Economics of Financial Markets: Take-Home Exam (Question 1)

Gaspar Suliansky and Santiago Greloni

Introduction

For this question, we are dealing with daily and monthly stock prices for 88 Italian companies, starting from January 1st, 2015 to June 24th, 2022, present in "data for exam 2022.xlsx". For the computations, which can be found in the files "Code-Daily.ipynb" and "Code-Monthly.ipynb", we used the Python programming language.

Returns on Stocks: Statistics

Firstly, we start by computing the returns for each data point, both for daily and monthly data:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

This results can be found in the "Take-Home Exam.xlsx" Excel file for daily and monthly data. We also compute returns descriptive statistics (mean, standard deviation, variance, skewness, and kurtosis), which are shown below in Table 1 for daily frequency, and Table 2 for monthly frequency:

$$\mu = \frac{1}{T}\sum_{t=1}^{T} r_t$$

$$\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2}$$

$$\sigma^2 = \frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2$$

$$S = \frac{\frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^3}{[\frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2]^{\frac{3}{2}}}$$

$$K = T \frac{\sum_{t=1}^{T} (r_t - \mu)^4}{[\sum_{t=1}^{T} (r_t - \mu)^2]^2}$$

Company	μ	σ	σ^2	**Skewness**	**Kurtosis**
LDO	0.0396%	0.0238	0.000566	-0.282	12.227
ECK	0.0382%	0.0324	0.001050	2.617	19.447
LRZ	0.0355%	0.0314	0.000983	1.106	11.834
PIRL	-0.0156%	0.0226	0.000512	-0.127	6.545
STL	0.0786%	0.0251	0.000628	-0.458	5.416
PINF	0.0282%	0.0372	0.001382	-0.541	88.580
BRE	0.0459%	0.0195	0.000381	0.124	4.067
ISP	0.0106%	0.0215	0.000464	-0.786	12.733
ILLB	0.0620%	0.0193	0.000373	0.235	8.217
UCG	-0.0097%	0.0276	0.000764	-0.055	7.288
BANC	0.0293%	0.0194	0.000376	-0.418	5.872
BPE	0.0031%	0.0295	0.000869	0.213	7.655
FCBK	0.0691%	0.0203	0.000411	-0.095	4.033
CPR	0.0828%	0.0162	0.000263	-0.254	9.068
SPAC	0.0055%	0.0245	0.000601	0.354	7.919
CALT	0.0486%	0.0170	0.000290	0.206	5.233
AST	-0.0464%	0.0387	0.001495	0.284	24.329
ENEL	0.0333%	0.0159	0.000252	-1.186	15.349
ARN	0.1561%	0.0259	0.000672	1.746	12.609
A2A	0.0353%	0.0164	0.000268	-1.028	12.151
TRN	0.0448%	0.0139	0.000194	-0.692	9.273
ACE	0.0374%	0.0158	0.000251	-0.208	7.369
DEA	-0.0066%	0.0154	0.000236	-1.061	11.858
BMED	0.0317%	0.0201	0.000405	-0.430	5.889
BIM	-0.0584%	0.0490	0.002401	1.090	16.182
TIPS	0.0660%	0.0163	0.000265	0.204	11.928
MB	0.0360%	0.0216	0.000465	-0.853	11.890
EQUI	0.0277%	0.0168	0.000283	0.965	11.801
ANI	0.0267%	0.0245	0.000602	0.135	5.074
TITR	-0.0244%	0.0237	0.000563	0.134	19.898
TIT	-0.0335%	0.0241	0.000580	0.588	19.018
ENV	0.0181%	0.0186	0.000346	0.932	16.667
VAL	-0.0018%	0.0183	0.000334	0.678	6.656
CLT	0.0184%	0.0197	0.000386	2.947	28.966
HER	0.0302%	0.0154	0.000236	-0.537	15.013
IRE	0.0566%	0.0157	0.000246	-0.646	7.199
IG	0.0346%	0.0152	0.000231	-0.613	7.169
ELN	0.1316%	0.0243	0.000589	0.307	3.541

Company	μ	σ	σ^2	**Skewness**	**Kurtosis**
AMP	0.1124%	0.0198	0.000394	-0.496	7.072
DLG	0.0329%	0.0207	0.000427	0.110	3.958
BOR	0.0775%	0.0283	0.000799	4.772	144.205
CNHI	0.0609%	0.0229	0.000525	-0.390	4.784
FD	0.0197%	0.0326	0.001064	2.754	22.644
IP	0.0801%	0.0191	0.000367	-0.169	3.021
IKG	0.0519%	0.0215	0.000462	0.527	4.430
ENAV	0.0202%	0.0157	0.000246	1.323	19.345
PST	0.0331%	0.0180	0.000324	-1.327	17.727
CASS	0.0314%	0.0218	0.000476	3.110	55.001
RCS	0.0161%	0.0263	0.000693	0.985	10.778
CAI	-0.0300%	0.0211	0.000444	0.406	4.530
MON	-0.0437%	0.0262	0.000686	2.386	22.307
GAMB	-0.0275%	0.0333	0.001109	3.198	26.864
UNI	0.0290%	0.0221	0.000488	-0.242	8.671
G	0.0098%	0.0160	0.000255	-0.730	12.633
SRG	0.0307%	0.0148	0.000219	-1.273	18.218
ENI	0.0056%	0.0182	0.000331	-1.034	18.690
TOD	-0.0180%	0.0225	0.000507	0.885	9.812
REC	0.0758%	0.0175	0.000306	0.000	13.664
RN	0.0724%	0.0352	0.001240	1.449	9.140
BRI	0.0187%	0.0235	0.000550	0.553	7.591
FUL	0.0105%	0.0332	0.001105	2.468	19.894
AISW	0.0874%	0.0303	0.000917	1.857	14.529
AGL	0.0368%	0.0238	0.000568	1.314	29.681
JUVE	0.0786%	0.0267	0.000713	0.249	8.536
SSL	0.0703%	0.0259	0.000670	0.235	8.992
CLE	-0.0826%	0.0308	0.000948	1.591	15.113
B	-0.0383%	0.0234	0.000548	1.255	11.174
CEM	0.0341%	0.0212	0.000450	0.190	2.662
US	0.0161%	0.0163	0.000267	-0.131	4.811
BZU	0.0428%	0.0204	0.000418	-0.021	4.306
CE	0.0115%	0.0185	0.000342	0.020	2.923
DAN	0.0213%	0.0210	0.000440	0.603	9.709
ITM	0.0686%	0.0175	0.000308	2.410	36.099
ZUC	0.0360%	0.0376	0.001414	2.919	45.934
IPG	-0.0053%	0.0249	0.000622	-0.290	11.461
VIN	0.0115%	0.0174	0.000302	0.230	3.627
EDNR	0.0288%	0.0144	0.000209	-0.406	14.299
RAT	0.0357%	0.0200	0.000399	1.018	14.845
GAB	0.0558%	0.0283	0.000800	1.137	7.629
MS	-0.0257%	0.0240	0.000575	1.267	24.572
ERG	0.0758%	0.0173	0.000299	-0.156	14.073

Company	μ	σ	σ^2	**Skewness**	**Kurtosis**
CMB	0.0614%	0.0179	0.000319	0.078	3.894
SAB	0.0574%	0.0187	0.000350	0.434	5.314
BE	0.0239%	0.0240	0.000578	2.029	14.325
SOL	0.0601%	0.0177	0.000315	0.411	1.940
DAL	0.0212%	0.0244	0.000594	0.439	7.922
BSS	0.0572%	0.0279	0.000776	-0.159	5.774
SAFI	-0.0356%	0.0295	0.000872	0.426	10.079

Table 1: Daily returns statistics

Company	μ	σ	σ^2	**Skewness**	**Kurtosis**
LDO	0.961%	0.114	0.013	0.193	2.800
ECK	2.170%	0.314	0.098	7.514	65.225
LRZ	0.988%	0.177	0.031	3.536	20.782
PIRL	-0.273%	0.095	0.009	-0.547	1.440
STL	1.932%	0.121	0.015	-0.248	1.150
PINF	-0.317%	0.126	0.016	1.105	3.909
BRE	1.167%	0.091	0.008	-0.174	-0.075
ISP	0.288%	0.099	0.010	-0.406	2.101
ILLB	1.213%	0.093	0.009	-1.399	5.162
UCG	-0.196%	0.120	0.014	-0.334	1.707
BANC	0.821%	0.092	0.009	-0.648	1.032
BPE	0.042%	0.127	0.016	0.432	0.650
FCBK	1.529%	0.086	0.007	-0.124	0.194
CPR	1.747%	0.070	0.005	-0.333	0.572
SPAC	-0.068%	0.103	0.011	-0.023	1.257
CALT	1.116%	0.074	0.005	-0.101	0.832
AST	-1.139%	0.159	0.025	-0.364	5.106
ENEL	0.725%	0.062	0.004	-0.183	1.200
ARN	3.533%	0.133	0.018	1.751	3.889
A2A	0.936%	0.069	0.005	-1.071	2.746
TRN	0.920%	0.046	0.002	-0.148	-0.402
ACE	0.987%	0.076	0.006	-0.323	0.375
DEA	-0.014%	0.081	0.007	0.472	2.653
BMED	0.789%	0.092	0.009	-0.471	2.419
BIM	-2.275%	0.177	0.031	0.867	2.627
TIPS	1.497%	0.064	0.004	0.039	0.044
MB	0.860%	0.096	0.009	-0.700	2.192
EQUI	0.690%	0.070	0.005	0.008	2.145
ANI	0.758%	0.113	0.013	0.068	0.878
TITR	-0.531%	0.102	0.010	0.738	1.873
TIT	-0.704%	0.109	0.012	1.348	4.838

