STUDY GUIDE TO ACCOMPANY

SEARS / ZEMANSKY / YOUNG

University Physics

SEVENTH EDITION

JAMES R. GAINES/WILLIAM F. PALMER

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PREFACE

The emphasis in any calculus-based physics course for scientists and engineers is on problem solving. This Guide has been written to help you learn how to solve the kind of problems you will encounter in homework assignments and examinations.

Each chapter in the Guide corresponds to a chapter in the parent text, *University Physics*, Seventh Edition, by Francis W. Sears, Mark W. Zemansky, and Hugh D. Young, Addison-Wesley Publishing Co., Reading, 1987, referred to as SZY in the Guide. The organization of the chapters is as follows:

Objectives. All chapters begin with a statement of learning objectives, stressing application of basic physical principles to problems and calculations.

Review. Primary concepts, definitions, formulas, units, and physical constants are then reviewed and highlighted.

Supplement. Some chapters have supplementary material which builds the basis for or expands upon materials of the text. These supplements are primarily in the early chapters, where they are concerned with applying basic skills of vector manipulation and calculus to physical problems.

Problem-Solving Strategies. Many chapters have a section on problem-solving strategies, including step-by-step guidance on how to set up and work a problem through to a solution.

Examples and Solutions. In this section examples similar to the problems at the end of the chapter in SZY are worked out, step-by-step, with commentary on organization of given information, principles applied, set-up procedure, pitfalls, and alternative solutions.

Quiz. Each chapter has a short quiz, with answers. In addition, as described below, any of the solved examples may be used as a self-quiz.

The Guide is thus an organizer of study, review, and self-diagnostic activities, as well as a ready reference of solved problems. If difficulty is encountered with a homework problem, a similar one can usually be found in the Guide, fully worked out. Rather than stare at the homework problem, with no useful forward progress, you can refer to the similar one for guidance. This is one of the chief uses to which the Guide may be put. Solved exercises as a pedagogical tool are a tradition in physics. We have attempted to make the solutions clear and detailed. We have emphasized problems which demonstrate the main concepts, avoiding those which require only substitution into a formula of the text.

After reading the text and notes you may wish to read the objective and review sections, and then work through the examples, referring back to the text and supplement sections when an unfamiliar concept or method is met. This process may occur before, after, or during the process of attempting the homework.

A word of caution is in order when reading through solved problems. You may doze, nod vigorously and excitedly, or experience warm feelings inside as you peruse the solutions, but the acid test of understanding comes only when you challenge yourself by taking a self-administered examination. This occurs each time you sit down to do a homework problem, or you can make things more realistic by testing yourself with an example in the Guide under examination conditions: Do not peek at the solution. Use no books or notes. Then score yourself according to correct method and correct answer.

The results of such an exercise may be enlightening. Remember your instructor regards the statement 'I understand the material but don't know how to solve the problems' as self-contradictory. In a sense, the problems **are** the material. A complete mastery of the material consists in the ability to (1) recognize, in the physical situation set up by the problem, the appropriate concepts to apply, (2) apply these concepts with confidence and care to obtain a rigorous solution method, and (3) carry this solution through **to the end** with out making computational errors.

To that end we hope this Study Guide will make your way a little less rough. Good luck!

James. R. Gaines
William F. Palmer
Columbus, Nov. 1986

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1

UNITS, PHYSICAL QUANTITIES, AND VECTORS

OBJECTIVES

In the first part of this chapter you are introduced to physical quantities, the result of measurements, which are related to each other by the laws and formulas of physics. Your objectives are to:

Write down numerical quantities with the correct number of significant figures and the correct units.

Calculate absolute and percent uncertainties.

Treat units algebraically and make unit conversions.

Make estimates and find the order of magnitude of quantities.

In the second part of this chapter you are introduced to vectors and vector operations. Your objectives are to:

Plot a vector.

Add and subtract two vectors.

Resolve a vector into its rectangular components.

Add and subtract vectors using the component method.

Write a vector and sums and differences of vectors in terms of unit vectors.

Multiply vectors using the scalar product.

Multiply vectors using the vector product.

REVIEW

All physical quantities in the early parts of the text are given in terms of the basic mechanical units of measurement: meters (m) for length, kilograms (kg) for mass, and seconds (s) for time, in the SI system. For example a volume is given in terms of m^3; a density or mass per unit volume is given in terms of $kg \cdot m^{-3}$.

All quantities derived from measurement have <u>uncertainties</u> <u>or</u> <u>errors</u>. For example, if a length is known with some certainty to be between 10.1 m and 9.9 m, it is written as

$$L = 10.0 \pm 0.1 \text{ m}.$$

The absolute uncertainty is 0.1 m. The percent uncertainty is the absolute uncertainty divided by the measurement itself and then multiplied by 100. In this case:

$$\text{Percent uncertainty} = \frac{0.1 \text{ m}}{10.0 \text{ m}} \times 100 = 1\%.$$

In the last example the measurement is known to three <u>significant</u> <u>figures</u>. The mass (26.24 ± 0.02) kg is known to four significant figures. If it were written (26.243 ± 0.02) kg, the last digit in the measurement would not be significant. If a quantity is written down without an uncertainty, the last digit on the right must be a significant figure: 26.24 kg is the correct way of writing the mass in the last example.

The units on each side of an equation must be the same or multiples of each other: this affords a computational check of the algebra. In the equation $m = \rho V$ (mass = density x volume), if $\rho = 8\ g \cdot cm^{-3}$ and $V = 10\ cm^3$, we have:

$$m = (8\ g \cdot cm^{-3})(10\ cm^3) = 80\ g.$$

The result automatically comes out in the correct units of mass (g) when the units are cancelled as algebraic quantities.

To convert units from one unit of mass to another, use the conversion relations as algebraic substitutions: for example, since

$$1\ g = 10^{-3}\ kg \text{ (conversion relation)},$$

we know that

$$80\ g = 80 \times 10^{-3}\ kg = 0.080\ kg.$$

Similarly

$$1\ m = 10^{-3}\ km,$$

$$1\ s = (3600)^{-1}\ hr,$$

so that

$$20\ m \cdot s^{-1} = (20 \times 10^{-3}\ km)(3600\ (hr)^{-1})$$

$$20\ m \cdot s^{-1} = 72\ km \cdot hr^{-1}.$$

An <u>order of magnitude</u> estimate is the result of a rough calculation accurate to factors of about two or even ten.

A vector **A** is a quantity with a direction and a magnitude. The magnitude is written $|\mathbf{A}| = A$. The sum or resultant **R** of the vectors **A** and **B**,

$$\mathbf{A} + \mathbf{B} = \mathbf{R},$$

may be obtained by graphical construction or by other methods discussed in the supplementary material below.

A vector may be resolved into component vectors: $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$. In terms of <u>unit vectors</u> **i**, **j**, we may write the component vectors as

$$\mathbf{A}_x = A_x \mathbf{i}, \quad \mathbf{A}_y = A_y \mathbf{j},$$

and

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} = \mathbf{A}_x + \mathbf{A}_y$$

The components A_x and A_y may be positive or negative.

The scalar product of two vectors **A** and **B** is the scalar quantity

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta = A_x B_x + A_y B_y$$

where θ is the angle between **A** and **B**.

The vector product of two vectors **A** and **B** is the vector **C**,

$$\mathbf{C} = \mathbf{A} \times \mathbf{B},$$

whose magnitude is $AB|\sin \theta|$ and whose direction is perpendicular to **A** and **B**, with sense given by the right hand rule.

SUPPLEMENTARY MATERIAL

All quantities in the first portion of the text may be expressed algebraically in terms of quantities with dimension mass, length and time.

Some quantities are pure numbers, without dimension, such as the ratio of two lengths. These quantities do not have units.

A dimensional physical quantity is not specified unless its units are given. The equation s(length) = 10 is meaningless because it could mean 10 meters or 10 miles. In the table below the three basic quantities of length, time and mass and one derived quantity, speed, are analyzed according to their dimension and units in the S.I. and British systems.

	QUANTITY	DIMENSION	UNIT(S.I.)	UNIT(BRI.)
distance	s (basic)	L	meter	foot
time	t (basic)	t	second	second
mass	m (basic)	m	kilogram	pound-mass
speed	v (derived)	Lt^{-1}	meter(second)$^{-1}$	feet(second)$^{-1}$

Note the dimension is independent of the system of units.

Equations involving physical quantities must be dimensionally consistent. For example in the equation

$$x = x_0 + vt$$

if x and x_0 have dimension L, and v has dimension Lt^{-1}, then each side has the dimension L and the equation is dimensionally correct. This is a useful check in any result you derive.

The <u>units</u> of an equation, on the other hand, need not be identical on right and left:

$$1 \text{ inch} = 2.54 \text{ cm} = 2.54 \times 10^{-2} \text{ m}$$

is a correct relation expressing the dimensionless ratio

$$\frac{2.54 \text{ cm}}{1 \text{ inch}} = 1 = \frac{2.5 \times 10^{-2} \text{ m}}{\text{inch}}$$

Such a relation is useful in unit conversions when it is noted that units are algebraic quantities which combine and cancel, like numbers, according to the rules of algebra: Suppose it is known that $v = st^{-1}$ and s = 10 inch, t = 5 s. What is the speed v in $\text{cm}\cdot\text{s}^{-1}$?

$$v = \frac{10 \text{ inch}}{5 \text{ s}} = 2 \text{ inch}\cdot\text{s}^{-1} = 2 \text{ inch}\cdot\text{s}^{-1}\, 2.54 \ \frac{\text{cm}}{\text{inch}}$$

$$= 5.08 \text{ cm}\cdot\text{s}^{-1}.$$

Note we have simply inserted the factor $1 = 2.54 \text{ cm}\cdot\text{inch}^{-1}$ and cancelled the units.

When two numbers are combined algebraically the number of significant figures of the result is the number of significant figures of the least significant of the two:

$$2.145 \text{ m} + 3.0 \text{ m} = 5.1 \text{ m},$$

$$(2.3 \text{ m})(1.234 \text{ m}) = 2.8 \text{ m}^2.$$

Two vectors **A** and **B** are equal, **A** = **B**, if they have the same magnitude and direction, as shown in Fig. 1-1.

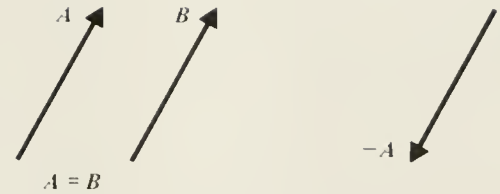


Figure 1-1

Note they need <u>not</u> be coincident. <u>For purposes of adding and subtracting, vectors may be moved about in space as long as the magnitude and direction are not changed</u>.

The negative of the vector **A**, denoted − **A**, is defined as a vector equal in magnitude but opposite in direction to **A**, as shown in Fig. 1-1.

Two vectors may be added according to the rule of placing the tail of one to the head of the other; the sum or resultant **R** is shown in Fig. 1-2. **R** is the vector drawn from the tail of the first to the head of the second, completing the triangle. The construction in Fig. 1-2 illustrates that the addition operation does <u>not</u> depend on the order, **A** + **B** = **B** + **A**.

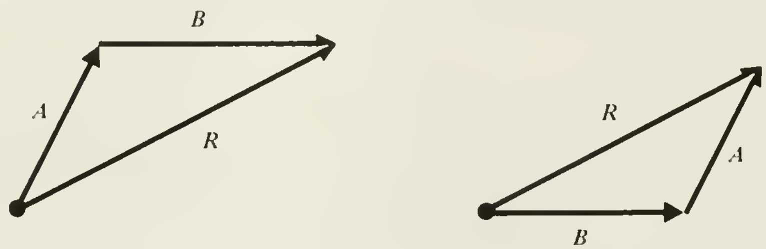


Figure 1-2

Vectors may be multiplied by numbers or scalars. a**A** is a vector with the same direction as **A** and a magnitude a|**A**|, if a is positive. If a is negative, a**A** is opposite in direction to **A**.

The sum **A** + **B** + **C** = (**A** + **B**) + **C** = **A** + (**B** + **C**), is shown in Fig. 1-3. Note the parentheses are unnecessary.

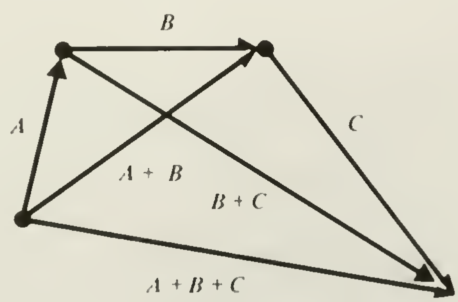


Figure 1-3

Subtraction of vectors is defined as follows

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}),$$

as shown in Fig. 1-4, where the vector − **B** is found from the given vector **B** by reversing its direction. The vector $-\mathbf{B}$ is then added to $\mathbf{A}$ by the usual procedure.

Figure 1-4

The rectangular components of a vector, as indicated in Fig. 1-5,

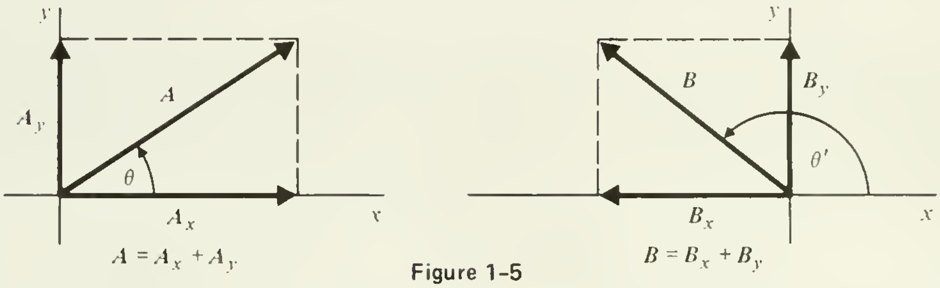


Figure 1-5

are the projections of $\mathbf{A}$ on the x and y axis. The magnitudes of the component vectors $\mathbf{A}_x$ and $\mathbf{A}_y$, A_x and A_y, are the called the <u>components</u> of $\mathbf{A}$, <u>with agreement that an x component is positive if it points in the positive axis direction</u>, with a similar rule for the y-component sign.

From Fig. 1-5 we see

$$\frac{A_x}{A} = \cos\theta \qquad \frac{A_y}{A} = \sin\theta$$

$$\frac{B_x}{B} = \cos\theta' \qquad \frac{B_y}{B} = \sin\theta'$$

Note the magnitudes of $\mathbf{A}, \mathbf{B}$ are always positive but the components may be positive (A_x, A_y and B_y) or negative (B_x) as shown in Fig. 1-5.

You can use various angles to describe the orientation of a vector, as shown in Fig. 1-6.

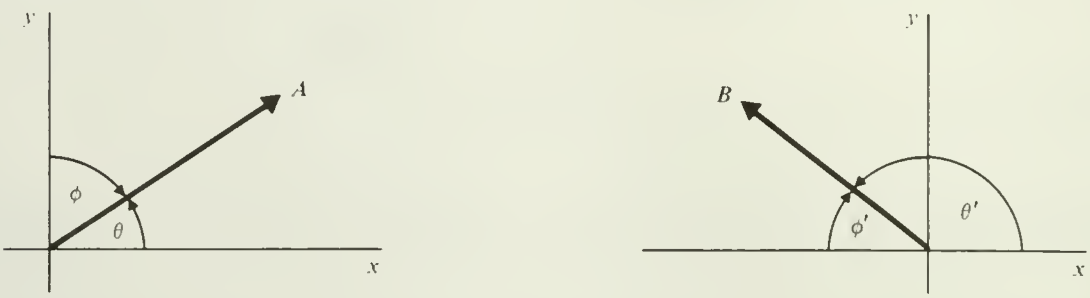


Figure 1-6

In terms of the angles ϕ and ϕ', complementary to θ and θ', we have

$$A_x = A \sin \phi, \qquad B_x = -B \cos \phi',$$

$$A_y = A \cos \phi, \qquad B_y = B \sin \phi',$$

and the formulas look different but yield the same numerical answer. For example, take $A = B = 2$, $\theta = 30^\circ$, $\phi = 60^\circ$, $\theta' = 150^\circ$, $\phi' = 30^\circ$.

Then by the first method

$$A_x = A \cos \theta = 2 \cos 30^\circ = 1.7,$$

$$A_y = A \sin \theta = 2 \sin 30^\circ = 1.0,$$

$$B_x = B \cos \theta' = 2 \cos (150^\circ) = -1.7,$$

$$B_y = B \sin \theta' = 2 \sin (150^\circ) = 1.0,$$

and by the second method

$$A_x = A \sin \phi = 2 \sin 60^\circ = 1.7,$$

$$A_y = A \cos \phi = 2 \cos 60^\circ = 1.0,$$

$$B_x = -B \cos \phi' = -2 \cos 30^\circ = -1.7,$$

$$B_y = B \sin \phi' = 2 \sin 30^\circ = 1.0.$$

Given the components of a vector you can reverse the process and calculate its magnitude and direction. From Fig. 1-5, the magnitude **A** and orientational angle θ are:

$$A = [A_x{}^2 + A_y{}^2]^{1/2},$$

$$\theta = \arctan \frac{A_y}{A_x}.$$

In the above numerical example

$$\theta = \arctan \frac{A_y}{A_x} = \arctan \frac{1.0}{1.7} = 30^\circ,$$

$$\theta' = \arctan \frac{B_y}{B_x} = \arctan \frac{-1.0}{1.7} = 150^\circ \text{ or } -30^\circ?$$

The last example illustrates the pitfalls of blind calculation. 150° and -30° have the same tangent; to decide which angle is correct you must refer to the diagram or examine the signs B_x and B_y to see which quadrant B lies in. By reference to Fig. 1-5 we see that $\theta = 150^\circ$.

To master the material in this text you must become efficient and accurate at component resolution. In later chapters you will often be required to sum forces. The component method is often the most convenient:

If **C** = **A** + **B**, then

$$\mathbf{C_x} = \mathbf{A_x} + \mathbf{B_x}, \qquad C_x = A_x + B_x,$$

$$\mathbf{C_y} = \mathbf{A_y} + \mathbf{B_y}, \qquad C_y = A_y + B_y.$$

Another useful way to write vectors is in terms of unit vectors, which have magnitude unity, are dimensionless, and point along the positive axes in a rectangular system, as shown in Fig. 1-7.

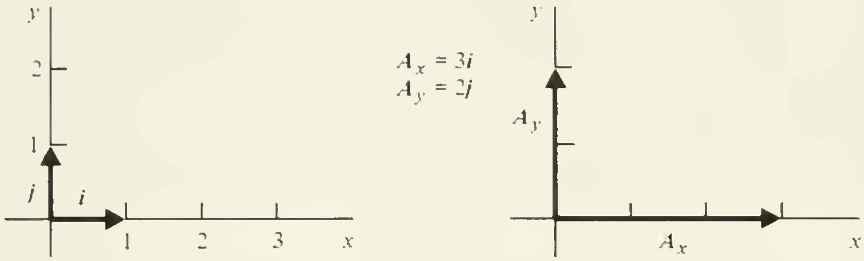


Figure 1-7

The component vectors are

$$\mathbf{A_x} = A_x\mathbf{i} \qquad \mathbf{A_y} = A_y\mathbf{j}$$

and the vector **A** is reconstructed as

$$\mathbf{A} = \mathbf{A_x} + \mathbf{A_y} = A_x\mathbf{i} + A_y\mathbf{j}$$

Extending this to three dimensions, we have

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},$$

where **k** points along the positive z axis. The usual formulas for vector addition and subtraction can be rewritten to utilize this new notation, e.g.

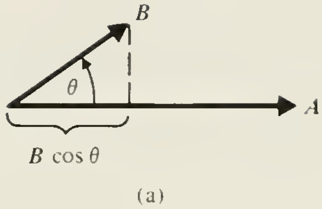
$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

Vectors may be 'multiplied' with each other but the operation is different from the simple multiplication of two scalars and must be carefully defined and distinguished. There are two useful products which will be used in the text, the scalar product and the vector product.

The scalar product **A·B** is a scalar found from **A** and **B** according to the rule

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$$

where θ is the angle between **A** and **B**, measured from **A** to **B**, as shown in the Fig. 1-8a and 1-8b.



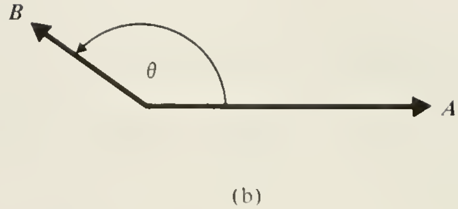


Figure 1-8

Since cos θ changes sign at $\theta = 90^\circ$, the scalar product is positive for $0 < \theta < 90^\circ$ and negative for $90^\circ < \theta < 180^\circ$ and vanishes at $\theta = 90^\circ$. As shown, the scalar product is the projection of **B** on **A** multiplied by the magnitude of A. In the example of Fig. 1-8 with A = 2, B = 1, and $\theta = 30^\circ$,

$$\mathbf{A}\cdot\mathbf{B} = AB \cos\theta = 2\cdot 1\cdot \cos 30^\circ = 1.73$$

If **A** and **B** are parallel, $\mathbf{A}\cdot\mathbf{B} = AB$; if they are anti-parallel, $\theta = 180^\circ$ and $\mathbf{A}\cdot\mathbf{B} = -AB$. If they are perpendicular $\mathbf{A}\cdot\mathbf{B} = 0$.

The scalar product obeys the law

$$(\mathbf{A} + \mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C} + \mathbf{B}\cdot\mathbf{C}$$

Thus with the rectangular resolution we can show

$$\mathbf{A}\cdot\mathbf{B} = (A_x\,\mathbf{i} + A_y\,\mathbf{j})\cdot(B_x\,\mathbf{i} + B_y\,\mathbf{j})$$
$$= A_xB_x + A_yB_y$$

because $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = 0$.

The <u>vector product</u> **C** of two vectors **A** and **B** is a vector obtained from **A** and **B** by the rule

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

Magnitude: $C = |AB \sin\theta|$

Direction: perpendicular to **A** and **B** with sense given by the right hand rule.

According to the right hand rule, if you curl the fingers of your right hand around the perpendicular to **A** and **B** in the direction of rotation of **A** into **B**, your thumb points toward **C**. As a consequence, the scalar product changes sign if the order of the factors is reversed:

$$\mathbf{B} \times \mathbf{A} = -\,\mathbf{A} \times \mathbf{B}$$

An important expression for **A** x **B** in terms of unit vectors is given by

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

$$= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

If **A** and **B** are parallel, **A** x **B** = 0 because $\sin\theta = 0$. If **A** and **B** are perpendicular, C = AB.

EXAMPLES AND SOLUTIONS

Example 1

In the equation,

$$x = x_o + v_ot + \frac{1}{2}at^2,$$

suppose we know that $x_o = 2.0$ m, $v_o = 3.0\ \mathrm{m \cdot s^{-1}}$, $a = -9.8\ \mathrm{m \cdot s^{-2}}$, and $t = 4.0$ s. Find the quantity x.

Solution:

$$x = 2.0\ \mathrm{m} + (3.0\ \mathrm{m \cdot s^{-1}})(4\ \mathrm{s}) + 1/2(-\ 9.8\ \mathrm{m \cdot s^{-2}})(4.0\ \mathrm{s})^2 = -\ 64\ \mathrm{m}$$

(to 2 significant figures)

Note how the units cancel to produce the unit of meters in each term and that the correct answer has a sign and a unit.

Example 2

Convert 60.0 mph to a speed in $\mathrm{km \cdot hr^{-1}}$, given 1 inch = 2.54 cm.

Solution:

$$\frac{60.0\ \text{mi}}{\text{hr}} = \frac{60.0\ \text{mi}}{\text{hr}} \left[\frac{5280\ \text{ft}}{\text{mi}} \right]\left[\frac{12\ \text{in}}{\text{ft}} \right] \times$$

$$\left[\frac{2.54\ \text{cm}}{\text{in}} \right]\left[\frac{10^{-5}\ \text{km}}{\text{cm}} \right] = \frac{96.6\ \text{km}}{\text{hr}}$$

(The bracketed expressions are all of magnitude unity and may be inserted anywhere in the expression. Note how the units cancel.)

Example 3

Consider the density of water to be exactly 1.00 $g \cdot cm^{-3}$. What is its value in pound-mass$\cdot ft^{-3}$ to three significant figures? (The 'pound-mass', a common household unit, is equal to 0.454 kg.)

Solution:

For this conversion you must know the relation between a gram (g) and a pound-mass, and the relation between a cubic centimeter (cm^3) and a cubic foot (ft^3). Since there are 12 inches in a foot and 10^3g in a kg,

$$\frac{1}{12}\ \text{foot} = 1\ \text{inch} = 2.54\ \text{cm},$$

$$1\ \text{pound-mass} = 0.454 \times 10^3 \text{g}.$$

In the last expression we have kept only three significant figures.

To convert units, multiply the original expression by the factors

$$1 = \frac{\text{pound-mass}}{.454 \times 10^3\ \text{g}} \quad \text{and } 1 = \left(\frac{2.54\ \text{cm}}{(1/12)\ \text{ft}} \right)^3$$

Then we have

$$1\ \text{g} \cdot \text{cm}^{-3} = 1\ \frac{\text{g}}{\text{cm}^3}\ \frac{1\ \text{pound-mass}}{.454 \times 10^3\ \text{g}}\ \frac{(2.54)^3\ \text{cm}^3}{(1/12)^3\ \text{ft}^3}$$

$$= 62.4\ \text{pound-mass ft}^{-3}.$$

Alternatively you may substitute for a gram in the original expression its value from the conversion equation

$$1 \text{ g} = \frac{1 \text{ pound-mass}}{.454 \times 10^3},$$

and correspondingly,

$$1 \text{ cm} = \frac{1}{2.54} \frac{1}{12} \text{ ft},$$

yielding

$$1 \text{ g}\cdot\text{cm}^{-3} = \frac{1 \text{ pound-mass}}{.454 \times 10^3} \left(\frac{1}{2.54} \frac{1}{12} \text{ ft}\right)^{-3}$$

$$= 62.4 \text{ pound-mass}\cdot\text{ft}^{-3}.$$

Example 4

Compute the number of km in a mile to three significant figures.

Solution:

$$1 \text{ mi} = 1 \text{ mi}\left[\frac{5280 \text{ ft}}{\text{mile}}\right]\left[\frac{12 \text{ inch}}{\text{ft}}\right]\left[\frac{2.54 \text{ cm}}{\text{inch}}\right]\left[\frac{10^{-5} \text{ km}}{\text{cm}}\right]$$

$$= 1.61 \text{ km}$$

Example 5

Compute the percent error of the approximation $(10)^{1/2}$ for $\pi = 3.142...$

Solution:

$$\text{absolute error} = (10)^{1/2} - \pi = 0.02;$$

$$\text{percent error} = \frac{(10)^{1/2} - \pi}{\pi} \times 100$$

$$= 0.6\%.$$

In this case both π and $(10)^{1/2}$ are known to any desired accuracy, as is the percent error, and it is a matter of judgement and convenience as to how many significant figures to present.

Example 6

Estimate the number of times you blink in a day.

Solution:

Counting your own or someone else's blinks, you might time the frequency of blinks to be <u>about</u> one per second (probably no greater than 3 per second nor less than one every ten seconds.) An order of magnitude estimate is thus

$$\begin{aligned} 1 \text{ blink per second} &= 60 \text{ blinks per minute} \\ &= (60)(60) \text{ blinks per hour} \\ &= (60)(60)(24) \text{ blinks per day} \\ &= \text{about } 10^5 \text{ blinks per day} \end{aligned}$$

For part of the day your eyes are closed in sleep and that reduces the result by about 1/3, but doesn't change the order of magnitude.

Example 7

A 'cloudburst' drops 2 cm of rain in one hour. Estimate the number of drops of rain that fall on your head if you are in the storm for one minute.

Solution:

A raindrop leaves a spot about one quarter of a centimeter in diameter. (You may argue that a typical drop is larger or smaller by a factor of two or three, and that uncertainty is part of our order-of-magnitude estimate.) The volume of the drop is

$$(4/3)\pi(\text{radius})^3 \sim 4(\text{cm}/8)^3 = (1/128)\ \text{cm}^3$$

In 10 minutes, <u>256</u> such drops must fall on 1 cm^2 to reach a height of 2 cm. If the top of your head is a square about 10 cm on a side, its area is 100 cm^2, and thus

$$(100\ \text{cm}^2)(256\ \text{drops}/\text{cm}^2) \sim 25600$$

drops fall on your head in an hour, or 427 drops in one minute.

Example 8

An example of a vector is a force with a magnitude measured in the unit N = newtons.
(a) Find the vector sum or resultant of the two forces in Fig. 1-9.
(b) Find their difference **D** = **A** − **B**.

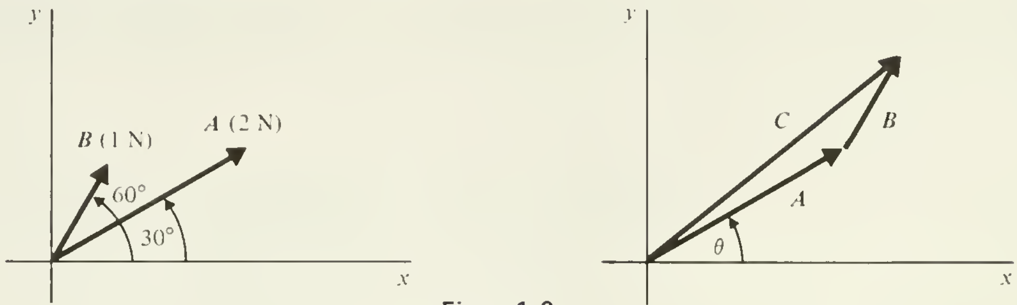


Figure 1-9

Solution:

(a) First resolve the individual vectors into their rectangular coordinates:

$$A_x = A \cos 30^\circ = 2 \cos 30^\circ = 1.7 \text{ N},$$

$$A_y = A \sin 30^\circ = 2 \sin 30^\circ = 1.0 \text{ N},$$

$$B_x = B \cos 60^\circ = 0.50 \text{ N},$$

$$B_y = B \sin 60^\circ = 0.87 \text{ N},$$

Then find the components, magnitude, and direction of **C**:

$$C_x = A_x + B_x = 2.2 \text{ N}$$

$$C_y = A_y + B_y = 1.87 \text{ N}$$

$$C = (C_x{}^2 + C_y{}^2)^{1/2} = [(2.2)^2 + (1.87)^2)]^{1/2} = 2.9 \text{ N}$$

$$\theta = \arctan \frac{C_y}{C_y} = \arctan \frac{1.87}{2.2} = 40^\circ$$

(For a check, compare these results with the graphical solution in Fig. 1-9. Even a rough sketch serves to check your analytical solutions for gross errors.)

We similarly find the vector difference **D** = **A** − **B**:

$$D_x = A_x - B_x = 1.7 - 0.5 = 1.2 \text{ N}$$

$$D_y = A_y - B_y = 1.0 - .87 = 0.03 \text{ N}$$

$$D = (D_x{}^2 + D_y{}^2)^{1/2} = [(1.2)^2 + (.03)^2]^{1/2} = 1.2\text{N}$$

D is a vector in the first quadrant ($D_x > 0$, $D_y > 0$) with

$$\theta = \text{arc tan}\ \frac{D_y}{D_x} = \arctan\ \frac{0.3}{1.2} = 1.43^\circ,$$

where θ is measured counter-clockwise from the positive x axis.

Example 9

Consider two vectors **A** and **B** in the x-y plane of Fig. 1-10. Find their cross product **A x B**.

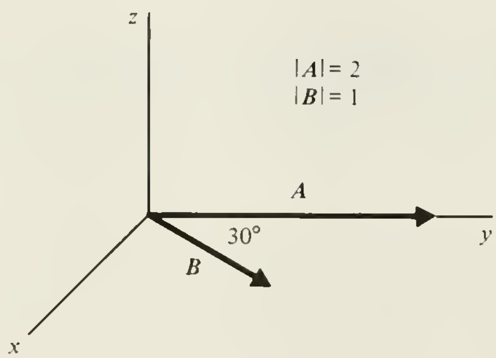


Figure 1-10

Solution:

The direction of **A x B** is along the negative z axis. (If your thumb points down along that axis your fingers indeed curl around the z axis from **A** to **B**.)

Alternatively, by the component method:

$A_x = 0$ $\quad\quad$ $A_y = 2$ $\quad\quad$ $A_z = 0$

$B_x = B \sin\theta = 1 \sin 30^\circ = 0.50$

$B_y = B \cos\theta = 1 \cos 30^\circ = 0.87$

$B_z = 0$

and

$C_x = A_yB_z - A_zB_y = 2(0) - 0(0.87) - 0$

$C_y = A_zB_x - A_xB_z = 0(0.5) - 0(0) = 0$

$C_z = A_xB_y - A_yB_x = 0(0.87) - 2(0.5) = -1.0.$

$C = 1$

Example 10

A car drives 3 km east and then 4 km north. Find the resultant displacement vectors:

Solution:

A: magnitude 3 km, direction east;

B: magnitude 4 km, direction north;

R = **A** + **B** = vector sum of **A** + **B**.

This sum may be found in several ways.

(a) Graphical Solution

This method requires graph paper, a protractor for measuring angles and a ruler for measuring lengths. You must also choose a scale, e.g., one box = 1 km. Then your plot will look like Fig. 1-11.

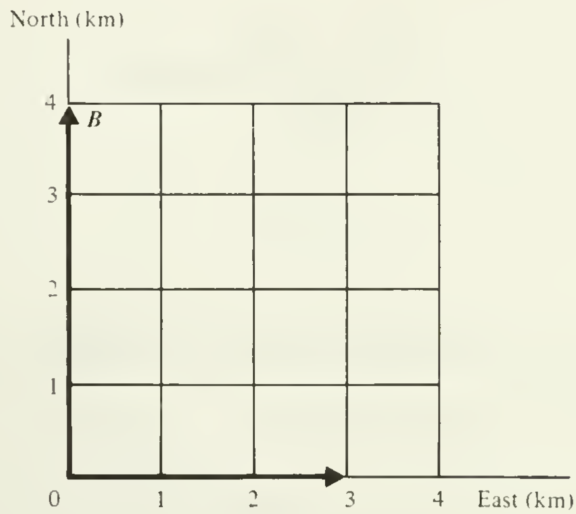


Figure 1-11

To add **A** and **B**, move A or B so that the head of one vector is coincident with the tail of the other, as shown in Fig. 1-12.

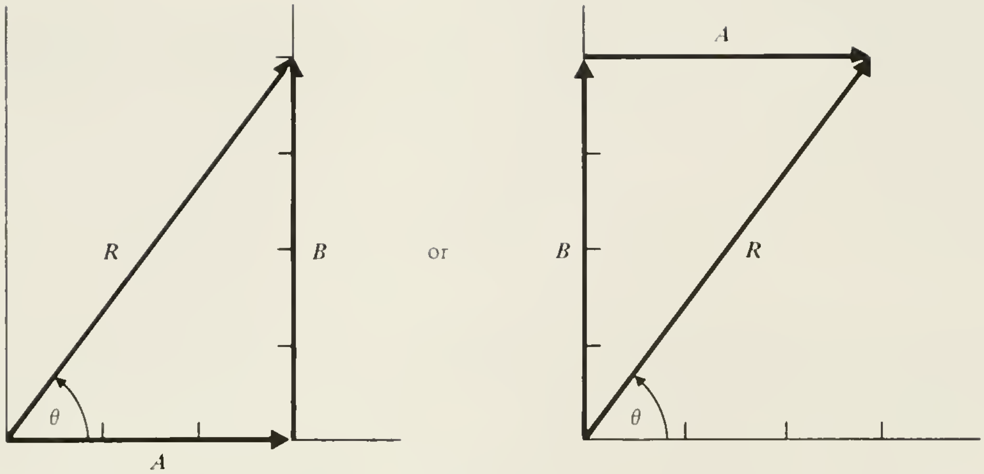


Figure 1-12

R is the vector from the open tail to the open head. Now measure the length of **R** in km, according to your scale. R ≃ 5 km. Measure the angle θ with a protractor: θ ≃ 53°. This graphical method inevitably suffers from errors of measurement.

