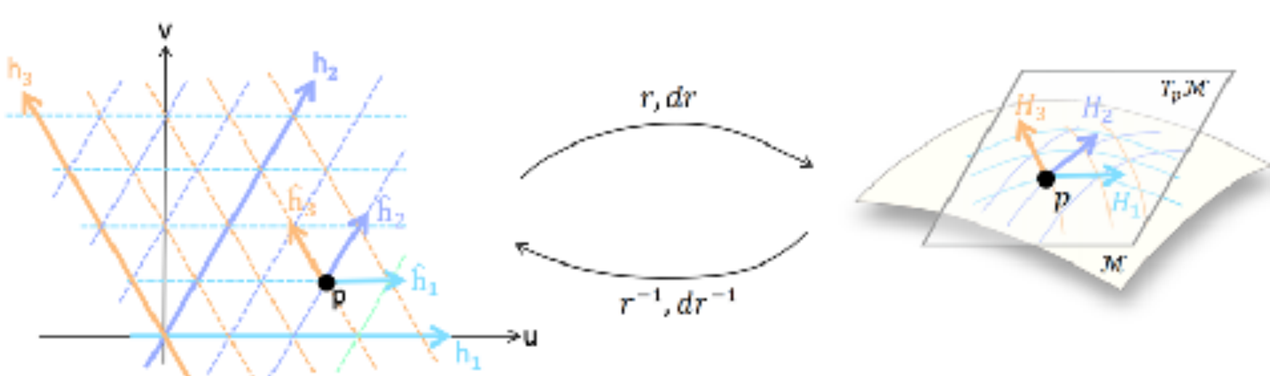
$$u_i^T II(p) u_j = 0$$

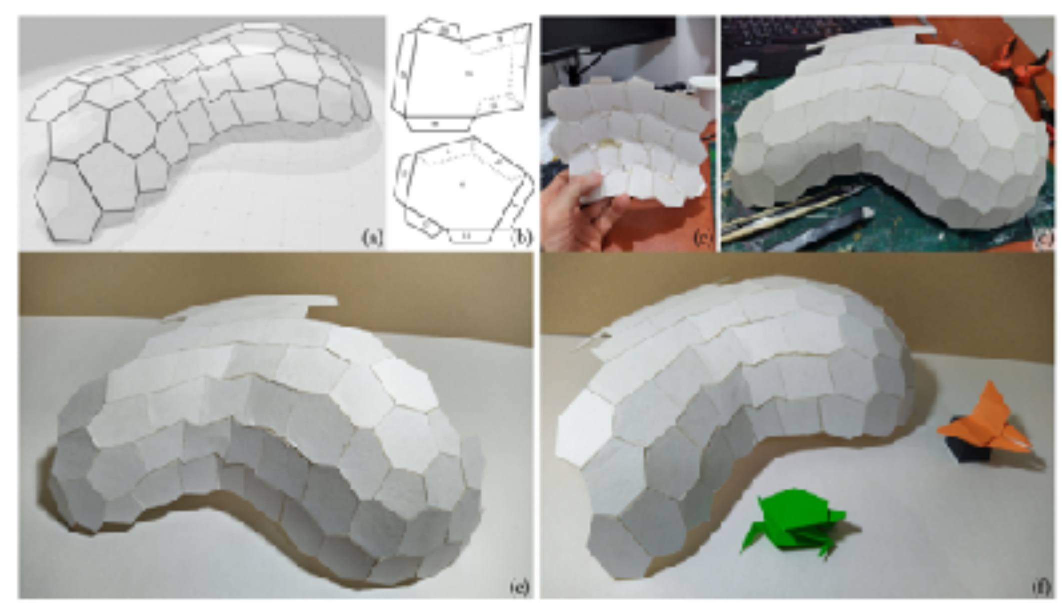


[Liu et al. 2011]



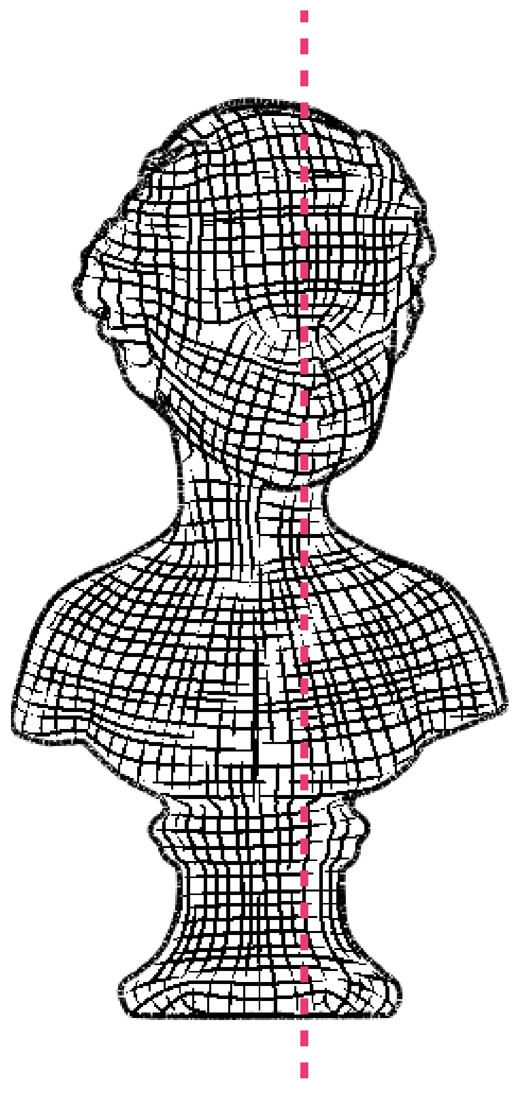
[Diamanti et al. 2014]