Company	μ	σ	σ^2	**Skewness**	**Kurtosis**
ENV	0.295%	0.066	0.004	-0.013	0.632
VAL	-0.044%	0.080	0.006	0.975	2.582
CLT	0.347%	0.071	0.005	2.026	6.892
HER	0.818%	0.061	0.004	-0.562	1.532
IRE	1.359%	0.072	0.005	-0.592	1.483
IG	1.064%	0.061	0.004	-0.274	0.834
ELN	3.152%	0.134	0.018	0.180	0.202
AMP	2.458%	0.084	0.007	-0.703	1.495
DLG	0.832%	0.088	0.008	-0.198	0.266
BOR	2.067%	0.181	0.033	3.942	23.944
CNHI	1.446%	0.097	0.009	-0.235	2.597
FD	1.028%	0.215	0.046	4.715	31.211
IP	1.891%	0.092	0.008	-0.584	0.250
IKG	1.134%	0.096	0.009	1.017	2.960
ENAV	0.449%	0.066	0.004	-0.300	2.640
PST	0.781%	0.071	0.005	-0.354	1.127
CASS	0.879%	0.121	0.015	0.914	3.924
RCS	0.445%	0.127	0.016	1.042	2.267
CAI	-0.486%	0.099	0.010	0.426	0.686
MON	-1.178%	0.099	0.010	1.819	8.572
GAMB	0.063%	0.235	0.055	5.448	39.700
UNI	0.718%	0.100	0.010	-0.547	1.784
G	0.276%	0.075	0.006	-0.226	2.155
SRG	0.658%	0.052	0.003	-0.307	-0.241
ENI	0.232%	0.074	0.006	0.818	3.137
TOD	-0.275%	0.108	0.012	1.376	4.349
REC	1.561%	0.072	0.005	0.028	0.783
RN	1.413%	0.171	0.029	1.346	3.327
BRI	0.336%	0.092	0.008	-0.183	1.397
FUL	0.472%	0.202	0.041	4.855	28.906
AISW	2.290%	0.166	0.027	2.524	14.050
AGL	0.930%	0.118	0.014	1.366	9.133
JUVE	2.074%	0.155	0.024	1.349	5.724
SSL	1.780%	0.140	0.020	0.866	3.846
CLE	-2.058%	0.117	0.014	0.855	2.756
B	-1.186%	0.061	0.004	1.216	4.577
CEM	0.800%	0.100	0.010	0.669	0.837
US	0.415%	0.074	0.005	-0.135	0.787
BZU	0.896%	0.076	0.006	-0.116	-0.115
CE	0.273%	0.078	0.006	0.704	4.423
DAN	0.508%	0.086	0.007	-0.183	0.836
ITM	1.499%	0.077	0.006	1.988	8.554
ZUC	-0.119%	0.099	0.010	1.598	5.996

Company	μ	σ	σ^2	**Skewness**	**Kurtosis**
IPG	-0.100%	0.109	0.012	0.776	1.717
VIN	0.135%	0.052	0.003	0.462	1.945
EDNR	0.631%	0.053	0.003	0.098	3.854
RAT	0.632%	0.061	0.004	0.172	1.850
GAB	1.896%	0.179	0.032	2.403	8.120
MS	-0.388%	0.124	0.015	3.413	20.887
ERG	1.658%	0.071	0.005	0.039	1.654
CMB	1.515%	0.084	0.007	-0.138	3.742
SAB	1.416%	0.097	0.009	0.149	0.296
BE	0.354%	0.094	0.009	1.525	4.705
SOL	1.318%	0.060	0.004	0.218	-0.639
DAL	0.563%	0.110	0.012	0.355	-0.103
BSS	1.464%	0.128	0.016	-0.209	0.410
SAFI	-0.586%	0.144	0.021	1.015	3.061

Table 2: Monthly returns statistics

Variance-covariance and correlation matrices for both daily and monthly frequencies returns can be found respectively in the "Take-Home Exam.xlsx" Excel file.

Securities sample selection

To execute a sample selection of 12 securities, the first criterion we choose is to select between those companies that have fully available data (non 'NaN' values). Moreover, the second criteria consists of an iterative greedy algorithm: First, we select the two securities with the least correlation value out of all. We then proceed with the following logic: For each of the $n-2$ non-selected companies, we take the highest correlation value between them and the 2 already-selected securities. Then, we choose the security for whose this maximum correlation value obtained is minimum among the rest. Afterwards, we continue with the algorithm: Take the $n-3$ non-selected companies, compute the correlations with the 3 already-selected companies and select the highest of the 3 values for each of them; choose the security whose maximum correlation value is minimum among the rest; and so forth until all 12 securities are chosen.

This approach allow us to have the 12 securities with the least maximum correlation between them. The motivation behind this criteria can be shown in Figure 1, which shows the curve of return as a function of volatility for a portfolio integrated by two assets, α_1 and α_2. As it can be seen, as the correlation coefficient ρ decreases, one can achieve higher return per unit of risk, due to diversification of idiosyncratic risk.

(Note that the selected securities through this method are not the same for daily and monthly data).

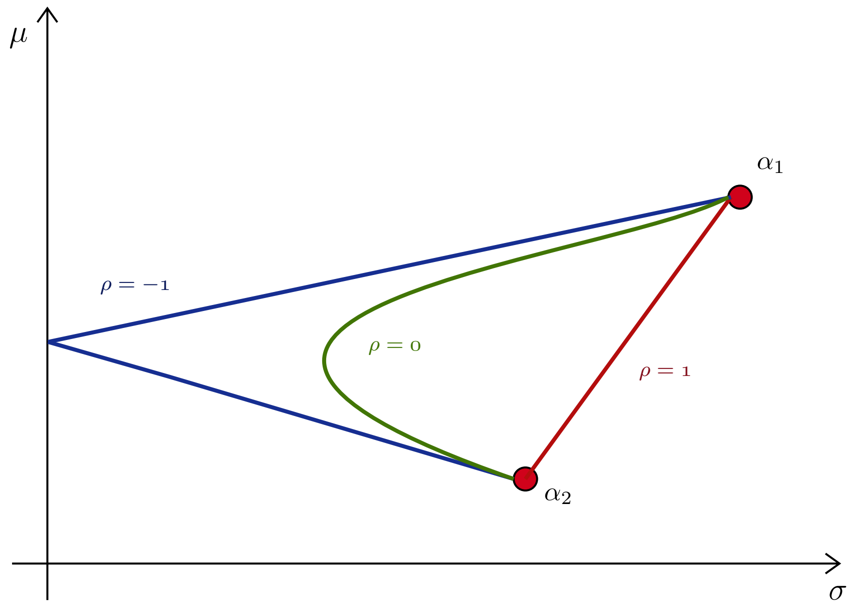

Figure 1: Efficient Frontier for a 2-asset portfolio and different values of correlation ρ

Securities behavior: Visualization analysis

To better familiarize ourselves with the selected securities, we perform a visualization analysis regarding securities prices, returns, and returns distributions during the entire length of the sample size. Figure 2, Figure 3, and Figure 4 show those variables respectively for daily data, and Figure 5, Figure 6, and Figure 7 follow suit for monthly frequency.