(b) Geometric or Triangle Method.

Here you need only draw a single sketch to guide the eye, as shown in Fig. 1-13.

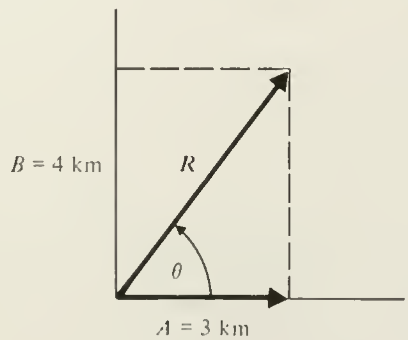


Figure 1-13

By the Pythagorean Theorem, the magnitude of the resultant is

$$R = (A^2 + B^2)^{1/2} = (3^2 + 4^2)^{1/2} = 5 \text{ km}$$

The angle between **R** and **A** is

$$\theta = \text{arc tan } \frac{B}{A} = \text{arc tan } \frac{4}{3} = 53.1°.$$

(Note these last two relations are true only when **A** and **B** are perpendicular.)

(c) Component Method--Rectangular Resolution.

To add **A** and **B** by this method, first resolve **A** and **B** into their x and y components:

$A_x = 3$ km, $A_y = 0$,

$B_x = 0$, $B_y = 4$ km.

Then

$$C_x = A_x + B_x = 3 + 0 = 3 \text{ km}$$

$$C_y = A_y + B_y = 0 + 4 = 4 \text{ km}$$

$$C = (C_x^2 + C_y^2)^{1/2} = (3^2 + 4^2)^{1/2} = 5 \text{ km}$$

$$\theta = \text{arc tan } \frac{C_y}{C_x} = \text{arc tan } \frac{4}{3} = 53.1^\circ$$

QUIZ

1. How many people standing hand-to-outstretched-hand are needed to span the width of the United States, coast to coast?

Answer: About 2.5 million

2. Find the sum (**C**) and difference (**D**) of vectors **A** and **B** shown in Fig. 1-14.

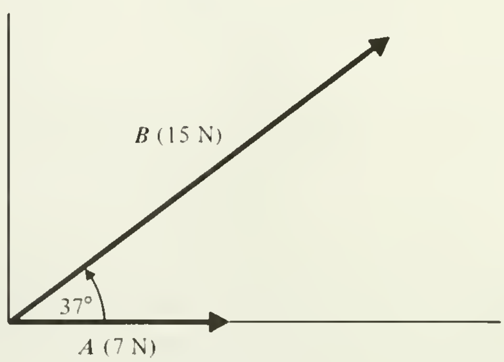


Figure 1-14

Answer: **C** = 21 N, 25° measured counter-clockwise from the x-axis.

D = 10 N, 241° measured counter-clockwise from the x-axis.

3. Two forces F_1 and F_2 act on a body at the origin as shown in Fig. 1-15. Find a third force F_3 acting at the origin such that the sum of all forces acting on

the body is zero.

Answer: F_3 = 12 N, θ = 187° measured counter-clockwise from the x-axis.

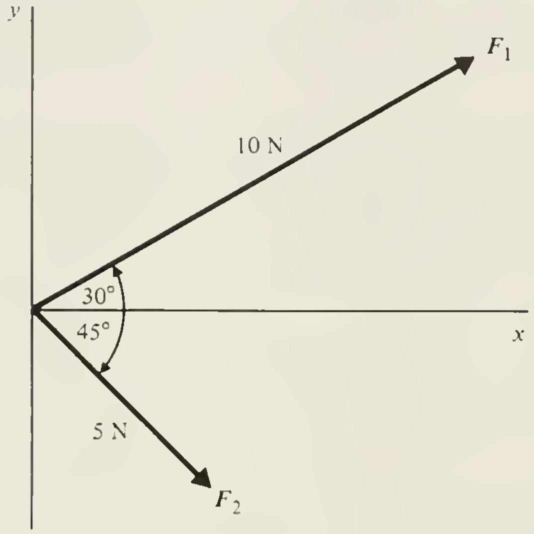


Figure 1-15

4. Find the scalar product of the vectors in Fig. 1-14

Answer: 84

5. What are the magnitude and direction of the vector product $\mathbf{F_1} \times \mathbf{F_2}$ in Fig. 1-15?

Answer: magnitude: 48; direction: down into page.

2
MOTION ALONG A STRAIGHT LINE

OBJECTIVES

In this chapter you will describe the motion of an object on a straight line by its position, velocity, and acceleration. A particularly important special case is when the acceleration is constant. Your objectives are to:

Calculate average and instantaneous velocity.

Calculate average and instantaneous acceleration.

Calculate instantaneous velocity when the position is given as a function of time by differentiation.

Calculate the position as a function of time when the instantaneous velocity is given as a function of time by integration.

Find the equations of motion for the case of constant acceleration.

Apply these equations of motion to a variety of problems involving constant acceleration, such as motion of a body falling in the earth's gravitational field.

Comments on the Objectives

The calculus techniques of differentiation and integration may be new concepts to many of you. Indeed, some of you will be seeing an integral for the first time. But don't panic: the entire chapter can be read without serious attention to the calculus techniques of differentiation and integration. Most of the problems at the end of the chapter can be solved without use of calculus. You will not be blown out of the water if you can't integrate or differentiate, at least not at this stage.

However, this is a calculus-based text and the introduction to these techniques comes most naturally here in connection with velocity and acceleration. You will be missing an important part of this chapter if you do not meet all of the above objectives. Eventually, sometime during the course, these objectives must be met.

REVIEW AND SUPPLEMENT

The <u>average velocity</u>, between time t_1 and time t_2, of a particle moving on a straight line from coordinate x_1 to coordinate x_2 is

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = \text{slope of the chord shown in Fig. 2-1}$$

During the interval between t_1 and t_2 the particle may speed up, slow down, even reverse its motion: the average velocity depends only on the initial and final positions. In a displacement x versus time t graph, Fig. 2-1a, the average velocity is the slope of the chord joining the points (x_1,t_1) and (x_2,t_2).

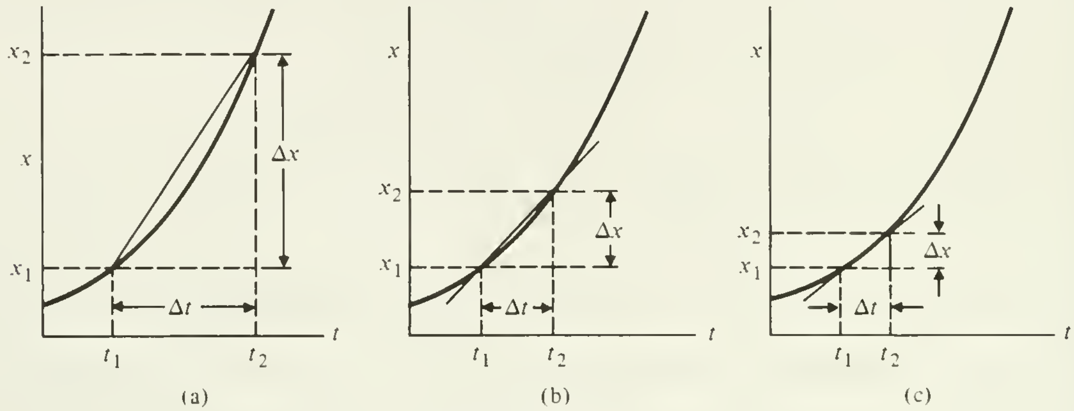


Figure 2-1

The last form may also be written

$$x_2 - x_1 = \bar{v}(t_2 - t_1)$$

When the initial time is $t_1 = 0$, the initial displacement is $x_1 = x_o$, and the final displacement is x,

$$x = x_o + \bar{v}t$$

If the initial point is the origin,

$$x_o = 0,\ x = \bar{v}t.$$

The <u>instantaneous velocity</u> is the limit of v as the increments Δx and Δt tend toward zero, that is, the limit of the <u>slope of the chord</u> in Fig. 2-1 as t_2 approaches t_1; in Fig. 2-1a, 2-1b, and 2-1c we have a sequence of successively smaller choices of $t_2 - t_1 = \Delta t$ and correspondingly smaller increments Δx. The smaller Δt, the more nearly is the chord tangent to the displacement curve as t_2 approaches t_1. The limit of the sequence of average velocities $\bar{v}$ for successively smaller Δx and Δt is called the derivative of x with respect to t and written dx/dt. The value of the derivative is the slope of the tangent curve. The instantaneous velocity is defined in terms of the limit as

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \text{slope of the tangent line.}$$

In Fig. 2-1 the coordinate x increases with increasing t, so the body is moving in the positive direction and has a positive velocity, corresponding to the positive slope and positive increment $\Delta x = x_2 - x_1$. In Fig. 2-2a the displacement x decreases with increasing time, the body is moving in the negative x direction and has a negative velocity corresponding to a negative slope and negative increment $\Delta x = x_2 - x_1$.

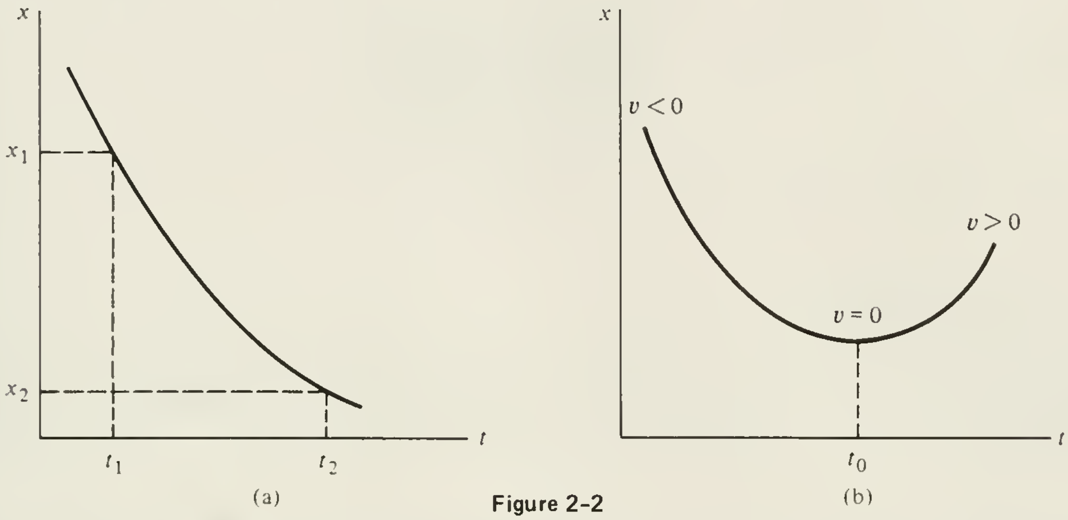


Figure 2-2

(By convention we take $t_2 > t_1$ so Δt is always positive.) In Fig. 2-2b the body starts out with negative velocity, which gets less negative, until it is zero at t_o where the tangent is horizontal and has zero slope. For $t > t_o$, v is positive.

In practice you can find the instantaneous velocity at a time t graphically by measuring the slope of the tangent to the x vs t graph at time t, or by differentialtion. If $x = ct^n$ where c is a constant, the drivative of x is

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(ct^n) = cnt^{n-1}$$

Note the derivative of a constant is zero.

Together with the fact that the derivative of a sum is the sum of the derivatives of the summands,

$$\frac{d}{dt}[f(t) + g(t)] = \frac{df}{dt} + \frac{dg}{dt}$$

this rule will enable you to differentiate any polynomial. For example, if $x = a + bt^2$, then $v = 2bt$.

Just as velocity is the rate of change of displacement with time, acceleration is the rate of change of velocity with time. The average acceleration a is defined by

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

and the instantaneous acceleration a by

$$a = \mathrm{Lim}_{\Delta t \to o} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Returning to the example $x = a + bt^2$, we found

$$v(t) = 2bt$$

If we wish to find the acceleration at t,

$$a = \frac{dv}{dt} = \frac{d}{dt}(2bt) = 2b$$

In Fig. 2-3a, b, c we show the coordinate $x = a + bt^2$ for $a > 0$, $b > 0$, the corresponding velocity $v = dx/dt = 2bt$, and the acceleration $a = dv/dt = 2b$.

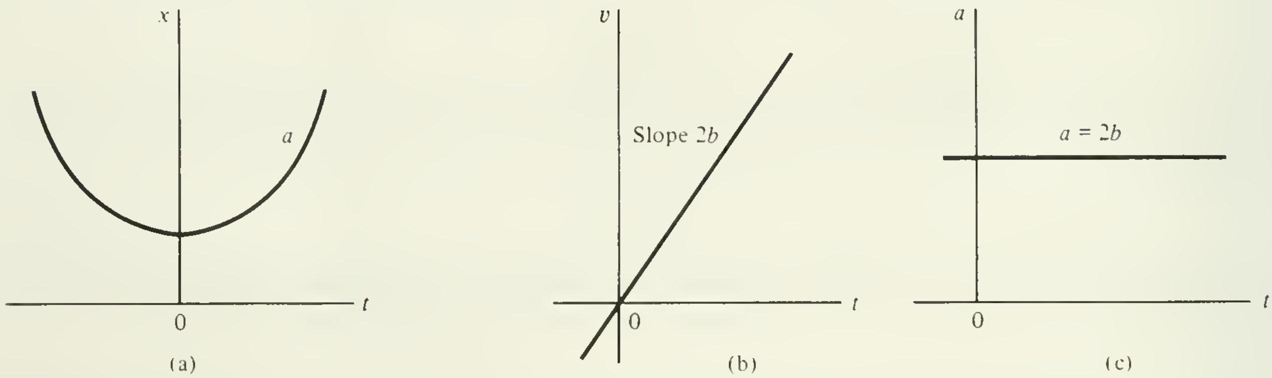


Figure 2-3

When the acceleration is constant, the displacement and velocity are given as functions of time by

$$x = x_o + v_o t + \frac{1}{2} at^2,$$

$$v = v_o + at.$$

The time may be eliminated in the two equations yielding a third equation, which although not independent of the other two, is very useful:

$$v^2 = v_o^2 + 2a(x - x_o)$$

Another way of deriving these and other relations is to use the method of integration and integral calculus. Consider the v versus time graph of Fig. 2-4.

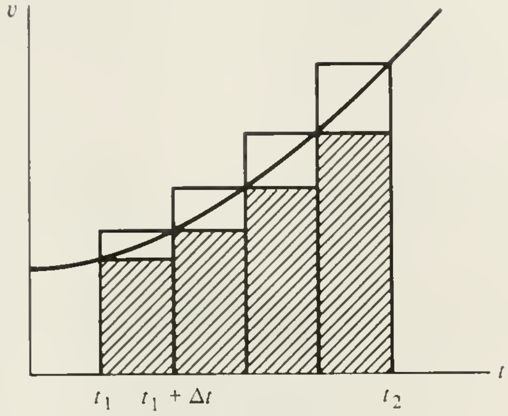


Figure 2-4

The time axis has been broken into small increments of width Δt. In each time increment, $\Delta x = v\Delta t$. If we sum all the increments Δx between $x_2 = x(t_2)$ and $x_1 = x(t_1)$ we have the interval length,

$$\sum \Delta x = x_2 - x_1 = \sum \bar{v}\Delta t.$$

In each interval the average velocity v lies somewhere between the highest and lowest instantaneous velocity in that interval. Thus, referring to Fig. 2-4, $\bar{v}\Delta t$ for each slice is an area somewhere between the smaller shaded slice and the larger slice, which includes the unshaded part. The sum $\sum \bar{v}\Delta t$ is thus bounded from above by the sum S_2 of the large slices and from below by the sum S_1 of the smaller slices,

$$S_1 < \sum \bar{v}\Delta t < S_2$$

As Δt is made smaller and smaller the common limit of S_1 and S_2 is the area under the v curve between t_2 and t_1, called the definite integral of v from t_1 to t_2, and denoted

$$\int_{t_1}^{t_2} v\,dt.$$

Thus we have

$$x_2 - x_1 = \int_{t_1}^{t_2} v\,dt = \int_{t_1}^{t_2} (v_o + at)dt$$

If $t_1 > t_2$ the area is counted negative when v is positive.

When $v = v_o + at$, as in Fig. 2-5,

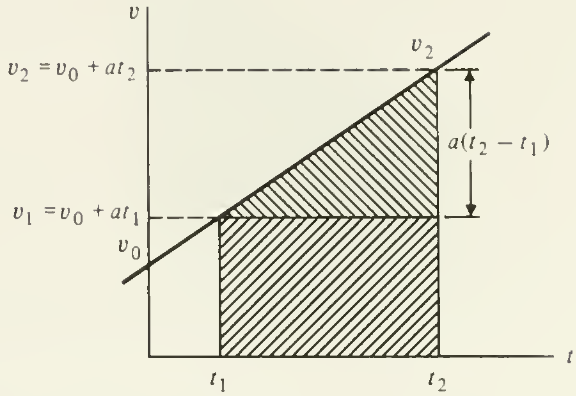


Figure 2-5

the area under the curve is the area of the box $= v_1(t_2 - t_1)$ plus the area of the triangle $1/2(t_2 - t_1)\, a(t_2 - t_1)$. Thus

$$x_2 - x_1 = v_1(t_2 - t_1) + \frac{a}{2}(t_2 - t_1)^2$$

When $t_1 = 0$, $v_1 = v_o$ and $x_1 = x_o$ we regain our earlier result.

$$x = x_o + v_o t + \frac{1}{2} at^2$$

For another demonstration of the power of these methods, consider the limit of

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{\Delta v}{\Delta x}\frac{\Delta x}{\Delta t} = \frac{\Delta v}{\Delta x} v$$

as all small increments tend to zero, when we obtain the instantaneous acceleration

$$a = \frac{dv}{dx} v$$

This is an example of the 'chain' rule. Then we have

$$\sum \bar{a}\Delta x = \sum v\Delta v$$

In the limit of small increments, these expressions define the integrals

$$\int_{x_1}^{x_2} a dx = \int_{v_1}^{v_2} v dv$$

These integrals respectively, are the shaded sreas in Fig. 2-6a and Fig. 2-6b when a = constant,

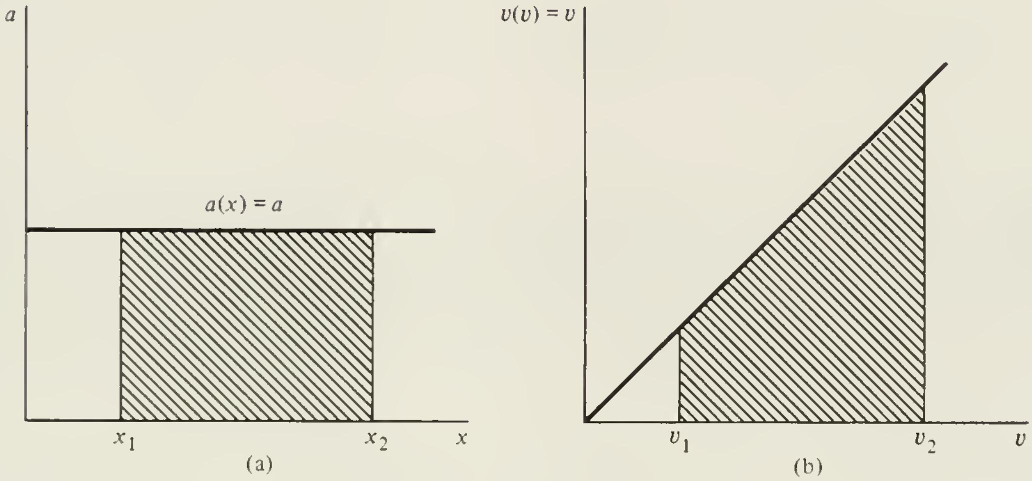


Figure 2-6

$$\int_{x_1}^{x_2} a dx = a(x_2 - x_1) \quad \text{and} \quad \int_{v_1}^{v_2} v dv = \frac{v_2^2}{2} - \frac{v_1^2}{2}$$

Thus we have

$$a(x_2 - x_1) = \frac{1}{2} (v_2^2 - v_1^2)$$

or, with $x = x_2$, $x_1 = x_o$, $v_2 = v$ and $v_1 = v_o$,

$$v^2 = v_o^2 + 2a(x - x_o).$$

You can not always easily calculate the areas under curves as in the simple examples above. Fortunately there is a simple theorem, true for sufficiently smooth but otherwise arbitrary curves f(x), that does the job:

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{if} \quad \frac{dF(x)}{dx} = f(x)$$

Thus one must find the 'anti-derivative' of f, that function which when differentiated yields f. The antiderivative of v is $v^2/2$ because

$$\frac{d}{dv}\left(\frac{v^2}{2} \right) = v.$$

Thus we find

$$\int_{v_1}^{v_2} v\,dv = \left(\frac{v^2}{2} \text{ at } v_2 \right) - \left(\frac{v^2}{2} \text{ at } v_1 \right)$$

$$\int_{v_1}^{v_2} v\,dv = \frac{v^2}{2} \Bigg|_{v_1}^{v_2} = \frac{v_2{}^2}{2} - \frac{v_1{}^2}{2}$$

yielding the area of Fig. 2-6b.

The acceleration of a body may be positive or negative depending on whether its positive velocity is increasing or decreasing. The constant acceleration formula applies to freely falling bodies in a constant gravitational field. The sign of displacements, velocities, and accelerations depends on the orientation of the coordinate system, a matter of choice. In Fig. 2-7a, up is positive; in Fig 2-7b, down is positive.

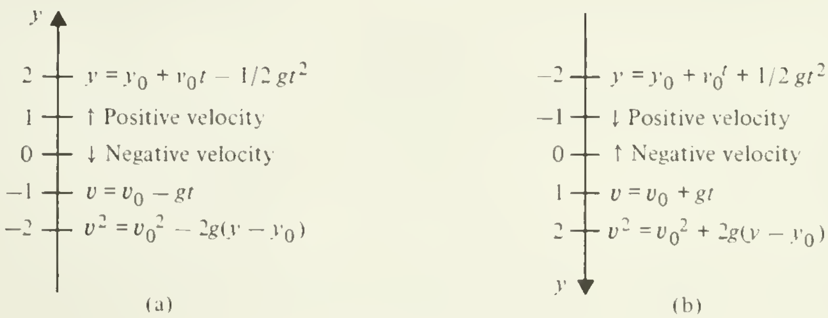


Figure 2-7

In Fig. 2-7a a particle moving upward (wherever it is, above or below the axis) has positive velocity; moving downward, negative velocity. The acceleration is negative: bodies moving upward are slowed down and bodies moving downward have velocities which are negative and are becoming more negative. The same physics is described in Fig. 2-6b, although the formulas look different because the axis is oriented to make downward velocities and accelerations positive.

In all of the above the motion has been referred to a single reference frame.

If there are two reference frames in motion with respect to (WRT) each other (say the earth E and a moving observer B) then if

V_{AE} = velocity of A WRT E

V_{AB} = velocity of A WRT B

V_{BE} = velocity of B WRT E

we have

$$V_{AE} = V_{AB} + V_{BE}$$

Note adjacent subscripts on the right are the ones which do not appear on the left. The first subscript refers to the body and the second to the reference frame.

HINTS AND PROBLEM-SOLVING STRATEGIES

A very systematic approach has been taken to solve the class of problems in this chapter involving constant acceleration:

(1) Decide on the origin and positive direction of a coordinate system. A sketch is often helpful.

(2) Collect the given input data such as initial or final positions, velocities, and accelerations. Take care that the units are consistent. (For example, if position is in m, the velocity should be in $m \cdot s^{-1}$ and the acceleration in $m \cdot s^{-2}$). Choose the signs of the input data in accordance with your choice in (1).

(3) Write down the equations of motion in terms of this input data.

$$x = x_0 + v_0 t + \frac{1}{2} at^2 \qquad v = v_0 + at$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

Note which quantities are known and which are unknown.

(4) Solve the equations, which may be simultaneous in several unknowns, for the desired quantities.

(5) Check that your solution makes physical sense by reference to the coordinate system and sketch of (1). For example, if your coordinate system's positive axis points upward, a ball thrown upward falls back to earth with a negative velocity.

EXAMPLES AND SOLUTIONS

Example 1

Calculate the average velocity between $t_1 = 1$ s and t_2 for the sequence $t_2 = 2$s, 1 1/2 s, 1 1/4 s, 1 1/8 s and 1 1/16 s for a particle whose position as a function of time is given by $x(t) = 1/2\ gt^2$, where $g = 9.8\ m/s^2$(see Fig. 2-8a).

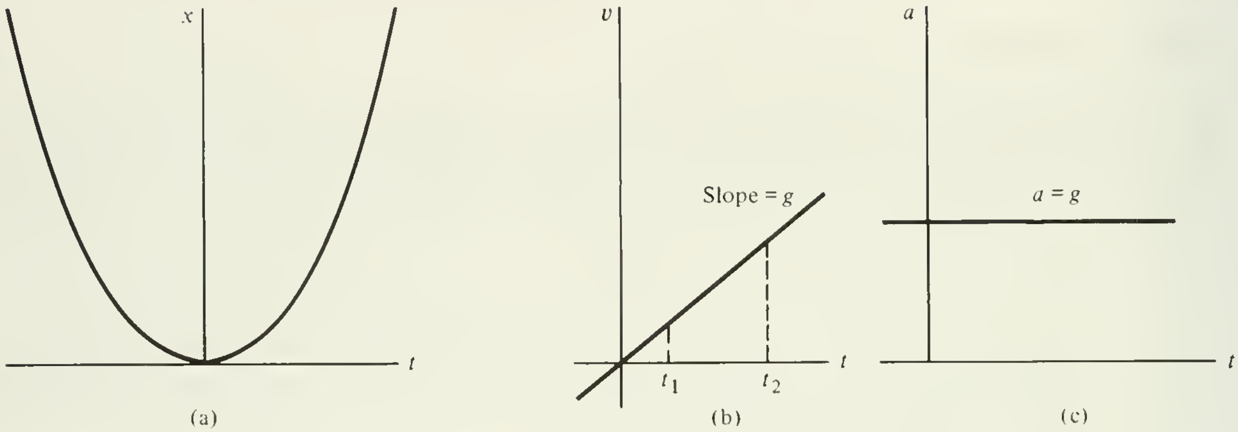


Figure 2-8

Solution:

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$= \frac{1/2\ gt_2^2 - 1/2gt_1^2}{t_2 - t_1}$$

$$= \frac{1}{2}\ g\ \frac{t_2^2 - t_1^2}{t_2 - t_1} = \frac{1}{2}\ g(t_1 + t_2)$$

for $t_2 = 2$s, $\quad v = \frac{1}{2}\ g\ (\ 2 + 1\ )\ s = (\ \frac{3}{2}\ g)\ s = 14.7\ m \cdot s^{-1}$

for $t_2 = 1\ \frac{1}{2}$ s, $\quad v = \frac{1}{2}g\ (\ \frac{3}{2} + 1\ )\ s = (\ \frac{5}{4}\ g)\ s = 12.2\ m \cdot s^{-1}$

for $t_2 = 1\ \frac{1}{4}$ s, $\quad v = \frac{1}{2}g\ (\ \frac{5}{4} + 1\ )\ s = (\ \frac{9}{8}\ g)\ s = 11.0\ m \cdot s^{-1}$

for $t_2 = 1\frac{1}{8}$ s, $v = \frac{1}{2}g\left(\frac{9}{8} + 1\right)$ s $= \left(\frac{17}{16}\, g\right)$ s $= 10.4\ \mathrm{m \cdot s^{-1}}$

for $t_2 = 1\frac{1}{16}$ s, $v = \frac{1}{2}g\left(\frac{17}{16} + 1\right)$ s $= \left(\frac{33}{32}\, g\right)$ s $= 10.2\ \mathrm{m \cdot s^{-1}}$

To find the limit of this sequence at $t_2 = t_1 = 1$ s, differentiate

$$x = \frac{1}{2}\, gt^2$$

and evaluate the result at $t = 1$ s. According to the rule

$$\frac{d}{dt}\,(ct^n) = cnt^{n-1}$$

we find

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{2}\, gt^2\right) = \frac{1}{2}\, g(2t) = gt$$

$$v(1\ \mathrm{s}) = g(1\ \mathrm{s}) = (9.8\ \mathrm{m \cdot s^{-2}}(1\ \mathrm{s}) = 9.8\ \mathrm{m \cdot s^{-1}}$$

Example 2

Calculate, in the last example, the average acceleration over the same intervals and the instantaneous acceleration at $t = 1$ s.

Solution:

From the previous result, $v(t) = gt$,

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$= \frac{v(2) - v(1)}{2 - 1} = \frac{2g - g}{1} = g \qquad \text{in the first interval}$$

$$a = \frac{v(1.5\ ) - v(1)}{1.5 - 1} = \frac{0.5\ g}{0.5} = g \qquad \text{in the second interval}$$

$$\bar{a} = g \qquad\qquad \text{in all intervals}$$

The average acceleration is the same in each interval. This is apparent from a sketch of v versus t, which is a straight line. See Fig. 2-8b. The chord joining v_1, t_1 to v_2, t_2 and the tangent to the curve v(t) are identical. In this special case the average acceleration is equal to the instantaneous acceleration, which is constant. To verify, calculate the derivative

$$a = \frac{dv}{dt} = \text{Lim}_{\Delta t \to o} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

$$= \text{Lim}_{\Delta t \to o} \frac{g(t + \Delta t) - g\,\Delta t}{\Delta t} = \text{Lim}_{\Delta t - o} \frac{g\Delta t}{\Delta t} = g$$

The acceleration is constant as shown in Fig. 2-8c.

Example 3

Find the position x(t) if v = b + at and $x(t_2) = x_2$, $x(t_1) = x_1$.

Solution:

$$v = \frac{dx}{dt} = b + at$$

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} (b + at)dt$$

The last line is shorthand for the limit of the sums

$$\sum \Delta x = \sum v\Delta t$$

which are areas under the curves, indicated in Fig. 2-9a for the left hand side, and Fig. 2-9b for the right hand side.

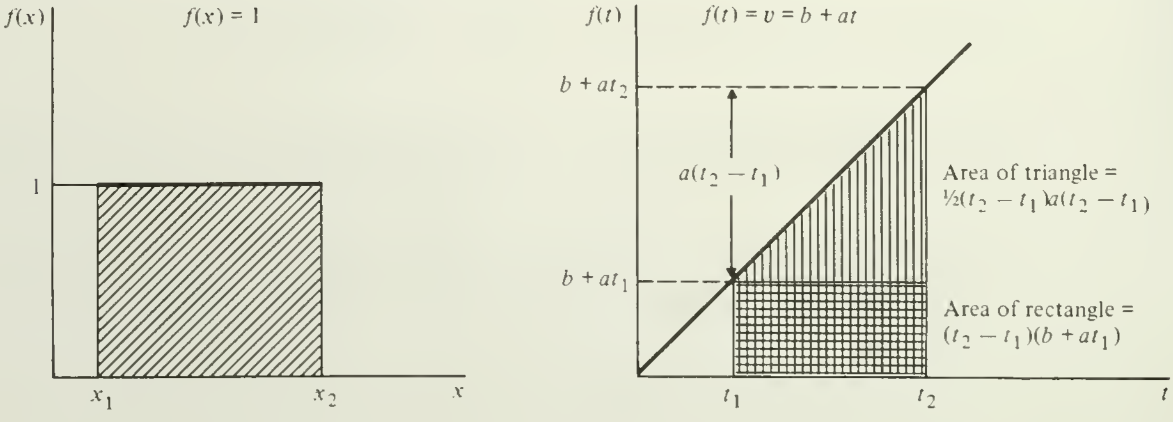


Figure 2-9

$$\text{Lim}_{\Delta x \to 0} \sum \Delta x = \int_{x_1}^{x_2} dx$$

$$= x_2 - x_1 = \text{area of rectangle in Fig. 2-9a.}$$

$$\text{Lim}_{\Delta t \to 0} \sum v\Delta t = \int_{t_1}^{t_2} vdt$$

$$= (t_2 - t_1)(b + at_1) + (\frac{1}{2} t_2 - t_1)\ a(t_2 - t_1)$$

$$= v(t_1)(t_2 - t_1) + \frac{1}{2} a(t_2 - t_1)^2$$

= area under curve in Fig. 2-9b.

If $t_1 = 0$, $x_1 = x_o$, $v(t_1) = v_o$, $t_2 = t$ and $x_2 = x$, we have

$$x = x_o + v_o t + \frac{1}{2} at^2$$

Example 4

Calculate $\int_{x_1}^{x_2} xdx$.

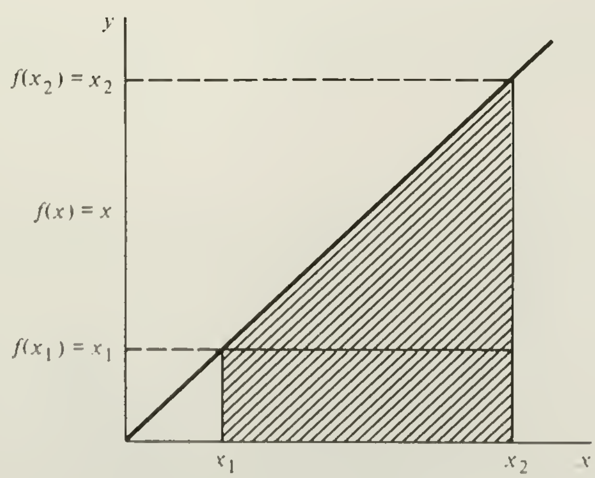


Figure 2-10

Solution:

This is the area under the curve in Fig. 2-10 between x_1 and x_2,

$$\int_{x_1}^{x_2} x\,dx = \left.\frac{x^2}{2}\right|_{x_1}^{x_2} = \frac{x_2^2}{2} - \frac{x_1^2}{2}$$

Example 5

If it takes 5 hours to drive from Boston to New York, a distance of 225 miles, what is the average velocity in mph and $m \cdot s^{-1}$?

Solution:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{225\ \text{mi}}{5\ \text{hr}} = 45\ \text{mph}$$

$$45\ \frac{\text{mi}}{\text{hr}} = 45\ \frac{\text{mi}}{\text{hr}} \cdot 1.6\ \frac{\text{km}}{\text{mi}} \cdot \frac{1}{3600\ \text{s}\cdot\text{hr}^{-1}} \cdot \frac{10^3\ \text{m}}{\text{km}}$$

$$= 20\ \text{m}\cdot\text{s}^{-1}$$

Example 6

Suppose x(t) is as given graphically in Fig. 2-11.
- (a) When is the velocity positive?
- (b) When is the acceleration positive?

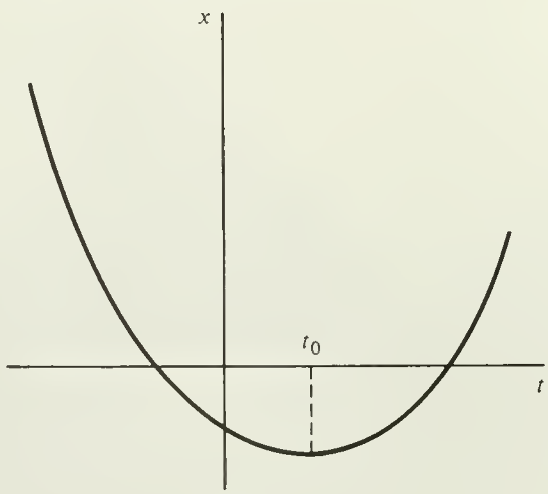


Figure 2-11

Solution:

The velocity is the slope of the tangent to the curve, which is positive for $t > t_0$. The acceleration is always positive. The velocity always increases as t increases. (The slope of the tangent curve always increases as x increases.)

