Notes for magic tricks project

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1 Objective function

Let $p \in \mathbb{R}^3$ be the point at which we want to balance the object, $\vec{n}$ be its unit normal and p_{com} the center of mass of the object. Then, if we define

$$\vec{v}_{\text{com}} = \frac{p_{\text{com}} - p}{|p_{\text{com}} - p|},$$

we know the scene achieves true balance if $\langle \vec{n}, \vec{v}_{\text{com}} \rangle \sim 1$. On the other hand, intuitive judgement seems to be better approximated by the center of mass of the convex hull of the object $p_{\text{com\_ch}}$. Therefore, if we let

$$\vec{v}_{\text{com\_ch}} = \frac{p_{\text{com\_ch}} - p}{|p_{\text{com\_ch}} - p|},$$

then we would like $\langle \vec{n}, \vec{v}_{\text{com\_ch}} \rangle$ to be as far away as possible from 1. Therefore, the first objective function we tried to minimize was

$$w_2 \langle \vec{n}, \vec{v}_{\text{com\_ch}} \rangle - w_1 \langle \vec{n}, \vec{v}_{\text{com}} \rangle .$$

However, the distance from the centers of mass to the point also play an important role, so we changed this objective function to

$$\boxed{w_1 |p_{\text{com}} - p|^2 \left(\langle \vec{n}, \vec{v}_{\text{com}} \rangle - 1\right)^2 + w_2 |p_{\text{com\_ch}} - p|^2 \left(\langle \vec{n}, \vec{v}_{\text{com\_ch}} \rangle + 1\right)^2}$$

2 Changing the density of a given region

Remember that a region $B \subset \mathbb{R}^3$ inside 3D space has center of mass

$$\text{com}(B) = \frac{1}{\text{mass}(B)} \int_B \vec{p}\, d\rho(\vec{p}) = \frac{1}{\text{mass}(B)} \left(\int_B x\, d\rho(\vec{p}), \int_B y\, d\rho(\vec{p}), \int_B z\, d\rho(\vec{p}) \right)$$

where $\vec{p} = (x, y, z)$, $\rho(\vec{p})$ represents the density at point $\vec{p}$ and $\text{mass}(B) = \int_B d\rho(\vec{p})$ is the total mass.

Notation. Given a region R in 3D space, keeping into account a density function ρ, its center of mass will be denoted $\text{com}(B)$. When the density is assumed to be constant equal to 1, $\rho \equiv 1$, it will be denoted $\text{com}_0(B)$. The same goes for $\text{mass}(B)$ and $\text{mass}_0(B)$.

Now consider a especial region $R \subset B$ inside the body and consider the density

$$\rho(\vec{p}) = \begin{cases} \alpha & \text{if } \vec{p} \in R \\ 1 & \text{if } \vec{p} \in B \setminus R \end{cases}.$$

Then

$$\begin{aligned} \text{mass}(B)\, \text{com}(B) &= \int_B \vec{p}\, d\rho(\vec{p}) = \alpha \int_R \vec{p}\, d\vec{p} + \int_{B \setminus R} \vec{p}\, d\vec{p} \\ &= (\alpha - 1) \int_R \vec{p}\, d\vec{p} + \int_B \vec{p}\, d\vec{p} \\ &= (\alpha - 1)\text{vol}(R)\, \text{com}_0(R) + \text{vol}(B)\, \text{com}_0(B) . \end{aligned}$$

So, using $\text{mass}(B) = \alpha \text{vol}(R) + \text{vol}(B \setminus R) = (\alpha - 1)\text{vol}(R) + \text{vol}(B)$ we deduce

$$\text{com}(B) = \frac{(\alpha - 1)\text{vol}(R)}{(\alpha - 1)\text{vol}(R) + \text{vol}(B)} \cdot \text{com}_0(R) + \frac{\text{vol}(B)}{(\alpha - 1)\text{vol}(R) + \text{vol}(B)} \cdot \text{com}_0(B)$$

which can be rewritten as

$$\boxed{\mathrm{com}\left(B\right) = (1-\theta)\cdot \mathrm{com}_0\left(R\right) + \theta\cdot \mathrm{com}_0\left(B\right), \quad \theta := \frac{\mathrm{vol}\left(B\right)}{(\alpha-1)\mathrm{vol}\left(R\right)+\mathrm{vol}\left(B\right)}}$$

Case 1. The especial case where $R\subset B$ is a hollow region of the body (i.e. $\alpha = 0$) we get

$$\boxed{\mathrm{com}\left(B\right) = (1-\theta)\cdot \mathrm{com}_0\left(R\right) + \theta\cdot \mathrm{com}_0\left(B\right), \quad \theta := \frac{\mathrm{vol}\left(B\right)}{\mathrm{vol}\left(B\right)-\mathrm{vol}\left(R\right)}}$$

Case 2. If we use ABS material with a metal ball on the inside then we get $\alpha\approx 8$ and

$$\theta\frac{\mathrm{vol}\left(B\right)}{(\alpha-1)\mathrm{vol}\left(R\right)+\mathrm{vol}\left(B\right)} = \frac{1}{C\frac{r^3}{\mathrm{vol}(B)}+1}.$$

Theta decreases A LOT the more we can increase r. Therefore, the bigger the ball, the more the COM moves.