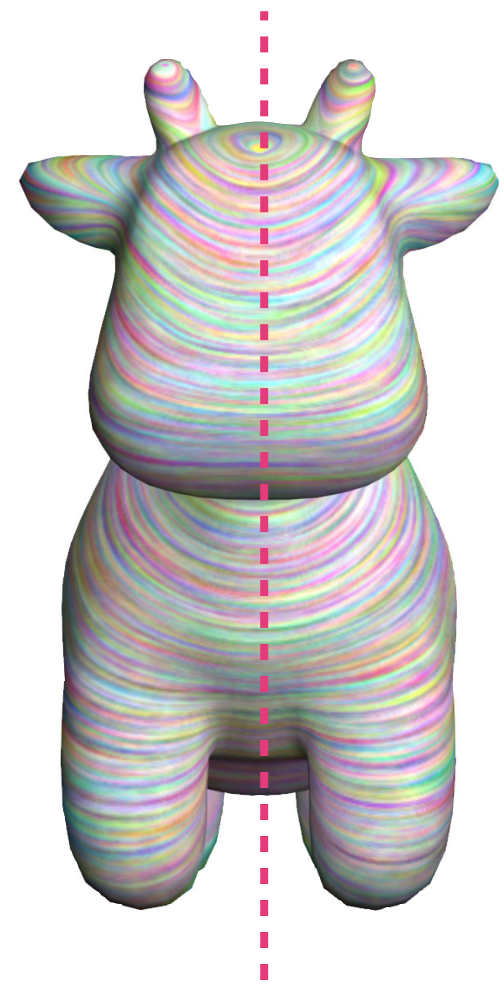


[Pluta *et al.* 2021]

CONSTRAINTS - SYMMETRY



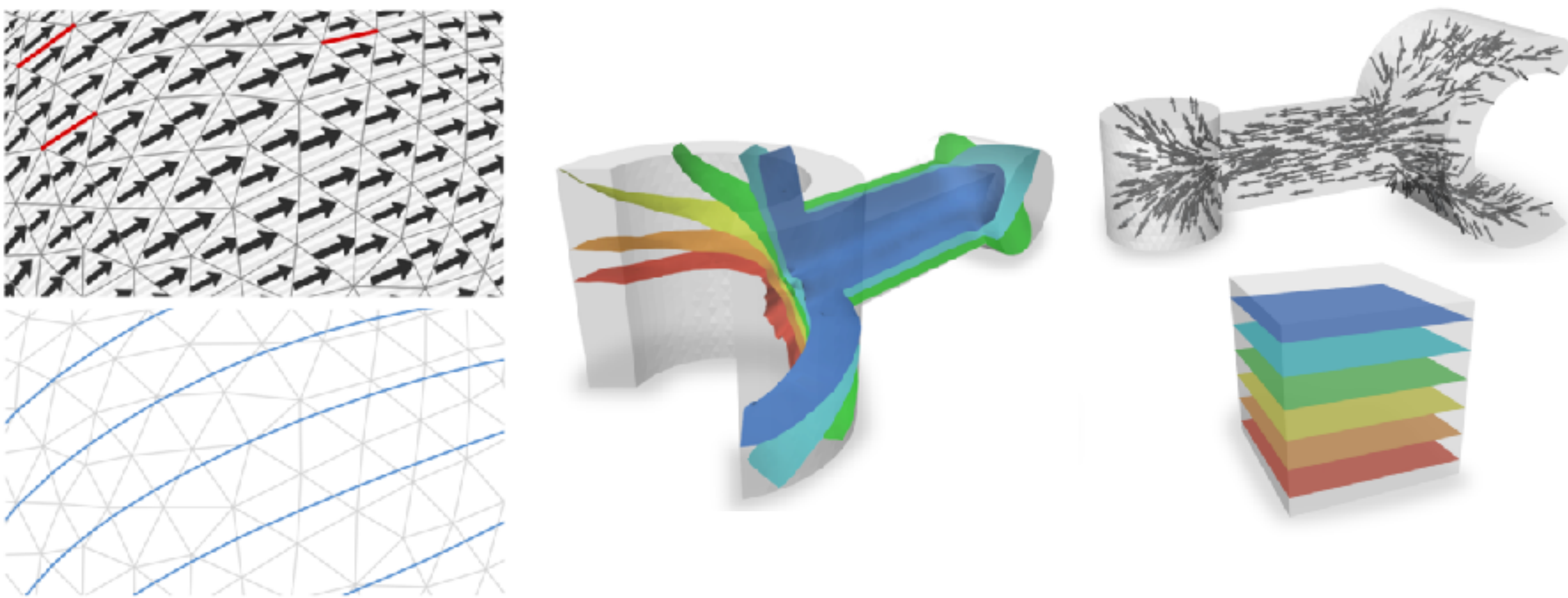
[Panozzo et al. 2012]



[Azencot et al. 2013]

CONSTRAINT: FOLIATION

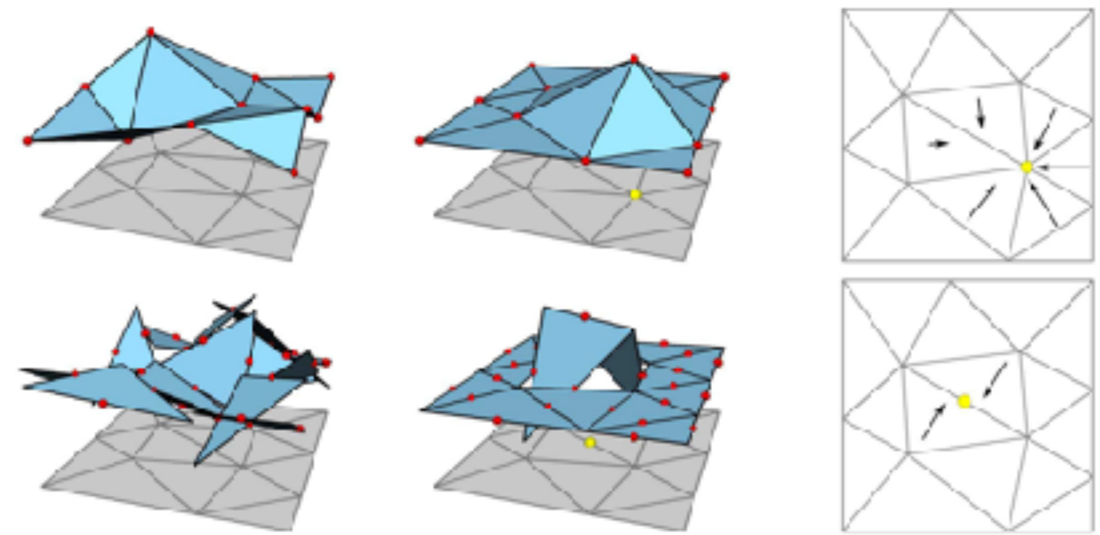
- The field should trace lines (and dual surfaces) that foliate space