Example 7

The position x(t) is given graphically in Fig. 2-12, with distance in meters and time in seconds.

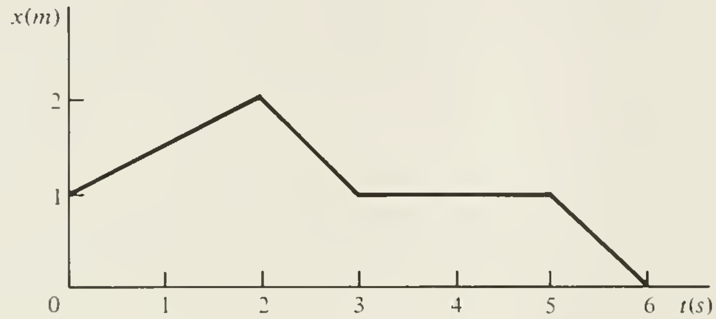


Figure 2-12

Find and plot v(t).

Solution:

Between t = 0 and t = 2 s

$$v = \frac{\Delta x}{\Delta t} = \frac{x(2) - x(0)}{2 - 0} = \frac{2m - 1m}{2\ s} = 0.5\ m \cdot s^{-1}$$

Between t = 3 s and t = 2 s

$$v = \frac{\Delta x}{\Delta t} = \frac{x(3) - x(2)}{3 - 2} = \frac{1\ m - 2\ m}{3\ s - 2\ s} = -1\ m \cdot s^{-1}$$

Between t = 5 s and t = 3 s

$$v = \frac{\Delta x}{\Delta t} = \frac{x(5) - x(3)}{5 - 3} = \frac{1\ m - 1\ m}{5\ s - 3\ s} = 0$$

Between t = 6 s and t = 5 s

$$v = \frac{\Delta x}{\Delta t} = \frac{x(6) - x(5)}{6 - 5} = \frac{0\ m - 1\ m}{1\ s} = -1\ m \cdot s^{-1}$$

A plot of v(t) is given in Fig. 2-13.

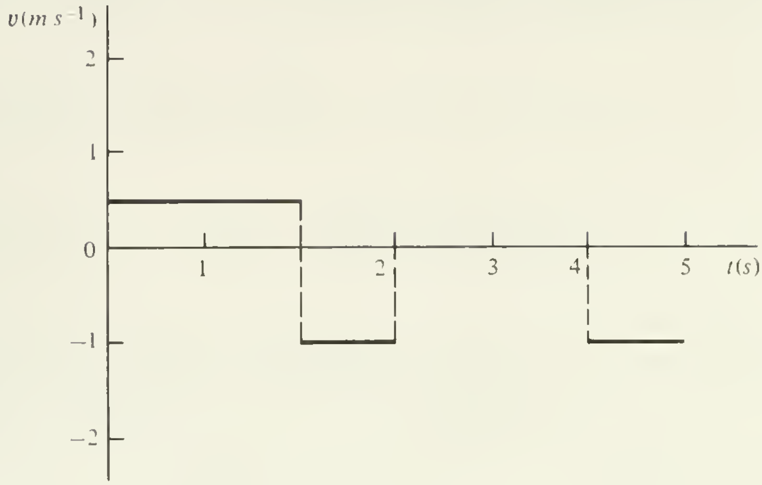


Figure 2-13

Example 8

A body undergoes an acceleration a(t) given in the graph of Fig. 2-14. Find and plot v(t) for $t > 0$. Take $v_o = 0$.

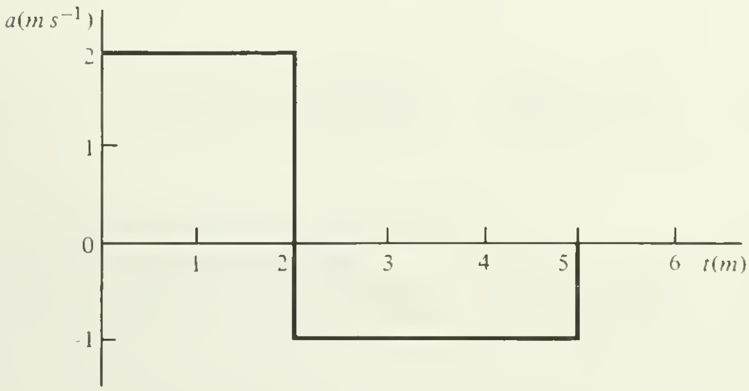


Figure 2-14

Solution:

In <u>each interval</u> of a single constant acceleration

$$v = v_o + at$$

where v_o is the initial velocity <u>in that interval</u> and t is the time measured <u>from the beginning of that interval</u>.

<u>$0 < t < 2s$</u> $\qquad a = 2m\cdot s^{-2} \qquad v_o = 0$

$v = v_o + at$

$v = 0 + 2\ m \cdot s^{-2}\ t = 2\ m \cdot s^{-2}\ t$ (in $m \cdot s^{-1}$ if t in s)

$v(2) = 2\ m \cdot s^{-2} \times 2\ s = 4 m \cdot s^{-1}$

<u>2s < t < 5s</u> $\qquad a' = -1 m \cdot s^{-2} \qquad v_o' = 4 m \cdot s^{-1}$

$v = v_o' + a't'$ $\qquad t' = t - 2\ s$ = time measured from beginning of second interval

$= 4 - 1\ (t - 2) = 6 - t$

$= 4\ m \cdot s^{-1} - 1\ m \cdot s^{-2}(t - 2\ s) = 6\ m \cdot s^{-1} - 1\ m \cdot s^{-2}\ t$

<u>5 s < t</u> $\qquad v_o'' = v(5) = 6\ m \cdot s^{-1} - 1\ m \cdot s^{-2}(5\ s) = 1\ m \cdot s^{-1}$

$v = v_o'' + a''t'' \qquad a'' = 0$

$v = 1\ m \cdot s^{-1}$

v(t) is plotted in Fig. 2-15

Figure 2-15

Example 9

A body undergoes the acceleration $a = 2\ m \cdot s^{-2}$ for $0 < t < 2$ and $a = 0$ for $t > 2s$. Find the position and velocity at $t = 10$ s.

Solution:

The initial position and velocity of the body are both zero, i.e., $x_o = 0$, $v_o = 0$.

For $0 < t < 2$ s

$$v = v_o + at = (2 m\cdot s^{-2})t$$

At the end of the acceleration interval, $t = 2$ s, and $v = (2\ m\cdot s^{-2})(2s) = 4\ m\cdot s^{-1}$. In this interval the position is given by

$$x = x_o + v_o t + \frac{1}{2}at^2 = \frac{1}{2}(2\ m\cdot s^{-2})t^2$$

At $t = 2s$, $x = \frac{1}{2}(2\ m\cdot s^{-2})(2\ s)^2 = 4\ m$.

Let us now reset the clock and the initial conditions to <u>start</u> the motion at $t = 2s$ where

$x_o' = 4$ m, the initial position in this interval

$v_o' = 4\ m\cdot s^{-1}$, the initial velocity for this interval

$a' = 0$ in this interval

$t' = t - 2$ s = time measured from beginning of this interval

In this interval we have

$$v = v_o' + a't' = 4\ m\cdot s^{-1}$$

$$x = x_o' + v_o't' + \frac{1}{2}a't'^2 = 4\ m + 4\ m\cdot s^{-1}t'$$

$$= 4\ m + 4\ m\cdot s^{-1}(t - 2\ s)$$

Thus at $t = 10$ s,

$$x = 4\ m + 4\ m\cdot s^{-1}\ (10\ s - 2\ s) = 36\ m$$

$$v = 4\ m\cdot s^{-1}$$

Example 10

An automobile starts from rest and reaches 100 $km\cdot hr^{-1}$ in 10 seconds.
(a) Find the average acceleration. State your answer as a multiple of the acceleration $g = 9.8\ m\cdot s^{-2}$.
(b) Assuming constant acceleration, find the distance covered.

Solution:

(a) $$a = \frac{\Delta v}{\Delta t} = \frac{(100 - 0)\ km\cdot hr^{-1}}{10\ s}$$

$$= \frac{(100)(10^3)(3600\ s)^{-1})}{(10\ s)}$$

$$= 2.78\ m\cdot s^{-2}$$

$$\frac{a}{g} = \frac{2.78\ m\cdot s^{-2}}{9.8\ m\cdot s^{-2}} = 0.28\ ; \quad a = 0.28\ g$$

(b) For $\bar{a} = a =$ constant and for initial position $x_0 = 0$ and initial velocity $v_0 = 0$,

$$x = x_0 + v_0 t + \frac{1}{2} at^2 = \frac{1}{2} at^2$$

$$x = \frac{1}{2} (2.78\ m\cdot s^{-2})(10\ s)^2 = 139\ m$$

Alternatively we may use the relation between displacement and the initial and final velocities:

$$v^2 = v_0^2 + 2ax = 2ax = [\frac{(100\ km)}{(hr)} \frac{(10^3\ m\cdot km^{-1})}{(3600\ s\cdot hr^{-1})}]^2$$

$$= [27.8\ m\cdot s^{-1}]^2$$

$$x = \frac{v^2}{2a} = \frac{(27.8\ m\cdot s^{-1})^2}{2\cdot 2.78\ m\cdot s^{-2}}$$

$$= 139\ m$$

Example 11

A motorist traveling 60 $km\cdot hr^{-1}$ brakes to a stop in 100 m.

(a) Find the acceleration assuming it to be constant.

(b) What would the stopping distance be if the acceleration were $a = -g = -9.8\ m\cdot s^{-2}$?

(c) What is the stopping time in case (a)?

Solution:

(a) $v^2 = v_0^2 + 2ax$ $\quad v_0 = 60\ km\cdot hr^{-1}$ $\quad v = 0$

$$a = \frac{-v_o^2}{2x} = \frac{-1}{2\cdot 100\ \text{m}} \left(\frac{60 \times 10^3\ \text{m}}{3600\ \text{s}} \right)^2 = -1.39\ \text{m}\cdot\text{s}^{-2}$$

Note that the speed had to be converted to $\text{m}\cdot\text{s}^{-1}$ to find the acceleration in $\text{m}\cdot\text{s}^{-2}$.

$$v^2 = v_o^2 + 2ax; \quad v_o = 60\ \text{km hr}^{-1}; \quad v = 0; \quad a = -9.8\ \text{m}\cdot\text{s}^{-2}$$

$$x = \frac{-v_o^2}{2a} = \frac{v_o^2}{2g} = \frac{1}{2(9.8\ \text{m}\cdot\text{s}^{-2})} \left(\frac{60 \times 10^3\ \text{m}}{3600\ \text{s}} \right)^2$$

$$= 14.2\ \text{m}$$

(c) $v = v_o + at \qquad v = 0 \qquad v_o = 60\ \text{km}\cdot\text{hr}^{-1}$

$a = -1.39\ \text{m}\cdot\text{s}^{-1}$

$$t = \frac{v - v_o}{a} = -\frac{v_o}{a} = \frac{(60 \times 10^3/3600)\text{m}\cdot\text{s}^{-1}}{1.39\ \text{m}\cdot\text{s}^{-2}} = 12\ \text{s}$$

Example 12

A car covers 150 m in 5 s, with constant acceleration. Its final velocity is 100 $\text{km}\cdot\text{hr}^{-1}$.

(a) What is the acceleration?
(b) What is the initial velocity?

Solution:

For constant acceleration,

$$x = x_o + v_o t + \frac{1}{2} at^2$$

$$v = v_o + at$$

x = final position

x_o = initial position with $x - x_o = 150$ m

v = final velocity = 100 $\text{km}\cdot\text{hr}^{-1}$ = 27.8$\text{m}\cdot\text{s}^{-1}$

v_o = initial velocity;

$t = 5$ s

The two equations involve only the unknowns v_o and a. We can solve for a by solving the second one for v_o,

$$v_o = v - at$$

and substituting in the first

$$x - x_o = (v - at)t + \frac{1}{2}at^2$$

$$= vt + a\left(\frac{t^2}{2} - t^2\right) = vt - at^2/2$$

Solving for a,

$$a = -\frac{2}{t^2}(x - x_o - vt)$$

$$= \frac{-2}{(5\ s)^2}[150\ m - 27.8\ m\cdot s^{-1}(5\ s)]$$

$$= -.88\ m\cdot s^{-2}$$

The initial velocity is

$$v_o = v - at = 27.8\ m\cdot s^{-1} + 0.88\ m\cdot s^{-2}(5\ s)$$

$$= 32.2\ m\cdot s^{-1}.$$

Example 13

A speeder traveling at a constant speed of 100 $km\cdot hr^{-1}$ passes a waiting police car which immediately starts from rest with a constant acceleration of 2.5 $m\cdot s^{-2}$.

(a) How long will it take the police car to catch the speeder?
(b) How far will the police car have traveled?
(c) How fast will the police car be traveling when it catches the speeder?

Solution:

Police car: $x_p = x_{op} + v_{op}t + \frac{1}{2}at^2 = \frac{1}{2}at^2$

$$v_p = v_{op} + at = at$$

$$a = 2.5 \text{ m}\cdot\text{s}^{-2}$$

Speeder: $$x_s = x_{os} + v_{os}t + \frac{1}{2} a_s t^2 = v_{os}t$$

$$v_s = v_{os} + a_s t = v_{os}$$

$$v_{os} = 100 \text{ km}\cdot\text{hr}^{-1} = 27.8 \text{ m}\cdot\text{s}^{-1}$$

The constants x_{op} and x_{os} are chosen so that the speeder and police car are at the same position at $t = 0$, ($x_{op} = x_{os} = 0$). We wish to know when this occurs again,

$$x_p = x_s \text{ implies } \quad \frac{1}{2} at^2 = v_{0s}t$$

The equation for this time has roots $t = 0$ and

$$t = \frac{2v_{os}}{a} = \frac{2(27.8 \text{ m}\cdot\text{s}^{-1})}{2.5 \text{ m}\cdot\text{s}^{-2}}$$

$$= 22.2 \text{ s}.$$

The velocity of the police car is then

$$v_p = at = (2.5 \text{ m}\cdot\text{s}^{-2})(22.2 \text{ s}) = 55.5 \text{ m}\cdot\text{s}^{-1}$$

$$= \frac{55.5 \times 10^{-3} \text{ km}}{(1/3600) \text{ hr}} = 200 \text{ km}\cdot\text{hr}^{-1}$$

The position is then

$$x_p = \frac{1}{2} at^2 = \frac{1}{2} (2.5 \text{ m}\cdot\text{s}^{-2})(22.2 \text{ s})^2 = 616 \text{ m}$$

Alternatively, for the police car

$$v_p^2 = v_{op}^2 + 2ax_p \qquad v_{op} = 0$$

$$x_p = \frac{v_p^2}{2a} = \frac{(55.5 \text{ m}\cdot\text{s}^{-1})^2}{2(2.5 \text{ m}\cdot\text{s}^{-2})} = 616 \text{ m}$$

Example 14

A body starts from an initial velocity of 1 $m \cdot s^{-1}$ and reaches, under constant acceleration, a final velocity of 5 $m \cdot s^{-1}$. Its initial position is 20 m and its final position is 100 m.

(a) What is its acceleration?

(b) How much time does it take to reach its final velocity?

Solution:

(a) Since initial and final velocities and positions are known, the appropriate equation to use is

$$v^2 = v_o^2 + 2a(x - x_o) \qquad v = 5\ m \cdot s^{-1}$$

$$v_o = 1\ m \cdot s^{-1}$$

$$x = 100\ m$$

$$x_o = 20\ m$$

$$a = \frac{v^2 - v_o^2}{2(x - x_o)} = \frac{(5\ m \cdot s^{-1})^2 - (1\ m \cdot s^{-1})^2}{2(100\ m - 20\ m)} = 0.15\ m \cdot s^{-2}$$

(b) $$v = v_o + at$$

$$t = \frac{v - v_o}{a} = \frac{5\ m \cdot s^{-1} - 1\ m \cdot s^{-1}}{0.15\ m \cdot s^{-2}} = 26.7\ s$$

Example 15

A car traveling 100 $km \cdot hr^{-1}$ is 200 m away from a truck traveling 50 $km \cdot hr^{-1}$ (in same direction). Assuming constant braking deceleration, what is the minimum deceleration the car must have if it is not to hit the truck?

Solution:

First write the equations of motion for the car and the truck, incorporating initial conditions.

Car: $$x_c = x_{oc} + v_{oc}t + \frac{1}{2}at^2 = v_{oc}t + \frac{1}{2}at^2$$

$$v_c = v_{oc} + at \qquad v_{oc} = 100\ km \cdot hr^{-1} = 27.8\ m \cdot s^{-1}$$

Truck: $x_T = x_{oT} + v_{oT}t + \frac{1}{2}a_T t^2 = x_{oT} + v_{oT}t$

$$v_T = v_{oT} + a_T t = v_{oT}$$

$$v_{oT} = 50\ \text{km}\cdot\text{hr}^{-1} = 13.9\ \text{m}\cdot\text{s}^{-1}$$

$$x_{oT} = 200\ \text{m}$$

If the collision is to be barely avoided, then when

$$x_c = x_T$$

the car and the truck will have zero relative velocity,

$$v_c = v_T .$$

Substituting for the car and truck final position and velocity, we have

$$v_{oc}t + \frac{1}{2}at^2 = x_{oT} + v_{oT}t$$

$$v_{oc} + at = v_{oT}$$

which are two equations in the unknowns a and t.

Solving the second for a and substituting in the first,

$$v_{oc}t + \frac{1}{2}\left(\frac{v_{oT} - v_{oc}}{t}\right)t^2 = x_{oT} + v_{oT}t$$

$$t = \frac{2x_{oT}}{v_{oc} - v_{oT}} = \frac{2(200\ \text{m})}{27.8\ \text{m}\cdot\text{s}^{-1} - 13.9\ \text{m}\cdot\text{s}^{-1}}$$

$$= 28.8\ \text{s}.$$

Thus

$$a = \frac{v_{oT} - v_{oc}}{t}$$

$$= -\ 0.48\ \text{m}\cdot\text{s}^{-1}.$$

Example 16

A ball is dropped from the roof of a building 100 m high. Find
(a) the velocity when it hits the ground and
(b) the time it takes to hit the ground.

Solution:

There are several ways of setting up this problem, as illustrated in Fig. 2-16.

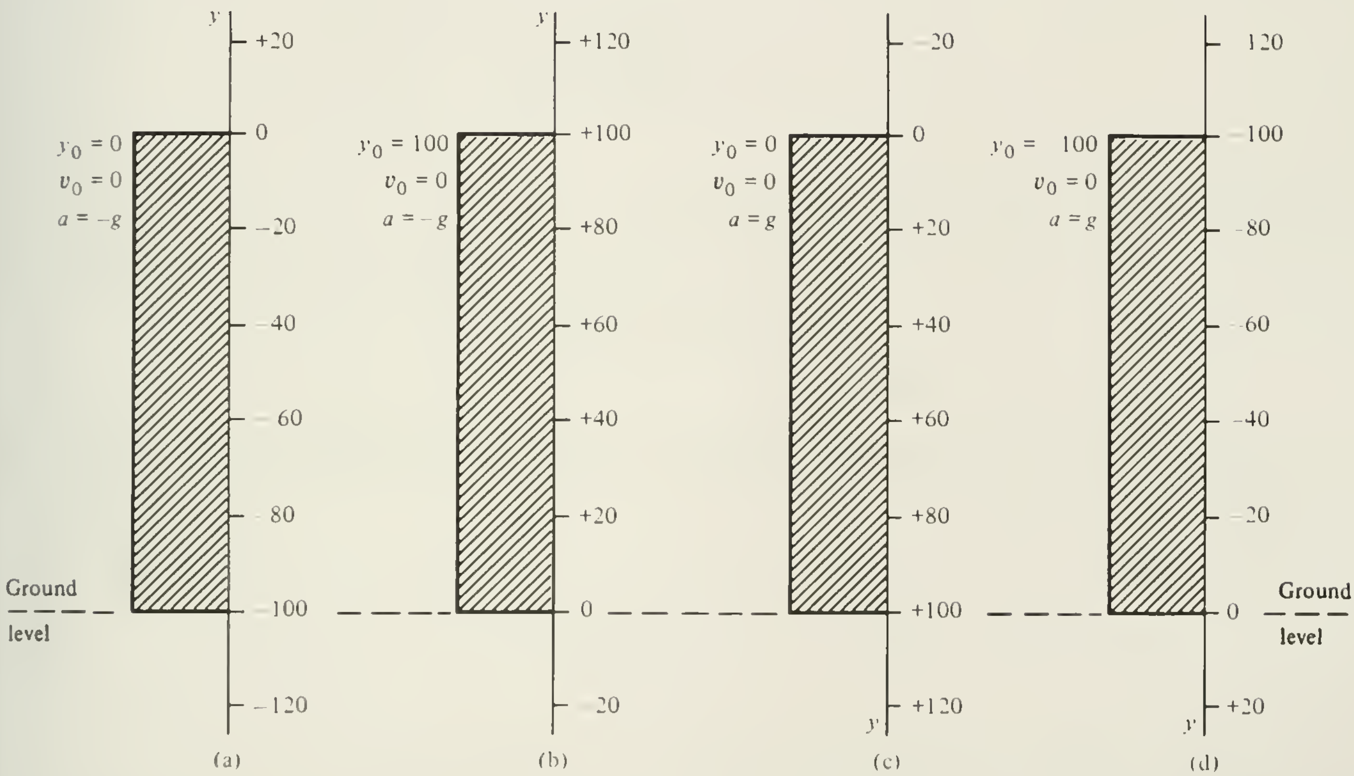


Figure 2-16

(a) $y = -\frac{1}{2}gt^2$ (b) $y = 100\text{ m} - \frac{1}{2}gt^2$

$v = -gt$ $v = -gt$

(c) $y = \frac{1}{2}gt^2$ (d) $y = -100\text{ m} + \frac{1}{2}gt^2$

$v = gt$ $v = gt$

In (a) we seek t when the ball is at $y = -100$ m:

$$-100\text{ m} = -\frac{1}{2}gt^2, \qquad t = 4.52\text{ s}.$$

At this time

$$v = -gt = -9.8\ m\cdot s^{-2}(4.52\ s) = -44.3\ m\cdot s^{-1}.$$

The velocity is negative because the ball is headed along the negative y axis, in this coordinate system.

In (b) we seek t when the ball is at y = 0,

$$0 = 100\ m - \frac{1}{2}gt^2 \qquad t = \left(\frac{2(100\ m)}{9.8\ m\cdot s^{-2}}\right)^{1/2} = 4.52\ s$$

$$v = -gt = -44.3\ m\cdot s^{-1}$$

In (c) we seek t when the ball is at +100 m,

$$100\ m = \frac{1}{2}gt^2 \qquad t = 4.52\ s$$

$$v = gt = 44.3\ m\cdot s^{-1}$$

The velocity is positive because the ball is headed in the positive y direction, in this coordinate system.

In (d) we seek t when y = 0,

$$0 = -100\ m + \frac{1}{2}gt^2 \qquad t = 4.52\ s$$

$$v = gt = 44.3\ m\cdot s^{-1}$$

Note the physical results are independent of the choice of origin and positive direction.

Example 17

A ball is thrown vertically upward from a 100 m high building with an initial velocity of 50 $m\cdot s^{-1}$.

(a) How high does it rise?
(b) How much time does it take to reach its maximum height?
(c) How long does it take to return to the top of the building?
(d) What is its velocity at this time?
(e) How long does it take to reach the ground?
(f) What is its velocity when it hits the ground?

Solution:

Using the coordinate system of Fig. 2-16b, we find

$$y = y_o + v_o t - \frac{1}{2} gt^2 = 100 \text{ m} + (50 \text{ m}\cdot\text{s}^{-1})\ t - \frac{1}{2}(9.8 \text{ m}\cdot\text{s}^{-2})t^2$$

$$v = v_o - gt = 50 \text{ m}\cdot\text{s}^{-1} - 9.8 \text{ m}\cdot\text{s}^{-2}\ t$$

$$v^2 = v_o^2 - 2g(y - y_o) = (50 \text{ m}\cdot\text{s}^{-1})^2 - 2(9.8 \text{ m}\cdot\text{s}^{-2})(y - y_o)$$

(a) $v^2 = 0$ at the topmost point, where

$$y - y_o = \frac{v_o^2}{2g} = \frac{(50 \text{ m}\cdot\text{s}^{-1})^2}{2(9.8 \text{ m}\cdot\text{s}^{-2})} = 128 \text{ m}$$

$$y = y_o + 128 \text{ m} = 228 \text{ m}$$

Alternatively, at the topmost point, $v = 0$, and the time when this point is reached may be obtained from

$$0 = v = v_o - gt = 50 \text{ m}\cdot\text{s}^{-1} - 9.8 \text{ m}\cdot\text{s}^{-2}\ t$$

$$t = \frac{50 \text{ m}\cdot\text{s}^{-1}}{9.8 \text{ m}\cdot\text{s}^{-2}} = 5.1 \text{ s}$$

At this time,

$$y = y_o + v_o t - \frac{1}{2} gt^2$$

$$= 100 + 50 \text{ m}\cdot\text{s}^{-1}\ (5.1 \text{ s}) - \frac{1}{2}\ 9.8 \text{ m}\cdot\text{s}^{-2}\ (5.1 \text{ s})^2 = 228 \text{ m}$$

(b) see (a)

(c) At this time $y = 100$ m, or

$$100 \text{ m} = 100 \text{ m} + 50 \text{ m}\cdot\text{s}^{-1}\ t - \frac{1}{2}\ 9.8 \text{ m}\cdot\text{s}^{-2}\ t^2$$

with roots $t = 0$ and $t = \dfrac{2(50 \text{ m}\cdot\text{s}^{-1})}{9.8 \text{ m}\cdot\text{s}^{-2}} = 10.2 \text{ s} = 2(5.1 \text{ s})$

The rise time is equal to the fall time.

(d) $v = 50 \text{ m}\cdot\text{s}^{-1} - 9.8 \text{ m}\cdot\text{s}^{-2}\ t = 50 \text{ m}\cdot\text{s}^{-1} - (9.8 \text{ m}\cdot\text{s}^{-2})(10.2)$

$= -50 \text{ m s}^{-1}$

(Same magnitude as initial velocity but opposite direction.)

(e) The ground is reached when $y = 0$

$$0 = y = 100\ \text{m} + 50\ \text{m}\cdot\text{s}^{-2}\ t - 4.9\ \text{m}\cdot\text{s}^{-2}\ t^2$$

This is a quadratic equation with roots

$$t = \frac{-b \pm [b^2 - 4ac]^{1/2}}{2a} = \frac{-50 \pm [50^2 - 4(-4.9)100]^{1/2}}{-2(4.9)}$$

$$= -1.71\ \text{s},\ 11.92\ \text{s}$$

The first root is 'extraneous' but has a simple physical interpretation. Consider a motion which starts the ball at a certain time from ground level such that it reaches the top of the building with a velocity of 50 $\text{m}\cdot\text{s}^{-1}$ at time $t = 0$, matching the initial conditions of this problem. That earlier time is -1.71 s.

Example 18

A ball is dropped from a tall building. 1 s later a ball is thrown with a velocity of 30 $\text{m}\cdot\text{s}^{-1}$ vertically down. Will the balls ever meet? If so, when and where? And what does this have to do with Superman?

Solution:

Using the coordinate system of Fig. 2-16(c) we have

$$y_1 = \frac{1}{2} at^2 = \frac{1}{2} gt^2 \qquad \text{First ball}$$

$$y_2 = y_{o2} + v_{o2}t' + \frac{1}{2} g(t')^2,\ t' = t - 1\ \text{s} \qquad \text{Second ball}$$

$$y_2 = y_{o2} + v_{o2}\ (t - 1\ \text{s}) + \frac{1}{2} g(t - 1\ \text{s})^2$$

$$= v_{o2}(t - 1\ \text{s}) + \frac{1}{2} g(t - 1\ \text{s})^2$$

We have adjusted the time t' for the second ball so that when $t = 1$ s it is at the origin, in accordance with the initial conditions. ($y_2 = 0$ at $t = 1$ s.)

For the balls to meet, $y_1 = y_2$. The time t when this happens is

$$\frac{1}{2} gt^2 = v_{o2} (t - 1 \text{ s}) + \frac{1}{2} g (t - 1 \text{ s})^2$$

or

$$t = \frac{v_{o2}(1 \text{ s}) - 1/2 \; g \; (1 \text{ s})^2}{v_{o2} - g(1 \text{ s})} = 1.24 \text{ s}. \; (v_{o2} = 30 \text{ m}\cdot\text{s}^{-1})$$

At this time,

$$y_1 = y_2 = \frac{1}{2} (9.8 \text{ m}\cdot\text{s}^{-2})(1.24 \text{ s})^2 = 7.57 \text{ m}.$$

More generally if Δt is the time interval between the first ball and the second ball,

$$y_1 = \frac{1}{2} gt^2$$

$$y_2 = v_{o2}(t - \Delta t) + \frac{1}{2} g(t - \Delta t)^2$$

Setting $y_1 = y_2$ and solving for t we have

$$0 = v_{o2}(t - \Delta t) - t\Delta t g + \frac{(\Delta t)^2}{2}$$

Superman performs a similar calculation when he jumps from a 300 m high building to rescue a student who has been falling for 5 seconds. In that problem, the second ball is Superman, and $y_1 = 300$ m, $\Delta t = 5$ s, and v_{o2} is the unknown. Thus from

$$y_1 = \frac{1}{2} gt^2, \qquad t = \left(\frac{2y_1}{g} \right)^{1/2}$$

$$= \left[\frac{2(300 \text{ m})}{9.8 \text{ m}\cdot\text{s}^{-2}} \right]^{1/2} = 7.82 \text{ s}$$

we find that Superman's initial velocity must be

$$v_{o2} = \frac{t\Delta t g - [(\Delta t)^2/2]g}{t - \Delta t} = 92 \text{ m}\cdot\text{s}^{-1}$$

The limiting height $\bar{y}$ in this problem, the height below which even an infinitely fast Superman misses the rescue, is the free fall distance in which student hits the ground as Superman jumps. This height is

$$\bar{y} = \frac{1}{2} g(\Delta t)^2 = \frac{1}{2} 9.8(5)^2 = 122.5 \text{ m}$$

Example 19

The motion of a particle is given by

$$x = 2\text{m} + (6 \text{ m}\cdot\text{s}^{-2})t^2 - (3 \text{ m s}^{-4})t^4$$

Find the position, velocity, and acceleration at t = 2s.

Solution:

$$x(2) = 2 \text{ m} + (6 \text{ m}\cdot\text{s}^{-2})(2 \text{ s})^2 - (3 \text{ m}\cdot\text{s}^{-4})(2 \text{ s})^4$$

$$= -22 \text{ m}$$

$$v = \frac{dx}{dt} = 6 \text{ m}\cdot\text{s}^{-2}(2t) - (3 \text{ m}\cdot\text{s}^{-4})(4t^3)$$

$$= 12 \text{ m}\cdot\text{s}^{-2}\ t - 12 \text{ m}\cdot\text{s}^{-4}\ t^3$$

$$v(2) = 12 \text{ m}\cdot\text{s}^{-2}(2 \text{ s}) - (12 \text{ m}\cdot\text{s}^{-4})(2 \text{ s})^3$$

$$= -72 \text{ m}\cdot\text{s}^{-1}$$

$$a = \frac{dv}{dt} = 12 \text{ m}\cdot\text{s}^{-2} - 12 \text{ m}\cdot\text{s}^{-4}\ 3t^2$$

$$= 12 \text{ m}\cdot\text{s}^{-2} - 36 \text{ m}\cdot\text{s}^{-4}\ t^2$$

$$a(2) = 12 \text{ m}\cdot\text{s}^{-2} - (36 \text{ m}\cdot\text{s}^{-4})(2 \text{ s})^2$$

$$= -132 \text{ m}\cdot\text{s}^{-2}$$

Example 20

The acceleration of a particle is given by

$$a = 4 \text{ m}\cdot\text{s}^{-4}\ t^2 = bt^2 = \frac{dv}{dt}$$

Find v and x as a function of t.

Solution:

$$\int_{v_o}^{v} dv = \int_{t_o}^{t} bt^2 dt$$

The integral on the left is $v - v_o$ by the area rule as shown in Fig. 2-6a. To do the integral on the right we use the anti-derivative rule

$$\int_{t_o}^{t} bt^2 dt = \frac{bt^3}{3}\Big|_{\text{at } t} - \frac{bt^3}{3}\Big|_{\text{at } t_o}$$

$$= \frac{b}{3}\,(t^3 - t_o{}^3)$$

Thus

$$v - v_o = \frac{b}{3}\,(t^3 - t_o{}^3)$$

or

$$v = \frac{dx}{dt} = v_o + \frac{b}{3}\,(t^3 - t_o{}^3)$$

If we put $v = 0$ at $t = 0$, v_o and t_o are zero, and then

$$v = \frac{b}{3}\,t^3 = \frac{dx}{dt}\,,$$

so that for $x = 0$ at $t = 0$, one has:

$$\int_0^x dx = \int_0^t vdt = \int_0^t \frac{bt^3}{3}$$

$$x = \frac{b}{3} \cdot \frac{t^4}{4}$$

QUIZ

1. An astronaut on the moon's surface throws a ball vertically upward at a velocity of 10 $m \cdot s^{-1}$ and catches it 12 s later. What is the acceleration of

gravity on the moon?

Answer: 1.7 $m \cdot s^{-2}$

2. A motorist traveling 60 $km \cdot hr^{-1}$ brakes to a gentle stop in 200 m.
   a) Find his braking acceleration, assuming it to be constant.
   b) What would his stopping distance be if his acceleration were $- g = - 9.8$ $m \cdot s^{-2}$?
   c) What is his stopping time in case (a)?

Answer: $- 0.7$ m s^{-2}, 14.2 m, 24 s

3. A ball is thrown vertically upward from the top of a building 160 ft high. It hits the ground 5 s later.
   a) What is its initial velocity?
   b) How long does it take to reach its maximum height?
   c) What is its maximum height?

Answer: 48 ft s^{-1}, 1.5 s, 196 ft

4. An arrow shot vertically upward travels at 30 $m \cdot s^{-1}$ when it first reaches a height of 40 m. How long after this does it take to reach the ground?

Answer: 7 s.

3
MOTION IN A PLANE

OBJECTIVES

In the last chapter you described the motion of objects moving along in a straight line. Taking that straight line to be the x axis, you needed then to consider only one component (the x component) of the position, velocity, and acceleration. In this chapter you treat the more general case in which the motion is in a plane and the position must be described by two coordinates, the x and y components. Your objectives are to:

> Describe the position of a particle by a position vector **r**, its instantaneous velocity by a vector **v**, and its instantaneous acceleration by a vector **a**.
>
> Resolve the vectors into rectangular components, and into components parallel (or tangent, t) and perpendicular (or normal, n) to the motion.
>
> Analyse the motion of projectiles near the surface of the earth, objects with a constant downward acceleration.
>
> Analyse objects in circular motion in horizontal and vertical planes.
>
> Find the velocity of objects relative to different frames of reference.

Comments on the Objectives

A review of the section of Chapter 1 concerning vectors may be necessary to achieve these objectives. A review of Chapter 2 may be necessary before attempting the projectile problems.

REVIEW AND SUPPLEMENT

The motion of a particle in a plane is described by its position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

where (x,y) are its rectangular coordinates, and **i**, **j** are unit vectors pointing in the positive x and y directions, as indicated in Fig. 3-1a. As the particle moves, the head of the position vector **r** traces out the trajectory curve illustrated in Fig. 3-1b.

