DAY 7

INTEGRATION, PROPERTIES OF INTEGRATION

DEFINITE AND INDEFINITE INTEGRATION

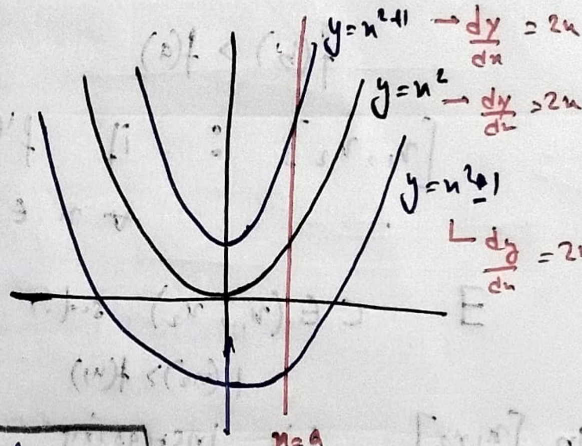
→ INTEGRAL – inverse of differentiation

– we need to find the func^n whose differential is given

→ $\int f(x)\,dx = F(x) + C$ (indefinite integral) └ arbitrary const.

NOTE: Derivative of a func^n is unique but a func^n can have infinite integrals (anti derivative)

Geometrical Interpretation.



PROPERTIES OF INDEFINITE INTEGRAL

i) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

ii) For any real num k, $\int k f(x)\,dx = k \int f(x)\,dx$

iii) in general if $f_1, f_2, f_3 \ldots f_n$ are functions & $k_1, k_2, k_3, \ldots k_n$ are real nos. then

$$\int [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\,dx = k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \ldots$$

$$\ldots + k_n \int f_n(x)\,dx$$

BASIC FORMULAE

i) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$

ii) $\int a^x\,dx = \frac{a^x}{\log a} + C$

iii) $\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$

(iv) $\int \sin x\,dx = -\cos x + C$

(v) $\int \cos x\,dx = \sin x + C$

(vi) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

(vii) $\int \cot x\,dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(viii) $\int \sec x\,dx = \log|\sec x + \tan x| + C = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$

(ix) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan \frac{x}{2}\right| + C$

ix) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

x) $\int \sec x \tan x \, dx = \sec x + C$

(xiii) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

xi) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(xiv) $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

xii) $\int \sec^2 x \, dx = \tan x$

(xv) $\int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x + C$

xvi) $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

(xvii) $\int \frac{-1}{1+x^2} \, dx = \cot^{-1} x + C$

xviii) $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$

(xix) $\int \frac{-1}{x\sqrt{x^2-1}} \, dx = \operatorname{cosec}^{-1} x + C$

xx) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{c} + C$

(xxi) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$

xxii) $\int \frac{dx}{x^2+a^2} = \log\left|x + \sqrt{x^2+a^2}\right| + C$

(xxiii) $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(xxiv) $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(xxv) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(xxvi) $\int \sqrt{x^2-a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + C$

(xxvii) $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(xxviii) $\int \sqrt{x^2+a^2} \, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + C$

(xxix) $\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \quad n \neq -1$

(xxx) $\int e^x[f(x) + f'(x)] \, dx = f(x)\, e^x + C$

METHODS OF INTEGRATION

- **Integration by Substitution** – changing the variable simplifies the integral:

If $I = \int f(x)\,dx$, put $x = g(z) \Rightarrow dx = g'(z)dz$

Then $I = \int f[g(z)] \cdot g'(z)\,dz$

- Choose a derivative whose derivative already exist in the integral.
- Express the result back in terms of the og variable

Ex. $\int \sin x\,dx = -\cos x + C$

$\int (\sin x + \cos x)\,dx = -\cos x + \sin x + C$

Q. Find $\int \tan x\,dx$.

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$ so, $\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$

$du = -\sin x\,dx$

Substitute $u = \cos x$:

$\int \tan x\,dx = -\ln|\cos x| + C$

$= \ln|\cos^{-1} x| + C$

$= \ln|\sec x| + C$

- **Integration by Partial Fraction**
  - Express a ration no. as sum of simpler fraction, with liner irreducible quad deno.
  - simplifies integral, diff, & algebraic manipul^n

If $R(x) = \frac{P(x)}{Q(x)}$ and

$Q(x) = (x-a_1)^{m_1}(x-a_2)^{n} \cdots (x^2+bx+c)^n$ factored form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \dots + \frac{B_1x+C_1}{x^2+bx+c} + \dots$$

cond: A_i, B_i, C_i are found algebraically.