[Campen *et al.* 2017]

CONSTRAINT: STRUCTURE-PRESERVATION

- Discrete setting must respect:



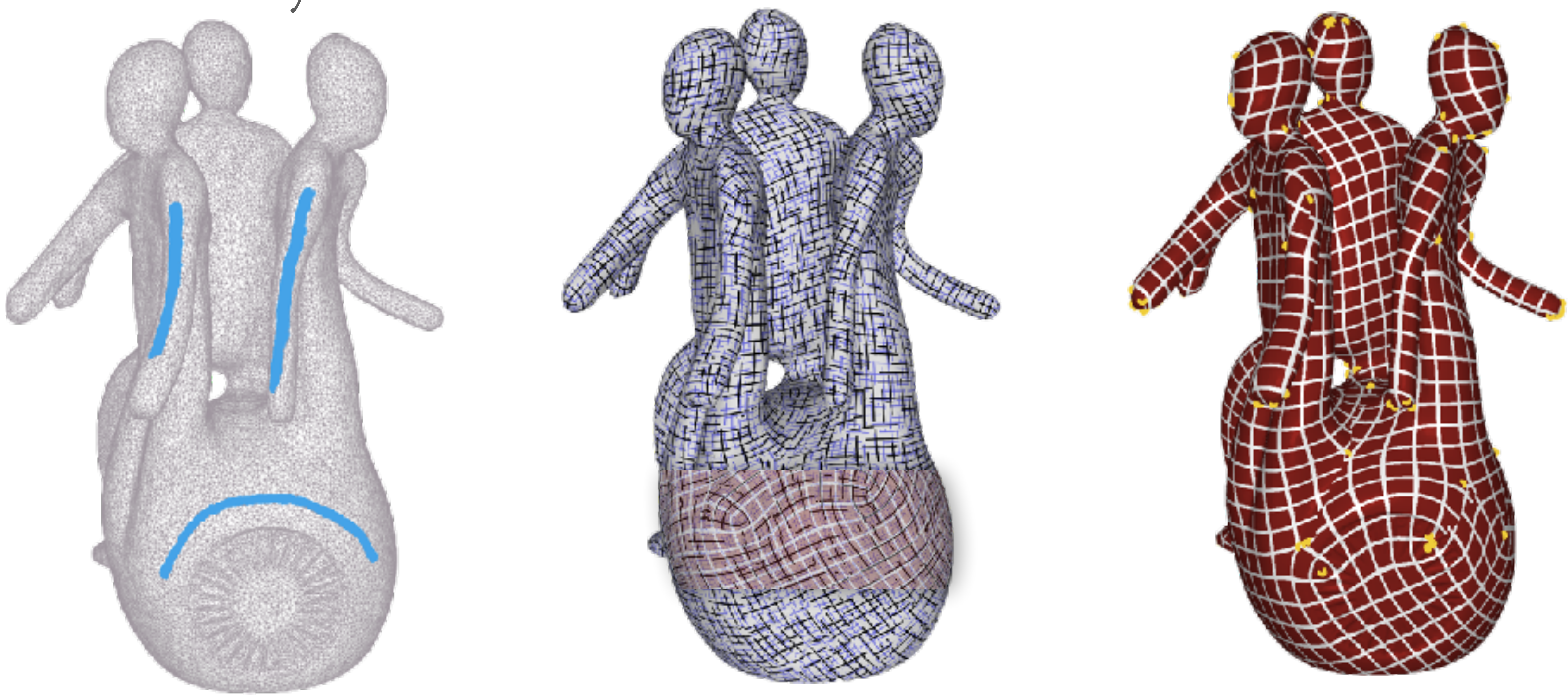
$$v = \nabla f$$

$$v = \nabla \times g$$

- By mixing conf/non-conf elements [Wardetzky 2006].
- Allowing for the existence and uniqueness of the Hodge decomposition.

CONSTRAINTS - INTEGRABILITY

- We want field to serve as *candidate gradients* for solving parameterizations.
- Necessary condition: to be curl free.