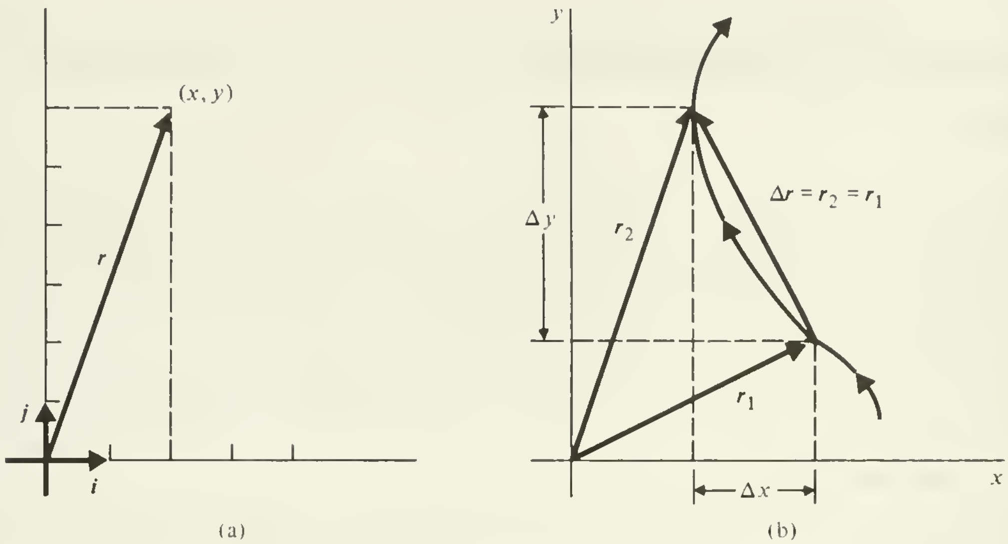


Figure 3-1

If the particle is at position $\mathbf{r}_1$ at t_1 and $\mathbf{r}_2$ at t_2, its displacement between times t_1 and t_2 is

$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \Delta x\mathbf{i} + \Delta y\mathbf{j}$$

and its average velocity is the vector

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t}$$

The instantaneous velocity **v** is the limit of the average velocity as Δt (and hence the length of Δ**r**) tends to zero,

$$\mathbf{v} = \mathrm{Lim}_{\Delta t\to o} \frac{\Delta\mathbf{r}}{\Delta t} = \mathrm{Lim}_{\Delta t\to o} \left(\frac{\Delta x}{\Delta t} \mathbf{i} + \frac{\Delta y}{\Delta t} \mathbf{j}\right)$$

$$= \frac{dx}{dt} \mathrm{i} + \frac{dy}{dt} \mathrm{j}$$

$$= v_x\mathbf{i} + v_y\mathbf{j} = \frac{d\mathbf{r}}{dt}$$

The velocity **v** is always tangent to the trajectory curve as indicated in Fig. 3-2a.

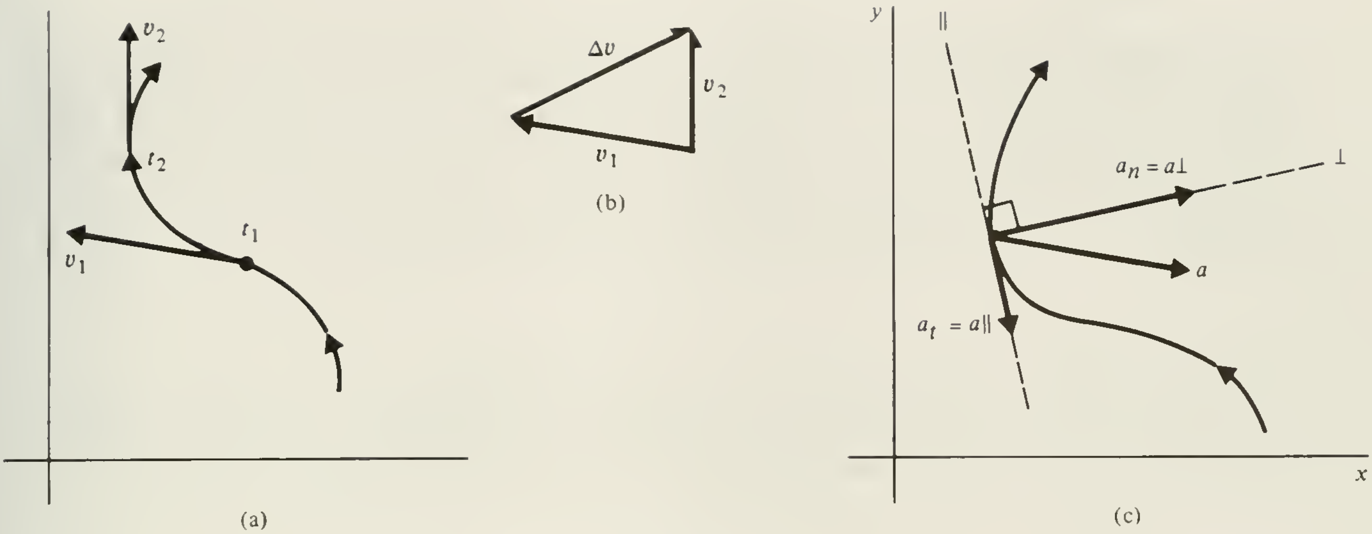


Figure 3-2

If the particle has velocity $\mathbf{v}_1$ at t_1 and $\mathbf{v}_2$ at t_2, the change in vector velocity is (Fig. 3-2b)

$$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = (v_{2x} - v_{1x})\mathbf{i} + (v_{2y} - v_{1y})\mathbf{j}$$

$$= \Delta v_x \mathbf{i} + \Delta v_y \mathbf{j}$$

and the average acceleration is

$$\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\,\mathbf{i} + \frac{\Delta v_y}{\Delta t}\,\mathbf{j}.$$

The instantaneous acceleration $\mathbf{a}$ is the limit of $\mathbf{a}$ as Δt becomes small

$$\mathbf{a} = \text{Lim}_{\Delta t \to o} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\,\mathbf{i} + \frac{dv_y}{dt}\mathbf{j}$$

$$= a_x \mathbf{i} + a_y \mathbf{j}.$$

Referring to Fig. 3-2, the velocity difference always points to the concave side of a curving trajectory; its limiting direction as $\mathbf{v}_2 \to \mathbf{v}_1$ is the direction of the acceleration $\mathbf{a}$. $\mathbf{a}$ in general is <u>not</u> tangent to the path.

In the above, you have resolved the acceleration and velocity vectors into rectangular components referred to a fixed coordinate system. Another useful resolution is in terms of a direction parallel or tangential (t) and a direction perpendicular or normal (n) to the trajectory. This coordinate system moves with the particle. The velocity vector is always parallel to the trajectory

$$v_t = v$$

$$v_n = 0$$

but the acceleration may have both parallel and perpendicular components as illustrated in Fig. 3-2c. In this case, since $|v_2| < |v_1|$ the particle slowed down in the interval and a_t is negative. (The positive sense here for a_t is the velocity direction.) The a_t component indicates that the particle is speeding up or slowing down along the trajectory. The a_n component indicates that it is changing direction. (The positive sense for a_n is toward the concave side of the curve.) $a_n = 0$ only when the motion is in a straight line.

Projectile Motion

A body given an initial velocity and then released, and acted upon only by the earth's gravity and perhaps air resistance, is called a projectile. When there is no air resistance the acceleration of the projectile is downward,

$$\mathbf{a} = -g\mathbf{j}$$

where the positive y (or $\mathbf{j}$) direction has been taken to be up. The equations of motion in component form are

$$a_x = 0$$

$$a_y = -g$$

The x and y motions are independent since a_x is independent of y and a_y is independent of x. We have here two equations of motion already considered in Chapter 2, namely uniform motion ($a_x = 0$) and motion with constant acceleration ($a_y = -g$). The general solution, following the methods of Chapter 2, is

$$x = x(t) = x_o + v_{ox}t \qquad v_x = v_{ox}$$

where x_o is the x coordinate at $t = 0$ and v_{ox} is the x component of the velocity at $t = 0$. v_x is constant because there is no acceleration in the x direction. For the y motion we have

$$y = y(t) = y_o + v_{oy}t - 1/2(gt^2)$$

$$v_y = v_y(t) = v_{oy} - gt$$

where y_o is the y coordinate at $t = 0$ and v_{oy} is the y component of the velocity at $t = 0$. As in Chapter 2, t may be eliminated between the last two equations, resulting in

$$v_y^2 = v_{oy}^2 - 2g(y - y_o)$$

The coordinate functions x(t) and y(t) when plotted for various values of time t trace out the trajectory; eliminating t between them yields the trajectory equation y(x),

$$y = y_o + v_{oy} \left(\frac{x - x_o}{v_{ox}} \right) - \frac{1}{2} g \left(\frac{x - x_o}{v_{ox}} \right)^2$$

which is a concave down parabola, as shown in Fig. 3-3a for $x_o = 0$, $y_o > 0$, and $v_{oy} > 0$.

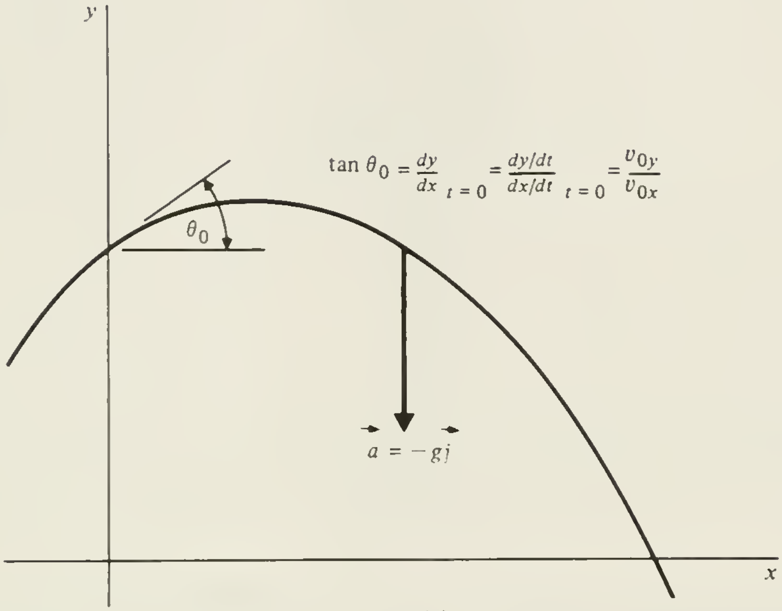


Figure 3-3a

When $y_o = 0$ and $x_o = 0$

$$y = \left(\frac{v_{oy}}{v_{ox}} \right) x - \frac{1}{2} \frac{x^2}{v_{ox}^2} = (\tan\theta_o) x - \frac{g x^2}{2 v_o^2 \cos^2\theta_o}$$

where $v_{ox} = v_o\cos\theta_o$, $v_{oy} = v_o\sin\theta_o$ and θ_o is the angle of initial projection as shown in Fig. 3-3a. Various motions possible are indicated in Fig. 3-3b. Since the trajectory equation does not change when v_{oy} is replaced by $-v_{oy}$ and v_{ox} is replaced by $-v_{ox}$, the same curve is followed, in the opposite direction, when the initial velocity vector is reversed. (See Fig. 3-3b.)

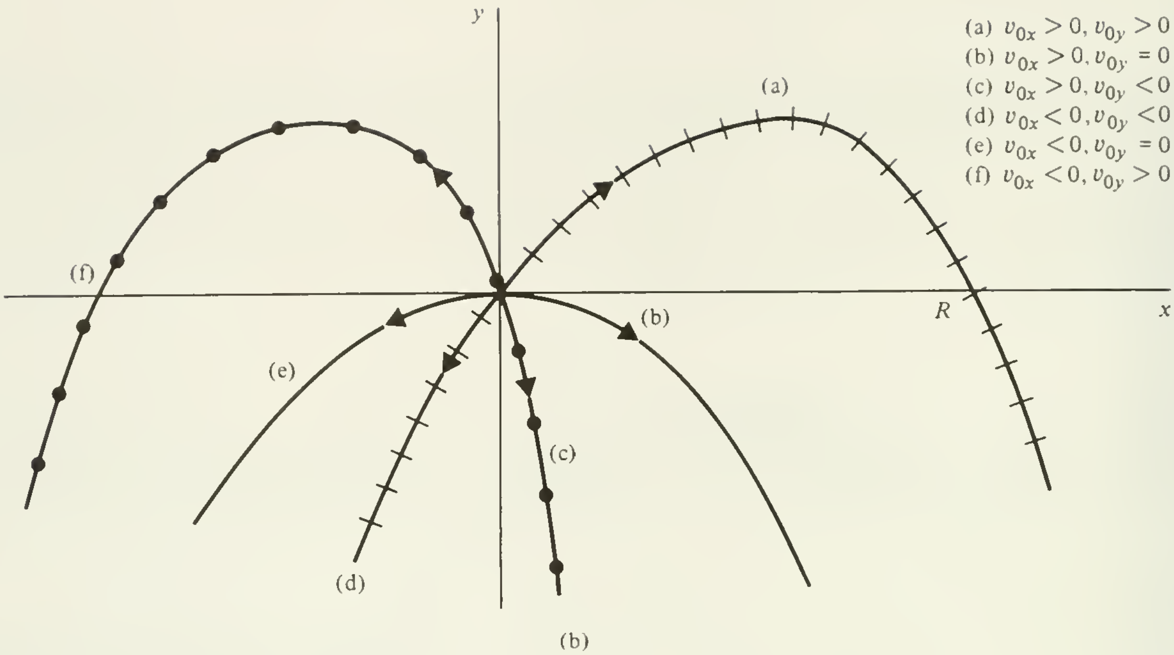


Figure 3-3b

To find the horizontal range R in Fig. 3-3b, set $y = 0$ in the last equation, resulting in the roots $x = 0$ and

$$x = R = (2v_0^2/g) \cos\theta \sin\theta = (v_0^2/g) \sin 2\theta$$

Circular Motion

When a body moves in a circle, the perpendicular or centripetal component of its acceleration is given by

$$a_n = \frac{v^2}{R} ,$$

directed inward towards the center. a_n is called the centripetal acceleration.

(This equation is also true for an arbitrary curve if R = radius of curvature.) The parallel component of acceleration is $a_t = dv/dt$ where v is the <u>magnitude</u> of the velocity.

If the circular motion is 'uniform', v = constant and $a_t = 0$. Then the only acceleration is a_n, acting to change the direction but not the magnitude of $\mathbf{v}$.

If a body in uniform circular motion of radius R completes one revolution in the <u>period</u> τ, its velocity is

$$v = \frac{2\pi R}{\tau}$$

Relative Velocity

Consider the velocity of a body as viewed from a frame E and a frame F. With WRT meaning 'with respect to', we have,

$\mathbf{v}_{FE}$ = the velocity of F WRT E

$\mathbf{v}_{AE}$ = the velocity of the body WRT E

$\mathbf{v}_{AF}$ = the velocity of the body WRT F

Then

$$\mathbf{v}_{AE} = \mathbf{v}_{AF} + \mathbf{v}_{FE}$$

Note the subscipt index of the intermediate frame F is repeated on the right and does not appear on the left. The first and last indices on the right appear on the left in the same order. This is the vector form of the similar one dimensional relation considered in the second chapter.

HINTS AND PROBLEM-SOLVING STRATEGIES

As in earlier chapters, a diagram and a coordinate system choice are always useful.

Write down all given information, in particular any knowledge you may have of the initial coordinates and velocities.

Write down the equations of motion and incorporate the initial coordinate and velocity information.

Solve for the remaining unknowns.

EXAMPLES AND SOLUTIONS

Example 1

A particle's coordinates are given by

$$x = x(t) = (2\ \mathrm{m \cdot s^{-1}})t$$

$$y = y(t) = 2\mathrm{m} - (4.9\ \mathrm{m \cdot s^{-2}})t^2$$

Find the position, velocity, and acceleration at t = 0.5s.

Solution:

The constants in the equation are chosen so that if t is given in s, x is in m. The position at time $t = 0.5$ s is

$$x(0.5s) = 2\ m\cdot s^{-1}(0.5\ s) = 1\ m$$

$$y(0.5s) = 2\ m - (4.9\ m\cdot s^{-2})(0.5\ s)^2 = 0.78\ m$$

$$\mathbf{r} = 1\ m\ \mathbf{i} + 0.78\ m\ \mathbf{j}$$

$$r = (x^2 + y^2)^{1/2} = 1.3\ m;$$

$$\theta = \text{arc tan}\ y/x = 38^\circ \text{ (above the x axis)}$$

To find the velocities you must differentiate x and y with respect to t,

$$v_x = v_x(t) = \frac{dx}{dt} = 2\ m\cdot s^{-1}$$

$$v_y = v_y(t) = \frac{dy}{dt} = -4.9\ m\cdot s^{-2}\ (2t) = -9.8\ m\cdot s^{-2}t$$

$$\mathbf{v} = (2\ m\cdot s^{-1})\ \mathbf{i} - (9.8\ m\cdot s^{-2})t\ \mathbf{j}$$

At $t = 0.5$ s,

$$v_x = 2\ m\cdot s^{-1}$$

$$v_y = -\ 9.8\ m\cdot s^{-2}(0.5\ s) = -\ 4.9\ m\cdot s^{-1}$$

$$v = (v_x{}^2 + v_{y2})^2 = 5.3\ m\cdot s^{-1}$$

$$\theta = \text{arc tan}\ \frac{v_y}{v_x} = -68^\circ \text{ (below this x-axis)}$$

To find the acceleration you must differentiate the velocities with respect to time

$$a_x = \frac{dv_x}{dt} = 0$$

$$a_y = \frac{dv_y}{dt} = -9.8\ m\cdot s^{-2}$$

$$\mathbf{a} = -g\ \mathbf{j} \qquad |\mathbf{a}| = 9.8\ m\cdot s^{-2} = g \text{ (down)}$$

Example 2

A ball rolls off the edge of a tabletop 1 m above the floor with a horizontal velocity of 1 $m \cdot s^{-1}$. Find

(a) the time it takes to hit the floor,
(b) the horizontal distance it covers, and
(c) the velocity when it hits the floor.

Solution:

A convenient choice of coordinate system is illustrated in Fig. 3-4 in which a free body diagram has also been supplied, as well as a sketch of the expected motion and the values of the initial position and velocity as culled from the statement of the problem. The trajectory is tangent to the x axis because the initial velocity is horizontal.

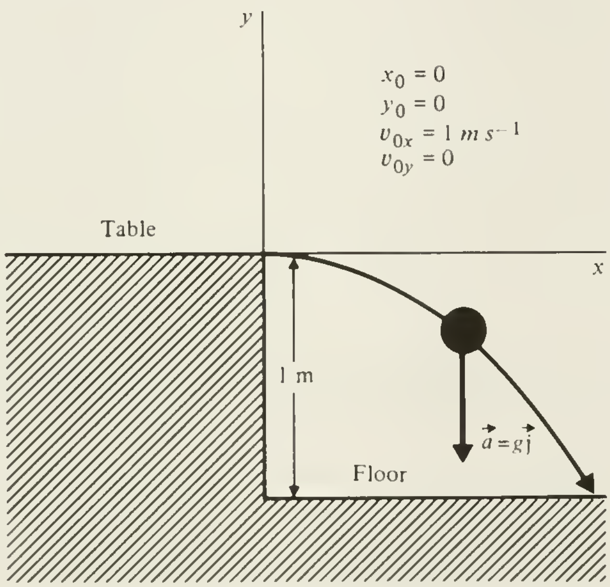


Figure 3-4

The equations of motion are $a_x = 0$, with the solution

$$x = x_o + v_{ox}t = v_{ox}t; \quad v = v_{ox} + a_x t = v_{ox}$$

and

$$a_y = -g$$

with the solution

$$y = y_o + v_{oy}t - \frac{1}{2}gt^2 = -\frac{1}{2}gt^2$$

$$v_y = v_{oy} - gt = -gt$$

(a) The time t when the ball hits the floor occurs when $y = -1$ m.

$$y = -1 \text{ m} = -(1/2)(9.8 \text{ m}\cdot\text{s}^{-2})t^2$$

$$t = \left(\frac{2 \text{ m}}{9.8 \text{ m}\cdot\text{s}^{-2}}\right)^{1/2} = 0.45 \text{ s}$$

(b) In this time it covers a horizontal distance

$$x = v_{ox}t = (1 \text{ m}\cdot\text{s}^{-1})(0.45 \text{ s}) = 0.45 \text{ m}$$

(c) The velocity at this time is

$$v_x = v_{ox} = 1 \text{ m}\cdot\text{s}^{-1}$$

$$v_y = v_{oy} - gt = -(9.8 \text{ m}\cdot\text{s}^{-2})(0.45 \text{ s}) = -4.4 \text{ m}\cdot\text{s}^{-1}$$

$$v = (v_x^2 + v_y^2)^{1/2} = 4.5 \text{ m}\cdot\text{s}^{-1}$$

$$\theta = \text{arc tan } \frac{v_y}{v_x} = -77^\circ \text{ (below x-axis)}$$

Alternatively in (a) you could have solved for the trajectory by eliminating t in $x = v_{ox}t$; $y = -1/2(gt^2)$, resulting in

$$y = -\frac{1}{2} g \frac{x^2}{v_{ox}^2}$$

$$x = \left(\frac{-2y\, v_{ox}^2}{g}\right)^{1/2} = \left(\frac{2(1 \text{ m})(1 \text{ m}\cdot\text{s}^{-1})^2}{9.8 \text{ m}\cdot\text{s}^{-2}}\right)^{1/2}$$

$$= 0.45 \text{ m}$$

Example 3

An archer shoots an arrow into the air at an angle $\theta = 30^\circ$ above the horizontal. It lands on a building 100 m away at a height of 20 m. What were the initial speed (in $\text{km}\cdot\text{hr}^{-1}$) and the time of flight?

Solution:

Referring to Fig. 3-5,

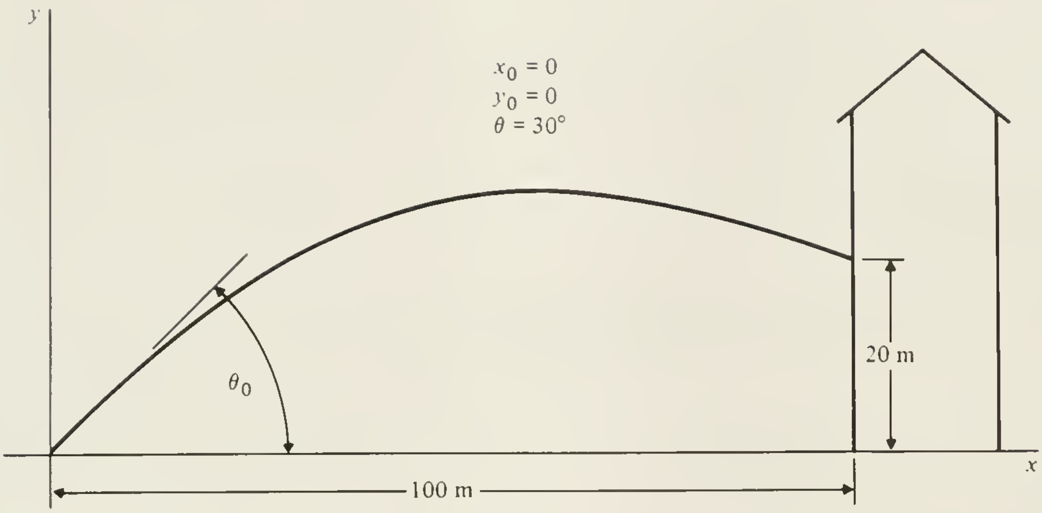


Figure 3-5

we see the pertinent information sketched on the figure. The solution to the equations of motion is

$$x = x_o + v_{ox}t = v_{ox}t = v_o \cos\theta_o t$$

$$y = y_o + v_{oy}t - \frac{1}{2} gt^2 = v_o(\sin\theta_o)t - \frac{1}{2} gt^2$$

Eliminating t, we have the trajectory equation

$$y = x \tan\theta_o - \frac{1}{2} g \left(\frac{x^2}{v_o^2 \cos^2 \theta_o} \right)$$

which can be solved for the initial velocity

$$v_o = \left[\frac{gx^2}{2\cos^2 \theta_o(x \tan \theta - y)} \right]^{1/2}$$

in terms of x and y.

This equation is satisfied by all x,y on the trajectory, in particular by the terminal point x = 100 m, y = 20 m.

$$v_o = \left[\frac{9.8 \text{ m}\cdot \text{s}^{-2}(100 \text{ m})^2}{2(.87)^2(100 \text{ m}(.58) - 20 \text{ m})} \right]^{1/2}$$

$$= 41 \ m\cdot s^{-1} = 41 \ m\cdot s^{-1} \cdot 10^{-3} \ \frac{km}{m} \cdot 3600 \ \frac{s}{hr}$$

$$= 150 \ km\cdot hr^{-1}$$

The time of flight is

$$t = \frac{x}{v_o \cos \theta_o} = \frac{100 \ m}{41 \ m\cdot s^{-1} \ (.87)} = 2.8 \ s$$

Example 4

A soccer ball is kicked 25 m. Its maximum height is 6 m. Find the initial velocity and the time of flight.

Solution:

Referring to Fig. 3-6, we have

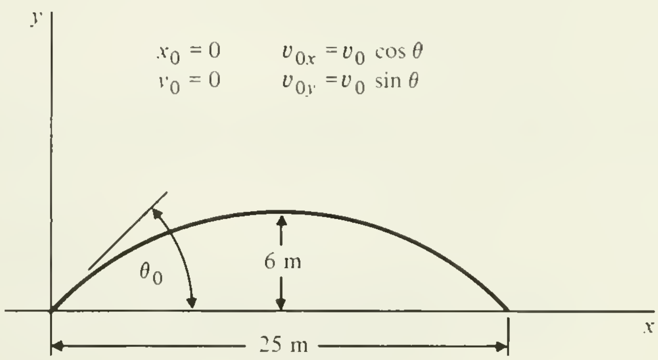


Figure 3-6

$x_o = 0$ $\qquad v_{ox} = v_o \cos \theta$

$y_o = 0$ $\qquad v_{oy} = v_o \sin \theta$

$$x = v_{ox}t = v_o \ (\cos \theta_o)t$$

$$v_x = v_{ox}$$

$$y = v_{oy}t - \frac{1}{2} gt^2 = v_o(\sin \theta_o)t - \frac{1}{2} gt^2$$

$$v_y = v_{oy} - gt = v_o \sin \theta - gt$$

At the maximum height y = 6 m and $v_y = 0$; substituting these in the last two equations we have

$$6 \text{ m} = v_o (\sin \theta_o)t - \frac{1}{2} gt^2$$

$$0 = v_{oy} - gt = v_o (\sin \theta_o) - gt$$

These equations may be solved simultaneously for t and $v_o \sin \theta_o$:

$$v_o \sin \theta_o = 10.8 \text{ m}\cdot\text{s}^{-1}$$

$$t = 1.1 \text{ s}$$

The other information is that at maximum range x = 25 m. Since the orbit, a parabola, is symmetric about its top most point, the time of flight is twice the time to reach the midpoint, or 2.2 s. Thus from $x = v_o(\cos \theta_o)t$ we have

$$v_o \cos \theta_o = \frac{x}{t} = \frac{25 \text{ m}}{2.2 \text{ s}} = 11.3 \text{ m}\cdot\text{s}^{-1}$$

Combining this with

$$v_o \sin \theta_o = 10.8 \text{ m}\cdot\text{s}^{-1}$$

yields

$$\theta_o = \arctan \frac{10.8}{11.3} = 44^\circ$$

and then

$$v_o = \frac{10.9 \text{ m}\cdot\text{s}^{-1}}{\sin \theta_o} = 15.6 \text{ m}\cdot\text{s}^{-1}$$

Alternatively the time of flight can be calculated by setting y = 0 in the equation of motion for y,

$$y = 0 = [v_o (\sin \theta_o)t - \frac{1}{2} gt^2] = (10.8 \text{ m}\cdot\text{s}^{-1})t - (4.9 \text{ m}\cdot\text{s}^{-2})t^2$$

This quadratic equation for t has two roots, the starting time t = 0 and time of flight,

$$t = 2.2 \text{ s}$$

$$= 41\ m\cdot s^{-1} = 41\ m\cdot s^{-1} \cdot 10^{-3}\ \frac{km}{m} \cdot 3600\ \frac{s}{hr}$$

$$= 150\ km\cdot hr^{-1}$$

The time of flight is

$$t = \frac{x}{v_o \cos \theta_o} = \frac{100\ m}{41\ m\cdot s^{-1}\ (.87)} = 2.8\ s$$

Example 4

A soccer ball is kicked 25 m. Its maximum height is 6 m. Find the initial velocity and the time of flight.

Solution:

Referring to Fig. 3-6, we have

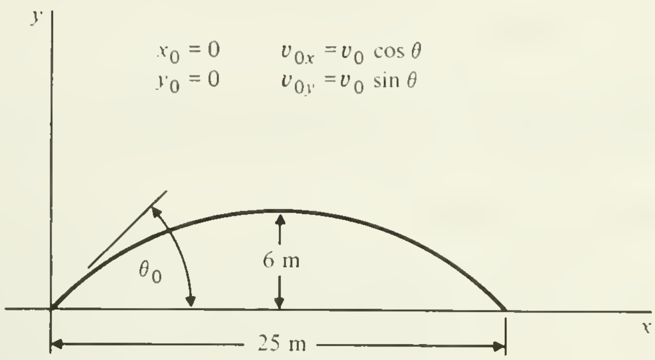


Figure 3-6

$$x_o = 0 \qquad v_{ox} = v_o \cos \theta$$

$$y_o = 0 \qquad v_{oy} = v_o \sin \theta$$

$$x = v_{ox}t = v_o (\cos \theta_o)t$$

$$v_x = v_{ox}$$

$$y = v_{oy}t - \frac{1}{2} gt^2 = v_o(\sin \theta_o)t - \frac{1}{2} gt^2$$

$$v_y = v_{oy} - gt = v_o \sin \theta - gt$$

At the maximum height y = 6 m and $v_y = 0$; substituting these in the last two equations we have

$$6\ \mathrm{m} = v_o (\sin\theta_o)t - \frac{1}{2} gt^2$$

$$0 = v_{oy} - gt = v_o (\sin\theta_o) - gt$$

These equations may be solved simultaneously for t and $v_o \sin\theta_o$:

$$v_o \sin\theta_o = 10.8\ \mathrm{m\cdot s^{-1}}$$

$$t = 1.1\ \mathrm{s}$$

The other information is that at maximum range x = 25 m. Since the orbit, a parabola, is symmetric about its top most point, the time of flight is twice the time to reach the midpoint, or 2.2 s. Thus from $x = v_o(\cos\theta_o)t$ we have

$$v_o \cos\theta_o = \frac{x}{t} = \frac{25\ \mathrm{m}}{2.2\ \mathrm{s}} = 11.3\ \mathrm{m\cdot s^{-1}}$$

Combining this with

$$v_o \sin\theta_o = 10.8\ \mathrm{m\cdot s^{-1}}$$

yields

$$\theta_o = \arctan\frac{10.8}{11.3} = 44^\circ$$

and then

$$v_o = \frac{10.9\ \mathrm{m\cdot s^{-1}}}{\sin\theta_o} = 15.6\ \mathrm{m\cdot s^{-1}}$$

Alternatively the time of flight can be calculated by setting y = 0 in the equation of motion for y,

$$y = 0 = [v_o (\sin\theta_o)t - \frac{1}{2} gt^2] = (10.8\ \mathrm{m\cdot s^{-1}})t - (4.9\ \mathrm{m\cdot s^{-2}})t^2$$

This quadratic equation for t has two roots, the starting time t = 0 and time of flight,

$$t = 2.2\ \mathrm{s}$$

Example 5

A boy throws a ball from a cliff at a departure angle of $\theta = 30^\circ$ and an initial velocity 10 $m \cdot s^{-1}$. It lands 100 m from the base of the cliff.

(a) How high is the cliff?

(b) What is the time of flight?

(c) Find its velocity upon impact. See Fig. 3-7.

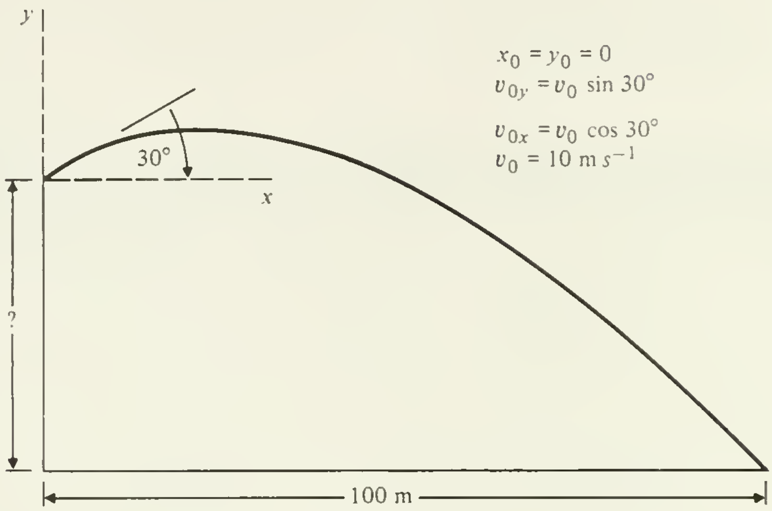


Figure 3-7

Solution:

The initial data is

$x_o = y_o = 0 \qquad v_{ox} = v_o \cos 30^\circ$

$v_{oy} = v_o \sin 30^\circ \qquad v_o = 10 \ m \cdot s^{-1}$

By methods which should now be familiar, the trajectory is found to be

$$y = x \tan \theta - \frac{1}{2} g\left(\frac{x^2}{v_o^2 \cos^2\theta} \right)$$

At impact x = 100 m and

$$y = (.58)(100 \text{ m}) - \frac{(9.8 \text{ ms}^{-2})(100 \text{ m})^2}{2(10 \text{ ms}^{-1})^2(.75)}$$

$$= 58 \text{ m} - 653 \text{ m} = -595 \text{ m}$$

(a) The cliff is 595 m high. (b) The time of flight is given, from $x = v_0 \cos \theta t$, by

$$t = \frac{x}{v_0 \cos \theta} = \frac{100 \text{ m}}{(10 \text{ m}\cdot\text{s}^{-1}).87} = 11.5 \text{ s}$$

(c) The velocity at impact is

$$v_x = v_0 \cos \theta = (10 \text{ m}\cdot\text{s}^{-1})(.87) = 8.7 \text{ m}\cdot\text{s}^{-1}$$

$$v_y = v_0 \sin \theta - gt = (10 \text{ m}\cdot\text{s}^{-1})(0.5) - (9.8 \text{ ms}^{-2})(11.5 \text{ s})$$

$$= -107 \text{ m}\cdot\text{s}^{-1}$$

$$v = (v_x^2 + v_y^2)^{1/2} = 108 \text{ m}\cdot\text{s}^{-1}$$

$$\theta = \arctan \frac{-107}{8.7} = -85^\circ \text{ (}85^\circ\text{ below x axis)}$$

Example 6

Lois playfully pushes Clark Kent horizontally off a 500 m building, giving him an initial velocity of 15 $\text{m}\cdot\text{s}^{-1}$. Clark must save himself by changing into his Superman suit before hitting the ground. How much time does Clark have?

Solution:

$$y = -\frac{1}{2} t^2 = -500 \text{ m}$$

$$t = \left[\frac{2(500 \text{ m})}{9.8 \text{ m}\cdot\text{s}^{-2}} \right]^{1/2} = 10.1 \text{ s}$$

Note: Lois' playful horizontal push is irrelevant, as long as she does not impart to Clark a vertical component of velocity.