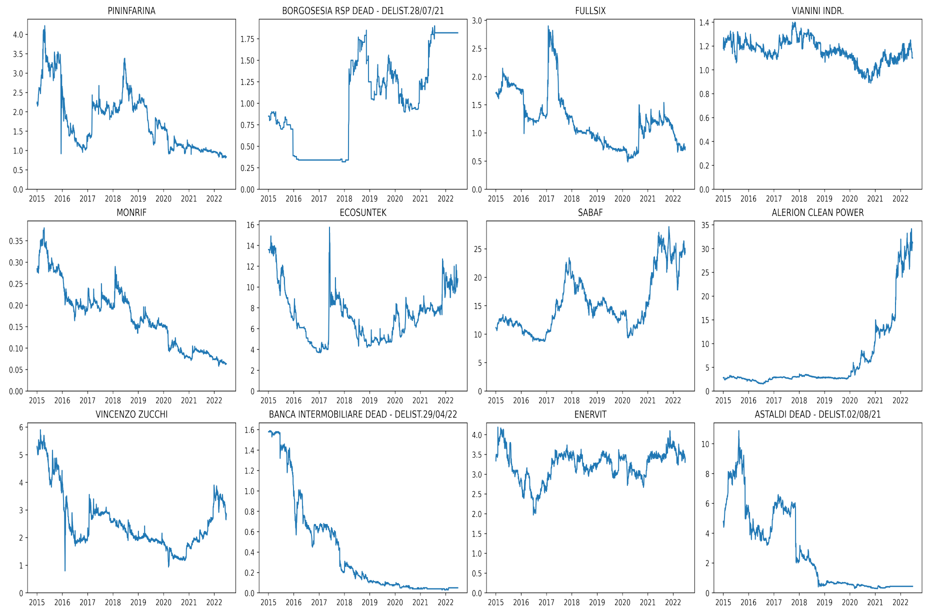

Figure 2: Stocks daily prices over time

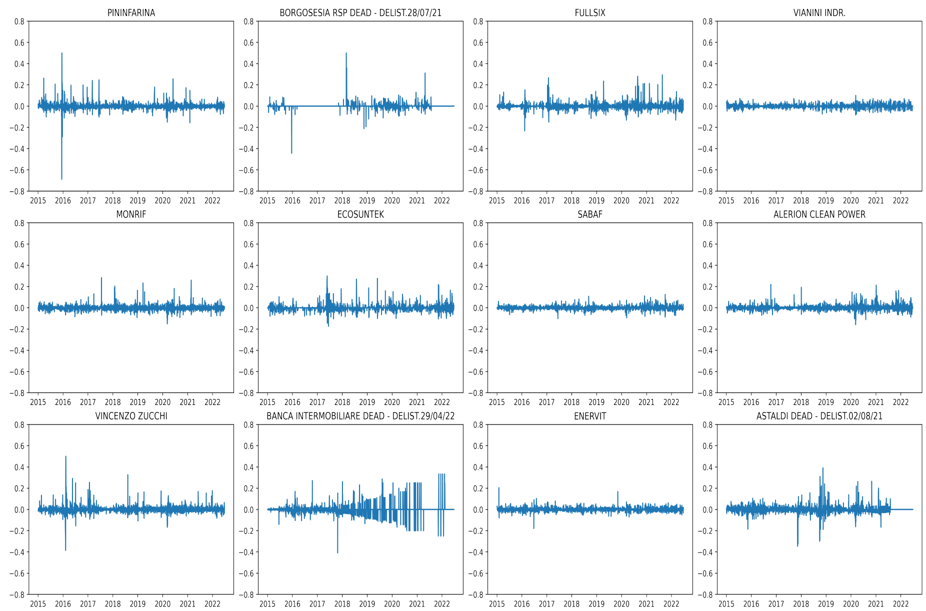

Figure 3: Stocks daily returns over time

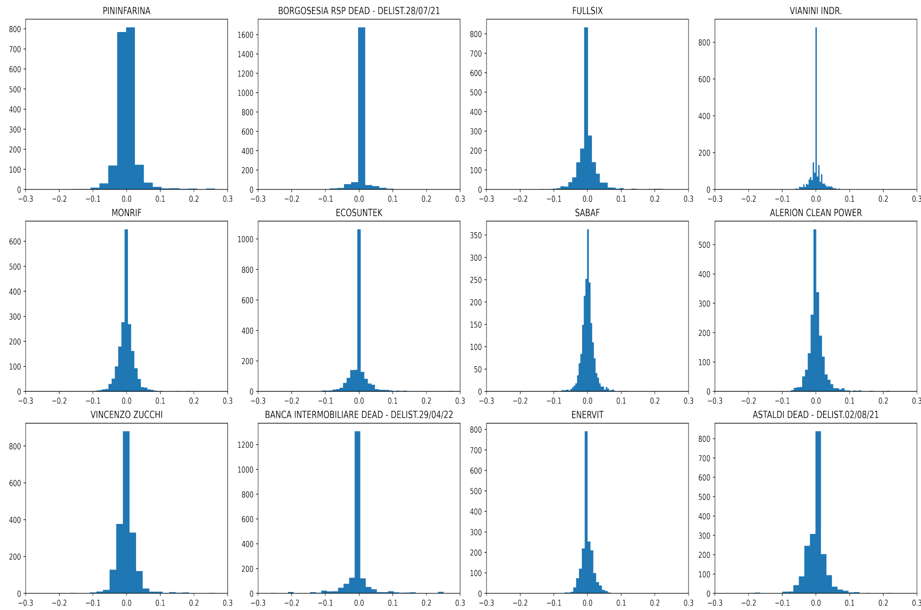

Figure 4: Distribution of stocks daily returns

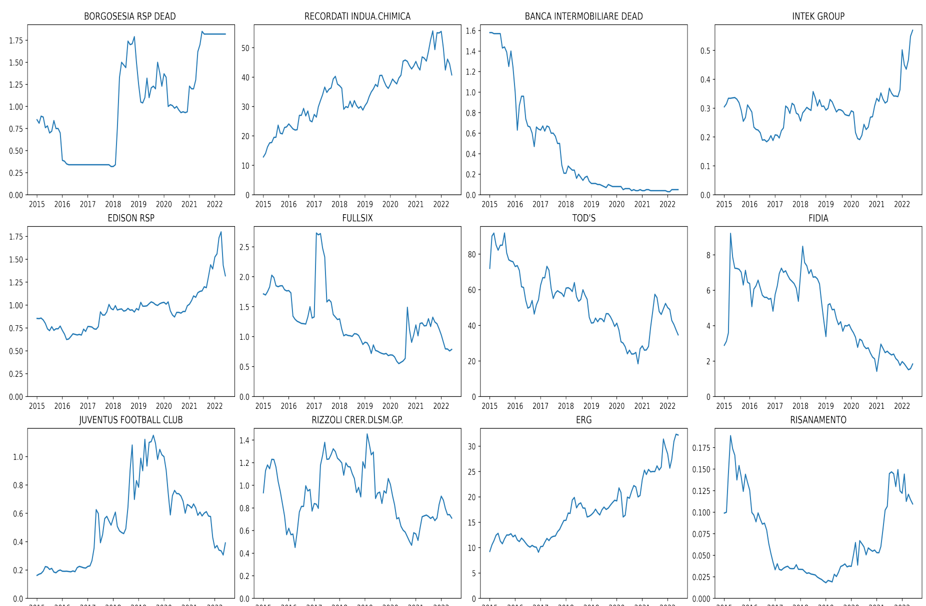

Figure 5: Stocks monthly prices over time

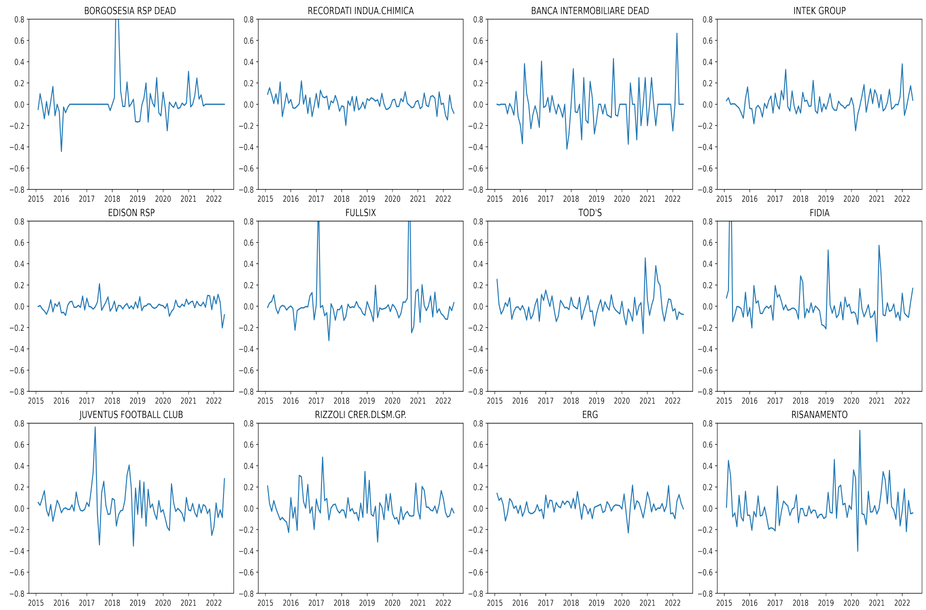

Figure 6: Stocks monthly returns over time

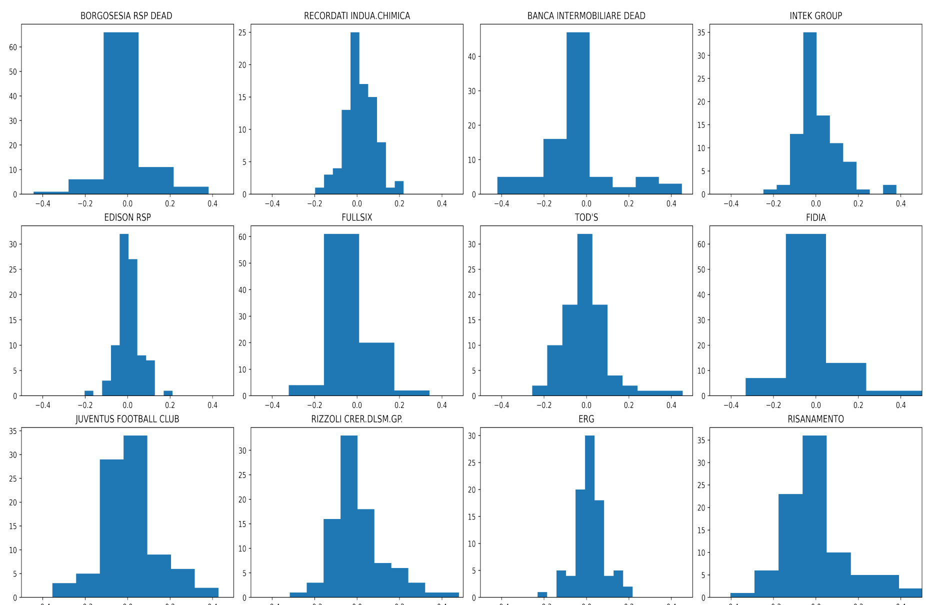

Figure 7: Distribution of stocks monthly returns

Mean-Variance Portfolio Allocation

Consider N risky securities:

- Returns vector: $\underset{Nx1}{R} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix}$

- Expected returns vector: $\underset{Nx1}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$

- Vector of ones: $\underset{Nx1}{\iota} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

- Variance-covariance matrix: $\underset{NxN}{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_N^2 \end{bmatrix}$

- Weights vector: $\underset{Nx1}{W} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$

The Mean-Variance Frontier optimization problem looks like the following:

$$\underset{W}{min}\, \sigma_p^2 = W^\intercal \Sigma W \tag{1}$$

subject to

$$\mu_p = W^\intercal \mu \tag{2}$$

$$W^\intercal \iota = 1. \tag{3}$$

Under no short-selling constraints, $W_i \; \forall i = 1, ..., N$ can take any value—positive meaning a long position, and negative meaning a short position.

