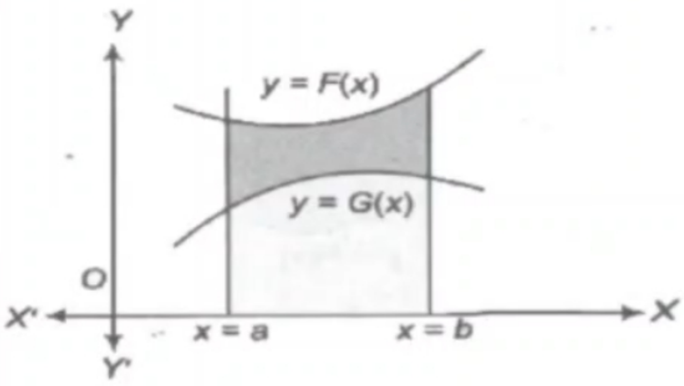
Thank You

Area Between Curves

- Area bounded by two curves y = F (x) and y = G (x) between x = a and x = b is given by..



$$\int_a^b \{F(x) - G(x)\}\, dx$$

Properties of Definite Integrals

properties of Definite Integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

   Generalization

   If $a < c_1 < c_2 < \ldots < c_{n-1} < c_n < b$, then

   $$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx$$

4. $\int_a^a f(x)\,dx = 0$
5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
6. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

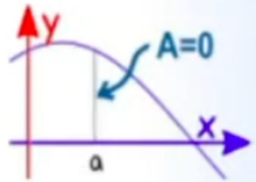
Definite Integrals

In

Interval of zero length

When the interval starts and ends at the same place, the result is zero:

$$\int_a^a f(x)\,dx = 0$$



Adding intervals

We can also add two adjacent intervals together:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Summary

The Definite Integral between **a** and **b** is the Indefinite Integral at **b** minus the Indefinite Integral at **a**.

Definite Integrals

Adding Functions

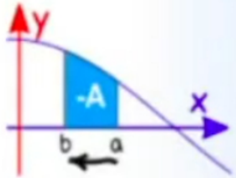
The integral of **f+g** equals the integral of **f** plus the integral of **g**:

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Reversing the interval

Reversing the direction of the interval gives the negative of the original direction.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$



Interval of zero length

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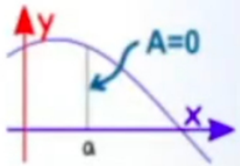
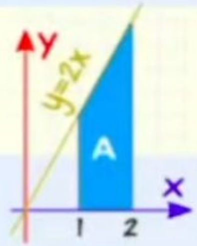


Definite Integrals

Example: What is

$$\int_1^2 2x\,dx$$



We are being asked for the **Definite Integral**, from 1 to 2, of **2x** dx

First we need to find the ***Indefinite* Integral**.

Using the Rules of Integration we find that $\int 2x\,dx = x^2 + C$

Now calculate that at 1, and 2:

- At x=1: $\int 2x\,dx = 1^2 + C$
- At x=2: $\int 2x\,dx = 2^2 + C$

Subtract:

$(2^2 + C) - (1^2 + C)$

$2^2 + C - 1^2 - C$

$4 - 1 + C - C = 3$

And "C" gets cancelled out ... so with **Definite Integrals we can ignore C**.

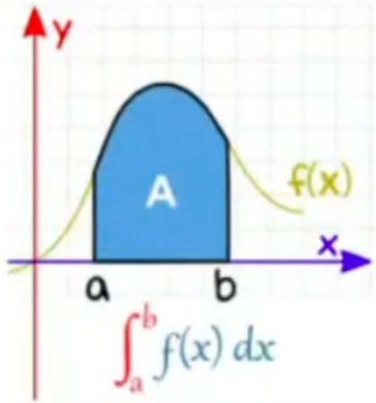
Result:

Definite Integrals

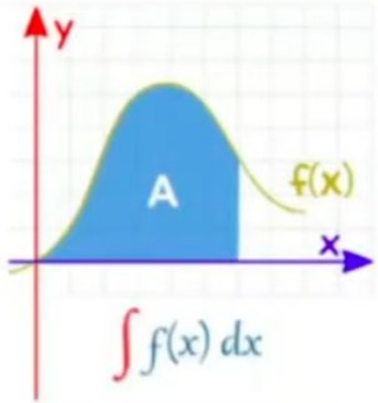
Definite Integral

A **Definite Integral** has start and end values: in other words there is an **interval** [a, b].

a and b (called limits, bounds or boundaries) are put at the bottom and top of the "S", like this:



Definite Integral
(from **a** to **b**)



Indefinite Integral
(no specific values)

Integration By Parts

Example: What is $\int x \cos(x)\, dx$?