Example 7

A particle's coordinates are given by

$$x = A \cos \omega t$$

$$y = A \sin \omega t$$

(a) Find the trajectory equation.
(b) Find v_x, v_y, and v.
(c) Find a_x, a_y, and a.

Solution:

(a) $$r^2 = x^2 + y^2 = A^2(\cos^2\omega t + \sin^2\omega t) = A^2$$

Hence

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = A(\cos \omega t\ \mathbf{i} + \sin \omega t\ \mathbf{j})$$

traces out a circle of radius $r^2 = x^2 + y^2 = A^2$, describing uniform circular motion, with

(b) $$v_x = \frac{dx}{dt} = -A\omega \sin \omega t$$

$$v_y = \frac{dy}{dt} = A\omega \cos \omega t$$

The velocity vector

$$\mathbf{v} = -A\omega \sin \omega t\ \mathbf{i} + A\omega \cos \omega t\ \mathbf{j}$$

has constant magnitude

$$v^2 = \omega^2 A^2 \sin^2\omega t + \omega^2 A^2 \cos^2\omega t$$

$$= A^2\omega^2 = r^2\omega^2; \quad v = r\omega.$$

(c) $$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-\omega A \sin \omega t) = -\omega^2 A \cos \omega t$$

$$= -\omega^2 x$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(\omega A \cos \omega t) = -\omega^2 A \sin \omega t$$

$$= -\omega^2 y$$

$$\mathbf{a} = (-\omega^2)x\mathbf{i} + (-\omega^2)y\mathbf{j} = (-\omega^2)\mathbf{r}$$

$$|\mathbf{a}| = a = \omega^2 r = v^2/r$$

Note the velocity and position vector have zero inner product, that is,

$$\mathbf{r} \cdot \mathbf{v} = r_x v_x + r_y v_y = x v_x + y v_y$$

$$= A \cos \omega t(-A\omega \sin \omega t) + A \sin \omega t(A\omega \cos \omega t)$$

$$= 0$$

indicating that **r** and **v** are always perpendicular. See Fig. 3-8.

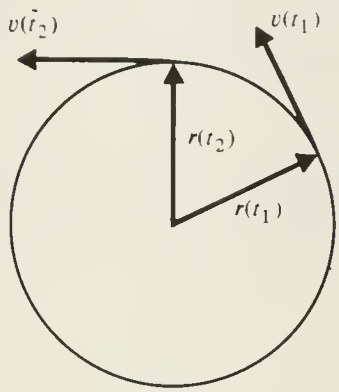


Figure 3-8

Example 8

You are a ship of the line steaming due north, in formation, 1000 m astern of the flagship, at 10 $km(hr)^{-1}$. You are ordered to come 1000 m abeam of her. Your maximum speed is 20 $km(hr)^{-1}$. What is your course to come to station in best time?

Solution:

Referring to Fig. 3-9, where you are A, and F is the flagship, we see that

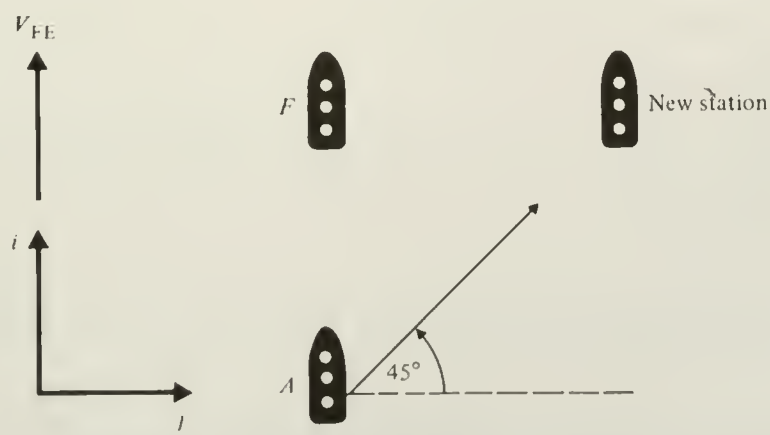


Figure 3-9

$$\mathbf{v}_{AF} = v_{AF} \sin 45^\circ \mathbf{i} + v_{AF} \cos 45^\circ \mathbf{j}$$

where $\mathbf{v}_{AF}$ is your velocity relative to the flagship F, with v_{AF} the unknown magnitude.

The velocity of the flagship relative to the earth is

$$\mathbf{v}_{FE} = 10 \text{ km(hr)}^{-1} \mathbf{i} = v_{FE}\mathbf{i}$$

If $\mathbf{v}_{AE}$ is your velocity relative to the earth, then

$$\mathbf{v}_{AE} = \mathbf{v}_{AF} + \mathbf{v}_{FE}$$

$$= \frac{v_{AF}}{[2]^{1/2}} \mathbf{i} + \frac{v_{AF}}{[2]^{1/2}} \mathbf{i} + v_{FE} \mathbf{i}$$

$$\mathbf{v}_{AE} = \left(\frac{v_{AF}}{[2]^{1/2}} + v_{FE}\right) \mathbf{i} + \frac{v_{AF}}{[2]^{1/2}} \mathbf{j}.$$

To use best speed, set the magnitude of this vector equal to your maximum speed

$$v_{AE}^2 = \left(\frac{v_{AF}}{[2]^{1/2}} + v_{FE}\right)^2 + \left(\frac{v_{AF}}{[2]^{1/2}} \right)^2 = [20 \text{ km(hr)}^{-1}]^2$$

Since $v_{FE} = 10 \text{ km(hr)}^{-1}$ is known, this is a quadratic equation for v_{AF} with the solution

$$v_{AF} = 11 \text{ km(hr)}^{-1}.$$

Thus

$$v_{AE} = \left[\left(\frac{11}{[2]^{1/2}} + 10\right) \mathbf{i} + \frac{11}{[2]^{1/2}} \mathbf{j} \right] \text{ km(hr)}^{-1}$$

$$= 18 \text{ km(hr)}^{-1}\mathbf{i} + 8 \text{ km(hr)}^{-1}\mathbf{j}$$

Your true heading is thus

$$\theta = \arctan \frac{8}{18} = 24^\circ \text{ east of north.}$$

Example 9

A ship wishes to reach a point 10 km due north. It steams at 20 km(hr)^{-1}. The current is 5 km(hr)^{-1} due east. What should its heading be?

Solution:

Defining the directions $\mathbf{i}$ = north and $\mathbf{j}$ = east and the velocities

$$\mathbf{v}_{AE} = \text{ship relative to earth}$$

$$\mathbf{v}_{AW} = \text{ship relative to water}$$

$$\mathbf{v}_{WE} = \text{water relative to earth}$$

we have

$$\mathbf{v}_{AE} = \mathbf{v}_{AW} + \mathbf{v}_{WE}$$

We seek a solution in which the velocity $\mathbf{v}_{AE}$ is due north

$$\mathbf{v}_{AE} = v_{AE}\mathbf{i}$$

and in which $v_{AW} = 20\ \text{km(hr)}^{-1}$,

$$\mathbf{v}_{AW} = v_{AW} \cos\theta\ \mathbf{i} + v_{AW} \sin\theta\ \mathbf{j}$$

where θ is the heading angle. The current is

$$\mathbf{v}_{WE} = 5\ \text{km(hr)}^{-1}\ \mathbf{j}$$

Thus we have

$$v_{AE}\mathbf{i} = v_{AW} \cos\theta\ \mathbf{i} + v_{AW} \sin\theta\ \mathbf{j} + v_{WE}\mathbf{j}$$

$$v_{AE}\mathbf{i} = 20\ \text{km(hr)}^{-1} \cos\theta\ \mathbf{i} + [5\ \text{km(hr)}^{-1} + 20\ \text{km(hr)}^{-1} \sin\theta]\ \mathbf{j}$$

Equating the x and y components separately on each side,

$$v_{AE} = 20\ \text{km(hr)}^{-1} \cos\theta$$

$$5\ \text{km(hr)}^{-1} + 20\ \text{km(hr)}^{-1} \sin\theta = 0$$

$$\sin\theta = -\frac{5}{20} = -\frac{1}{4} \qquad \theta = -14.5^{\circ}$$

Since θ is negative, the heading is 14.5° away from $\mathbf{i}$ in the negative $\mathbf{j}$ direction, or 14.5° west of north. See Fig. 3-10.

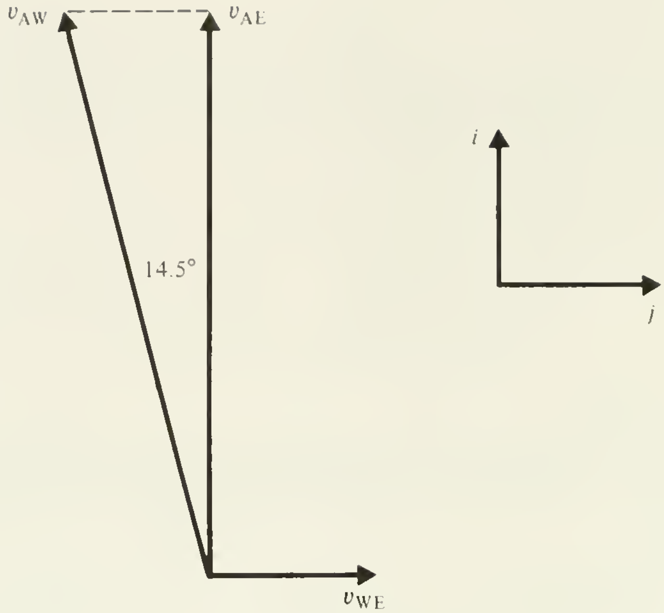


Figure 3-10

QUIZ

1. A batter hits a ball which lands on the top of a 15.3 m high fence 122 m away. The angle of departure is 45°.
   (a) Find the initial velocity.
   (b) Find the time of flight.
   (c) Find the maximum height of the ball.

Answer: 36.9 $m \cdot s^{-1}$, 4.7 s, 34.8 m

2. What is the centripetal acceleration of a person standing on the equator of the earth?

Answer: 0.03 $m \cdot s^{-2}$

3. A marble rolls on a laboratory table 1 m above the floor and falls off the table, landing a horizontal distance of 0.5 m from where it fell off. What was its initial velocity as it fell off the table?

Answer: 1.1 $m \cdot s^{-1}$

4. An artillery shell is fired at an elevation angle of 30° and lands 1000 m away on level ground. What was its maximum height?

Answer: 140 m

4

NEWTON'S LAWS OF MOTION

OBJECTIVES

In Chapters 2 and 3 you studied the acceleration of a body moving along a line or in a plane, without specifying the cause of the acceleration. In Chapter 4, <u>force</u>, a vector quantity is identified as the means by which bodies influence each others motion and the cause of acceleration. Your objectives are to:

Identify the <u>forces</u> acting between bodies and calculate the resultant or total force, the vector sum of all forces acting on a single body.

Define <u>equilibrium</u> as a state in which the total or net force acting on a body is zero.

Observe that a body in equilibrium is at rest or in uniform motion along a straight line (<u>Newton's first law</u>.)

Define <u>mass</u> as the proportionality constant relating force and acceleration.

Determine the <u>acceleration</u> $\mathbf{a}$ of a body in terms of the total force $\sum \mathbf{F}$ acting on it: $\sum \mathbf{F} = m\mathbf{a}$. (<u>Newton's second law</u>.)

Calculate forces, masses, and accelerations in the SI, cgs, and engineering system of <u>units</u>.

Relate the downward force or <u>weight</u> w of objects near the surface of the earth to the acceleration of gravity g.

Observe that the mutual forces between two bodies are equal in magnitude and opposite in direction. (<u>Newton's third law</u>.)

Apply Newton's laws to <u>simple systems</u>; more complicated ones will be encountered in Chapters 5 and 6.

Recognize an <u>inertial frame of reference</u>.

REVIEW

According to Newton's first law, a body moves in a straight line with constant velocity, that is, with zero acceleration, if the net force acting on it is zero. If the net force is not zero, the body accelerates. By Newton's second law, the acceleration is proportional to the force, according to the vector relation,

$$\mathbf{F} = m\mathbf{a}$$

where the constant m is called the mass of the body. The units ot mass are those of force divided by acceleration. In the SI the unit of force is the newton (N) and the unit of mass is

$$N / (m \cdot s^{-2}) = \frac{N \cdot s^2}{m} = kg \text{ (kilogram)}$$

$$= 10^3 g \text{ (gram)}$$

In the British system the unit of force is the pound (lb) and the unit of mass is:

$$\frac{lb}{ft \cdot s^{-2}} = \frac{lb \cdot s^2}{ft} = slug$$

Newton's third law states that when body A exerts a force $\mathbf{F}$ on body B, body B must exert a force on A equal in magnitude but opposite in direction: $-\mathbf{F}$. Newton's third law is useful when analysing the forces acting between different parts of a mechanical system. Detailed examples are given in the next chapter of this Study Guide.

Forces, if unbalanced, tend to change the state of motion of a body. A particle is said to be in equilibrium if it is at rest or moves at constant speed in a straight line. The condition for equilibrium of a particle is that the total force acting on a body be zero,

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots = \sum \mathbf{F} = 0$$

This condition is a statement of Newton's First Law of motion, 'Every body continues in its state of rest, or of uniform motion, in a straight line, unless it is compelled to change that state by forces impressed on it.'

This law is true in 'inertial frames of reference'. The classroom, fixed with respect to the earth, is approximately an inertial system; a car moving smoothly at constant speed in a straight line with respect to the earth is another inertial system. Any system in uniform motion in a straight line with respect to an inertial system is another inertial system.

An accelerating car or a car moving at constant speed around a curve is not an inertial system; a marble on a smooth horizontal table will not remain at rest in these non-inertial systems even when there are no horizontal forces acting on it.

As for the first Law, Newton's second law is valid only in an <u>inertial frame of reference</u>: if Newton's laws are valid to observer A, then they are also valid in the frame of reference of observer B who moves with constant velocity with respect to A.

The force that makes projectiles accelerate downward is the force of their weight,

$$w = mg$$

All bodies on the surface of the earth are subject to the downward force of their weight.

EXAMPLES AND SOLUTIONS

Example 1

A wagon is pushed up a steep driveway with an incline angle of 20° by a horizontal force of 25 N. What is the component of the pushing force in the direction of motion?

Solution

As shown in Fig. 4-1, we choose a coordinate system with the x-axis along the driveway. Then the component of **F** along the driveway is

$$F_x = F \cos \theta = 25 \text{ N} \cos 20^\circ = 23 \text{ N}$$

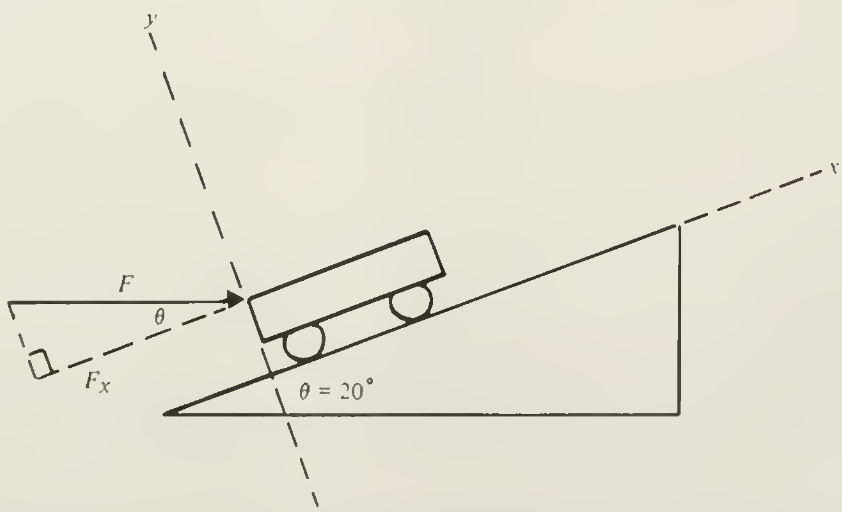


Figure 4-1

Example 2

A cat weighs 20 N on the surface of the earth, where the acceleration of gravity is 9.8 $m \cdot s^{-2}$. What is its mass?

Solution:

$$w = mg \text{ so } m = \frac{w}{g} = \frac{20 \text{ N}}{9.8 \text{ m} \cdot \text{s}^{-2}} = 2.04 \text{ kg}.$$

Example 3

A rock of mass 4 kg is suspended by a wire from a tree branch. When a horizontal force of 29.4 N is applied to the rock, it assumes an equilibrium position when the wire makes an angle θ <u>with the vertical</u>. Find the angle θ and the force the wire exerts on the rock.

Solution:

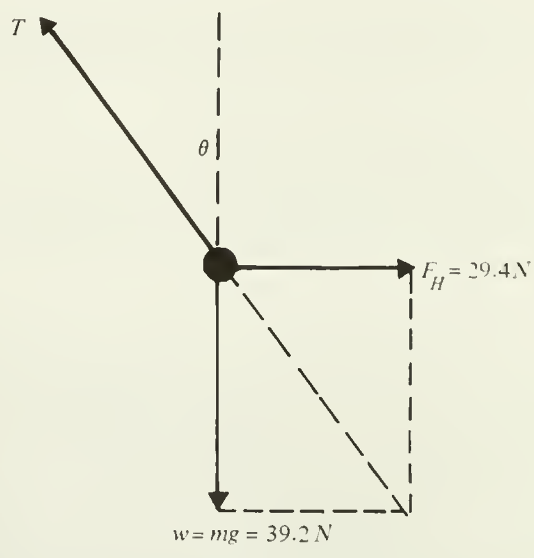


Figure 4-2

Referring to Fig. 4-2, we see that the rock is in equilibrium under the action of three forces: its weight

$$w = mg = 4 \text{ kg } (9.8 \text{ m} \cdot \text{s}^{-2}) = 39.2 \text{ N},$$

the horizontal force F_H, and the tension T. In equilibrium, the sum of all forces, and force components, is zero,

$$\sum F_x = 0 \qquad \sum F_y = 0$$

For the horizontal or x components we find

$$0 = \sum F_x = F_H - T \sin \theta = 29.4 \text{ N} - T \sin \theta$$

For the vertical or y components we find

$$0 = \sum F_y = T \cos \theta - 39.2 \text{ N}$$

or

$$T \sin \theta = 29.4 \text{ N}$$

$$T \cos \theta = 39.2 \text{ N}$$

Dividing these last two equations we find

$$\frac{\sin \theta}{\cos \theta} = \tan \theta = \frac{29.4}{39.2} = 0.75$$

$$\theta = \tan^{-1}(0.75) = 37^\circ$$

Thus the tension is

$$T = \frac{29.4 \text{ N}}{\sin 37^\circ} = 49 \text{ N}$$

Example 4

(a) An iceskater pushes off the wall of the rink. What is the reaction to the force of the skater against the wall?

(b) A book rests on a table. What is the reaction to the upward force of the table on the book?

(c) In Example 3, what is the reaction to the tension force on the rock?

Solution:

(a) The wall pushes against the skater, setting her in motion.

(b) The book exerts a downward force on the table.

(c) The wire is stretched between the rock and the tree branch. The branch pulls on the rock. The reaction of this force is the rock pulling the branch.

Example 5

A 3.1 g bullet traveling at 300 $m \cdot s^{-1}$ penetrates a block of wood to a depth of 15 cm. Assuming a constant retarding force, find the magnitude of the force and the time required for the bullet to come to rest.

Solution:

When the force is constant, the acceleration is constant, by Newton's second law: F = ma. Thus we may use our earlier (Chapter 2) results for constant acceleration:

$$v^2 = v_o^2 + 2ax$$

where v is the final velocity (v = 0), v_o is the initial velocity (v_o = 300 $m \cdot s^{-1}$), and x is the distance traveled between initial and final times, x = 5 cm. Thus the acceleration is

$$a = \frac{1}{2x}(v^2 - v_o^2) = \frac{1}{2(0.15\ m)}[-(300\ m \cdot s^{-1})^2]$$

$$= -\ 3 \times 10^5\ m \cdot s^{-2}$$

Newton's second law indicates that the force is

$$F = ma = (3.1 \times 10^{-3}\ kg)(-\ 3 \times 10^5\ m \cdot s^{-2})$$

$$= -\ 930\ N$$

The stopping time is given by the constant acceleration relation

$$v = v_o + at$$

with the final velocity v = 0:

$$t = \frac{v - v_o}{a} = \frac{-\ 300\ m \cdot s^{-1}}{-\ 3 \times 10^5\ m \cdot s^{-2}} = 10^{-3}\ s$$

Example 6

The coordinates of a body are given by:

$$x = A \cos \omega t, \quad y = \text{constant},$$

where $A = 0.2$ m and $\omega = 1\ \text{rad}\cdot\text{s}^{-1} = 1\ \text{s}^{-1}$. Find the total force on the body when $t = 0.5$ s.

Solution:

Since $F = ma$, we write for the components,

$$F_x = ma_x = m\frac{dv_x}{dt} = m\frac{d^2x}{dt^2}$$

$$F_y = ma_y = m\frac{dv_y}{dt} = m\frac{d^2y}{dt^2} = 0.$$

Thus the force is in the x-direction with

$$F_x = m\frac{d^2}{dt^2}(A\cos\omega t) = m\frac{d}{dt}\left[\frac{d}{dt}A\cos\omega t\right]$$

$$= m\frac{d}{dt}(-\omega A\sin\omega t) = -\omega^2 mA\cos\omega t$$

$$= -(1\ \text{s}^{-1})^2(2\ \text{kg})(0.2\ \text{m})\cos(1\ \text{s}^{-1}\cdot 0.5\ \text{s})$$

$$= 0.35\ \text{N}.$$

Note in the expression $\cos\omega t$, ωt is in radians. The expression

$$\frac{d}{dt}(\sin\omega t) = \omega\cos\omega t$$

is true only when ωt is in radian measure.

Example 7

A 1000 kg elevator rises with an acceleration equal to g. What is the tension in the supporting cable? (See Fig. 4-3)

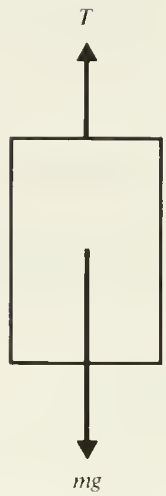


Figure 4-3

Solution:

If the up direction is taken as positive, we have

$$T - mg = ma = mg$$

$$T = 2\ mg = 2(1000\ kg)9.8\ m \cdot s^{-2}$$

$$= 19{,}600\ N$$

Example 8

An elevator weighing 4000 lbs falls with a downward acceleration of magnitude (1/2)g. What is the tension in the supporting cable?

Solution:

Referring to Fig. 4-3, we set the total force equal to the mass times the acceleration

$$T - mg = ma = m\left(\frac{-g}{2}\right)$$

Thus the tension is

$$T = mg - \frac{mg}{2} = \frac{mg}{2} = \frac{w}{2}$$

$$T = \frac{4000 \text{ lbs}}{2} = 2000 \text{ lbs.}$$

Example 9

A block, starting from rest, is acted upon by gravity and an upward force of 5 N. Its acceleration is 3 $m \cdot s^{-2}$, upward.

(a) What is its mass? (See Fig. 4-3, with T the upward force.)

(b) If the upward force acts for 5 s, how high does the block rise before it starts its descent?

Solution:

(a) $$T - mg = ma; \qquad \text{so } m(a + g) = T,$$

$$m = \frac{T}{a + g} = \frac{5 \text{ N}}{(3 + 9.8) m \cdot s^{-2}} = 0.39 \text{ kg}$$

(b) While the upward force acts, the acceleration is in an upward direction,

$$x = x_o + v_o t + (1/2)at^2$$

where $x_o = 0$, $v_o = 0$, and $a = 3 \ m \cdot s^{-2}$.

At $t = 5$ s,

$$x = \left(\frac{1}{2}\right)at^2 = \frac{(3 \ m \cdot s^{-2})(5 \ s)^2}{2} = 37.5 \text{ m},$$

and

$$v = v_o + at = (3 \ m \cdot s^{-2})(5 \ s) = 15 \ m \cdot s^{-1}.$$

At this point the upward force ceases to act and only a downward weight acts, producing a downward acceleration of magnitude g. The above results form the initial conditions for the subsequent motion: After $t = 5$ s, $a = -g$, and the coordinate and velocity are given by

$$x = x_o + v_o t - (1/2)gt^2 \text{ with } x_o = 37.5 \text{ m},$$

$$v = v_o - gt \text{ where } v_o = 15 \ m \cdot s^{-1}.$$

At the maximum height, $v = 0$ and

$$t = v_o g^{-1} = \frac{15 \ m \cdot s^{-1}}{9.8 \ m \cdot s^{-2}} = 1.53 \text{ s.}$$

$$x = 37.5 \text{ m} + (15 \ m \cdot s^{-1})(1.53 \ s) - (9.8 \ m \cdot s^{-2})(1.53 \ s)^2$$

$$x = 49 \text{ m}$$

Alternatively, use

$$v^2 = v_o^2 + 2a(x - x_o) = v_o^2 - 2g(x - x_o) = 0$$

with $v = 0$ at the maximum height. Solving for $x - x_o$ we have

$$x - x_o = \frac{v_o^2}{2g} = \frac{(15 \text{ m}\cdot\text{s}^{-1})^2}{2(9.8 \text{ m}\cdot\text{s}^{-2})} = 11.5 \text{ m}.$$

$$x = 37.5 \text{ m} + 11.5 \text{ m} = 49 \text{ m}.$$

Example 10

A block of mass 10 kg is suspended by a string. What is the string tension when:
(a) The block is at rest;
(b) The block is accelerating upward at 5 $\text{m}\cdot\text{s}^{-2}$;
(c) The block is accelerating downward at 5 $\text{m}\cdot\text{s}^{-2}$;
(d) The block is in free fall.

Solution: (See Fig. 4-3.)

(a) $T - mg = ma = 0$; so $T = mg = (10 \text{ kg})(9.8 \text{ m}\cdot\text{s}^{-2}) = 98 \text{ N}$.

(b) $T - mg = ma$; so $T = m(a + g) = (10 \text{ kg})(5 + 9.8)\text{m}\cdot\text{s}^{-2}$

$T = 148 \text{ N}$

(c) $T - mg = ma$; so $T = m(a + g) = (10 \text{ kg})(-5 + 9.8)\text{m}\cdot\text{s}^{-2}$

$T = 48 \text{ N}$

(d) $T - mg = ma = -mg$; so $T = 0$. ('Free fall' means that only the gravitational force acts and $a = -g$.)

Example 11

A 100 kg man stands on a bathroom scale in an elevator. What is the acceleration of the elevator when the scale reading is
(a) 150 kg.
(b) 100 kg.
(c) 50 kg.

Solution:

Referring to Fig. 4-4, F_s is the force of the scale on the man and mg is the force of gravity on the man. The acceleration of the man is equal to the acceleration of the elevator a,

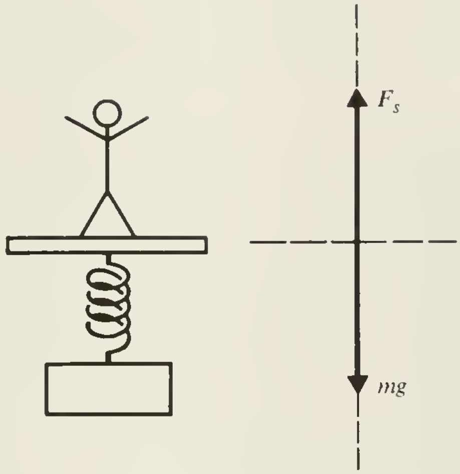


Figure 4-4

$$\sum F = F_s - mg = ma$$

$$a = \frac{F_s - mg}{m} .$$

The scale measures force but is calibrated so that it reads 1 kg when the force is equal to the weight of a 1 kg mass, that is, $F_s = m_s g$ where m_s is the <u>scale reading</u> as given above. Thus

$$a = \frac{m_s g - mg}{m} = \left(\frac{m_s}{m} - 1 \right) g$$

(a) $$a = \left(\frac{150}{100} - 1 \right) g = 4.9 \ m \cdot s^{-2}$$

(b) $$a = \left(\frac{100}{100} - 1 \right) g = 0$$

(c) $$a = \left(\frac{50}{100} - 1 \right) g = -4.9 \ m \cdot s^{-2}$$

Example 12

A ball slides without friction off the elevated inclined plane shown in Fig. 4-5a, starting from rest at the top of the plane. Find the distance $\bar{x}$ in Fig. 4-5a.

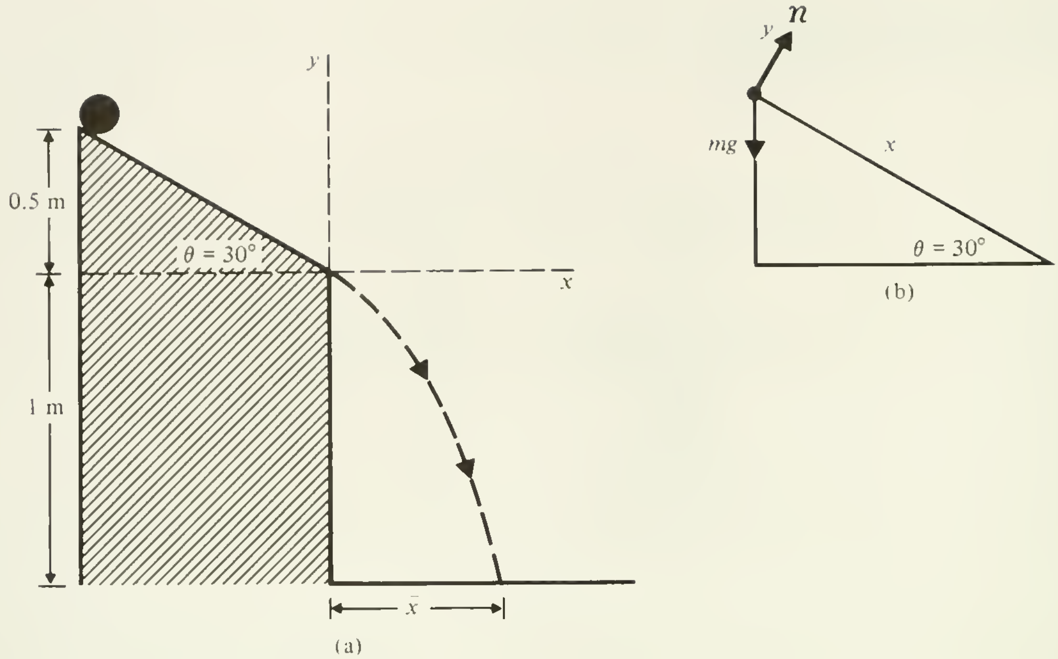


Figure 4-5a,b

Solution:

This problem is really two problems: first, to find the velocity at the bottom of the inclined plane, and second, to use this velocity as an initial velocity for the projectile part of the motion. Refer to Fig. 4-5b for the first part of the motion (while the ball is on the plane). During this time, the ball is acted upon by two forces, its weight and the normal force of the plane. By Newton's second law, the total force along the plane gives the mass times the acceleration along the plane:

$$F_x = mg \sin \theta = ma_x \; ; \; a_x = g \sin \theta$$

$$F_y = N - mg \cos \theta = 0$$

Thus while on the plane we have constant acceleration in the tilted x-direction,

$$x = x_o + v_{ox}t + \frac{1}{2} a_x t^2 = \frac{1}{2} g(\sin \theta)t^2,$$

$$t = \left(\frac{2x}{g \sin \theta} \right)^{1/2}$$

$$v_x = v_{ox} + a_x t = g \sin \theta \; t$$

where the initial conditions are $v_{ox} = 0$, $x = 0$ for the <u>tilted coordinate system</u> with origin at the top of the plane. The time to reach the bottom of the plane at $x = 0.5$ m/sin θ is

$$t = \left(\frac{2\,x}{g\,\sin\theta}\right)^{1/2} = \left[\frac{2(0.5\text{ m})}{9.8\text{ m}\cdot\text{s}^{-2}\,\sin^2\theta}\right]^{1/2}$$

$$= \frac{0.32\text{ s}}{\sin\theta}$$

and hence the velocity is

$$v_x = g\,\sin\theta\;t = (9.8\text{ m}\cdot\text{s}^{-2})(.32\text{ s})$$

$$= 3.1\text{ m}\cdot\text{s}^{-1}.$$

Alternatively we can use

$$v_x{}^2 = v_{ox}{}^2 + 2a_x(x - x_o).$$

With $x = .5\text{ m}(\sin\theta)^{-1}$, $v_{ox} = 0$ and $a_x = g\,\sin\theta$ we have

$$v_x{}^2 = 2(g\,\sin\theta)(0.5\text{ m})(\sin\theta)^{-1}$$

$$v_x = 3.1\text{ m}\cdot\text{s}^{-1}$$

independent of the incline angle θ.

You are now ready to do the second part of the problem, resetting your clock, coordinate system, and initial conditions, as shown in the x,y system of Fig. 4-5a. The initial conditions are

$$x_o = 0 \qquad\qquad y_o = 0$$

$$v_{ox} = v_o\cos\theta \qquad\qquad v_{oy} = -v_o\sin\theta$$

$$v_o = 3.1\text{ m}\cdot\text{s}^{-1}$$

and thus

$$x = x_o + v_{ox}t = v_o\cos\theta\;t$$

$$y = y_o + v_{oy}t + \frac{1}{2}(a_yt^2) = -v_o\sin\theta\;t - \frac{1}{2}gt^2$$

The time the ball hits the floor is when $y = -1$ m. Solving the last equation for t,

$$\frac{1}{2}gt^2 + v_o\sin\theta\;t + y = 0 = at^2 + bt + c$$

5

APPLICATIONS OF NEWTON'S LAWS—I

OBJECTIVES

In this chapter you will apply Newton's laws of motion to objects in equilibrinm (at rest or in uniform motion) and not in equilibrium (in accelerated motion.) Your objectives are to:

> Analyze simple systems by indentifying the forces acting on each component. Examples are blocks on inclined planes, and systems of weights and pulleys.
>
> Draw free-body diagrams illustrating all forces acting on a body.
>
> Recognize common mechanical forces such as tension, contact, and weight.
>
> Use Newton's third law to relate forces acting between two bodies.
>
> Use Newton's first law to insure equilibrium of a body.
>
> Use Newton's second law to find the acceleration of, or the force on, a body.

REVIEW AND SUPPLEMENT

As noted in Chapter 4, forces, such as **F**, $\mathbf{w}$, $\mathscr{n}$, or $\mathscr{F}$ in Fig. 5-1, arevector quantities, with magnitude and direction, exerted on a body at a point, along some line of action. Fig. 5-1 illustrates four common examples of forces you will repeatedly encounter in the next two chapters. F is a pull exerted by, say, a man on the box as he attempts to drag it to the right. $\mathbf{w}$ is the weight of the box, a downward force due to the gravitational attraction of the earth on the box. $\mathscr{n}$ is the normal force of the earth pushing upward on the box. It is always normal (perpendicular) to the two surfaces in contact. $\mathscr{F}$ is the frictional force or drag exerted on the box by the earth, opposing the motion which the force F tends to create if the system is initially at rest. The frictional force is parallel to the surface, perpendicular to the normal force. $\mathscr{n}$ and $\mathscr{F}$ are components of the total contact force.

It is crucial in this and later chapters to distinguish between (1) forces exerted on a body by its surroundings and (2) forces exerted on its surroundings by the body. In the example of Fig. 5-1 the indicated forces are those acting on the box. Fig. 5-1a is a 'free body diagram'.

As noted in the last chapter, the unit of force in the SI is a newton (N) and in the British system a pound. The conversion factor is 1 pound = 4.448 N.