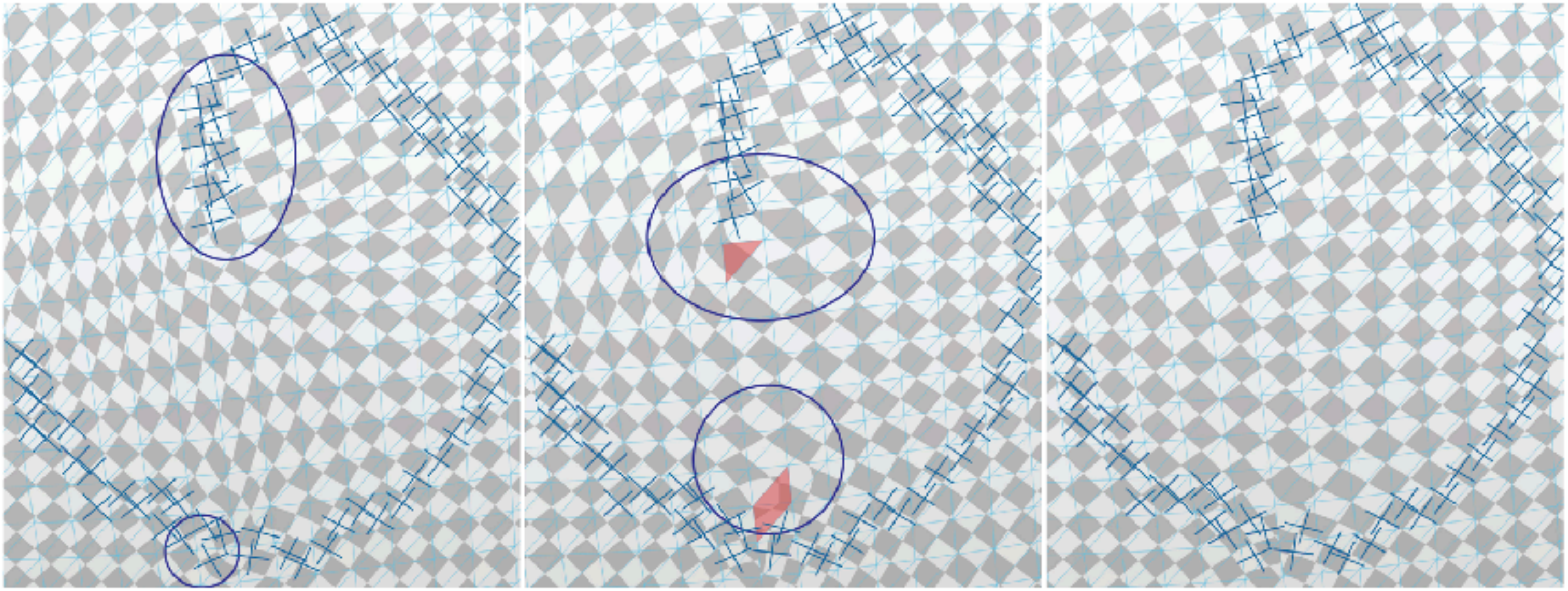


[Diamanti et al. 2015, Sageman-Furnas et al. 2019, Lu et al. 2020, Meekes et al. 2021…]

INTEGRABILITY

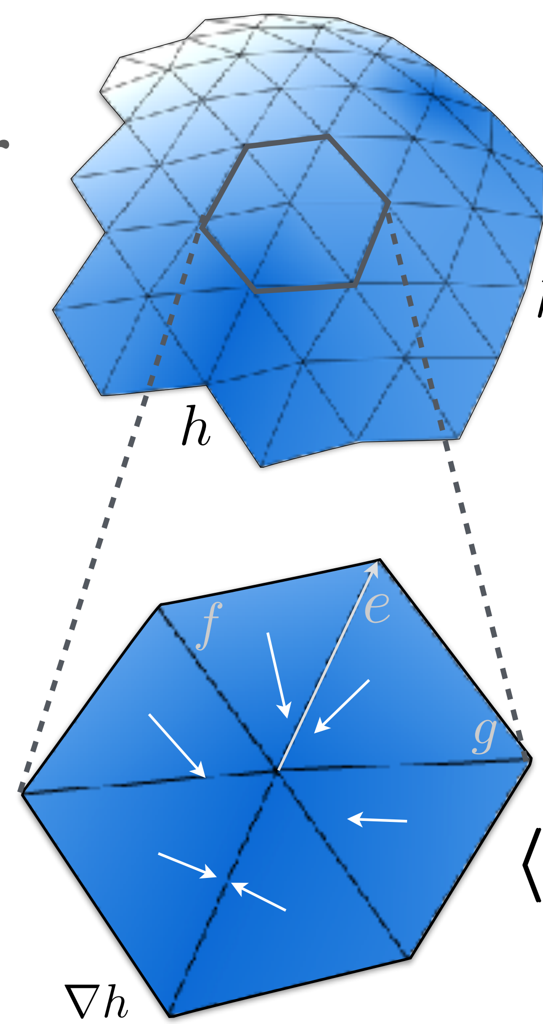
- Field to parameterization with minimum alignment error.

Non-integrable: misalignment Non-integrable: flip-overs Integrable: perfect alignment

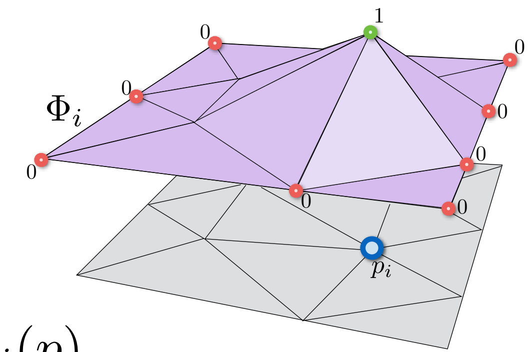


INTEGRABILITY AND DISCRETE CURL

Piecewise linear function



$$h(p) = \sum_{i \in V} h_i \Phi_i(p)$$



Gradient

$$\langle \nabla \boldsymbol{h_f}, \boldsymbol{e} \rangle = \langle \nabla \boldsymbol{h_g}, \boldsymbol{e} \rangle$$

Discrete curl (per edge)
for vector field u

$$\langle u_f, e \rangle - \langle u_g, e \rangle$$

[Polthier and Preuß 2003]

[Kälberer *et al.* 2007]