OK, we have **x** *multiplied by* **cos(x)**, so integration by parts is a good choice.

First choose which functions for **u** and **v**:

- $u = x$
- $v = \cos(x)$

So now it is in the format $\int$ **u v dx** we can proceed:

Differentiate **u**: $u' = x' = 1$

Integrate **v**: $\int v\, dx = \int \cos(x)\, dx = \sin(x)$ *(see Integration Rules)*

Now we can put it together:

$$\int x \cos(x)\, dx$$

$$x \sin(x) - \int 1\, (\sin(x))\, dx$$

Integration By Parts

Integration by Parts

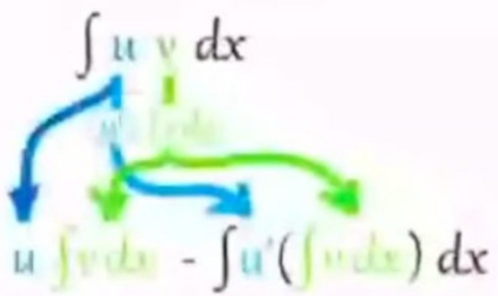
Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

You will see plenty of examples soon, but first let us see the rule:

$$\int u\,v\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$$

- **u** is the function u(x)
- **v** is the function v(x)
- **u'** is the derivative of the function u(x)

As a diagram:



Methods of Integration

- **Integration by Parts**
  For a given functions f(x) and q(x), we have
  $\int[f(x)\, q(x)]\, dx = f(x)\int g(x)dx - \int\{f'(x) \int g(x)dx\}\, dx$
  Here, we can choose the first function according to its position in ILATE, where
  I = Inverse trigonometric function
  L = Logarithmic function
  A = Algebraic function
  T = Trigonometric function
  E = Exponential function
  [The function which comes first in ILATE should taken as first function and other as second function]

- Note
  (i) Keep in mind, ILATE is not a rule as all questions of integration by parts cannot be done by above method.
  (ii) It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int\sqrt{x}\, \sin x\, dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}\, \sin x$.
  (iii) Observe that while finding the integral of the second function, we did not add any constant of integration.

$$\int \frac{2x+3}{x^2-1}\,dx.$$

Factor the denominator: $x^2 - 1 = (x-1)(x+1)$. Decompose

$$\frac{2x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Multiply through:

$$2x+3 = A(x+1) + B(x-1).$$

Set $x = 1 :\ 2(1)+3 = 5 = 2A \Rightarrow A = \frac{5}{2}$.
Set $x = -1 :\ 2(-1)+3 = 1 = -2B \Rightarrow B = -\frac{1}{2}$.

So the integral becomes

$$\int \left(\frac{5/2}{x-1} - \frac{1/2}{x+1} \right) dx = \frac{5}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

Partial Fractions

Methods

(i) $$\frac{p(x)+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b},\ a\neq b$$ ✓

(ii) $$\frac{px+q}{(x-a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$$ ✓

(iii) $$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$$

(iv) $$\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$$

(v) $$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+c}{x^2+bx+c},$$

Methods of Integration

- **Integration by Partial Fractions**

Partial fractions is a method of expressing a rational function (a fraction where both numerator and denominator are polynomials) as a sum of simpler fractions whose denominators are powers of **linear** or **irreducible quadratic** factors.

These simpler fractions are easier to integrate, differentiate, or simplify in algebraic manipulations.

Formal definition:

If

$$R(x) = \frac{P(x)}{Q(x)}$$

is a rational function, and

$$Q(x) = (x - a_1)^{m_1}(x - a_2)^{m_2} \dots (x^2 + bx + c)^n$$

is factored into linear and irreducible quadratic factors over the reals, then $R(x)$ can be written as

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{(x - a_1)^2} + \cdots + \frac{B_1x + C_1}{x^2 + bx + c} + \dots$$

where the constants A_i, B_i, C_i are determined algebraically.

- Find $\int \tan x \, dx$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x$, then $du = -\sin x \, dx$.

So:

$$\int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{u} \, du$$

$$= -\ln |u| + C$$

Now substitute back $u = \cos x$:

$$\int \tan x \, dx = -\ln |\cos x| + C$$

Methods of Integration

- **Integration by Substitutions**

  Substitution method is used, when a suitable substitution of variable leads to simplification of integral.

  If $I = \int f(x)dx$, then by putting $x = g(z)$, we get

  $I = \int f[g(z)]\, g'(z)\, dz$

  Note: Try to substitute the variable whose derivative is present in the original integral and final integral must be written in terms of the original variable of integration.