Bathroom scales measure force, for example, the weight of your body, in units of pounds. Similarly spring balances may be calibrated in newtons to measure the forces such as the force **F** in Fig. 5-1.

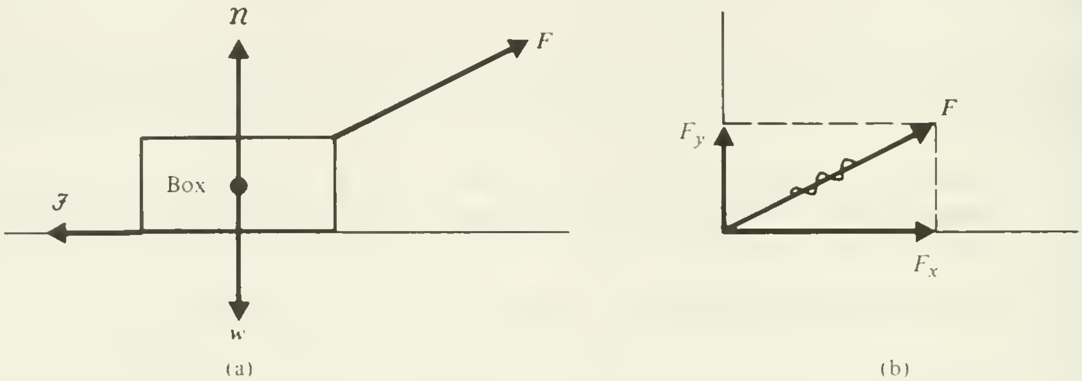


Figure 5-1

It is often useful to resolve forces into their rectangular coordinates as in Fig. 5-1b. Experimentally $\mathbf{F_x}$ and $\mathbf{F_y}$ acting at the same point of application as **F** have the same effect as **F**, consistent with the equality $\mathbf{F_x} + \mathbf{F_y} = \mathbf{F}$. Note in Fig. 5-1b we have followed the text's convention of drawing a snake over **F** to indicate that it is not a force additional to its components: work either with **F** or with $\mathbf{F_x}$ and $\mathbf{F_y}$, but not with all three.

Similarly if two forces $\mathbf{F_1}$ and $\mathbf{F_2}$ act on an object at the same point, their vector sum $\mathbf{R} = \mathbf{F_1} + \mathbf{F_2}$ has the same effect as $\mathbf{F_1}$ and $\mathbf{F_2}$.

The two techniques of vector summation and resolution of forces into components will be central to most of the problems of the next few chapters. The rectangular system of coordinates need not be the familiar vertical and horizontal axes; in the inclined plane example, you will indeed find it more convenient to resolve the forces into components along the plane and perpendicular to the plane, as shown in Fig. 5-2 and Fig. 5-3

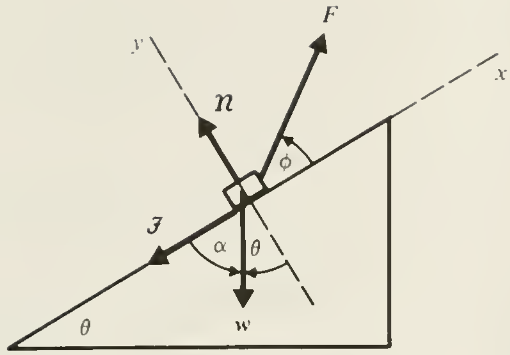


Figure 5-2

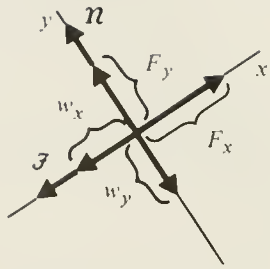


Figure 5-3

in which a force **F** drags a box up the plane against the friction force $\mathcal{F}$. Note the incline angle θ of the plane is equal to the angle between **w** and the negative y axis because they are complements of the same angle α. We have the components

$$F_x = F \cos \phi \qquad w_x = -w \sin \theta$$

$$F_y = F \sin \phi \qquad w_y = -w \cos \theta$$

$$n_x = 0 \qquad \mathcal{F}_x = -\mathcal{F}$$

$$n_y = n \qquad \mathcal{F}_y = 0$$

We have stressed above that forces may be divided into those exerted on a body A by its surroundings and those exerted by the surroundings on the body. Its surroundings are simply other bodies (B, C, D,...). By Newton's third law, if B exerts a force on A ($\mathbf{F}_{AB}$) then A exerts a force on B, which is equal in magnitude and opposite in direction, $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$. The equal and opposite forces occur on different bodies (A on B, B on A) and are called action, reaction pairs. They are equal and opposite whether or not the bodies are in equilibrium.

The application of Newton's Third Law will be critical in your analysis of the simple mechanical systems of the next few chapters. Return to our original example of the man pulling the box in Fig. 5-1, where we showed all forces acting on the box. To each 'action' **F**, n, **w**, $\mathcal{F}$ corresponds an equal and opposite 'reaction', a force of the box on the components of its surroundings. Some action-reaction pairs of forces are given in the table below and illustrated in Fig. 5-4.

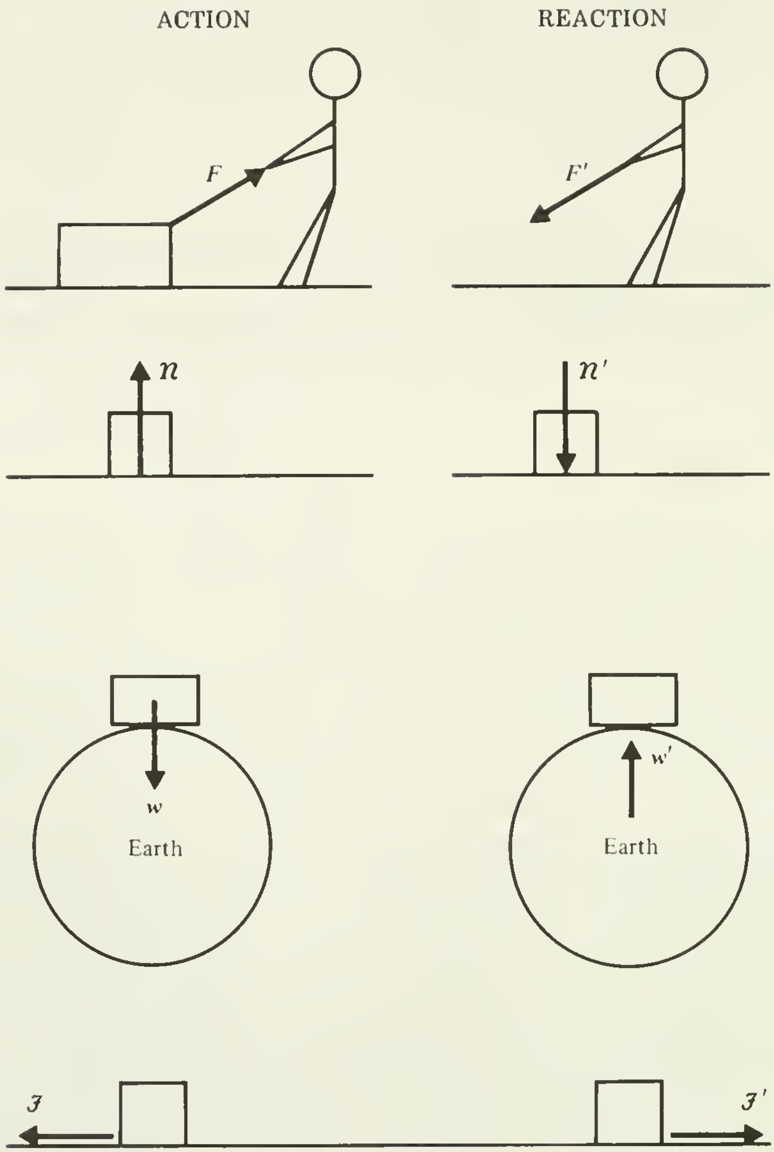


Figure 5-4

'Action'	'Reaction'
$\mathbf{F}$ (force of man on box)	$\mathbf{F}' = -\mathbf{F}$ (force of box on man)
$\mathcal{N}$ (force of contact of earth's surface against box)	$\mathcal{N}' = -\mathcal{N}$ (force of contact of box against earth's surface)
$\mathbf{w}$ (gravitational force of earth on box)	$\mathbf{w}' = -\mathbf{w}$ (gravitational force of box on earth)
$\mathcal{F}$ (friction force of earth against box tending to prevent box from moving to right)	$\mathcal{F}' = -\mathcal{F}$ (force of box on earth tending to drag earth to right)

Like inclined planes, pulleys and ropes are components of systems useful in the next few chapters to illustrate basic principles.

The tension T at a point in a rope is the magnitude of the force of one part of a rope on the other part as shown in Fig. 5-5, where T_{AB} and T_{BA} are action-reaction pairs.

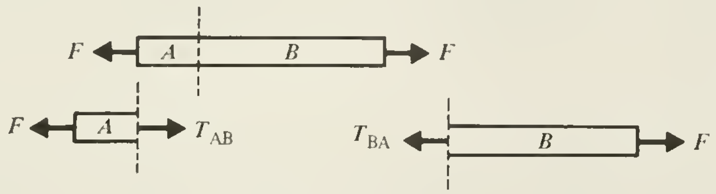


Figure 5-5

A rope, and any two of its segments A and B, are in equilibrium only if there are equal and opposite forces at their ends. The segment A is in equilibrium; hence the force of B on A at the division, the tension T, is equal in magnitude to F. (Note that the tension is NOT 2F.) The tension in this example is uniform, that is, the same at any point because our argument did not depend on where we choose to divide the rope into segments. This is true in an ideal massless rope whether or not it is in equilibrium. You will usually, consider rope of this kind; it transmits force without change of magnitude, even around corners as with pulleys (provided the pulley is massless or not accelerating) as indicated in Fig. 5-6.

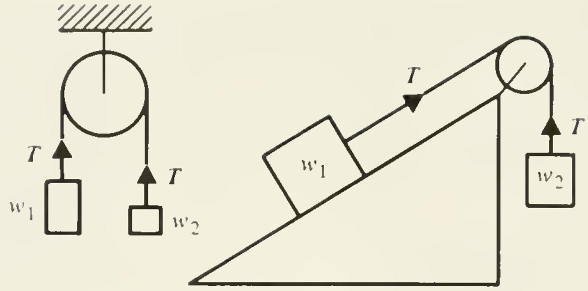


Figure 5-6

For an accelerating rope with mass (not in equilibrium) the tension is not uniform. For a stationary massive rope hanging in a gravitational field the tension is also non-uniform.

The friction force $\mathcal{F}$ between two surfaces is directed along the surfaces in a direction to oppose the relative motion which is occurring (kinetic friction) or the motion which would occur if the friction were not there (static friction.)

The static friction force is equal to or less than some maximum value. This maximum value is proportional (for simple surfaces) to the normal force $\mathcal{N}$ between the surfaces:

$$\mathcal{F}_s < \mu_s \mathcal{N}$$

The proportionality constant μ_s, the coefficient of static friction, is characteristic of the surfaces. When the external force, tending to move the surfaces over each other, exceeds $\mu_s\eta$, the surfaces slip and the <u>kinetic</u> friction force is now given by

$$\mathcal{F}_k = \mu_k \cdot \mathcal{N}$$

We will discover in the examples that $\mu_k < \mu_s$ for a simple pair of surfaces. Note there is no reason why $\mu < 1$.

HINTS AND PROBLEM-SOLVING STRATEGIES

A very systematic approach has been used to solve all the problems met in this chapter. This approach is summarized in the following recipe:

(1) Draw a sketch of the problem situation. Isolate the various bodies in the problem and make a FREE-BODY DIAGRAM for each body showing all the forces acting <u>on</u> the body.

(2) Choose a suitable coordinate system (i.e. y axis is vertical and x axis is horizontal except in situations like the inclined plane where a different choice is simpler) and resolve the various force vectors into their x and y components.

(3) Apply the Newton's law condition for translational equilibrium and write $\sum F_x = 0$; $\sum F_y = 0$; and $\sum F_z = 0$, if the body is at rest or moving uniformly.

(4) Apply Newton's second law in the case of general motion, $\sum F_x = ma_x$, $\sum F_y = ma_y$, and $\sum F_z = ma_z$.

(5) Solve the set of equations obtained in step (3) or (4) for the unknown quantities.

EXAMPLES AND SOLUTIONS

Example 1

Consider a block of weight w at rest or in uniform motion in a straight line on a smooth frictionless surface. Find all forces acting on the block.

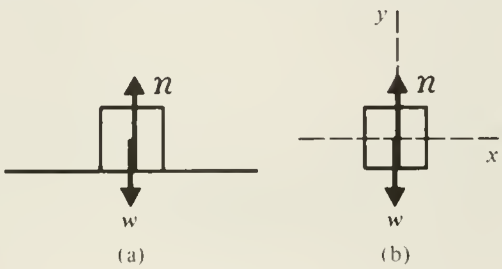


Figure 5-7

Solution:

The block is in equilibrium, and hence $\sum \mathbf{F} = 0$. Draw a diagram of the block showing all forces acting <u>on</u> it (Fig. 5-7). This will be called a 'free-body diagram'.

In order to avoid the temptation of adding an extraneous force acting on other bodies, such as the downward contact force of the block on the earth, it is helpful to 'isolate' the body as in Fig. 5-7b, where we have also drawn a system of coordinate axes useful in resolving components. We have the equilibrium condition $\sum \mathbf{F} = 0 = \mathbf{w} + \eta$. In rectangular components,

$$\sum F_x = 0$$

$$\sum F_y = 0 = \eta - w$$

and we learn that the normal force is equal and opposite to the weight. <u>This is not always the case</u>, as we will see below, in Example 3.

Example 2

Consider now the same block pulled with constant velocity along a surface with friction, as shown in Fig. 5-8, by a force **F**. Find the force F in terms of μ_k and w.

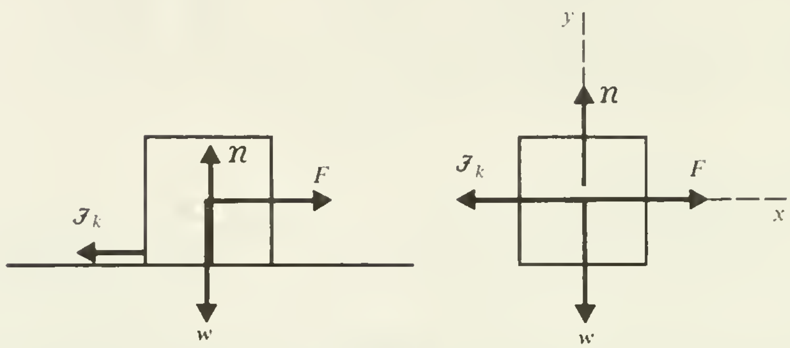


Figure 5-8

Solution:

In Fig. 5-8b we have moved all vectors so that their tails are at the origin (see rules for addition, Chapter 1). The point of application of the force and its line of action are relevant but not in connection with the condition of equilibrium for particles, $\sum \mathbf{F} = 0$.

Since the box is in equilibrium,

$$\sum F_x = 0 = F - \mathcal{J}_k = F - \mu_k n$$

$$\sum F_y = 0 = n - w \; .$$

Solving the first equation for F, and substituting $\mu = w$ we have

$$F = \mu_k n = \mu_k w$$

Note we have considered here the general case of arbitrary weight w and coefficient of friction μ_k, in terms of which we found the force F. The analysis answers every possible variation on this problem, such as

(i) What is the coefficient of friction μ_k if the weight is w = 30 N and the external force F necessary to make the box move uniformly is 10 N?

$$\mu_k = \frac{F}{w} = \frac{10}{30} = \frac{1}{3}$$

(ii) What is force F if the coefficient of friction is 0.25 and weight is 30 N?

$$F = \mu_k w = 0.25\ (30\ \text{N}) = 7.5\ \text{N}$$

(iii) How heavy is the body if the force F = 15 N and $\mu_k = 0.20$?

$$w = \frac{F}{\mu_k} = \frac{15}{0.2} = 75 \text{ N}$$

Example 3

Consider now the case when the block is moving uniformly against friction but the force F is not horizontal, as in Fig. 5-9. Find F given the weight w and the coefficient of friction μ_k.

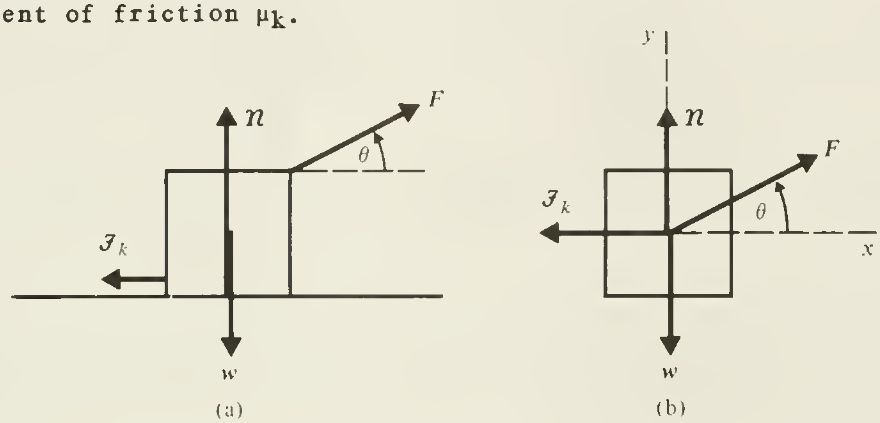


Figure 5-9

Solution:

Applying the conditions of equilibrium, insuring that the total force is zero, we find

$$\sum F_x = 0 = F \cos\theta - \mathcal{F}_k = F \cos\theta - \mu_k n$$

$$\sum F_y = 0 = n + F \sin\theta - w.$$

Notice that $n \neq w$. Solving the second equation for n and substituting the result in the first equation, we have

$$0 = F \cos\theta - \mu_k(w - F \sin\theta)$$

$$= F(\cos\theta + \mu_k \sin\theta) - \mu_k w$$

or

$$F = \mu_k \frac{w}{\cos\theta + \mu_k \sin\theta}$$

You can check this result by putting $\theta = 0$. In this limit it should reduce to the previous result, $F = \mu_k w$, as it does. Note also $F = 0$ if $\mu_k = 0$.

Example 4

Consider now a body of weight w = 10 N moving uniformly up a plane inclined at 30° with coefficient of friction $\mu_k = 0.30$, under the influence of an external force F pulling parallel to the plane. Find the forces F and $\mathcal{N}$.

Solution:

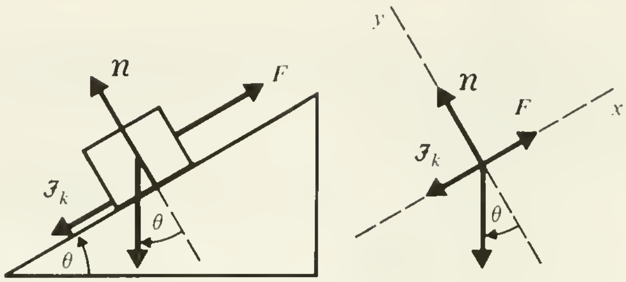


Figure 5-10

Again, we must first draw a sketch and a free body diagram indicating all forces acting on the body, as illustrated in Fig. 5-10. Here it is convenient to orient the x axis parallel to the incline. Imposing the equilibrium condition, we find

$$0 = \sum F_x = -\mathcal{F}_k + F - w \sin\theta = -\mu_k \mathcal{N} + F - w \sin\theta$$

$$0 = \sum F_y = \mathcal{N} - w \cos\theta$$

It is worthwhile to think about each sign in the above equations. Physics is as unforgiving as a bank concerning sign errors. The x component of the weight is <u>negative</u> because the projection of **w** on the x axis points down the plane in the negative x direction. Correspondingly the y component of the weight is negative.

Note also that the distinction between sin θ and cos θ is of essence. Return to chapter 1 for pertinent comments if you are not sure why $w_x = -w \sin\theta$ and $w_y = -w \cos\theta$ with θ defined in Fig. 5-10.

There is no point yet in substituting the specific values for θ, μ_k, or w.

We now can solve the two simultaneous equations for F and $\mathcal{N}$. The second yields

$$\mathcal{N} = w \cos\theta$$

substituting this result in the first yields

$$0 = -\mu_k w \cos\theta + F - w \sin\theta$$

or

$$F = w(\sin\theta + \mu_k \cos\theta)$$

yielding F and n for all w, θ, and μ_k.

In the specific case at hand,

$$n = 10\text{N} \cos 30^\circ = 8.7 \text{ N}$$

$$F = (10 \text{ N})(\sin 30^\circ + 0.30 \cos 30^\circ)$$

$$= 7.6 \text{ N}$$

Example 5

A block of weight w is at rest on a rough surface of coefficient of static friction μ_s under the influence of a horizontal force F. Find the range of values possible for F before the block moves.

Solution:

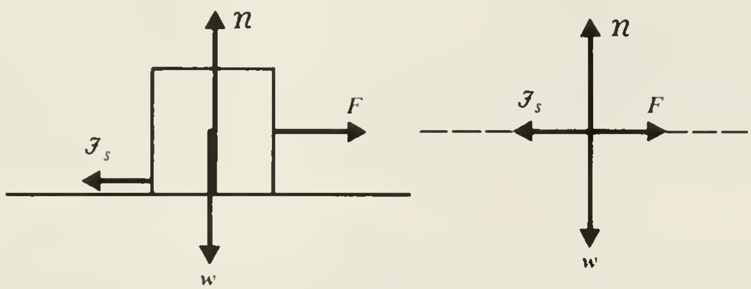


Figure 5-11

Since the values of w and μ_s have not been supplied, one must assume that the desired range of F is to be given algebraically in terms of w and μ_s. In Fig. 5-11 we have a sketch and free-body diagram. The equilibrium conditions are

$$\sum F_x = 0 = F - \mathcal{F}_s \quad \text{but} \quad \mathcal{F}_s \leq \mu_s n$$

$$\sum F_y = 0 = n - w$$

Note it is incorrect to conclude that $\mathcal{F}_s = \mu_s n$ The static friction $\mathcal{F}_s$ depends on F. If F = 0 the block just sits there and $\mathcal{F}_s = 0$. The correct conclusion is $\mathcal{F}_s = F$ when the block does not move, that is when

$$J_s \leq \mu_s n = \mu_s w$$

or

$$F \leq \mu_s w$$

Static equilibrium is maintained as long as F does not exceed $\mu_s w$. Beyond that point the block moves and the friction force is $\mathcal{F}_k = \mu_k \mathcal{N} = \mu_k w$. This force of kinetic friction must be smaller than the maximum static friction, or the block would never move, that is

$$\mu_k w < \mu_s w \quad \text{or} \quad \mu_k < \mu_s.$$

Example 6

Suppose a block of weight w is at rest on an adjustable inclined plane of angle θ and static coefficient of friction μ_s. Find the range of possible values of θ for which the block remains at rest.

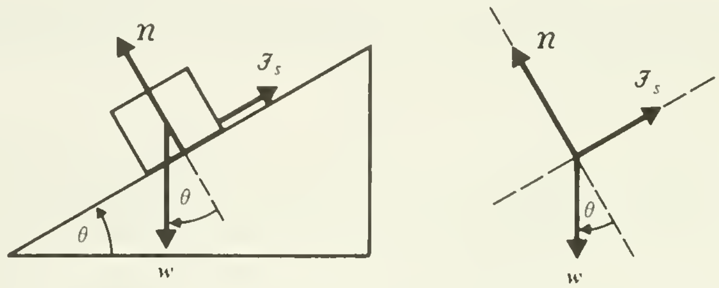


Figure 5-12

Solution:

The obligatory sketch and free-body diagram are indicated in Fig. 5-12. The conditions of equilibrium are

$$0 = \sum F_x = \mathcal{F}_s - w \sin\theta$$

$$0 = \sum F_y = \mathcal{N} - w \cos\theta$$

The condition $\mathcal{F}_s < \mu_s \mathcal{N}$ yields,

$$w \sin\theta < \mu_s w \cos\theta$$

or

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \leq \mu_s$$

Alternately we may observe that the critical case occurs when $\mathcal{F}_s = \mu_s \mathcal{N}$ or w sin θ = μ_s w cos θ ; μ_s = tan θ.

At the angle of impending motion, tan θ = μ_s, and $\mathcal{F}_s = \mu_s$ w sin θ is at its maximum.

Example 7

In the pulley and weight system shown (Fig. 5-13a) the pulley is frictionless and the rope is massless. What force P applied to the free end of the rope is required to support the weight w?

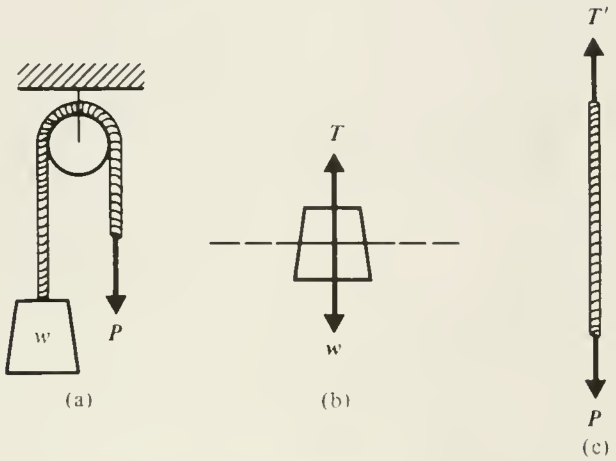


Figure 5-13

Solution:

In Fig. 5-13b we have a free-body diagram of the weight. In Fig. 5-13c we have a free-body diagram of a segment of the rope. Since the rope is massless (see supplementary section) $T' = T$. Since the segment of rope is in equilibrium $P = T'$. Hence $P = T' = T = w$.

Example 8

Consider the system shown in Fig. 5-14 where the weight $w_1 = 5$ N moves at constant speed on a rough table under the influence of the weight $w_2 = 2$ N. What is the coefficient of kinetic friction?

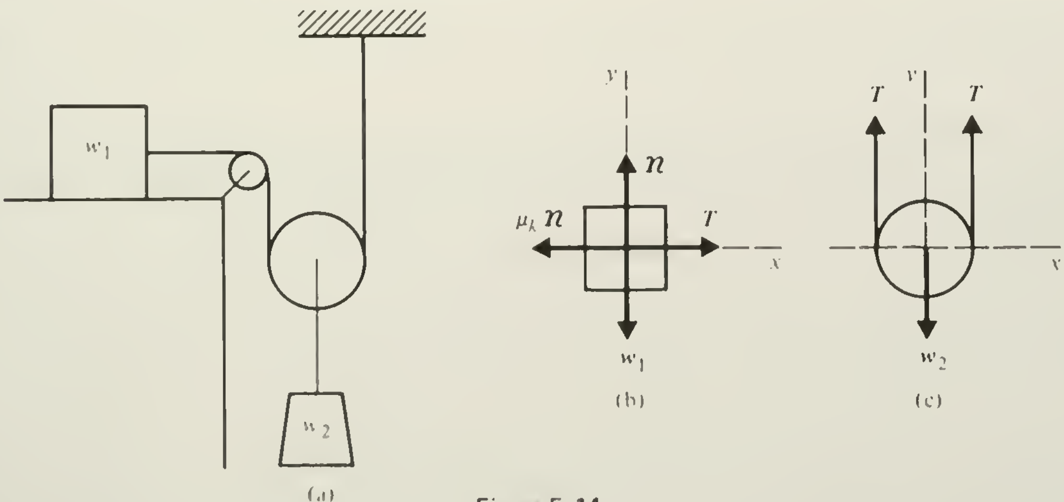


Figure 5-14

Solution:

In Fig. 5-14b and 5-14c we have the free-body diagrams for the forces acting on the two components of the system, w_1 and w_2 respectively. For w_1 the equilibrium conditions are

$$0 = \sum F_x = T - \mu_k n$$

$$0 = \sum F_y = n - w_1$$

from which we learn $T = \mu_k w_1$. For w_2 we have

$$0 = \sum F_x$$

$$0 = \sum F_y = 2T - w_2$$

or $T = w_2/2$ from which we conclude

$$\mu_k = \frac{T}{w_1} = \frac{w_2}{2w_1} = \frac{2\ N}{2(5\ N)} = 0.20$$

Alternately, you may wish to substitute immediately the input data $w_1 = 5$ N and $w_2 = 2$ N. Then we have

$$0 = T - \mu_k n$$

$$0 = n - 5\ N\ ; \quad n = 5\ N$$

for w_1 and

$$0 = 2\ T - w_2 = 2\ T - 2\ N\ ; \quad T = 1\ N$$

for w_2. Then the first relation yields

$$\mu_k = \frac{T}{n} = \frac{1\ N}{5\ N} = 0.20$$

It is usually a labor saving device to carry through an algebraic solution, plugging in the data only at the end.

This problem is one of many coupled systems you will deal with in the next few chapters. The motion of and forces on w_1 are related to those of w_2 by the coupling through the rope and pulley. A similar system is considered in the next problem.

Example 9

Consider the rough incline and pulley system of Fig. 5-15. Find the weight w_2 necessary to keep w_1 moving up the plane at constant velocity.

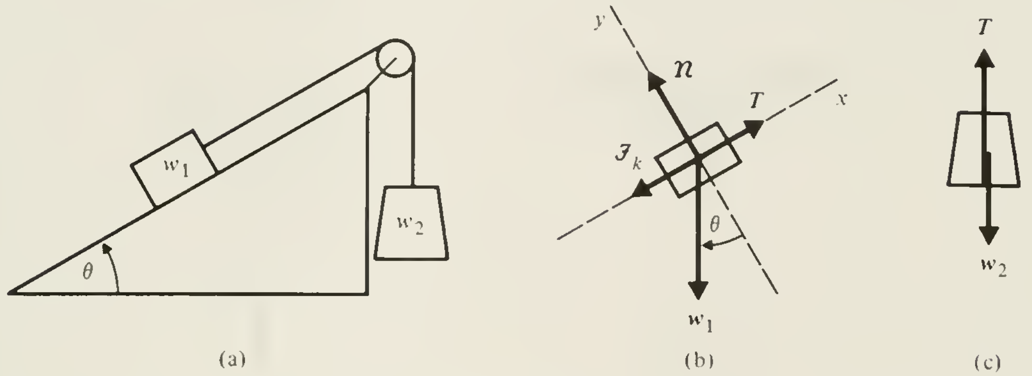


Figure 5-15

Solution:

Since w_1 moves up the plane, the force of kinetic friction points down the plane as indicated in w_1's free-body diagram (Fig. 5-15b). The equilibrium condition for w_1 is

$$0 = \sum F_x = T - \mathcal{F}_k - w_1 \sin\theta = T - \mu_k \eta - w_1 \sin\theta$$

$$0 = \sum F_y = \eta - w_1 \cos\theta \qquad \text{Note: } \eta \neq w_1$$

and for w_2 (here we use a different coordinate system from the one used for w_1)

$$0 = \sum F_x$$

$$0 = \sum F_y = T - w_2$$

using the last result in the first two,

$$0 = w_2 - \mu_k \eta - w_1 \sin\theta$$

$$0 = \eta - w_1 \cos\theta$$

Eliminating η, we have

$$w_2 = \mu_k w_1 \cos\theta + w_1 \sin\theta$$

which, with $w_1 = w$ should be compared to Example 4. w_2 corresponds to F.

Example 10

Consider the weight w hung from the ceiling as in Fig. 5-16. Find the tension in each string.

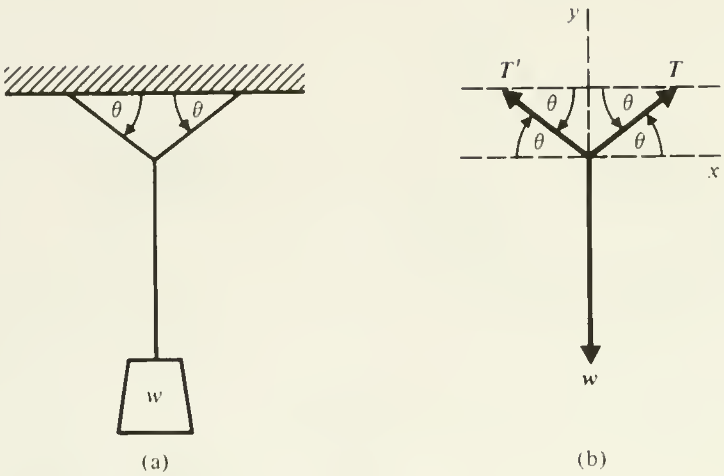


Figure 5-16

Solution:

The free-body diagram for the knot where the 3 ropes join is shown in Fig. 5-16b. The tension in the vertical string is the weight w. The knot's equilibrium conditions are then

$$0 = \sum F_x = T \cos \theta - T' \cos \theta$$

$$0 = \sum F_y = T \sin \theta + T' \sin \theta - w$$

The first tells us $T = T'$, which could have been guessed from the symmetry of the geometry; the system doesn't know left from right. The second yields

$$T = \frac{w}{2 \sin \theta}$$

As θ approaches 0, T approaches ∞. Infinite tension is required to keep a loaded cable without sag.

Example 11

Consider the trigonometrically more challenging case in Fig. 5-17 where the two arms are not of equal length. Find the tensions T_1 and T_2.

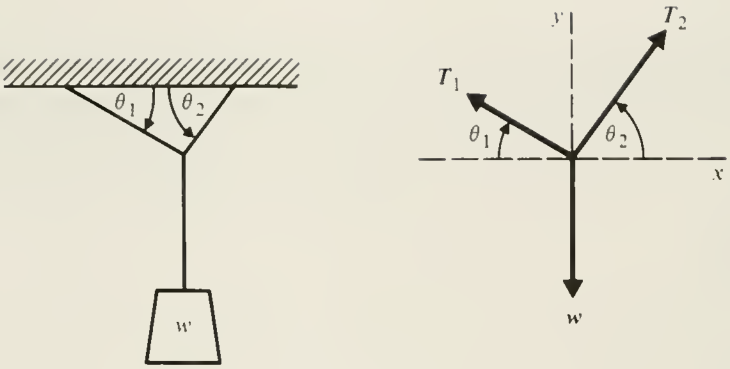


Figure 5-17

Solution:

The equilibrium equations are

$$0 = \sum F_x = T_2 \cos \theta_2 - T_1 \cos \theta_1$$

$$0 = \sum F_y = T_2 \sin \theta_2 + T_1 \sin \theta_1 - w$$

The first equation tells us the ratio of tensions

$$\frac{T_2}{T_1} = \frac{\cos \theta_1}{\cos \theta_2}.$$

The second then yields

$$T_1 = \frac{w}{\sin \theta_1 + (\cos \theta_1/\cos \theta_2)\sin \theta_2}$$

$$= \frac{w \cos \theta_2}{\sin(\theta_1 + \theta_2)}$$

If $\theta_1 = \theta_2 = \theta$,

$$T_1 = T_2 = \frac{w}{2 \sin \theta}$$

as in the last problem.

If $\theta_1 + \theta_2 = 90°$ then

$$T_1 = w \sin \theta_1 = w \cos \theta_2$$

$$T_2 = w \cos \theta_1 = w \sin \theta_2$$

Example 12

Consider the weightless boom and cable configuration in Fig. 5-18. Suppose that the only force of the wall on the boom is a perpendicular force η, as if the boom and wall were unattached and perfectly frictionless. The system is in equilibrium. Find the tension T and normal force η if $\theta = 60^{\circ}$ and $w = 100$ N.