PARTIAL FRACTION METHODS

(i) $$\frac{p(x)+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} \quad a \neq b$$

(ii) $$\frac{px+q}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$$

(iii) $$\frac{px^2+qx+c}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

(iv) $$\frac{px^2+qx+c}{(x-a)^2(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$$

(v) $$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$$

E.g. $$\int \frac{2x+3}{x^2-1}\,dx$$

Factor the denominator $x^2-1 = (x-1)(x+1)$ Decompose

$$\frac{2x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Multiply through: $2x+3 = A(x+1) + B(x-1)$

set($x = 1$): $2(1)+3 = 5 = 2A \Rightarrow A = \frac{5}{2}$

set($x=-1$): $2(-1)+3 = 1 = -2B \Rightarrow B = \frac{-1}{2}$

so the integral becomes:

$$\int \left(\frac{5/2}{x-1} - \frac{1/2}{x+1}\right) dx = \frac{5}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

* INTEGRATION BY PARTS

for a given func^n $f(x)$ & $g(x)$.

$$\int (f(x)\, g(x))\, dx = f(x)\int g(x)\, dx - \int \left\{ f'(x) \int g(x)\, dx \right\} dx$$

We choose the first func^n acc to : ILATE.

I = Inverse Trigonometric Func^n
L = Logarithmic function
A = Algebraic function
T = Trigonometric function
E = Exponential function

[The func^n which comes first in ILATE should taken as first func^n and other as 2nd func^n]

→ ILATE – only a guideline, not always perfect for every q.
→ In some cases, int. by parts may not help.
→ when you choose u, make sure its derivative is simpler
→ Integrate the func^n assigned as v.

example: Integration by parts

$$\int u \cdot v\, dx = u\int v\, dx - \int u'\left(\int v\, dx\right) dx$$

$\int x \cos x\, dx$

$u' = x' = 1$

Let $u = x$
$v = \cos x$, $\int v\, dx = \int \cos x\, dx = \sin x$

$\int x \cos x\, dx$

$x\int \cos x\, dx - \int x' \left(\int \cos x\, dx\right) dx$

$x(\sin x) - \int 1\, (\sin x)\, dx$

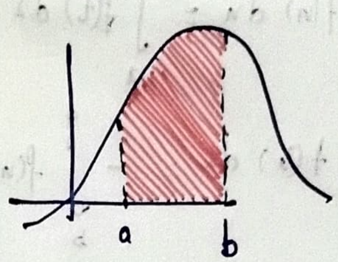
$x \sin x - \int \sin x\, dx$

$x \sin x + \cos x + c$

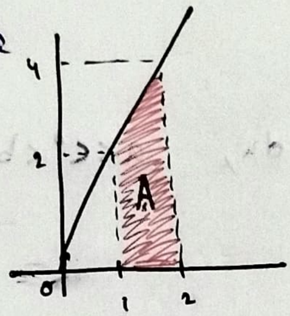
DEFINITE INTEGRALS

→ has start and end values → Interval $[a, b]$

a, b → (limit, bounds) $\quad I = \int_a^b f(x)\,dx$



e.g. $\int_1^2 2x\,dx =$



$\int 2x\,dx = x^2 + C$

at $x = 1$: $\int 2x\,dx = 1^2 + C$

$x = 2$: $\int 2x\,dx = 2^2 + C$

subtract: $(2^2 + C) - (1^2 + C) = 2^2 + C - 1^2 - C = 4 - 1 + C - C = 0$ → C gets cancelled out

∴ with Definite Integral we can ignore "C"

Adding Functions

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Reversing the Interval

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Interval of Length "0"

$$\int_a^a f(x)\,dx = 0$$

Adding of Intervals :

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Summary

The definite integral b/w a & b is the Indefinite Integral at b minus the Indefinite integral at a

PROPERTIES OF DEFINITE INTEGRATION

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b$

- Generalization

$a < c_1 < c_2 < \cdots < c_{n-1} < c_n < b$, then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx$$

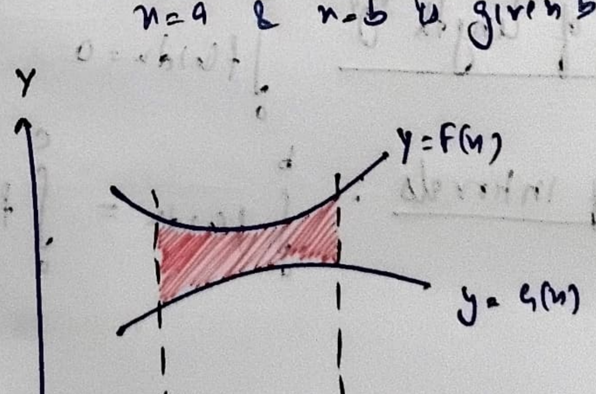
4. $\int_a^a f(x)\,dx = 0$

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

6. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Area b/w Curves

Area bounded by two curves $y = F(x)$ & $y = G(x)$ b/w $x = a$ & $x = b$ is given by



$$\int_a^b [F(x) - G(x)]\,dx$$