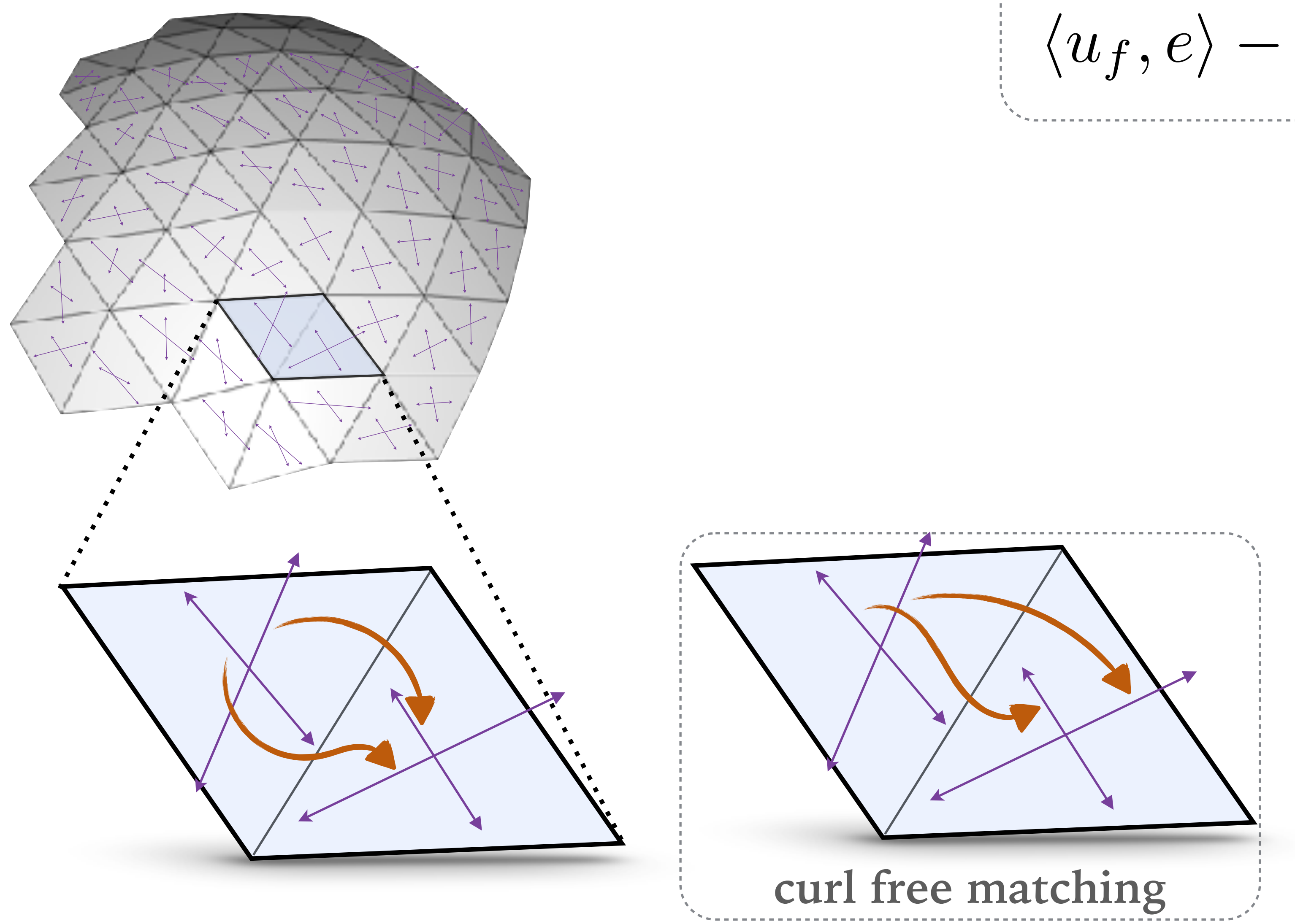
CURL-FREE FRAME FIELDS

Discrete curl

$$\langle u_f, e\rangle - \langle u_g, e\rangle$$

frame field

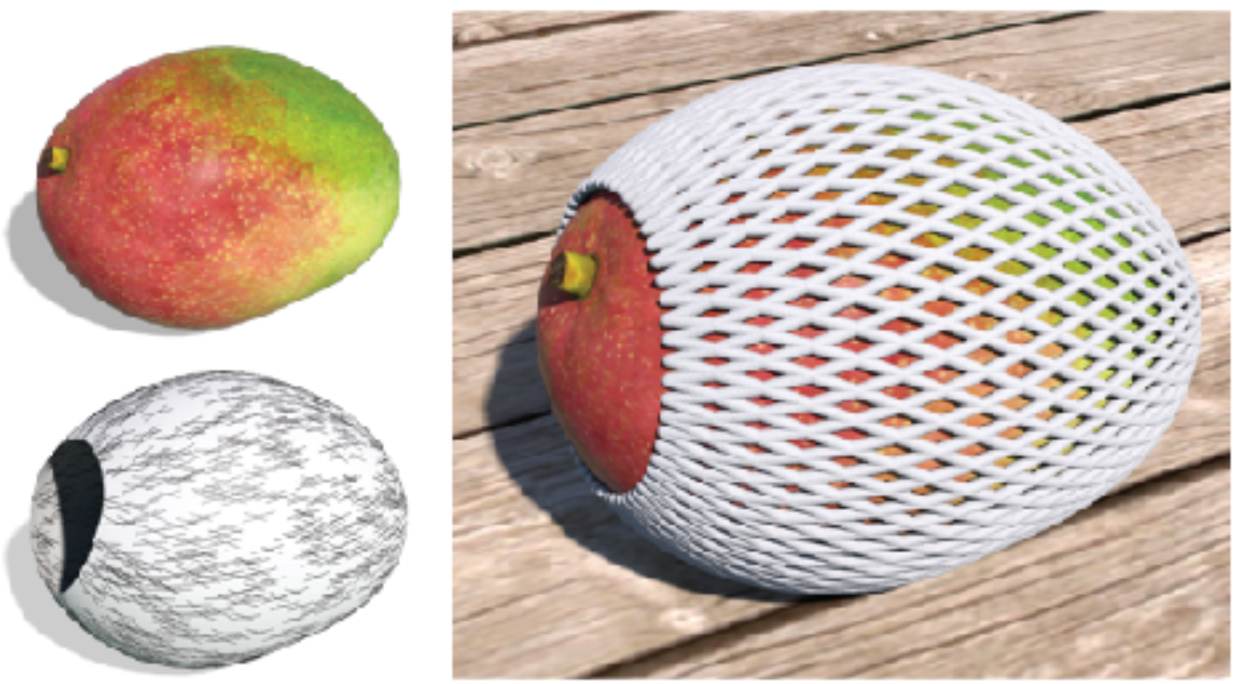
matchings



Curl-free directional field → **Zero-error seamless parameterization!***

CONSTRAINTS – UNIFORM (UNIT) LENGTH

- $|u_i| = 1$
- Application: Chebyshev nets

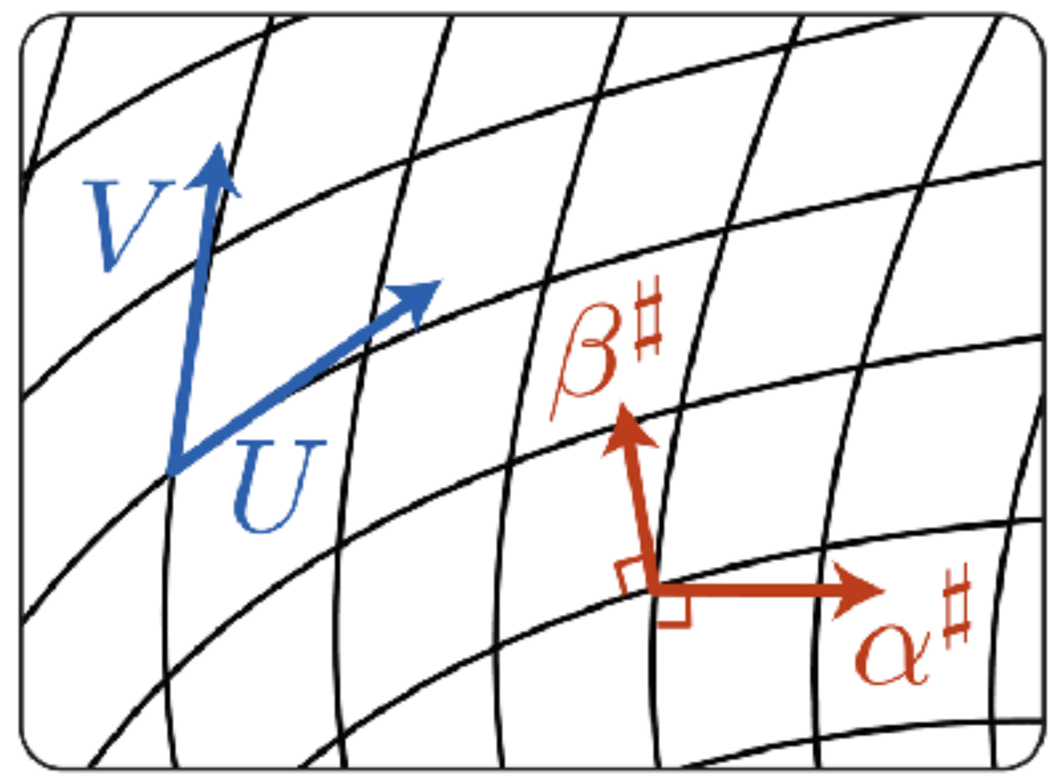


[Sageman-Furnas et al. 2019]