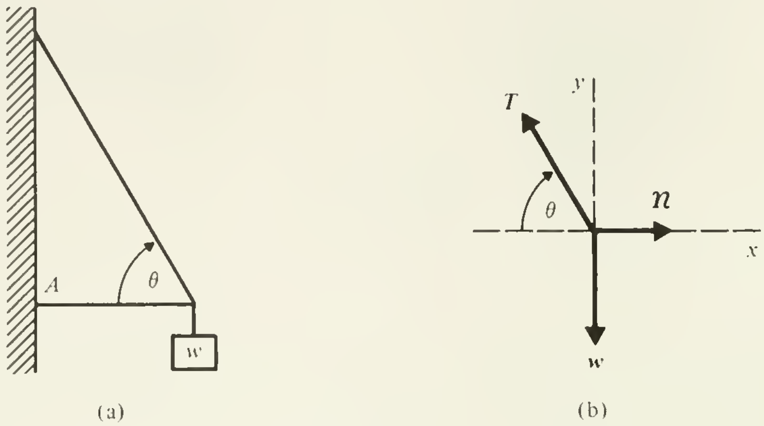


Figure 5-18

Solution:

In Fig. 5-18b the normal force of the wall on the strut, the tension T of the cable on strut, and the weight w have been indicated in the free-body diagram for the boom. Alternatively Fig. 5-18b can be interpreted as the free-body diagram for the point on the boom where the cables meet, with the understanding that the normal force of the wall on the boom is transmitted by the compressive forces within the boom (opposite of tension) to the cable joint. The equilibrium equations are

$$0 = \sum F_x = n - T \cos \theta$$

$$0 = \sum F_y = T \sin \theta - w$$

from which we conclude

$$n = T \cos \theta = \frac{w \cos \theta}{\sin \theta} = \frac{w}{\tan \theta} = \frac{100 \text{ N}}{\tan 60} = 58 \text{ N}$$

$$T = \frac{W}{\sin \theta} = \frac{100 \text{ N}}{0.866} = 115 \text{ N}$$

Example 13

A body of mass 10 kg, initially at rest on a smooth horizontal surface, experiences a horizontal force of 20 N.

(a) Find its acceleration.
(b) How far does it move in 5 s?
(c) What is its velocity at t = 10 s?

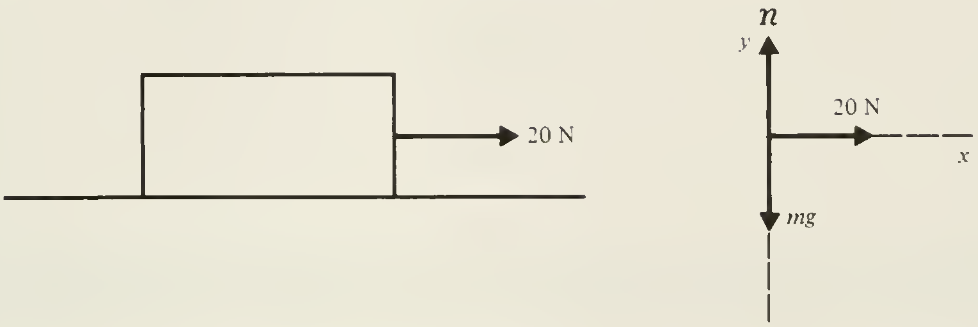


Figure 5-19

Solution:

Referring to Fig. 5-19 we see that the vertical forces cancel because the body can't accelerate in the vertical direction, indicating that

$$F_y = n - mg = ma_y = 0.$$

(a) The equation of motion for the horizontal motion ($a_x \equiv a$) is

$$F_x = ma_x = ma$$

$$a = \frac{F_x}{m} = \frac{20\ \text{N}}{10\ \text{kg}} = 2\ \text{m}\cdot\text{s}^{-2}.$$

(b) $$x = (1/2)at^2 = (1/2)(2\ \text{m}\cdot\text{s}^{-2})(5\ \text{s})^2 = 25\ \text{m}$$

(c) $$v = at = (2\ \text{m s}^2)\ 10\ \text{s} = 20\ \text{m}\cdot\text{s}^{-1}$$

Example 14

A body is dragged by a constant force F <u>equal to its weight</u> over a rough surface with coefficient of friction $\mu = 0.25$. Find its acceleration.

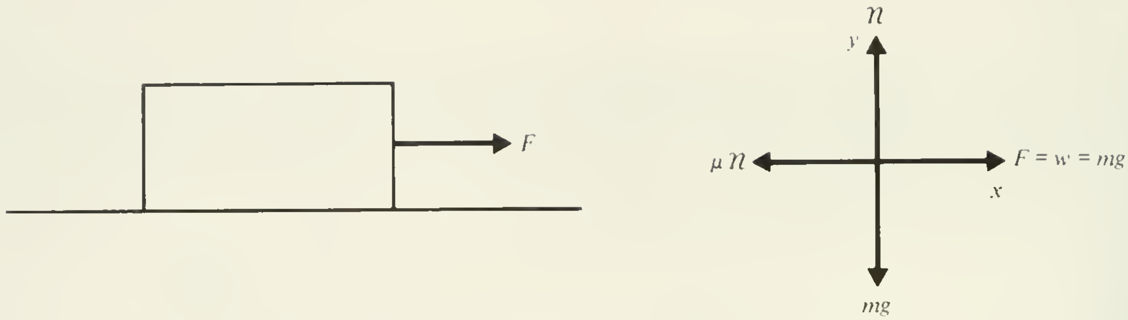


Figure 5-20

Solution:

Referring to the free body diagram and coordinate system of Fig. 5-20, we note the force F is equal to the body's weight and thus F = w = mg. Thus

$$\sum F_x = ma = F - \mu n \qquad a = (F - \mu n)/m$$

$$\sum F_y = 0 = n - mg; \qquad n = mg.$$

yielding

$$a = \frac{F - \mu n}{m} = \frac{mg - \mu mg}{m}$$

$$= (1 - \mu)g = (3/4)g$$

$$= 7.35\ m\cdot s^{-2}.$$

If $\mu > 1$ it appears from this answer that a is negative and the box accelerates in a direction opposite to the applied force. This conclusion is false. If $\mu > 1$ the force necessary to drag the box, i.e. break the static friction, is greater than the body's weight, and the motion described above is not possible with the given force. Note that nothing in principle excludes $\mu > 1$.

Example 15

A 5 kg body slides down a smooth inclined plane of angle $\theta = 30°$. Find the normal force on the body and its acceleration. (See Fig. 5-21)

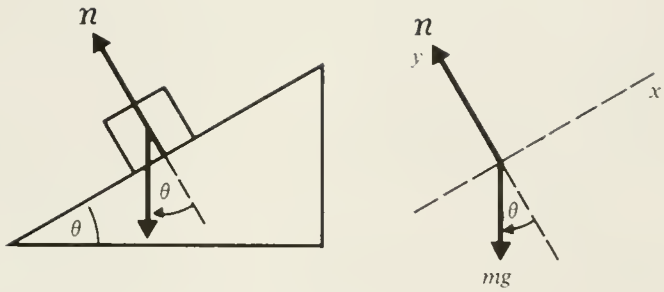


Figure 5-21

Solution:

For convenience the axes have been chosen along the perpendicular to the incline, as shown in Fig. 5-21. It is extremely tedious to work this problem with any other system of coordinates. The equations of motion are:

$$\sum F_x = ma = -mg \sin \theta$$

$$\sum F_y = 0 = n - mg \cos \theta.$$

The first equation may be solved for a and the second one for n:

$$a = -g \sin \theta = -(9.8\ m \cdot s^{-2})(0.5) = -\ 4.9\ m \cdot s^{-2}$$

$$n = mg \cos \theta = (5\ kg)(9.8\ m \cdot s^{-2})(0.87) = 42.4\ N$$

The acceleration is negative because the coordinate system has a positive x axis that points up the plane. If the positive x axis had been chosen to point down the plane, the acceleration would be positive.

Example 16

A body slides down a rough inclined plane of angle $\theta = 30^{\circ}$ with acceleration $a = 4\ m \cdot s^{-2}$. Find the coefficient of kinetic friction μ.

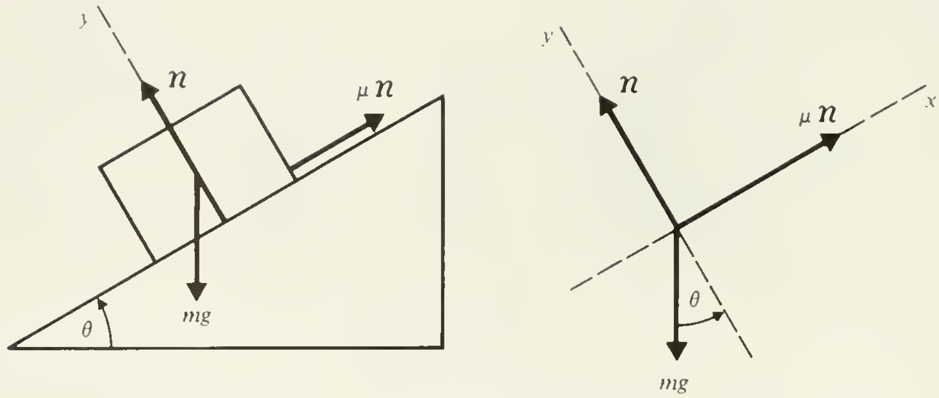


Figure 5-22

Solution:

Referring to the free body diagram and coordinate system of Fig. 5-22, we have

$$\sum F_x = \mu n - mg \sin \theta = ma$$

$$\sum F_y = n - mg \cos \theta = 0; \qquad n = mg \cos \theta.$$

The last equation may be solved for η and the result substituted in the first:

$$\mu \, mg \cos \theta - mg \sin \theta = ma$$

$$\mu = \frac{1}{g \cos \theta} (g \sin \theta + a).$$

With the x axis pointing up the plane, the acceleration down is <u>negative</u> so

$$\mu = \frac{1}{(9.8 \text{ m}\cdot\text{s}^{-2})(.87)} [(9.8 \text{ m}\cdot\text{s}^{-2})(0.5) - 4 \text{ m}\cdot\text{s}^{-2}]$$

$$= 0.11.$$

Had we chosen the x axis in Fig. 5-22 to point down, we would have obtained

$$\sum F_x = mg \sin \theta - \mu n = ma$$

$$\sum F_y = n - mg \cos \theta; \qquad N = mg \cos \theta$$

and thus

$$mg \sin\theta - \mu\, mg \cos\theta = ma$$

or

$$\mu = \frac{g \sin\theta - a}{g \cos\theta} .$$

With respect to this choice, $a = 4\ m\cdot s^{-2}$ and the final result for μ is unchanged.

Example 17

A packing case is kicked, giving it an initial velocity of $2\ m\cdot s^{-1}$. It slides across a rough floor and comes to a stop 1 m from its initial position. Find the coefficient of friction. (See Fig. 5-23.)

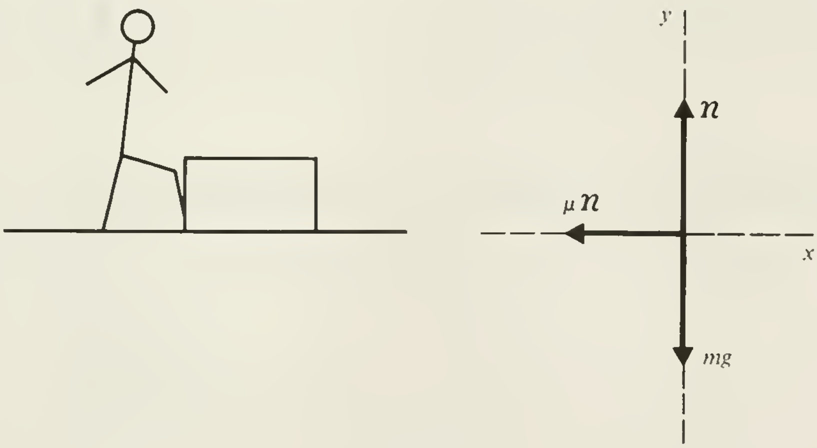


Figure 5-23

Solution:

$$\sum F_x = -\mu n = ma; \qquad a = \frac{\mu n}{m}$$

$$\sum F_y = n - mg = 0; \quad n = mg$$

The acceleration is thus $a = -\mu g$. From the previous chapter

$$v^2 = v_o^2 + 2ax;$$

With input data,

$$a = -\mu g,\ x = 1\ m,\ v = 0,\ v_o = 2\ m\cdot s^{-1},$$

we have,

$$a = \frac{-v_o^2}{2\ x} = -\ \mu g$$

$$\mu = \frac{v_o^2}{2 \cdot g} = \frac{(2\ m \cdot s^{-1})^2}{2(1\ m)(9.8\ m \cdot s^{-2})} = 0.2$$

Example 18

A body of mass 10 kg moves with constant velocity 5 $m \cdot s^{-1}$ on a horizontal surface under the influence of a horizontal force of 20 N.

(a) Find the coefficient of kinetic friction.

(b) If the horizontal force is removed (see Fig. 5-24), how long will it take to stop?

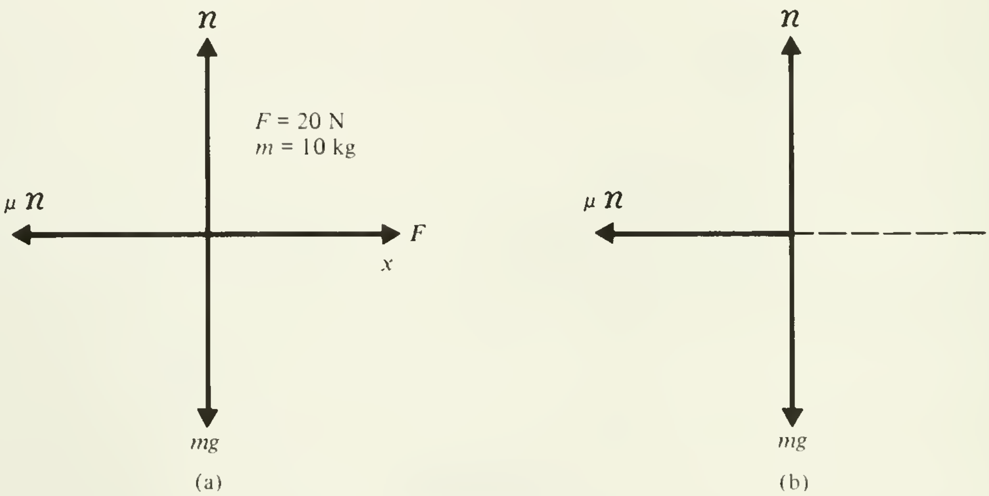


Figure 5-24

Solution:

(a) The body is in equilibrium because a = 0, and v is constant. Hence (see Fig. 5-24a)

$$\sum F_x = F - \mu n = 0$$

$$\sum F_y = n - mg = 0$$

and we have

$$\mu n = -\ \mu mg = F.$$

$$\mu = \frac{F}{mg} = \frac{20\ \text{N}}{(10\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})} = 0.20$$

(b)

$$\sum F_x = -\mu\mathcal{N} = ma; \qquad a = \frac{-\mu\mathcal{N}}{m} = -\mu\ g$$

$$v = v_o + at \qquad v = 0 \qquad v_o = 5\ \text{m}\cdot\text{s}^{-1}$$

$$t = \frac{-v_o}{a} = \frac{-v_o}{-\mu g} = \frac{5\ \text{m s}^{-1}}{0.2(9.8\ \text{m s}^{-2})} = 2.5\ \text{s}$$

Example 19

A chain of mass 10 kg and length 10 m lies on a smooth horizontal surface and is pulled by a force at one end which gives it an acceleration of 1 $\text{m}\cdot\text{s}^{-2}$. Find the tension in the chain at any point along its length.

Solution:

Referring to Fig. 5-25a, mathematically divide the chain into two segments A and B and consider the equation of motion of segment A (see Fig. 5-25b) acted upon by the pulling force F and the tension T at x in the chain.

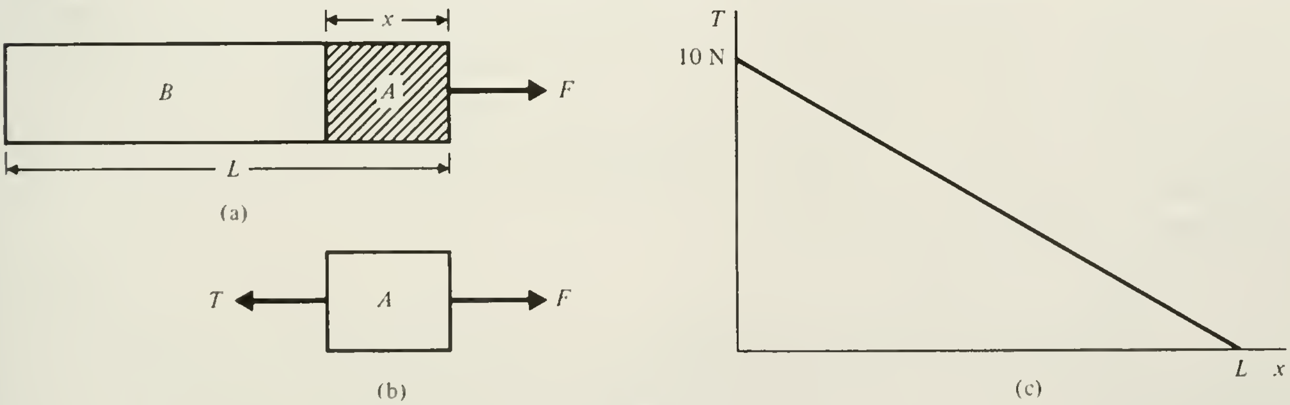


Figure 5-25

If m is the mass of the entire chain, $F = ma = (10\ \text{kg})(1\ \text{m}\cdot\text{s}^{-2}) = 10\ \text{N}$.

With the notation, $m_A = \frac{x}{L}\,m$, $L = 10$ m, and $m = 10$ kg, the equation of motion for A is:

$$F - T = m_A a \text{ or } T = F - m_A a = F - \frac{x}{L}\,ma = F\left(1 - \frac{x}{L}\right) = 10\ \text{N}\left(1 - \frac{x}{L}\right)$$

The tension is greatest at the pulling end ($x = 0$) and drops linearly to zero at the free end, as shown in Fig. 5-25c.

Example 20

Find the acceleration of the frictionless system in Fig. 5-26.

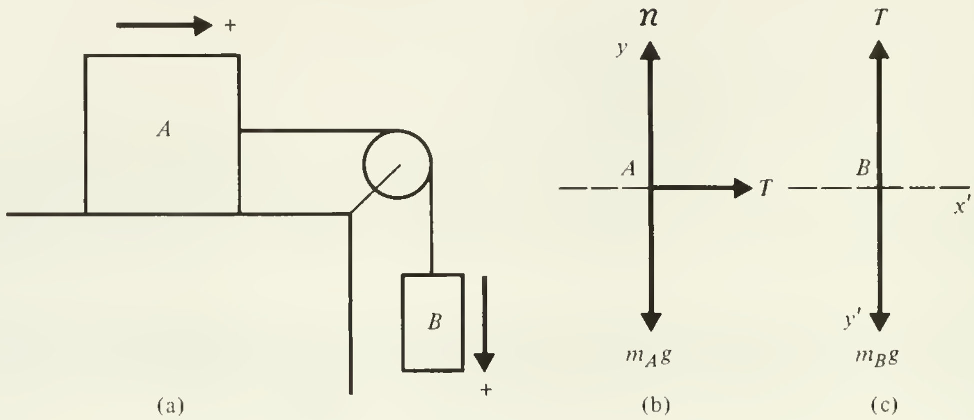


Figure 5-26

Solution:

This is a coupled system. If A moves a distance Δl in a time t, a length Δl of the rope passes over the pulley and B moves the same distance in the same time. Thus $v_A = \Delta l/\Delta t = v_B$. The signs of the velocities are the same provided, as indicated in Fig. 5-26a by the arrows, and in Fig. 5-26b, c by the positive axis orientations, an appropriate sign convention is chosen. Since the velocities v_A and v_B are always the same, the accelerations a_A and a_B are also the same, $a_A = a_B$. Since the rope is massless, the tension is uniform. The equations of motion for A are:

$$\sum F_x = T = m_A a$$

$$\sum F_y = n - m_A g = 0; \qquad n = m_A g$$

and for B is

$$\sum F_{y'} = m_B g - T = m_B a.$$

The last two equations may be solved simultaneously for a and T. (Solve one for T and substitute it into the other; this yields a. Then use this result to find T from either equation, giving,)

$$a = \frac{m_B}{m_A + m_B} g$$

$$T = \frac{m_A m_B g}{m_A + m_B} .$$

Note a is always less than g and T is always less that the weight $m_B g$. (A common error is to assume $T = m_B g$. That is only true if B is in equilibrium and it isn't.)

Example 21

Find the acceleration of the frictionless system in Fig. 5-27.

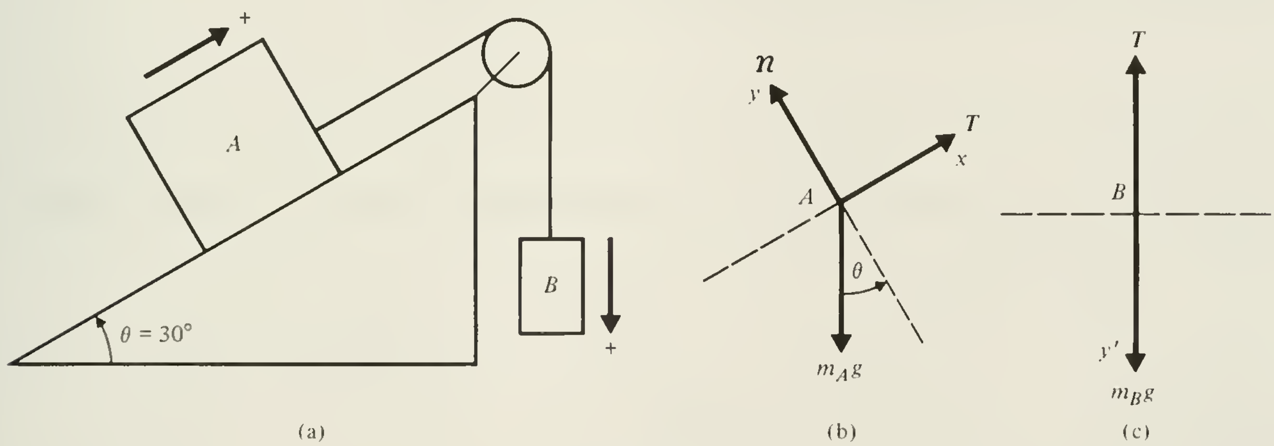


Figure 5-27

Solution:

Referring to Fig. 5-27 b and c, the equations of motion for A are:

$$\sum F_x = T - m_A g \sin\theta = m_A a$$

$$\sum F_y = n - m_A g \cos\theta = 0$$

and for B is:

$$\sum F_{y'} = m_B g - T = m_B a.$$

The first and last equation may be solved simultaneously for a and T:

$$a = \frac{m_B - m_A \sin\theta}{m_A + m_B}\, g$$

$$T = \frac{(m_A m_B g)}{m_A + m_B}\,(1 + \sin\theta)$$

Again note $T \neq m_B g$.

If $\theta = 0$, the plane degenerates into the table of the previous example, verifying the results in Example 18.

If $m_A = 5$ kg, $m_B = 2$ kg, and $\theta = 30°$, then

$$a = \frac{2\text{kg} - (5\text{kg})(1/2)}{7\text{kg}}\, g = -\ 0.7\ \text{m}\cdot\text{s}^{-2}$$

The negative sign indicates that for this mass and this angle the system accelerates with A falling down the plane and B rising.

Example 22

Find the acceleration of the Atwood's machine in Fig. 5-28 if $m_A = 2$ kg and $m_B = 5$ kg.

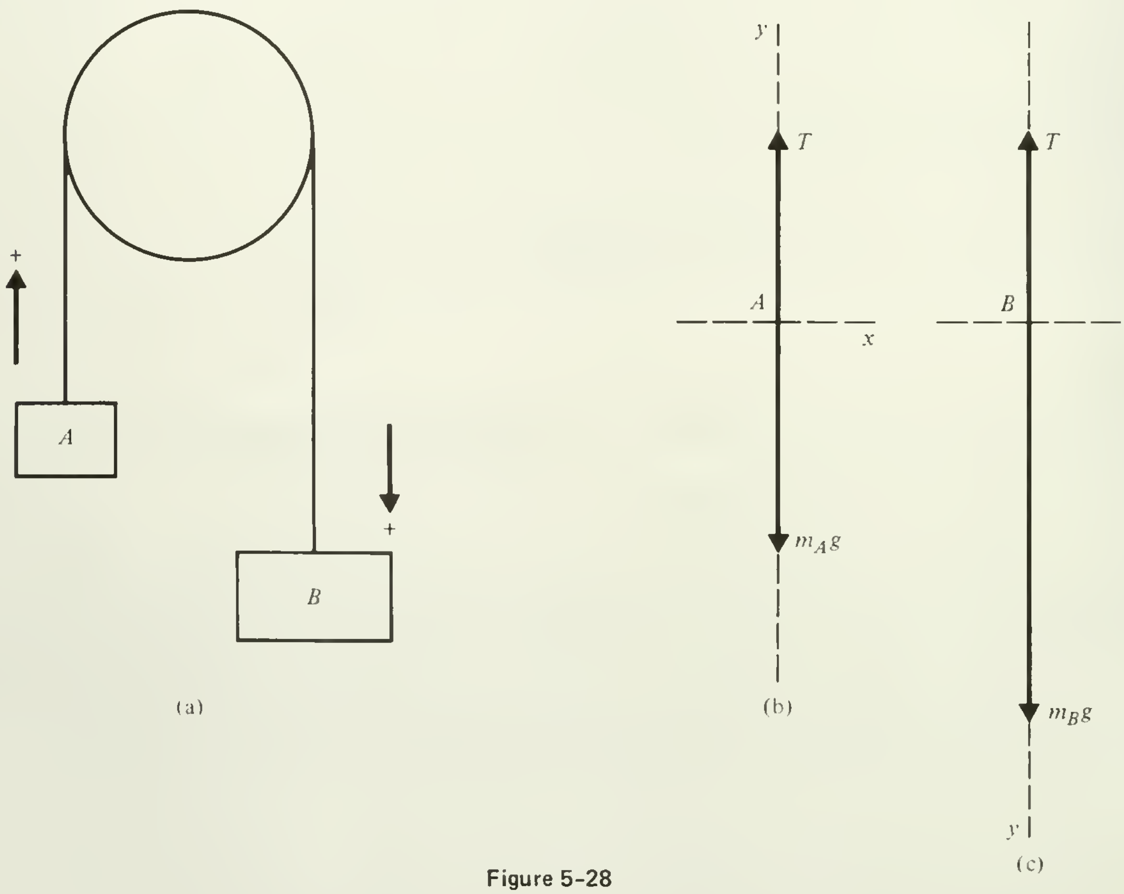


Figure 5-28

Solution:

A and B have the same accelerations if the positive directions are those indicated in Fig. 5-28b and c. The equations of motion for each mass are thus

$$T - m_A g = m_A a$$

$$m_B g - T = m_B a$$

Adding these equations eliminates T and yields,

$$a = \frac{m_B - m_A}{m_A + m_B}\, g.$$

Note this is the limit of Example 19 with $\theta = 90^\circ$.

$$a = \frac{5 - 2}{5 + 2}\,(9.8\ \mathrm{m\cdot s^{-2}}) = 4.2\ \mathrm{m\cdot s^{-2}}$$

Example 23

Find the tensions and acceleration in the frictionless system of Fig. 5-29.

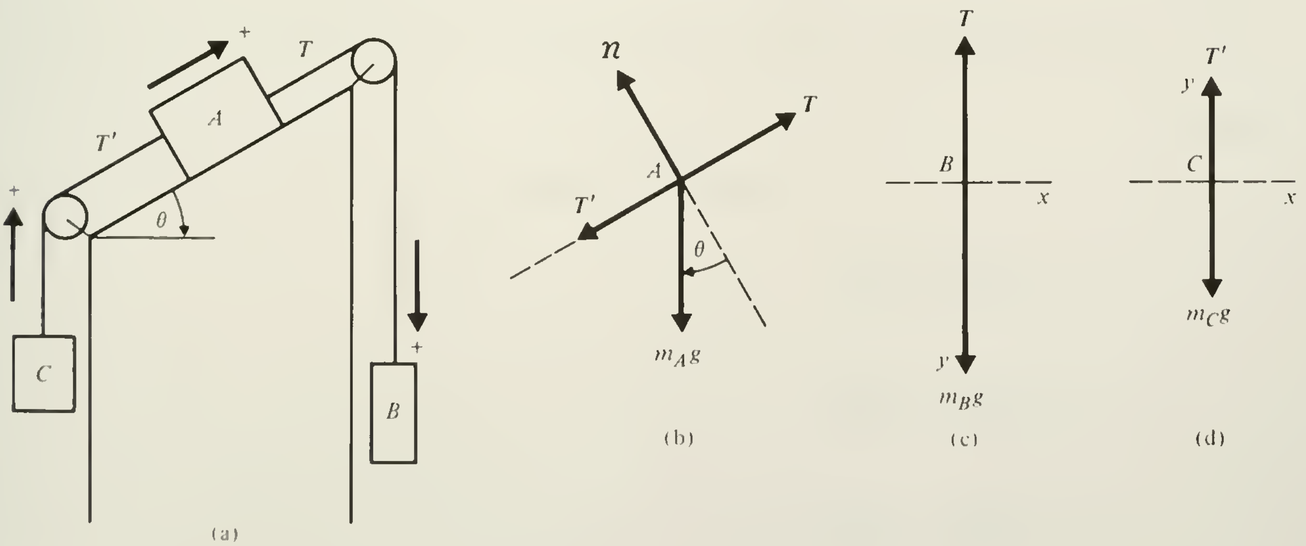


Figure 5-29

Solution:

The equations of motion are

A: $T - T' - m_A g \sin\theta = m_A a$ (Fig. 5-29b)

$$B: \quad m_B g - T = m_B a \qquad \text{(Fig. 5-29c)}$$

$$C: \quad T' - m_C g = m_C a \qquad \text{(Fig. 5-29d)}$$

Adding all three equations eliminates T and T' and yields

$$a = \frac{(m_B - m_A \sin\theta - m_C)g}{m_A + m_B + m_C} .$$

The remaining equations may be solved for T and T',

$$T' = m_C(a + g) = \frac{2m_C m_B + m_C m_A(1 - \sin\theta)}{m_A + m_B + m_C} \; g$$

$$T = m_B(g - a)$$

$$= \frac{m_A m_B(1 + \sin\theta) + 2m_C m_B}{m_A + m_B + m_C} \; g.$$

In the limit m_C approaches 0 we retrieve the results of Example 21.

Example 24

In the last example, if $\theta = 0°$ you should retrieve the results for a horizontal table. Take this limit and check your results.

Solution:

Setting $\theta = 0°$ in the above equations gives:

$$a = \frac{(m_B - m_C)g}{(m_A + m_B + m_C)}$$

$$T = \frac{(m_A m_B + 2m_C m_B)}{(m_A + m_B + m_C)} g$$

$$T' = \frac{(2m_C m_B + m_C m_A)}{(m_A + m_B + m_C)} \; g$$

Example 25

A wedge car as shown in Fig. 5-30 is accelerating to the left. The coefficient of static friction between the block and the inclined plane is μ. Find the maximum acceleration the wedge car can have before the block slides up the plane.

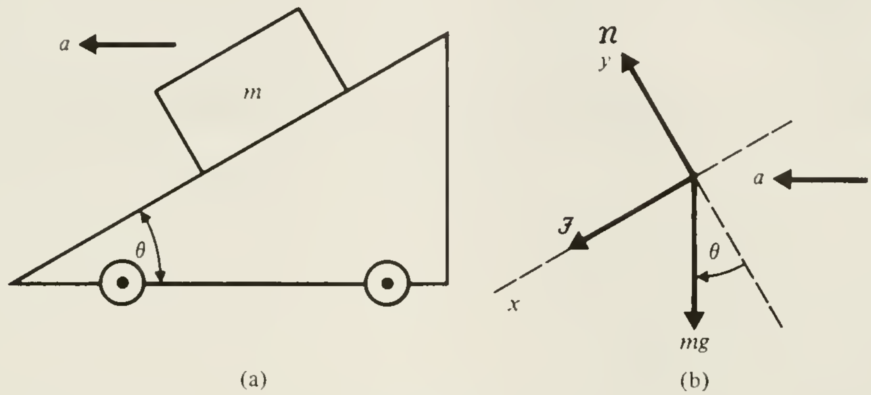


Figure 5-30

Solution:

Just before slipping up the plane the block feels a friction force pointing down the plane as shown in Fig. 5-30b. When not slipping, the block has the same acceleration as the car, resolved in the inclined system as,

$$a_x = +a \cos \theta$$

$$a_y = +a \sin \theta$$

The equations of motion are

$$\sum F_x = F + mg \sin \theta = ma_x = ma \cos \theta$$

$$F = ma \cos \theta - mg \sin \theta,$$

and

$$\sum F_y = n - mg \cos \theta = ma \sin \theta$$

$$n = ma \sin \theta + mg \cos \theta.$$

The no slip condition is $F \leq \mu n$ or

$$ma \cos \theta - mg \sin \theta \leq \mu(ma \sin \theta + mg \cos \theta)$$

$$a(\cos \theta - \quad \sin \theta) \leq g(\sin \theta + \mu\cos \theta)$$

$$a \leq g \frac{(\sin \theta + \mu \cos \theta)}{(\cos \theta - \mu \sin \theta)}.$$

Thus the acceleration is bounded so long as the denominator,

$$\cos\theta - \mu\sin\theta > 0$$

i.e., so long as

$$\mu < \frac{\cos\theta}{\sin\theta}$$

If μ is greater than this the block won't slip, no matter how great a is.

For $\theta = 45^\circ$ and $\mu < 1$,

$$a \leq g\,\frac{1+\mu}{1-\mu}\,.$$

As θ approaches 0 (flat car)

$$a \leq \mu\, g.$$

For $\mu = 0.3$, as an example, we must have $\theta \leq 73.3^\circ$ for there to be a maximum acceleration beyond which the box slips up the plane. If $\theta > 73.3^\circ$, the box never slips, no matter how great a is.

Example 26

The deflection of the accelerometer in Fig. 5-31 is 30°. Find the magnitude and direction of the cart's acceleration.

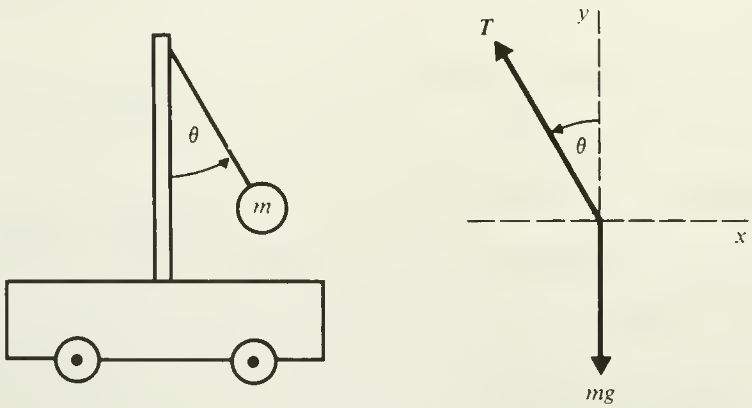


Figure 5-31

Solution:

The equations of motion are of the mass m are:

$$\sum F_x = -T \sin \theta = ma_x = ma$$

$$\sum F_y = T \cos \theta - mg = 0. \text{ } (a_y = 0 \text{ because } \theta \text{ is constant.})$$

Dividing the first by the second to eliminate T gives:

$$a = -g \tan \theta = -(9.8 \text{ m}\cdot\text{s}^{-2})(0.58) = -5.7 \text{ m}\cdot\text{s}^{-2}.$$

The acceleration is in the negative x direction.