The Lagrangian is then defined as

$$\mathcal{L} = \frac{1}{2} W^\intercal - \lambda_1 (W^\intercal \mu - \mu_p) - \lambda_2 (W^\intercal \iota - \iota) \tag{4}$$

Under the first order conditions (F.O.C.), we get

$$\frac{\partial \mathcal{L}}{\partial W} = -\lambda_1 \mu - \lambda_2 \iota = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = W^\intercal \mu - \mu_p = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = W^\intercal \iota - \iota = 0 \tag{7}$$

From (5) we get $W = \Sigma^{-1}(\lambda_1 \mu + \lambda_2 \iota)$, or equivalently,

$$\begin{bmatrix}\mu & \iota\end{bmatrix}^\intercal W = \underbrace{\begin{bmatrix}\mu & \iota\end{bmatrix}^\intercal \Sigma^{-1} \begin{bmatrix}\mu & \iota\end{bmatrix}}_{\text{Information Matrix } (I)} \begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} \tag{8}$$

It follows that

$$\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} = I^{-1} \begin{bmatrix}\mu\, W^\intercal \\ \iota\end{bmatrix} = I^{-1} \begin{bmatrix}\mu_p \\ \iota\end{bmatrix} \tag{9}$$

and

$$I = \begin{bmatrix}\mu^\intercal \Sigma^{-1} \mu & \mu^\intercal \Sigma^{-1} \iota \\ \iota^\intercal \Sigma^{-1} \mu & \iota^\intercal \Sigma^{-1} \iota\end{bmatrix} = \begin{bmatrix}A & B \\ B & C\end{bmatrix} \tag{10}$$

The determinant of the Information Matrix I is $|I| = AC - B^2 = D > 0$ and its inverse $I^{-1} = \frac{1}{D}\begin{bmatrix}C & -B \\ -B & A\end{bmatrix}$

Finally, after pre-multiplying by $\begin{bmatrix}\mu & \iota\end{bmatrix}^{\intercal -1}$ in (8) and substituting $\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}$ from (7) we end up with

$$\begin{aligned} W^* = \Sigma^{-1} \begin{bmatrix}\mu & \iota\end{bmatrix} \frac{1}{D} \begin{bmatrix}C & -B \\ -B & A\end{bmatrix} \begin{bmatrix}\mu_p \\ \iota\end{bmatrix} = \\ \frac{1}{D}\Sigma^{-1}(C\mu - B)\mu_p + \frac{\Sigma^{-1}\iota}{D}(A - B\mu) \end{aligned} \tag{11}$$

$$\sigma_p^2 = \frac{1}{D}(A - 2B\mu_p + C\mu_p^2) = W^\intercal \tag{12}$$

Equation (12) gives us a hyperbola where all possible efficient portfolio allocations lie [1]. Each point on the top part of this curve represents the minimum variance achievable for any possible given return. This is called the Mean-Variance Efficient Frontier (Figure 8, Figure 9, Figure 10 and Figure 11). Neither of these portfolios is more optimal than the others if no additional assumptions are made (be it a utility function, a performance metric, or the inclusion of a market portfolio). To choose our set of weights, we decide to pick the portfolio that minimizes the variances given a daily expected return of 0.001, since this is a return already achievable without needing to include a risk-free asset to leverage. We just plug in the formula the expected return and obtain

$$\mu_{MV} = \begin{bmatrix} 0.0002816 \\ 0.0007749 \\ 0.0001046 \\ 0.0001146 \\ -0.0004369 \\ 0.0003818 \\ 0.0005736 \\ 0.0015612 \\ 0.0003597 \\ -0.0005836 \\ 0.0001805 \\ -0.0004642 \end{bmatrix}$$

The variance-covariance matrix, Σ_{MV}, can be found in the "Take-Home Exam.xlsx" Excel file.

Under short-selling constraints, $W_i > 0 \ \forall i = 1, ..., N$. Constrained weights are computed by solving the optimization problem numerically with the PyPortfolioOpt Python package, instead of getting the analytical solution as we do in the unconstrained case. The resulting weights, both for the unconstrained allocation and after imposing the non-negativity constraint, are:

$$W_{MV(D)} = \begin{bmatrix} 0.030 \\ 0.164 \\ 0.025 \\ 0.130 \\ -0.089 \\ 0.057 \\ 0.247 \\ 0.360 \\ 0.026 \\ -0.037 \\ 0.141 \\ -0.055 \end{bmatrix} \qquad W_{MV(C)(D)} = \begin{bmatrix} 0.011 \\ 0.176 \\ 0.000 \\ 0.033 \\ 0.000 \\ 0.033 \\ 0.235 \\ 0.446 \\ 0.009 \\ 0.000 \\ 0.056 \\ 0.000 \end{bmatrix}$$

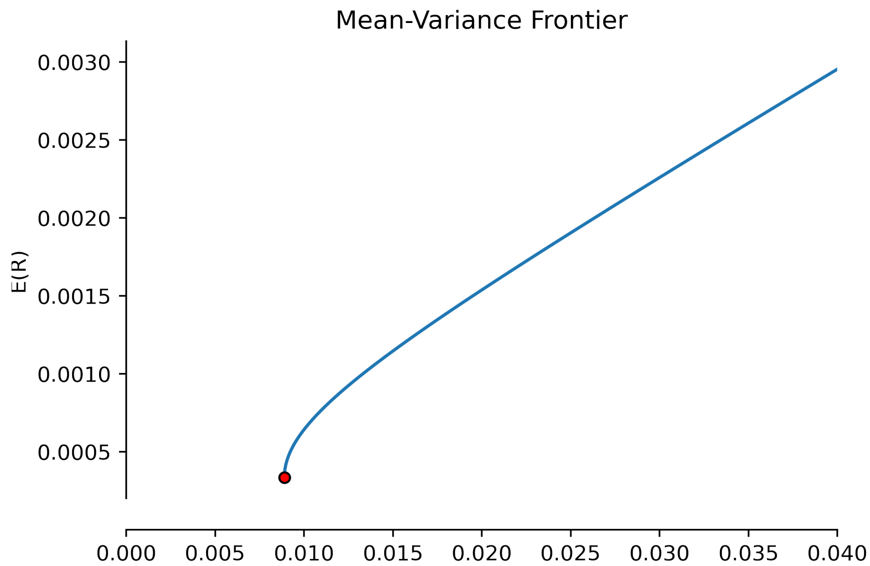

Figure 8: Mean-Variance efficient frontier (daily) (top part).

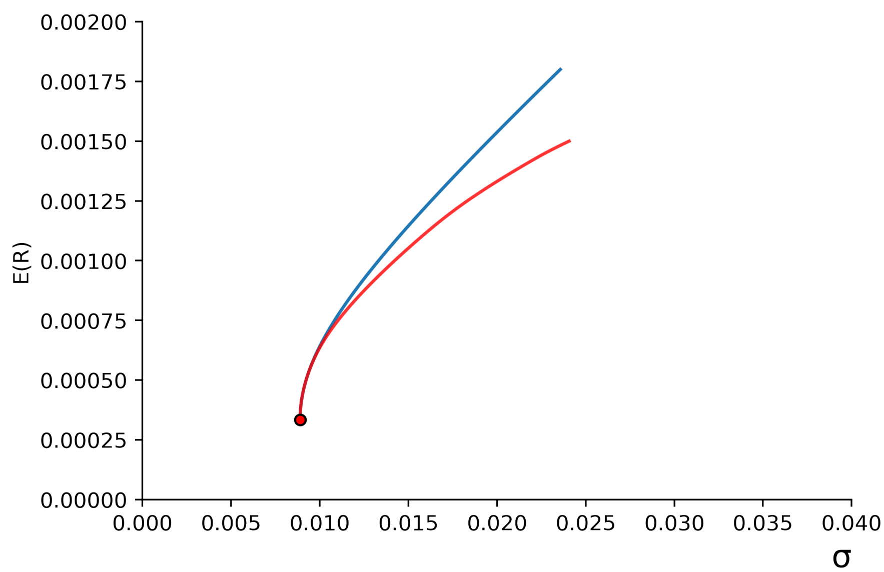

Figure 9: Mean-Variance efficient frontier (daily) with (red) and without (blue) non-negativity constraint on portfolio weights (top part).