(xxiv) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

(xxv) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

(xxvi) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(xxvii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$

(xxviii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(xxix) $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, n \neq -1$

(xxx) $\int e^x [f(x) + f'(x)] dx = f(x) e^x + C$

Basic Formulae

(xii) $\int \sec^2 x\,dx = \tan x + C$

(xiii) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(xiv) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

(xv) $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$

(xvi) $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

(xvii) $\int \frac{-1}{1+x^2}\,dx = \cot^{-1} x + C$

(xviii) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

(xix) $\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \operatorname{cosec}^{-1} x + C$

(xx) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$

(xxi) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log |x + \sqrt{x^2-a^2}\,| + C$

(xxii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log |x + \sqrt{x^2+a^2}\,| + C$

(xxiii) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Basic Formulae

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

(iii) $\int a^x dx = \frac{a^x}{\log a} + C$

(iv) $\int \sin x\, dx = -\cos x + C$

(v) $\int \cos x\, dx = \sin x + C$

(vi) $\int \tan x\, dx = -\log|\cos x| + C = \log|\sec x| + C$

(vii) $\int \cot x\, dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(viii) $\int \sec x\, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(ix) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

(x) $\int \sec x \tan x\, dx = \sec x + C$

(xi) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

Properties of Indefinite Integral

(i) $\int[f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(ii) For any real number k, $\int k\,f(x)\,dx = k\int f(x)dx$.

(iii) In general, if $f_1, f_2, \ldots\ldots\ldots, f_n$ are functions and $k_1, k_2, \ldots, k_n$ are real numbers, then

$\int[k_1 f_1(x) + k_2\,f_2(x) + \ldots + k_n f_n(x)]\,dx = k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \ldots + k_n \int f_n(x)\,dx$

Geometrical Interpretation

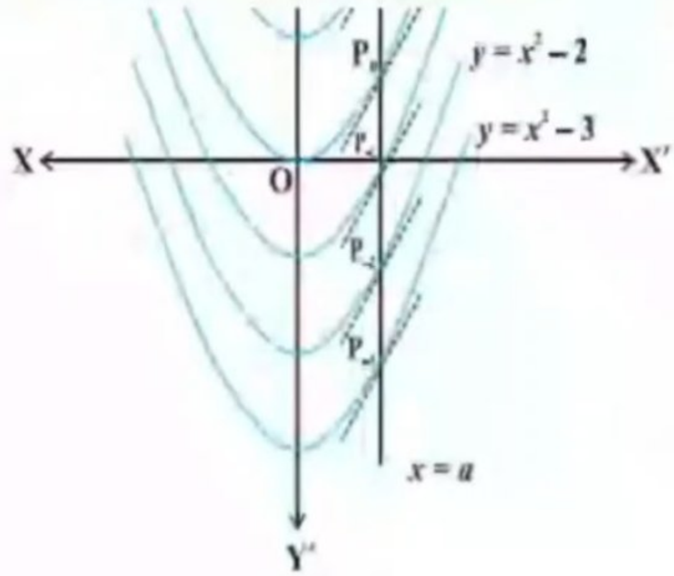


Fig 7.1

the tangents to all the curves $y = F_C(x)$, $C \in \mathbf{R}$, at the points of intersection of the curves by the line $x = a$, $(a \in \mathbf{R})$, are parallel.

Further, the following equation (statement) $\int f(x)\,dx = F(x) + C = y$ (say), represents a family of curves. The different values of C will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

What is Integral?

$y = 2x^2$; $\frac{dy}{dx} = 2 \times 2 \times x = 4x$

- Integration is the inverse process of differentiation.
- In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given.
- Then, $\int f(x)\,dx = F(x) + C$, these integrals are called indefinite integrals or general integrals. C is an arbitrary constant by varying which one gets different anti-derivatives of the given function.

$y = x^2 \rightarrow \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = 2x$ $\quad \frac{d}{dx}(\;)$

$\int f(x)dx \quad \int_a^b f(x)dx \quad \frac{d}{dx}$

definite

Note: Derivative of a function is unique but a function can have infinite anti-derivatives or integrals.

$\int 2x\,dx = \frac{2x^2}{2} + c = x^2 + c$

What is Integral?

Integration, Properties of Integration, Definite and Indefinite Integrals

CONTENT

What is Integration?

Geometrical Interpretation

Properties

Basic Formula

Methods of Integration

Integration By Parts

Definite Integrals

Integration, Properties of Integration, Definite and Indefinite Integrals

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