[Liu et al. 2020]

CHEBYSHEV PARAMETERIZATION

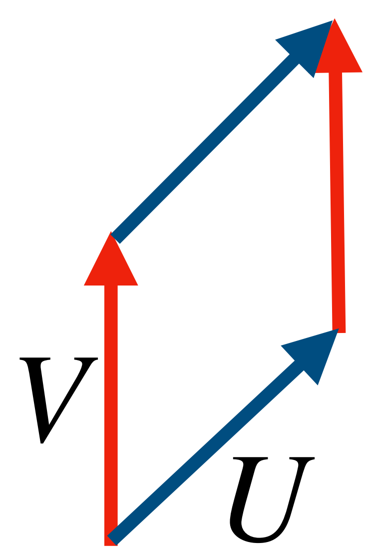
- **Problem:** traditional gradient design methods fail.
  - Gradients are normal to lines. **We need tangents**.
  - Gradients are not unit length!



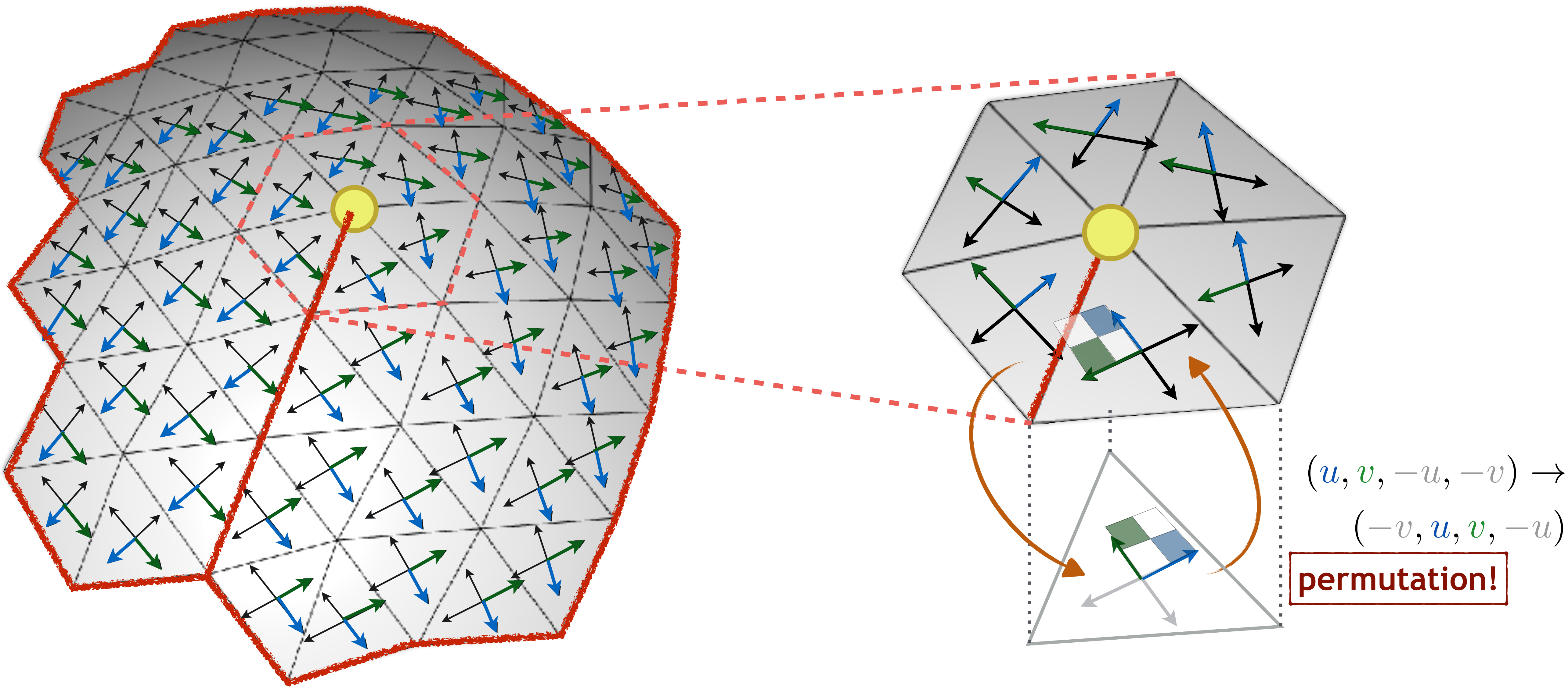
CONSTRAINTS: LIE-BRACKET INTEGRABILITY

- **Alternative:** Design unit-length tangents.
- **Integrability condition**: Optimize for zero Lie bracket:

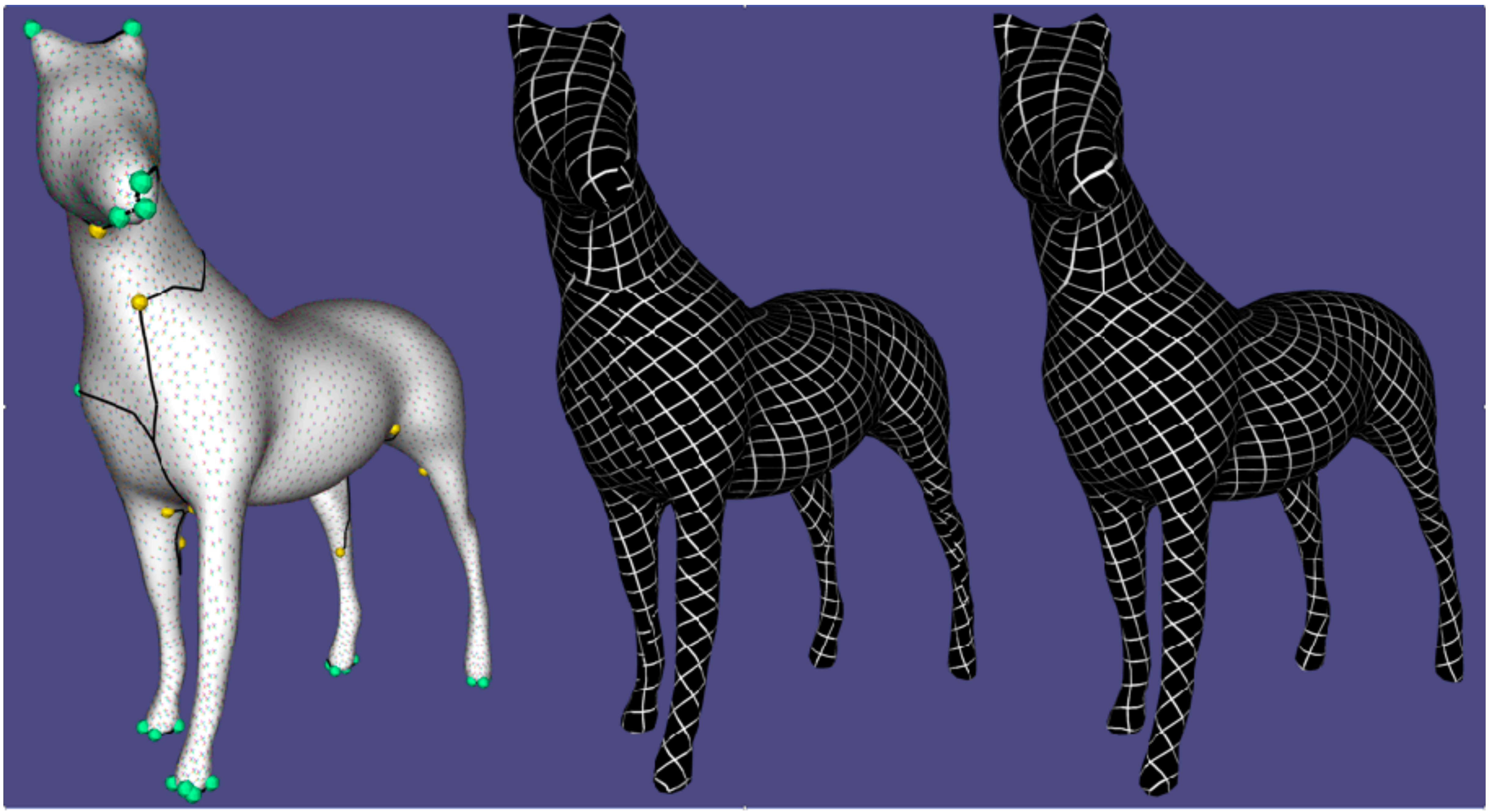
$$\nabla_U V - \nabla_V U = [U, V] = 0$$



RECOVERING THE SCALAR FUNCTIONS



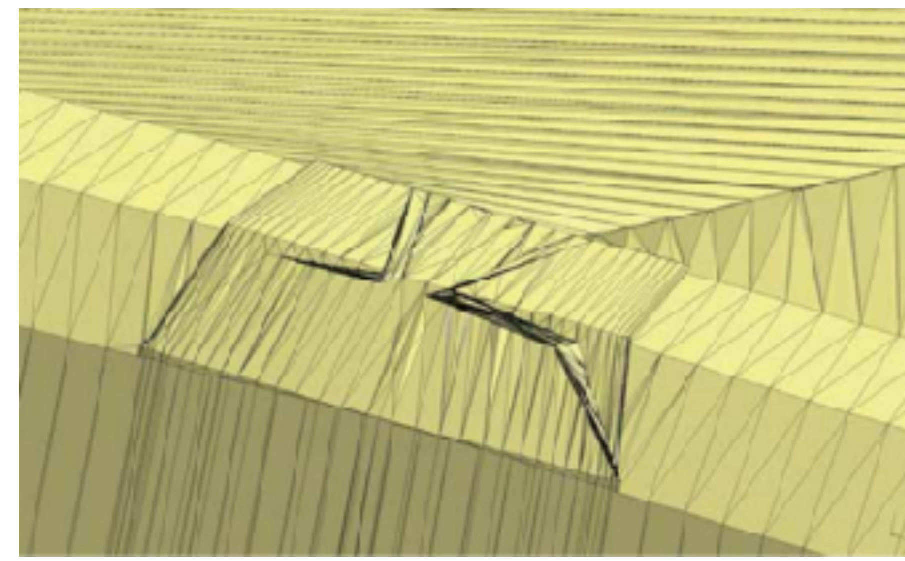
SEAMLESS PARAMETERIZATION



Field *Rot. Seamless* *Fully seamless*

CHALLENGES: DIRECTIONAL FIELDS "IN THE WILD"

- Many meshes are self-intersecting and not water-tight
- Most existing algorithms would fail!