For the monthly data, we choose a return of 0.01 and obtain the following unconstrained and constrained weights, respectively:

$$W_{MV(M)} = \begin{bmatrix} 0.067 \\ 0.251 \\ 0.032 \\ 0.044 \\ 0.423 \\ 0.031 \\ 0.019 \\ 0.009 \\ 0.043 \\ 0.019 \\ 0.087 \\ -0.025 \end{bmatrix} \qquad W_{MV(C)(M)} = \begin{bmatrix} 0.064 \\ 0.235 \\ 0.030 \\ 0.050 \\ 0.425 \\ 0.028 \\ 0.020 \\ 0.005 \\ 0.044 \\ 0.017 \\ 0.082 \\ 0.000 \end{bmatrix}$$

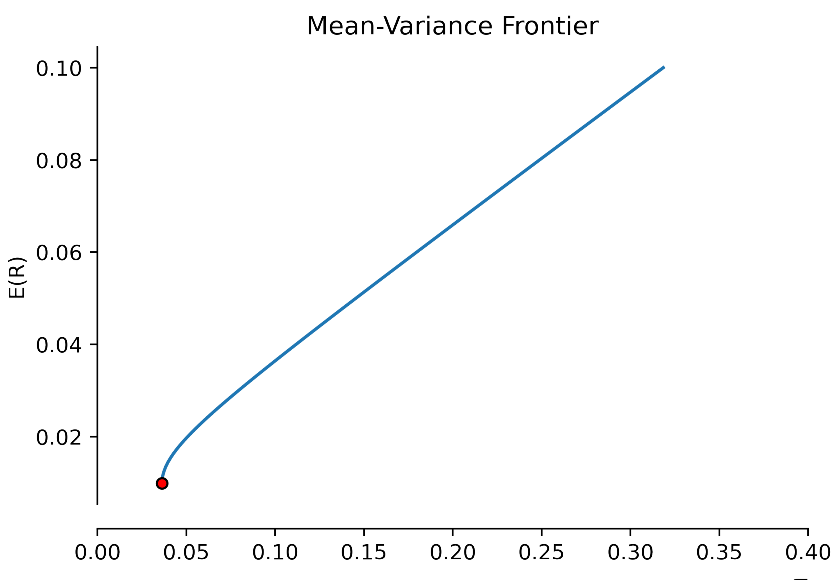

Figure 10: Mean-Variance efficient frontier (monthly) (top part).

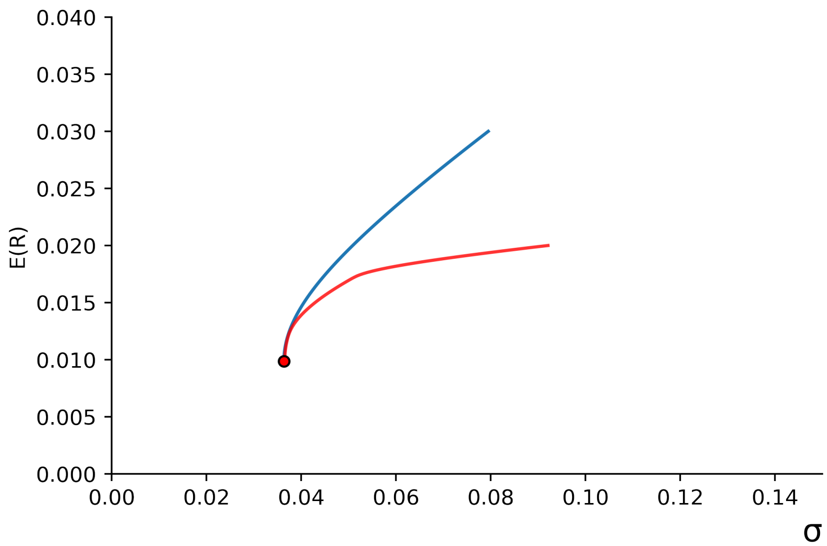

Figure 11: Mean-Variance efficient frontier (monthly) with (red) and without (blue) non-negativity constraint on portfolio weights (top part).

We now compute the main statistics for both portfolios (with and without non-negative restrictions), for daily and monthly data.

$$\mu_P = \mu^\intercal W$$

$$\sigma_P^2 = \mu^\intercal \Sigma \mu$$

$$S_P = W M_3 \left(W^\intercal \otimes W^\intercal\right)$$

with co-skewness matrix $M_3 = E[(r_t - \mu)(r_t - \mu)^\intercal \otimes (r_t - \mu)^\intercal]$

$$K = W M_4 \left(W^\intercal \otimes W^\intercal \otimes W^\intercal\right)$$

whit co-kurtosis matrix $M_4 = E[(r_t - \mu)(r_t - \mu)^\intercal \otimes (r_t - \mu)^\intercal \otimes (r_t - \mu)^\intercal]$

Descriptive statistics (mean, standard deviation, variance, skewness, and kurtosis) of the optimal Mean-Variance portfolio for both daily and monthly frequencies are shown below in Table 3.

	μ	σ	σ^2	**Skewness**	**Kurtosis**
Unconstr. (D)	0.001000	0.013322	0.000177	0.102048	0.711868
Constr. (D)	0.001000	0.014285	0.000202	0.155993	1.089773

	μ	σ	σ^2	**Skewness**	**Kurtosis**
Unconstr. (M)	0.010000	0.036361	0.001322	-0.046935	0.339556
Constr. (M)	0.010000	0.036565	0.001337	-0.057142	0.343310

Table 3: Mean-Variance portfolios statistics

Firstly, the mean return μ_P is the easiest to interpret: it is the expected return of the portfolio, so it is easy to see that higher values are better since investors look to maximize their profit. The variance and standard deviation (volatility) give the risk of the investment: Given the same expected return, lower variance portfolios are preferred since they tend to be more consistent and fluctuate less around the mean, thus carrying less risk; i.e., they are considered safer investments. Thirdly, positive skewness is a desired characteristic of a return distribution since it implies that gains will be higher than losses when they occur. This is due to the long tail at the right of the distribution, which gives a high probability of extreme positive values while maintaining a low probability for extremely negative returns. Finally, kurtosis, as the variance, is a form of measuring risk: Leptokurtic distributions (K>3) have fatter tails, giving higher probabilities of extreme returns, which are usually avoided by investors since they imply a higher risk, even if the volatility is the same. In this case, these statistics tend to be better for the unconstrained minimum variance portfolio. This is not surprising since having the possibility of shorting the securities with the least returns gives more flexibility to the conformation of the portfolio. More intuitively, an unconstrained portfolio could always match a constrained portfolio (by not taking selling positions), but the opposite is not always true. This relation can be also seen in Figure 9 and Figure 11, where unconstrained Mean-Variance portfolios (blue line) give a greater return per unit of risk than their constrained peers (red line).

Global Minimum Variance portfolio

For the Minimum Variance Portfolio, from (12) we get

$$\frac{\partial \sigma_p^2}{\partial \mu_p} = \frac{1}{D}(-2B + 2C)\mu_p = 0 \to \mu_{MV} = \frac{B}{C} \to \sigma_{MV}^2 = \frac{1}{C} \to W_{MV} = \frac{\Sigma^{-1}\iota}{\iota^\intercal \Sigma^{-1}\iota}$$

The resulting weights are:

$$W_{GMV(D)} = \begin{bmatrix} 0.023 \\ 0.090 \\ 0.045 \\ 0.244 \\ 0.076 \\ 0.058 \\ 0.147 \\ 0.081 \\ 0.024 \\ 0.015 \\ 0.181 \\ 0.016 \end{bmatrix} \qquad W_{GMV(M)} = \begin{bmatrix} 0.067 \\ 0.251 \\ 0.034 \\ 0.043 \\ 0.425 \\ 0.031 \\ 0.021 \\ 0.009 \\ 0.042 \\ 0.019 \\ 0.084 \\ -0.026 \end{bmatrix}$$

And we obtain the following statistics:

	μ	σ	σ^2	**Skewness**	**Kurtosis**	**Sharpe Ratio**
Daily	0.000334	0.008894	0.000079	-0.01023	0.227249	0.03737
Monthly	0.009861	0.036358	0.001322	-0.047196	0.341165	0.270633

Table 4: Global Minimum Variance statistics

Market Portfolio

For the following analysis, we include the FTSE Italia All-Share total return and consider it representative of the market portfolio. We then compute its main statistics for both daily and monthly frequencies (Table 5), and compare them with the mean-variance portfolios previously obtained.

	μ	σ	σ^2	**Skewness**	**Kurtosis**
Daily	0.000187	0.014032	0.000197	-1.391548	15.830972
Monthly	0.004804	0.057897	0.003352	-0.537919	3.445321

Table 5: FTSE statistics

Surprisingly, the expected return of the market portfolio is (much) lower than the returns of our portfolio while having almost the same volatility and lower skewness. However, we cannot jump to any conclusions without thinking about how we defined our portfolio and computed its statistics. Recall that we constructed our mean-variance portfolio ex-post with historical data, optimizing the weights for that same data. To see if our portfolio would "beat the market", performance analysis would need to be done with future data instead and current results would remain valid if and only if historical returns were a good proxy of future returns for each security.

Betas (β)

After computing the main statistics for the market portfolio, we can obtain the β of our portfolios to obtain the non-diversifiable or systematic risk (both for daily and monthly data). This value, according to the Capital Asset Pricing Model (CAPM), is the only type of risk that drives expected return and is obtained by

$$\beta_i = \left(\frac{Cov(R_i, R_M)}{Var(R_M)} \right)$$

where R_i indicates security return and R_m market return.

We obtain the β_i for each of the 12 securities and our Mean-Variance portfolio. The results are shown below in Table 6 for daily frequency and in Table 7 for monthly frequency:

	β
Portfolio	0.2878
PINF	0.6577
BOR	0.0982
FUL	0.3868
VIN	0.1105
MON	0.3187
ECK	0.2794
SAB	0.3973
ARN	0.421
ZUC	0.6645
BIM	0.5235
ENV	0.2774
AST	0.8462

Table 6: Mean-Variance portfolio and securities daily betas

	β
Portfolio	0.4082
BOR	0.0848
REC	0.4666
BIM	0.7934
IKG	0.6307
EDNR	0.3024
FUL	0.2862
TOD	0.7845
FD	1.0864

	β
JUVE	0.8157
RCS	0.884
ERG	0.5408
RN	1.1171

Table 7: Mean-Variance portfolio and securities monthly betas

An intuitive explanation of why this β drives returns is by imposing a concave utility function over consumption levels, i.e., a utility function where the more the agent can consume, the less an additional unit of consumption increases its satisfaction. In this framework, high betas, which imply high returns when the market is up and vice-versa, entail that profits would usually come when the marginal utility is lower, and losses would come when the marginal utility is higher.

Security Market Line

The inclusion of a risk-free asset allows us to work under the Capital Asset Pricing Model (CAPM). In this case, we assume an annual risk-free rate of 0.005. With this rate, we can compute the Capital Market Line for any given security (including our portfolio). This line gives us every volatility and expected return pair obtainable in an investment by combining the risk-free asset and the given security.

$$E(R_i) = r_f + \frac{(R_p - r_f)}{\sigma_p}\sigma_i$$

where:

- $E(R_i)$ is the expected return of the investment
- R_p is the expected return of the security (without leverage or lending)
- r_f is the risk-free rate
- σ_p is the volatility of the security (without leverage or lending)
- σ_i is the volatility of the investment

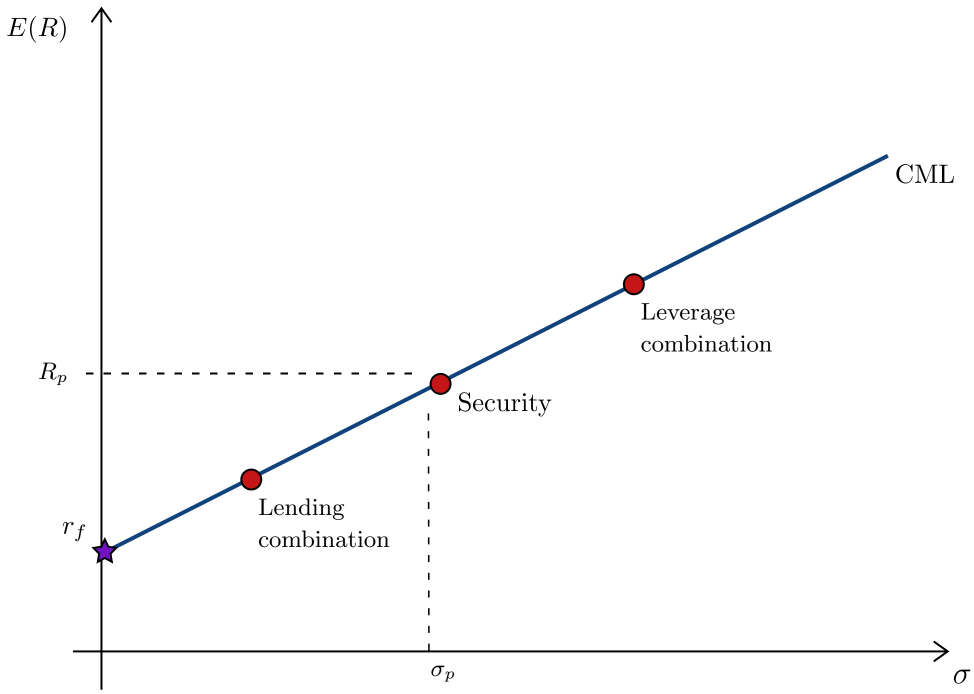

Figure 12: Capital Market Line

The greater the slope, the best return per unit of risk the possible investment will have. This is given by the Sharpe Ratio ($SR = \frac{R_p - r_f}{\sigma_p}$), which is usually used as a performance metric for a given security. If we do the same procedure but accounting only for the systematic risk β_i instead of the volatility σ_i, we arrive at the Security Market Line, which has the formula:

$$E(R_i) = r_f + (R_M - r_f) \cdot \beta_i$$

where R_M is the expected return of the Market Portfolio.

The SML gives us the possibility to price assets: If a security is above the SML, the asset return is too high with respect to equilibrium, i.e., asset is underpriced and one should take a long position. Contrarily, below the SML the asset return is too low, i.e., asset is overpriced and one should take a short position. We calculate the equilibrium return for the stocks of companies Pininfarina and Fullsix as well as the SML return of our portfolio.

	μ	**SML**
Portfolio	0.001000	0.0000548
PINF	0.000282	0.0001235
FUL	0.000105	0.0000732

	μ	**SML**

Table 8: Portfolio and securities daily returns and Security Market Line

As we can see from Table 8 and Figure 13 below, Fullsix returns are relatively consistent with equilibrium price, verifying the Security Market Line, while Pininfarina returns, and especially our portfolio returns, are very much underpriced since they yield a much greater return.

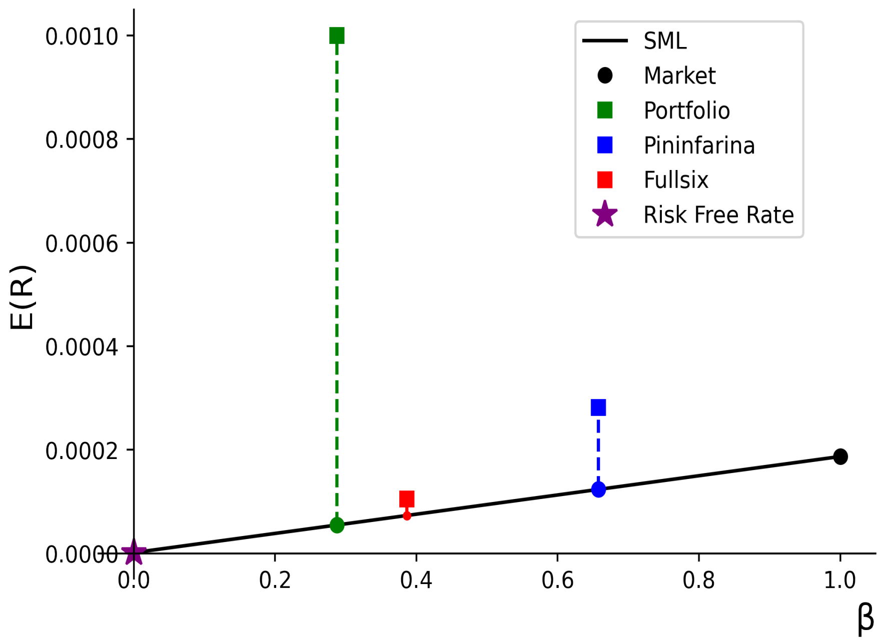

Figure 13: Security Market Line, portfolio and securities (daily)

Similar results are obtained for the monthly data:

	μ	**SML**
Portfolio	0.01000	0.00197
ERG	0.01658	0.00261
FUL	0.00472	0.00139

Table 9: Portfolio and securities monthly returns and Security Market Line

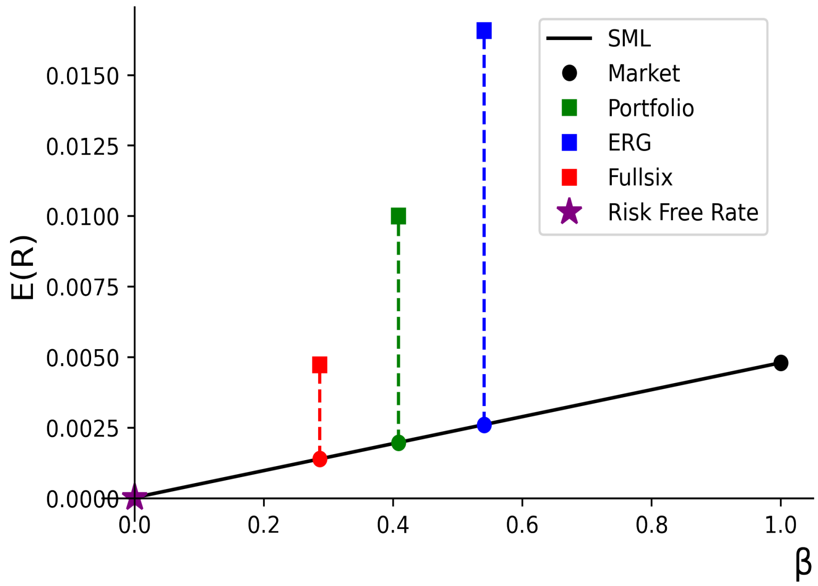

Figure 14: Security Market Line, portfolio and securities (monthly)

Black-Litterman

We have seen that even if at times the CAPM and the mean-variance standard portfolio asset allocation are a simple and useful way to understand the behavior of a set of securities and easily get a portfolio, it has been shown that Maximum Likelihood estimators of the expected returns vector and variance-covariance matrix are usually heavily biased. Furthermore, the allocation weights of the Markowitz Mean-Variance portfolio are extremely sensitive to small changes in the estimation of μ and Σ and often give very risky allocations. This problem is so big that in many studies (see for example [6]) even a naive approach of setting uniform priors for all the assets has beaten the Markowitz portfolio. Finally, investors can have valuable knowledge and intuition about the market and the securities which the Mean-Variance optimization does not account for.