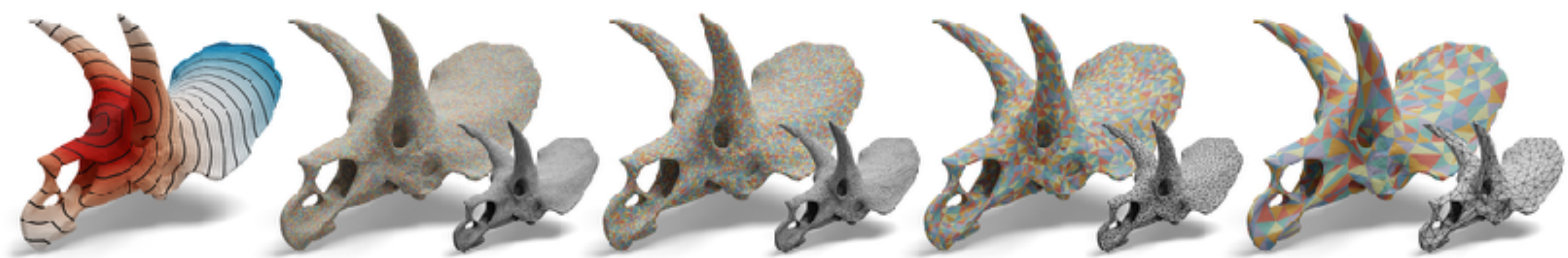


[Hu *et al.* 2019]

- Interesting directions:
  - Envelopes [Hu et al. 2019]
  - Tufted covers [Sharp and Crane 2020]
  - Level sets

CHALLENGES: BIG GEOMETRIC DATA

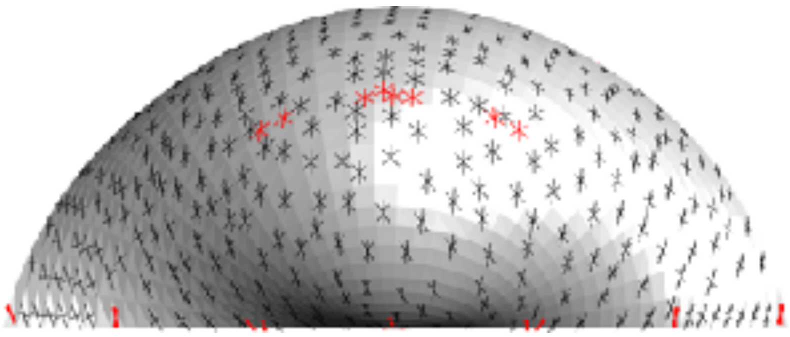
- Currently relying on sparse linear systems for design.
- Computing mostly smooth solutions.
- Interesting direction: multigrid.

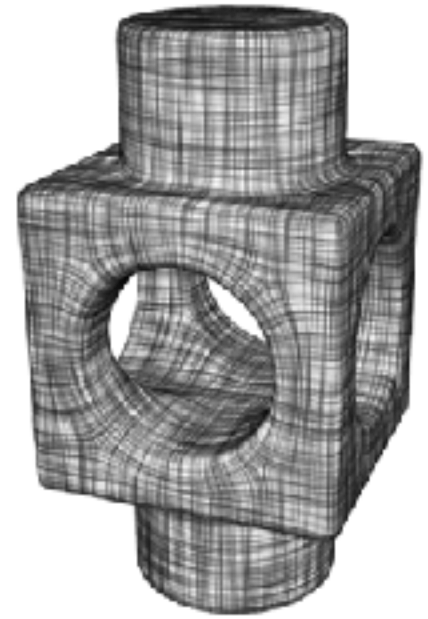


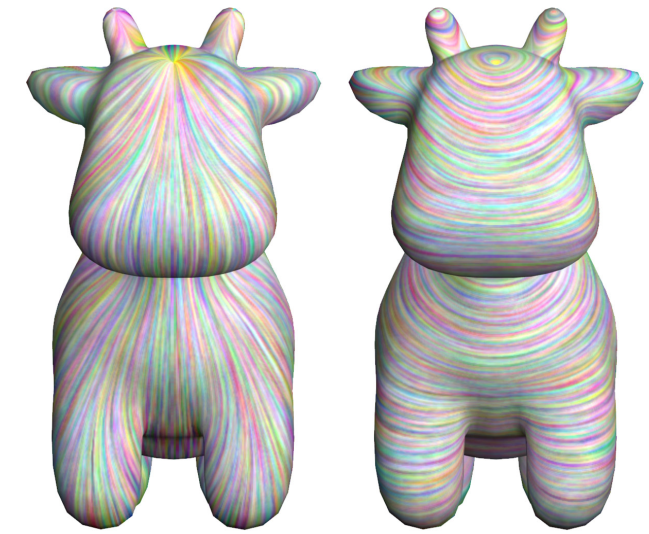
[Liu *et al.* 2021]

DIRECTIONAL-FIELD PROCESSING SOURCES

- Vector field processing on triangle meshes: https://authors.library.caltech.edu/73395/1/a27-degoes.pdf
- Directional: https://github.com/avaxman/Directional
- Geometry Central: http://geometry-central.net/tutorials/direction_fields/
- DGTal: https://dgtal.org/doc/stable/packageDEC.html
- CoMISo: https://www.graphics.rwth-aachen.de/software/comiso/
- …Code for all individual papers mentioned!







Symmetry Anti-symmetry

THANK YOU!