To address these three issues, many other (more conservative) ways to compute the return vector and the variance-covariance matrix have arose in the literature. This is the case of Bayesian methods and, in particular, the Black-Litterman model.

The Black-Litterman model, through a Bayesian framework, makes it possible to include the investor's views, combining them with the Maximum Likelihood estimation according to the confidence in each estimate. More formally, consider N assets and K views:

- Expected returns vector:

$$\underset{Nx1}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

- Picking matrix:

$$\underset{KxN}{P} = \begin{bmatrix} p_{1,1} & \cdots & p_{1,N} \\ \vdots & \ddots & \vdots \\ p_{K,1} & \cdots & p_{K,N} \end{bmatrix}$$

- Views vector:

$$\underset{Kx1}{\nu} = \underset{KxN}{P} \underset{Nx1}{\mu} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_N \end{bmatrix}$$

Consider the prior on μ

$$f_{PR}(\mu) \sim N(\mu_{eq}, \tau\Sigma) \tag{13}$$

where τ is a scalar tuning constant specifying the degree of confidence with respect to the equilibrium allocation.

The likelihood:

$$f(\nu|\mu) \propto exp\,\{-\frac{1}{2}(\nu - P\mu)^\intercal\Omega^{-1}(\nu - P\mu)\} \tag{14}$$

The posterior is defined as

$$f_{PO}(\mu|\nu) \propto exp\,\{-\frac{1}{2}\mu^\intercal[(\tau\Sigma)^{-1} + P^\intercal\Omega^{-1}P]\mu + \mu^\intercal[(\tau\Sigma)^{-1}\mu_{eq} + P^\intercal\Omega^{-1}\nu\} \tag{15}$$

It follows that

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P^\intercal\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\mu_{eq} + P^\intercal\Omega^{-1}\nu] \tag{16}$$

$$\mu_{BL} = \mu_{eq} + \tau\Sigma P^\intercal[(\tau\Sigma)^{-1} + P^\intercal\Omega^{-1}P]^{-1}\underbrace{(\nu - P\mu_{eq})}_{d} \tag{17}$$

where $d = \nu - P\mu_{eq}$ is a correction term that represents the difference between what one thinks and what the market "thinks" about your own views.

Then,

$$\Sigma_{BL} = [(\tau\Sigma)^{-1} + P^\intercal \Omega^{-1} P]^{-1} \tag{18}$$

$$\Sigma_{BL} = \tau\Sigma - \tau\Sigma P^\intercal [(\tau\Sigma)^{-1} + P^\intercal \Omega^{-1} P]^{-1} P^\intercal \tau\Sigma \tag{19}$$

We now use this framework to compute new estimates of the returns vector and variance-covariance matrix. We start by specifying our own pick matrix (P) and Views (ν), both for daily and monthly data (the views are the same but converted into the appropriate capitalization time-period):

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.3 & 0.6 & 0 & 0.4 & 0 & -0.7 & 0 & 0 \end{bmatrix}$$

$$\nu_{BL(D)} = \begin{bmatrix} 0.0009 \\ 0.0007 \\ 0.0009 \\ -0.0006 \\ 0.0015 \end{bmatrix} \qquad \nu_{BL(M)} = \begin{bmatrix} 0.027 \\ 0.021 \\ 0.027 \\ -0.018 \\ 0.046 \end{bmatrix}$$

Walters (2013) [9] gives a sensible default for $\tau = 0.05$. We asume Ω, the uncertainty over the views, to be diagonal (there is no covariance between views) and specify it as $\Omega = diag(P(\tau \cdot \Sigma)P^T)$

Plugging into (17) yields the Black-Litterman estimation of the returns vector:

$$\mu_{BL(D)} = \begin{bmatrix} 0.0000428 \\ 0.0007823 \\ 0.0000808 \\ 0.0001972 \\ -0.0005150 \\ 0.0006247 \\ 0.0005575 \\ 0.0015429 \\ -0.0001504 \\ -0.0005565 \\ -0.0002172 \\ -0.0005460 \end{bmatrix} \qquad \mu_{BL(M)} = \begin{bmatrix} 0.002254 \\ 0.013332 \\ -0.023571 \\ 0.011510 \\ 0.003344 \\ 0.024621 \\ 0.000753 \\ 0.012677 \\ 0.005273 \\ -0.004736 \\ -0.000199 \\ 0.011205 \end{bmatrix}$$

The Black-Litterman variance-covariance matrix estimation, Σ_{BL} (19), can be found in the "Take-Home Exam.xlsx" Excel file.

Once we have the estimation of the returns vector and variance-covariance matrix we need to compute the weights. As we have introduced the risk-free asset we can have lending and leveraging portfolios, allowing us to have weights that do not necessarily add up to one. Then, to choose the optimal weight allocation we assume a quadratic utility function with a risk aversion of $\gamma = 3$. Finally, we arrive at the formula

$$W_{BL} = \frac{1}{\gamma}\Sigma_{BL}^{-1}\mu_{BL} \tag{20}$$

which outputs the following weights for both daily and monthly data:

$$W_{BL(D)} = \begin{bmatrix} 0.007 \\ 0.280 \\ 0.017 \\ 0.189 \\ -0.253 \\ 0.161 \\ 0.451 \\ 0.709 \\ -0.070 \\ -0.092 \\ -0.199 \\ -0.131 \end{bmatrix} \qquad W_{BL(M)} = \begin{bmatrix} 0.098 \\ 0.737 \\ -0.260 \\ 0.432 \\ 0.632 \\ 0.245 \\ -0.133 \\ 0.108 \\ 0.070 \\ -0.275 \\ -0.410 \\ 0.038 \end{bmatrix}$$

	μ	σ	σ^2	**Skewness**	**Kurtosis**	**Sharpe Ratio**
Daily	0.002010	0.025885	0.000670	0.756987	7.937806	0.077602
Monthly	0.032734	0.104458	0.010911	0.087007	5.420333	0.313165

Table 10: Black-Litterman statistics

Looking at the statistics, the Black-Litterman allocation gives better mean returns, but is also riskier, since it has higher volatility and kurtosis. This is caused by the possibility of having weights that sum up to more than one, meaning that there is a leverage using the risk-free asset. However, all in all, the Black-Litterman model gives a better portfolio, since the rise in risk is not as big as the improvement on expected returns. Furthermore, the bigger skewness gives the Black-Litterman portfolio an additional attractive characteristic.

Standard Bayesian Asset Allocation

Anther way to estimate the variance-covariance matrix and vector of expected returns is the pure Bayesian:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \tag{21}$$

We start by defining a prior belief over the μ, μ_0, and Σ, Λ_0, of the data Y (how to pick this prior depends completely on the investor and the specific problem).

Prior distribution:

$$f_{pr}(Y) \propto exp \left\{ -\frac{1}{2}(\mu - \mu_0)^\intercal \Lambda_0^{-1} (\mu - \mu_0) \right\} \tag{22}$$

We compute the maximum likelihood estimators $\hat{\mu}$ and Σ. This gives us the likelihood of our data Y:

$$f_L(\mu|Y, \Sigma)) \propto exp \left\{ -\frac{1}{2}(\mu - \hat{\mu})^\intercal T\Sigma^{-1} (\mu - \hat{\mu}) \right\} \tag{23}$$

Combining this two together and ignoring constant terms we get the posterior distribution:

$$f_{PO}(Y|\mu, \Sigma) \propto exp \left\{ -\frac{1}{2}(\mu - \hat{\mu})^\intercal T\Sigma^{-1} (\mu - \hat{\mu}) - \frac{1}{2}(\mu - \mu_0)^\intercal \Lambda_0^{-1} (\mu - \mu_0) \right\} \tag{24}$$

$$\propto exp \left\{ -\frac{1}{2}[(\mu - \hat{\mu})^\intercal T\Sigma^{-1} (\mu - \hat{\mu}) + (\mu - \mu_0)^\intercal \Lambda_0^{-1} (\mu - \mu_0)] \right\} \propto \tag{25}$$

$$\begin{aligned} \propto exp \Big\{ -\frac{1}{2}[&\mu^\intercal T\Sigma^{-1}\mu - \hat{\mu}^\intercal \Sigma^{-1}\mu - \mu^\intercal T\Sigma^{-1}\hat{\mu} + \hat{\mu}^\intercal T\Sigma\hat{\mu} + \\ &\mu^\intercal \Lambda_0^{-1}\mu - \mu_0^\intercal \Lambda_0^{-1}\mu - \mu^\intercal \Lambda_0^{-1}\mu_0 + \mu_0^\intercal \Lambda_0^{-1}\mu_0] \end{aligned} \tag{26}$$

$$\propto exp \left\{ \frac{1}{2}[\mu^\intercal \underbrace{(T\Sigma^{-1} + \Lambda_0^{-1})}_{\text{A}}\mu - 2\underbrace{(\hat{\mu}^\intercal T\Sigma^{-1} + \mu_0^\intercal \Lambda_0^{-1})}_{a^\intercal}\mu + \underbrace{(\hat{\mu}^\intercal T\Sigma^{-1}\hat{\mu} + \mu_0 \Lambda_0^{-1})]}_{\text{C}} \right\} \tag{27}$$

Completing squares we get

$$f_{PO}(\mu|Y, \Sigma) \propto exp \left\{ -\frac{1}{2}[(\mu - A^{-1}a)^\intercal A(\mu - A^{-1}a)] \right\} \tag{28}$$

which is a Normal with $\mu_1 = A^{-1}a$ and $\Sigma_1 = A^{-1}$. Therefore,

$$\mu_1 = (T\Sigma^{-1} + \Lambda_0^{-1})^{-1}(T\Sigma^{-1}\hat{\mu} + \Lambda_0^{-1}\mu_0) \tag{29}$$

$$\Sigma_1 = (T\Sigma^{-1} + \Lambda_0^{-1})^{-1} \tag{30}$$

In our case we stated a prior distribution with $\mu_0 = \mu + \sigma$ and $\Lambda_0 = 2\Sigma$. Plugging them and the maximum likelihood into the formula for the posterior we got:

$$\mu_{B(D)} = \begin{bmatrix} 0.0002911 \\ 0.0007822 \\ 0.0001131 \\ 0.0001191 \\ -0.0004302 \\ 0.0003901 \\ 0.0005784 \\ 0.0015678 \\ 0.0003693 \\ -0.0005711 \\ 0.0001853 \\ -0.0004542 \end{bmatrix} \qquad \mu_{B(M)} = \begin{bmatrix} 0.021682 \\ 0.016009 \\ -0.021761 \\ 0.011875 \\ 0.006605 \\ 0.005850 \\ -0.002145 \\ 0.011479 \\ 0.021606 \\ 0.005166 \\ 0.016975 \\ 0.015089 \end{bmatrix}$$

The Bayesian variance-covariance matrix, Σ_B (31), can be found in the "Take-Home Exam.xlsx" Excel file.

Following the quadratic utility maximization approach we did for the previous section and maintaining $\gamma = 3$. Finally, we arrive at the formula

$$W_B = \frac{1}{\gamma}\Sigma_B^{-1}\mu_B \tag{31}$$

giving both for daily and monthly data the following weights:

$$W_{B(D)} = \begin{bmatrix} 94.511 \\ 579.534 \\ 38.497 \\ 177.388 \\ -513.660 \\ 158.201 \\ 850.169 \\ 1456.327 \\ 78.256 \\ -186.499 \\ 327.471 \\ -266.272 \end{bmatrix} \qquad W_{B(M)} = \begin{bmatrix} 29.600 \\ 74.445 \\ -22.978 \\ 21.650 \\ 73.104 \\ 7.055 \\ -29.869 \\ 1.049 \\ 28.686 \\ -7.248 \\ 65.838 \\ 3.631 \end{bmatrix}$$

The yielded portfolio results are shown below:

	μ	σ	σ^2	**Skewness**	**Kurtosis**	**Sharpe Ratio**
Daily	3.881	1.1374	1.2937	6086301924	1.27437E+14	3.4122
Monthly	5.0552	1.6851	1.2981	-3555.8166	1567578219	3.894

Table 11: Bayesian statistics

As we can see, nor the weights nor the statistics make any sense in this context. Here the Bayesian approach is giving a variance-covariance matrix with really small values, what means that the model is really confident in the returns. This causes the weights to blow up making huge leverages (both longs and shorts), giving an extremely risky portfolio.

Jorion Shrinkage Estimator

Within the Bayesian framework there are the so called "Shrinkage Estimators", presented by James and Stein (1962) [10]. Theses estimators, although biased, dominate maximum likelihood estimators when their performances are measured by a quadratic loss. Jorion (1986) [11] presents a Bayesian estimator based on these results that outperforms the sample mean-variance portfolio allocation. The Jorion estimator, rather than the sample mean of the returns, computes a shrunk version of it. The degree of shrinkage depends directly on an hyperparameter δ which, in turn, rest on the uncertainty of the data. In mathematical terms:

$$f_{PR}(\mu|\mu_{GMV}, h) \propto |\Sigma^{-1}| exp\,\{-\frac{1}{2}(\mu - \mu_{GMV} \cdot \iota)^\intercal (h\Sigma)^{-1}(\mu - \mu_{GMV} \cdot \iota)\} \tag{32}$$

$$\delta = \frac{N+2}{(N+2) + T(\hat{\mu} - \mu_{GMV} \cdot \iota)^\intercal\ \hat{\Sigma}^{-1}(\hat{\mu} - \mu_{GMV} \cdot \iota)} \tag{33}$$

$$h = \frac{(N+2)}{(\mu - \mu_{GMV} \cdot \iota)^\intercal (\hat{\Sigma})^{-1}(\mu - \mu_{GMV} \cdot \iota)} \tag{34}$$

$$\Sigma = \frac{T\hat{\Sigma}}{T-N-2} \tag{35}$$

And

$$\mu_J = (1-\delta)\hat{\mu} + \delta\mu_{GMV} \cdot \iota \tag{36}$$

$$\Sigma_J = \left(1 + \frac{1}{T+h}\right)\hat{\Sigma} + \frac{h}{T(T+1+h)}\frac{\iota \cdot \iota^\intercal}{\iota^\intercal \cdot \hat{\Sigma}^{-1} \cdot \iota} \tag{37}$$

Replacing with our data we get our Jorion estimations of the returns vectors and variance-covariance matrices, both for daily and monthly data:

$$\mu_{J(D)} = \begin{bmatrix} 0.0003137 \\ 0.0005033 \\ 0.0002457 \\ 0.0002495 \\ 0.0000376 \\ 0.0003522 \\ 0.0004259 \\ 0.0008055 \\ 0.0003437 \\ -0.0000188 \\ 0.0002749 \\ 0.0000271 \end{bmatrix} \qquad \mu_{J(M)} = \begin{bmatrix} 0.013137 \\ 0.011602 \\ -0.000020 \\ 0.010310 \\ 0.008785 \\ 0.008304 \\ 0.006040 \\ 0.009988 \\ 0.013158 \\ 0.008223 \\ 0.011897 \\ 0.011156 \end{bmatrix}$$

The Jorion variance-covariance matrix estimation, Σ_J (33), can be found in the "Take-Home Exam 2022.xlsx" Excel file.

Again, making the same assumptions as in the previous cases, we compute the weights with the formula

$$W_J = \frac{1}{\gamma}\Sigma_J^{-1}\mu_J \tag{38}$$

and arrive to the weights:

$$W_{JR(D)} = \begin{bmatrix} 0.038 \\ 0.190 \\ 0.046 \\ 0.242 \\ -0.036 \\ 0.080 \\ 0.291 \\ 0.354 \\ 0.035 \\ -0.024 \\ 0.219 \\ -0.039 \end{bmatrix} \qquad W_{JR(M)} = \begin{bmatrix} 0.172 \\ 0.558 \\ -0.021 \\ 0.120 \\ 0.802 \\ 0.060 \\ -0.055 \\ 0.014 \\ 0.137 \\ 0.004 \\ 0.308 \\ -0.027 \end{bmatrix}$$

Portfolio statistics are shown below:

	μ	σ	σ^2	**Skewness**	**Kurtosis**	**Sharpe Ratio**
Daily	0.000686	0.015125	0.000229	0.099654	1.380471	0.045284
Monthly	0.022521	0.086643	0.007507	-0.326895	6.080350	0.259679

Table 12: Jorion Rule statistics

Further Improvements: Chimera Approach

By looking at all the different ways to allocate the assets in the portfolio one could try to find which is the optimal one. Just looking at its Sharpe Ratio, this could be considered the Black-Littermann. However, it is considerably riskier than the Jorion and, mainly, the Global Minimum Variance approach, so maybe the former could be preferred in some contexts. Given that virtually no allocation is strictly better than the other, one could combine all of them in a single portfolio by making a "portfolio of portfolios". This approach is very common in Machine Learning models and helps to reduce the variance. An intuition behind this approach is that the pitfalls of every asset allocation method gets "diluted away" while the errors tend to compensate, reducing the average risk while maintaining a good expected return level.

Figure 15: This is a Chimera.

We implement this "chimera" approach by equally combining the weights for the Markowitz Mean-Variance, Global Minimum Variance, Jorion and Black-Littermann portfolios. The resulting weights are:

$$W_{CH(D)} = \begin{bmatrix} 0.024 \\ 0.181 \\ 0.033 \\ 0.201 \\ -0.075 \\ 0.089 \\ 0.284 \\ 0.376 \\ 0.004 \\ -0.035 \\ 0.086 \\ -0.052 \end{bmatrix} \qquad W_{CH(m)} \begin{bmatrix} 0.081 \\ 0.359 \\ -0.043 \\ 0.128 \\ 0.456 \\ 0.073 \\ -0.029 \\ 0.028 \\ 0.059 \\ -0.046 \\ 0.014 \\ -0.008 \end{bmatrix}$$

Yielding the following statistics:

	μ	σ	σ^2	**Skewness**	**Kurtosis**	**Sharpe Ratio**
Daily	0.001	0.014	0.000	0.124	0.997	0.077
Monthly	0.014	0.044	0.002	-0.048	0.504	0.326

Table 13: Chimera statistics

As we can see, the results are very decent, almost matching the Black-Litterman's Sharpe Ratio but maintaining a very low variance and kurtosis, thus making it less risky. We find this approach to be the more robust of all since it does not depend on any particular estimation of μ nor Σ but on everyone of them.

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