Example 27

A rowboat of mass 200 kg, including its rower, is rowed at a constant velocity of 4 $\text{km}\cdot\text{hr}^{-1}$. When the rower stops rowing he finds that he coasts to half his original velocity in 3 m. The drag of the water, he estimates, is proportional to the square of his velocity.

$$F_d = -bv^2$$

(a) Find the coefficient b.

(b) How far will the boat coast before reducing its speed to 1/10 the original value? 1/100 the original value?

(c) Find the force necessary to propel him at the constant speed of 4 $\text{km}\cdot\text{hr}^{-1}$.

Solution:

(a) The equation of motion when the rower is coasting is

$$F_x = -bv^2 = ma = m \frac{dv}{dt} = m \frac{dv}{dx} \frac{dx}{dt}$$

$$= mv \frac{dv}{dx}$$

We then integrate the equation of motion,

$$\int_{v_0}^{v} \frac{dv}{v} = -\int_{0}^{x} \frac{b}{m} dx$$

between the initial velocity v_0 when $x = 0$ to a later velocity when the displacement from the initial position is x. The result is

$$\ln \frac{v}{v_o} = -\frac{b}{m} x \quad \text{or} \quad \ln \frac{v_o}{v} = \frac{b}{m} x$$

thus

$$b = -\frac{m}{x} \ln \frac{v}{v_o} = \frac{m}{x} \ln \frac{v_o}{v} = \frac{(200 \text{ kg}) \ln (2)}{(3 \text{ m})}$$

$$b = 46.2 \text{ kg}\cdot\text{m}^{-1}$$

(b) Thus

$$x = \frac{m}{b} \ln \frac{v_o}{b} = \frac{200 \text{ kg}}{(46.2 \text{ kg}\cdot\text{m}^{-1})} \ln \frac{v_o}{v}$$

$$= (4.33 \text{ m}) \ln \frac{v_o}{v}$$

At $v = 0.1\ v_o$, $x = (4.3 \text{ m}) \ln 10 = 9.9$ m.

At $v = 0.01\ v_o$, $x = (4.3 \text{ m}) \ln 100 = 19.8$ m.

(c) At constant velocity, the total force is zero, $F_R + F_d = 0$ where F_R is the force supplied by the rower.

$$v = 4 \text{ km}\cdot\text{hr}^{-1} = \frac{4 \times 10^3}{3600 \text{ s}} \text{ m.}$$

$$F_R = -F_d = bv^2 = (46.2 \text{ kg}\cdot\text{m}^{-1})\left(\frac{4 \times 10^3}{3600}\right)^2 \frac{\text{m}^2}{\text{s}^2}$$

$$= 57 \text{ N}$$

QUIZ

1. A horizontal rope is stretched between two buildings 20 m apart. When a man weighing 800 N hangs from the center of the rope, the center is 4 m lower that the ends. What is the tension in the rope?

Answer: 1081 N

2. Two ropes attached to the ceiling support a 100 N weight. One rope makes an angle of 30° with the ceiling and the other makes an angle of 45° with the ceiling. Find the tension in each rope.

Answer: 73 N, 90 N

3. 20 lb and 10 lb weights attached by a cord hang from a frictionless pulley, the 20 lb weight 10 ft above the ground and the 10 lb weight on the ground. The system is released from rest.

(a) Find the acceleration.
(b) Find the tension in the rope.
(c) Find the time it takes the 20 lb weight to hit the floor.
(d) Find the velocity when it hits the floor.

Answer: 10.7 $ft \cdot s^{-1}$, 13.3 lb, 1.37 s, 14.6 s

4 . A box is kicked along a rough floor, giving it an initial velocity of 10 $m \cdot s^{-1}$. It comes to rest 20 m from the place where it was kicked. What is the coeficient of friction?

Answer: 0.25

6
APPLICATIONS OF NEWTON'S LAWS—II

OBJECTIVES

In this chapter you make further applications of Newton's laws of motion. You also are introduced to Newton's law of gravitation. Your objectives are to:

Analyze the force, velocity, and acceleration of objects in circular motion.

Apply this analysis to motion in a vertical circle.

Find the gravitational force between two bodies.

Determine the weight of a body on the surface of the earth in terms of this gravitational force.

Find the gravitational field near a point mass or near simple mass distributions.

Analyze the motion of satellites in circular orbits.

Determine the effect of the rotation of the earth on an object's apparent weight.

REVIEW AND SUPPLEMENT

When a body moves in a circle, the perpendicular (or normal) component of its acceleration is given by

$$a_n = \frac{v^2}{R}$$

directed toward the center.

(This equation is also true for an arbitrary curve if R = radius of curvature.) The parallel (or tangential) component of acceleration is a_t =

dv/dt where v is the magnitude of the velocity. The equations of motion are

$$F_t = m\frac{dv}{dt}$$

$$F_n = ma_n = \frac{mv^2}{R}$$

F is called the centripetal force and v^2/R the centripetal acceleration. They both point toward the center of the motion.

If the circular motion is 'uniform', v = constant and $a_t = 0$, $F_t = 0$. The only acceleration is a_n, acting to change the direction but not the magnitude of v.

Motion in a Vertical Circle

Referring to Fig. 6-1, an object on the end of a string of tension T is undergoing motion in a vertical circle of radius R.

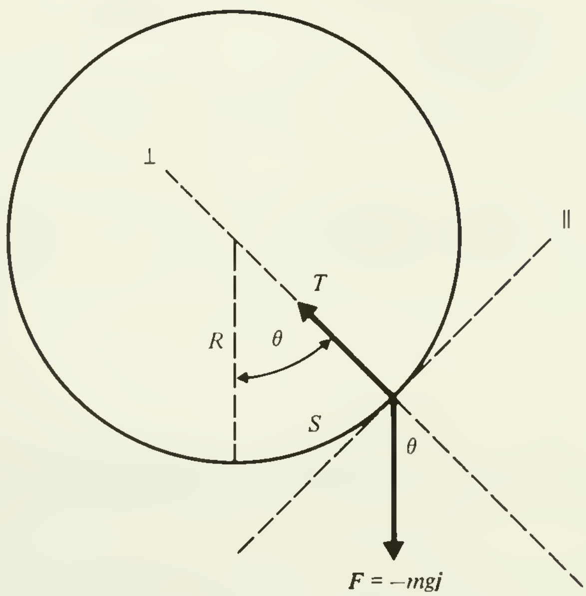


Figure 6-1

The equations of motion are

$$F_t = ma_t = m\frac{dv}{dt} = -mg\ \sin\theta$$

$$F_n = T - mg\cos\theta = \frac{mv^2}{R}$$

The tension T,

$$T = mg\cos\theta + \frac{mv^2}{R}$$

depends on the velocity v and the angle θ. Intuitively we expect the body to speed up at the bottom and slow down at the top, so that v is not a constant.

Newton's Law of Gravitation

Newton's Law of Universal Gravitation states that all small bodies attract each other according to the rule,

$$F_g = \frac{Gm_1m_2}{r^2}$$

where m_1 and m_2 are the masses of the bodies, r is their separation and G is a constant whose value is

$$G = 6.667 \times 10^{-11}\ \mathrm{N \cdot m^2 \cdot kg^{-2}}$$

in the SI. Strictly speaking the law is correct only for point masses but it is a good approximation for bodies whose dimensions are small compared to their separation. Additionally, if either of the bodies is a homogeneous sphere, or if it has spherical symmetry, then its gravitational force on the other particle is as if the sphere were a point particle.

The gravitational force of the earth (assumed to be spherically symmetric) on a body of mass m on its surface is its weight,

$$w = \frac{Gmm_E}{R^2} = mg,$$

where m_E is the mass of the earth and R is the radius of the earth. Dividing each side by the mass m gives the value of g as:

$$g = \frac{Gm_E}{R^2} = 9.8\ \mathrm{m \cdot s^{-2}} = 32\ \mathrm{ft \cdot s^{-2}}.$$

By Newton's Third Law, the body pulls on the earth with a force equal in magnitude and opposite in direction to its weight. The weight of a kilogram is

$$w(\text{of kilogram}) = (1\ \mathrm{kg})(9.8\ \mathrm{m \cdot s^{-2}}) = 9.8\ \mathrm{N}.$$

The weight of a slug is

$$w(\text{of slug}) = (1\ \text{slug})(32\ \text{ft}\cdot\text{s}^{-2}) = 32\ \text{lbs}.$$

Another unit of mass possible in the British system is the mass whose weight is one lb. Since $m = w/g$ this mass is

$$\text{lb-mass} = \frac{1\ \text{lb-force}}{32\ \text{ft}\cdot\text{s}^{-2}} = \frac{1}{32}\ \text{slug}.$$

Satellite and Planetary Motion in Circular Orbits

Referring to Fig. 6-2, the centripetal force in this case is supplied by the gravitational attraction between the two bodies. A possible motion is uniform circular motion as indicated in the figure.

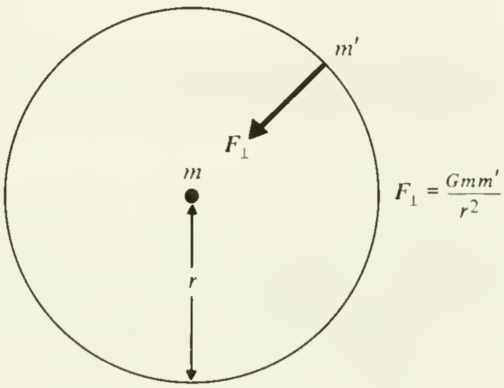


Figure 6-2

The equations of motion are

$$F_t = m'a_t = 0 = m\,\frac{dv}{dt}$$

$$F_n = m'\,\frac{v^2}{r} = \frac{Gmm'}{r^2}$$

where m' is the satellite mass and m the earth's mass. The first indicates that v is constant and the second that

$$v^2 = \frac{Gm}{r}$$

The period τ of the satellite is the time for a complete revolution $2\pi r$. Thus

$$v = \frac{2\pi r}{\tau} \qquad \tau = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{(Gm)^{1/2}}$$

If m is the earth, $m = m_E$, it is convenient to evaluate the last denominator as

$$g = \frac{Gm_E}{R^2} , \qquad Gm_E = R^2 g$$

where R is the radius of the earth, resulting in

$$\tau = \frac{2\pi r^{3/2}}{[gR^2]^{1/2}}$$

EXAMPLES AND SOLUTIONS

Example 1

A 4000 kg locomotive 'coasts' at a constant speed of 100 $km \cdot hr^{-1}$ about a circular section of track of radius 800 m.

(a) Find the force of the locomotive against the tracks.

(b) How does this force compare with the weight of the locomotive?

Solution:

(a) The force of the locomotive against the tracks is equal but opposite to the centripetal force of the tracks against the locomotive, of magnitude

$$F = ma = \frac{mv^2}{R}$$

where v is the speed of the locomotive and R the radius of the track:

$$F = 4000 \text{ kg} \frac{(100 \times 10^3 \text{ m}/3600 \text{ s})^2}{800 \text{ m}}$$

$$= 3900 \text{ N}$$

(b) The weight of the locomotive is

$$w = mg = 4000 \text{ kg}(9.8 \text{ m}\cdot\text{s}^{-2}) = 39{,}000 \text{ N},$$

ten times the centripetal force.

Example 2

The Unbanked Curve: A car is driven in a circle of radius R = 500 m on a horizontal track with coefficient of static friction $\mu = 0.5$ between the tires and the track. Find the maximum speed the car can have before skidding off the track.

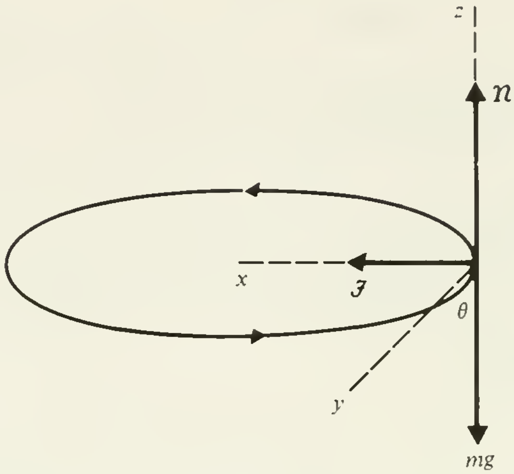


Figure 6-3

Solution:

Referring to Fig. 6-3, the equations of motion are

$$F_x = F_n = ma_n = \frac{mv^2}{R} = \mathcal{F}_s$$

$$F_y = F_t = ma_t = 0$$

$$F_z = ma_z = 0 = \mathcal{N} - mg; \quad \mathcal{N} = mg$$

where $\mathcal{F}_s$ is the static force of friction of the track on the tires of the car. The static friction condition $\mathcal{F}_s \leq \mu\mathcal{N}$ is here

$$\frac{mv^2}{R} \leq \mu mg$$

or

$$v \leq (\mu Rg)^{1/2}$$

If v exceeds this limit the lateral static friction force of the road against the tires is not strong enough to supply the centripetal acceleration, and the car skids out of the circle.

Example 3

The Banked Curve: To avoid the previous mishap, curves are banked so that a component of the normal force helps in supplying the centripetal force. Find the maximum velocity before skidding for a track of radius R banked at an angle θ.

Solution:

Referring to Fig. 6-4, we bank the track, viewing the car end-on.

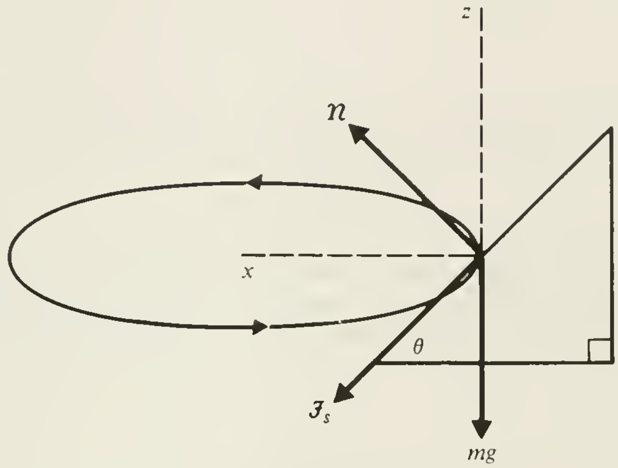


Figure 6-4

The equations of motion in the xz plane are

$$F_x = F_n = \frac{mv^2}{R} = \mathcal{n} \sin\theta + \mathcal{F}_s \cos\theta$$

$$F_z = \mathcal{n} \cos\theta - \mathcal{F}_s \sin\theta - mg = 0$$

Note the x component is the centripetal component in this case. We have assumed the velocity is large enough that $\mathcal{F}_s$ points down the plane. These equations may be solved for $\mathcal{n}$ and $\mathcal{F}_s$,

$$\mathcal{n} = \frac{mv^2}{R} \sin\theta + mg\cos\theta$$

$$\mathcal{F}_s = \frac{mv^2}{R} \cos\theta - mg\sin\theta$$

The condition that $\mathcal{F}_s \leq \mu_s \mathcal{n}$ then implies

$$v^2 \leq Rg \ \frac{(\mu \cos\theta + \sin\theta)}{(\cos\theta - \mu \sin\theta)}$$

Even when there is no friction at all ($\mu = 0$) a judicious banking angle will provide the right centripetal acceleration for a given velocity,

$$\frac{mv^2}{R} = \mathcal{n} \sin \theta$$

$$\mathcal{n} \cos \theta - mg = 0$$

implying

$$v^2 = Rg \tan \theta$$

Example 4

A plastic bottle of negligible mass contains one liter of water. The bottle is tied to a string of length 0.8 m and swung in a verticle circle, reaching a speed of 2 $m \cdot s^{-1}$ at the bottom of its swing. Find the tension in the string at the bottom of the swing.

Solution:

The total force on the bottle at the bottom of the swing is the upward tension minus the downward weight,

$$F = T - mg \qquad \text{direction: up}$$

At the bottom of the swing the acceleration of the pail is centripetal (directed upward, toward the center of the circle) and of magnitude v^2/R. By Newton's second law we have

$$T - mg = \frac{mv^2}{R}$$

or

$$T = mg + \frac{mv^2}{R}$$

$$= (1 \text{ kg})(9.8 \text{ m}\cdot\text{s}^{-2}) + \frac{(1 \text{ kg})(2 \text{ m}\cdot\text{s}^{-1})^2}{0.8 \text{ m}}$$

$$= 14.8 \text{ N}$$

Example 5

The apparent weight w' of an object is the upward force exerted by its supporting surface. A passenger in a Ferris wheel rotates at 3 $m \cdot s^{-1}$ in a vertical circle of radius 5 m. Find the ratio of his apparent weight to his true weight at the top and the bottom of the circle.

Solution:

Referring to Fig. 6-5,

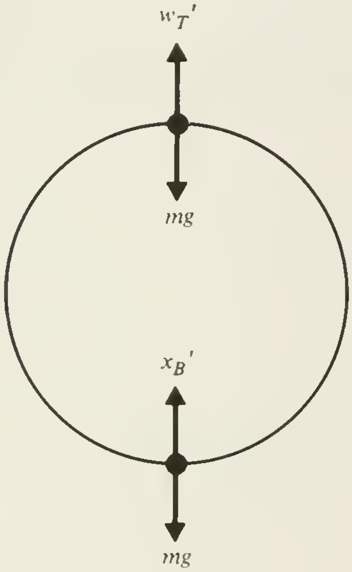


Figure 6-5

we write the equations of motion as

$$\text{Top:} \quad -w_T' + mg = F_n = \frac{mv^2}{R}$$

$$\text{Bottom:} \quad w_B' - mg = F_n = \frac{mv^2}{R}$$

yielding

$$w_T' = mg - \frac{mv^2}{R}$$

$$\frac{w_T'}{w} = 1 - \frac{v^2}{gR} = 1 - \frac{(3\ m\cdot s^{-1})^2}{(9.8\ m\cdot s^{-2})(5\ m)} = 0.82$$

$$w_{B'} = mg + \frac{mv^2}{R}$$

$$\frac{w_{B'}}{w} = 1 + \frac{v^2}{gR} = 1.18$$

Example 6

A pilot pulls a plane out of a dive by flying in an arc of a vertical circle of radius 1 km at a speed of 500 km hr^{-1}. What is his apparent weight at the bottom of the circle?

Solution:

Referring to Fig. 6-6,

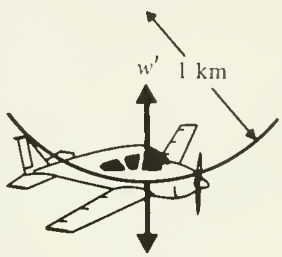


Figure 6-6

we write the radial equation of motion as

$$F_n = w' - mg = \frac{mv^2}{R} \qquad w' = mg + \frac{mv^2}{R}$$

$$\frac{w'}{w} = \frac{w'}{mg} = 1 + \frac{v^2}{gR}$$

$$= 1 + \left(\frac{500 \times 10^3 \text{ m}}{3600 \text{ s}}\right)^2 \frac{1}{(9.8 \text{ m}\cdot\text{s}^{-2})(10^3 \text{ m})}$$

$$= 3.0$$

The pilot is said to be experiencing 'forces' of 2g, that is, his apparent weight is as if the acceleration of gravity had been increased by 2g from g to 3g.

Example 7

A 100 kg man stands on the outer edge of a carousel of 4 m radius. The coefficient of friction of his shoes against the floor is $\mu = 0.3$. How small a period of rotation may the carousel have before the man slips off?

Solution:

Referring to Fig. 6-7,

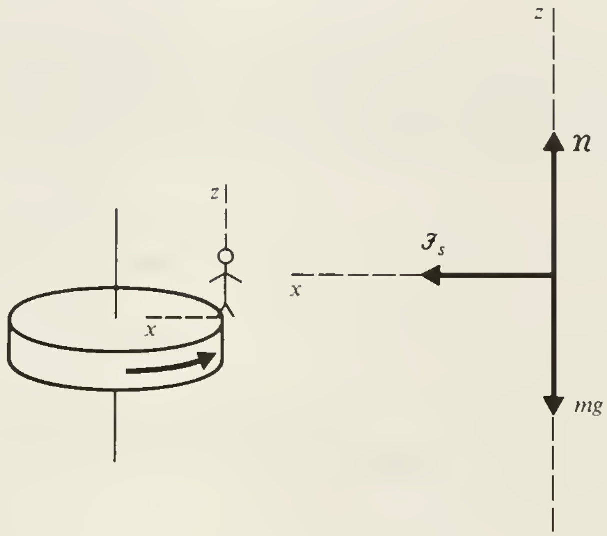


Figure 6-7

we write the equations of motion as

$$F_x = F = \frac{mv^2}{R} = \mathcal{F}$$

$$F_z = n - mg = 0$$

The static friction condition is

$$\mathcal{F}_s \le \mu_s n$$

$$\frac{mv^2}{R} \le \mu_s mg \qquad \text{or} \qquad v^2 < \mu_s gR$$

$$v \le (\mu_s gR)^{1/2} = (0.3 \cdot 9.8\ \text{m} \cdot \text{s}^{-2} \cdot 4\ \text{m})^{1/2} = 3.4\ \text{m} \cdot \text{s}^{-1}$$

The period is $T = 2\pi R/v$ and thus

$$T \geq \frac{2\pi \cdot 4\ \mathrm{m}}{3.4\ \mathrm{m \cdot s^{-1}}} = 7.4\ \mathrm{s}.$$

Example 8

In the previous problem the carousel speeds up and the man grabs a post. His maximum gripping strength for any reasonable length of time is twice his weight. What is the maximum velocity before he falls off? What is the period and frequency of rotation at this point?

Solution:

The gripping force $f \lesssim 2\ mg$ so the radial equation of motion yields

$$\frac{mv^2}{R} = f \lesssim 2mg$$

$$v^2 < 2Rg; \quad v < (2 \cdot 4\ \mathrm{m} \cdot 9.8\ \mathrm{m \cdot s^{-2}})^{1/2} = 8.8\ \mathrm{m \cdot s^{-1}}$$

The period at the critical point is

$$T = \frac{2\pi R}{v} = \frac{2\pi (4\ \mathrm{m})}{8.8\ \mathrm{m \cdot s^{-1}}} = 2.9\ \mathrm{s}$$

and the frequency of rotation is

$$f = \frac{1}{T} = 0.35\ \mathrm{s^{-1}}.$$

Example 9

What force does a mass of 50 kg exert on the earth?

Solution:

By Newton's third law, the force of the mass on the earth is equal and opposite to the force of the earth on the mass:

$$F = mg = (50\ \mathrm{kg})(9.8\ \mathrm{m \cdot s^{-2}}) = 490\ \mathrm{N} \quad \text{(upward)}$$

Example 10

A body weighs 20 N on the surface of the earth. What does it weigh on the surface of the moon?

Solution:

The weight of a body of mass m on the earth is equal to the force of attraction of the earth on the body

$$w = \frac{Gm_E m}{R_E^2}$$

where m_E and R_E are the earth's mass and radius respectively. The weight, w', of the body on the moon (m_M and R_M are the moon's mass and radius) is

$$w' = \frac{Gm_M m}{R_M^2}$$

In forming the ratio of the two weights, G cancels out:

$$\frac{w'}{w} = \frac{m_M}{m_E} \left[\frac{R_E}{R_M} \right]^2 = \frac{7.36 \times 10^{22} \text{ kg}}{5.95 \times 10^{24} \text{ kg}} \left[\frac{6.37 \times 10^6 \text{ m}}{1.74 \times 10^6 \text{ m}} \right]^2$$

$$= 0.166$$

$$w' = (0.166)(20 \text{ N}) = 3.32 \text{ N}$$

Example 11

The sun is $R = 1.48 \times 10^{11}$ m from the earth. Its mass is 1.99×10^{30} kg = 330,000 m_E. Find the ratio of the sun's gravitational force to the earth's gravitational force <u>for an object on the earth's surface</u>.

Solution:

$$F_s = \frac{Gmm_s}{R^2} = \text{force of sun on m}$$

$$F_E = \frac{Gmm_E}{R_E^2} = \text{force of earth on m}$$

$$\frac{F_s}{F_E} = \frac{m_s}{m_E} \frac{R_E^2}{R^2} = 330{,}000 \left[\frac{6.37 \times 10^6 \text{ m}}{1.49 \times 10^{11} \text{ m}} \right]^2 = 6.03 \times 10^4$$

Example 12

Three 1000 kg masses are at the vertices of an equilateral triangle r = 1 m on its side. Find the force of any two of the masses on the third. (See Fig. 6-8.)

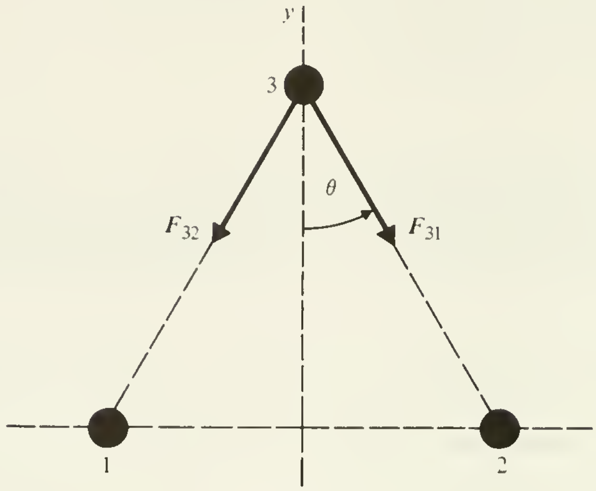


Figure 6-8

Solution:

$$\mathbf{F} = \mathbf{F}_{31} + \mathbf{F}_{32} \qquad |F_{31}| = |F_{32}| = \frac{GM^2}{R^2}$$

$$F_x = F_{31,x} + F_{32,x} = |F_{31}| \sin\theta - |F_{32}| \sin\theta = 0$$

$$F_y = F_{31,y} + F_{32,y} = |F_{31}| \cos\theta - |F_{32}| \cos\theta = 0$$

$$= - \frac{2GM^2 \cos\theta}{R^2}$$

$$= - 2(6.67 \times 10^{-11}\ \mathrm{N\ m^2})(\mathrm{kg}^{-2})\ \frac{(10^3\ \mathrm{kg})^2(0.87)}{(1\ \mathrm{m})^2}$$

$$= - 1.16 \times 10^{-4}\ \mathrm{N}$$

Example 13

A mass attached to a vertical post by two strings, rotates in a circle at constant speed v. (See Fig. 6-9.) At high enough speeds both strings are taut but below a critical velocity the lower string slackens. The mass is 1 kg and the radius is 1.5 m. Find the critical velocity and the tension in the upper string at the critical velocity.

Solution:

Referring to Fig. 6-9,

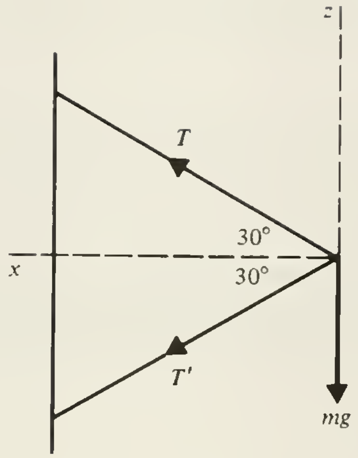


Figure 6-9

the equations of motion are

$$F_x = F_n = \frac{mv^2}{R} = (T + T')\cos 30^\circ$$

$$F_z = (T - T')\sin 30^\circ - mg = 0.$$

When v^2 is known these are two equations which can be solved for T and T':

$$T = \frac{mv^2}{2R\cos 30^\circ} + \frac{mg}{2\sin 30^\circ}$$

$$T' = \frac{mv^2}{2R\cos 30^\circ} - \frac{mg}{2\sin 30^\circ}$$

If the strings were rods, capable of delivering both compressive ($T, T' < 0$) and tensile forces ($T, T' > 0$) then any velocity is possible. A string however must have $T > 0$, $T' > 0$ or it slackens. Then we require

$$T' > 0 \text{ or } v^2 > gR(\tan\theta)^{-1}$$

$$v > \left(\frac{9.8\ \text{m}\cdot\text{s}^{-2} \times 1.5\ \text{m}}{0.58}\right)^{1/2} = 5\ \text{m}\cdot\text{s}^{-1}$$

At the critical velocity $T' = 0$, $v^2 = gR(\tan\theta)^{-1}$, and

$$T = \frac{m}{2R\cos 30^\circ}\frac{gR}{\tan 30^\circ} + \frac{m\,g}{2\sin 30^\circ}$$

$$= \frac{mg}{\sin 30^\circ} = \frac{(1\ \mathrm{kg})(9.8\ \mathrm{m\cdot s^{-2}})}{0.5} = 20\ \mathrm{N}.$$

Example 14

Communications satellites revolve in orbits over the earth's equator adjusted so that their period of rotation is the same as the period of rotation of the earth about its axis. An observer rotating with the earth thus sees the satellite fixed in the sky. Find the distance of these satellites above the surface of the earth.

Solution:

Let

r = distance of satellite from the center of earth;
R = radius of earth, 6.38×10^6 m; and
T = 1 day = (24)(60)(60 s).

Then the period of the satellite is:

$$T = \frac{2\pi r^{3/2}}{[gR^2]^{1/2}}$$

Solving for r,

$$r = \left[\frac{T(g)^{1/2}R}{2\pi}\right]^{2/3}$$

$$= \left[\frac{(24)(3600\ \mathrm{s})(9.8\ \mathrm{m\cdot s^{-2}})^{1/2}\,(6.38 \times 10^6\ \mathrm{m})}{2\pi}\right]^{2/3}$$

$$= 4.23 \times 10^7\ \mathrm{m} \simeq 7 \text{ earth radii}$$

Distance above earth $= r - R = 4.23 \times 10^7\ \mathrm{m} - 6.38 \times 10^6\ \mathrm{m}$

$= 3.6 \times 10^7$ m

$= 36{,}000$ km

QUIZ

1. A turntable rotates at 45 $rev \cdot min^{-1}$. A coin placed on the turntable will revolve without slipping provided it is less than 2.25 inches from the center. Find the coefficient of friction.

Answer: 0.13

2. A 60 kg person stands on a bathroom scale on a rotating Ferris wheel of radius 5 m. How many revolutions per minute is the Ferris wheel making if the person's apparent 'weight' is 65 kg at the bottom of the rotation?

Answer: 3.8 rpm

3. What would be the period of a satellite in circular orbit just above the earth's surface if air resistance could be neglected?

Answer: 1.4 hours

7
WORK AND ENERGY

OBJECTIVES

In this chapter Newton's laws are used to develop the ideas of work and power, kinetic and potential energy, and the conservation of mechanical energy. This will provide you with a powerful tool for analysing the motion ot mechanical systems. Your objectives are to:

Calculate the <u>work</u> done by constant and varying forces.

Define and calculate the change in <u>kinetic energy</u> of a body, given the work done by the total force acting on the body.

Identify and distinguish between <u>conservative</u> and <u>dissipative</u> forces.

Define the <u>potential energy</u> of a body acted upon by conservative forces. Calculate potential energy for the force of gravity and the elastic force of a spring.

Formulate the <u>principle of conservation of mechanical energy</u> and apply it to a variety of conservative systems.

Discuss the <u>qualitative motion</u> of a system, given its potential energy function.

Define and calculate the <u>power</u> associated with motion under the influence of force.

REVIEW AND SUPPLEMENT

<u>Work and Kinetic Energy</u>

The work done by a constant force $\mathbf{F}$ acting on a body which moves in a straight line through a displacement $\mathbf{s}$ is

$$W = \mathbf{F}\cdot\mathbf{s} = Fs\cos\theta$$

where θ is the direction between F and s as shown in Fig. 7-1a. The SI units of work are N·m = joules; the British units are ft-lbs = 1.356J. Work is a <u>scalar</u>

quantity, and can be positive or negative.

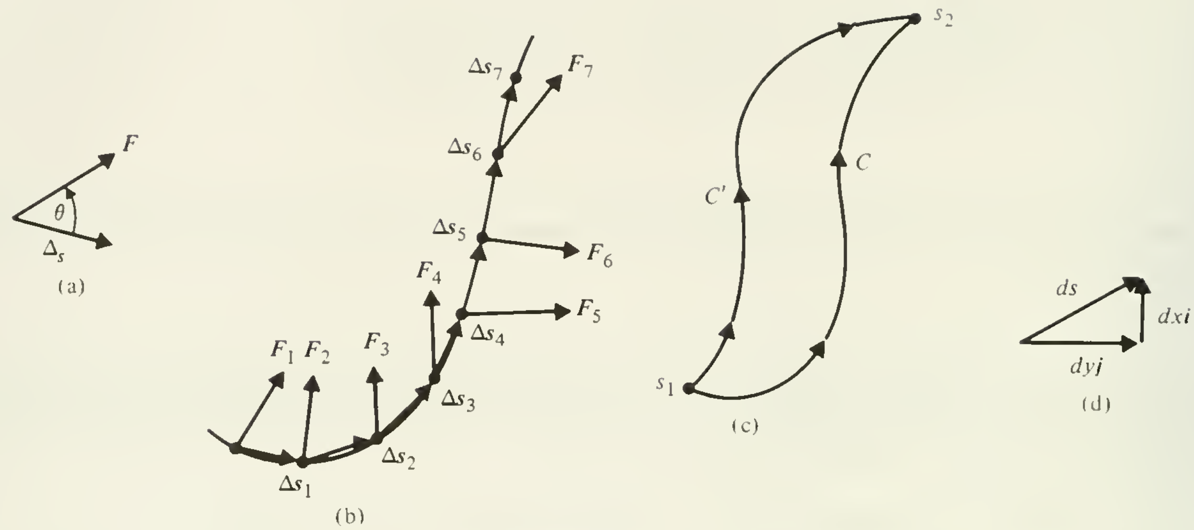


Figure 7-1

The work is positive if the force acts in the direction of displacement, $\theta < 90^\circ$, and negative if $90^\circ < \theta < 180^\circ$. The work is zero if **F** and **s** are perpendicular. If **F** is the total force on the body, then W is the total work done <u>on the body</u> by **F**. If $\mathbf{F} = \mathbf{F}_A + \mathbf{F}_B$, $W = \mathbf{F}_A \cdot \mathbf{s} + \mathbf{F}_B \cdot \mathbf{s} = W_A + W_B$; the total work is the sum of the partial work due to $\mathbf{F}_A$ and $\mathbf{F}_B$.

If the force varies or the displacement changes direction, then the displacement curve must be divided into small increments $\Delta \mathbf{s}_i$ as shown in Fig. 7-1b. In each increment **F** does not vary significantly if the increments are small enough, and the work within that displacement increment is well approximated by $\mathbf{F}_i \cdot \Delta \mathbf{s}$. The total work is the sum of all such increments

$$W = \sum_i \mathbf{F}_i \cdot \mathbf{s}_i$$

The limit of this sum as $|\Delta \mathbf{s}_i|$ approaches 0 defines the work in terms of a <u>line integral</u>,

$$W = \text{Lim}_{\Delta s_i \to 0} \sum \mathbf{F}_i \cdot \Delta \mathbf{s}_i = \int_C \mathbf{F} \cdot d\mathbf{s}$$

$$= \int_C F \cos \theta \, ds$$

The work in general depends on the path C connecting the end points of the motion; its value may be different for the path C′ in Fig. 7-1c.

The <u>kinetic energy</u> K of a body with mass m moving with velocity v is defined as

$$K = \frac{1}{2} mv^2.$$

The total work W done <u>on a body</u> is equal to the increase in kinetic energy of the body,

$$W = \Delta K = \frac{1}{2} mv_2{}^2 - \frac{1}{2} mv_1{}^2 = K_2 - K_1$$

where v_1 is the initial and v_2 the final velocity.

A force acting in the direction of displacement does positive work and hence increases the kinetic energy: an example is a ball falling in a gravitational field. Here the force of gravity and the displacement are in the same direction. The work is positive. The kinetic energy increases.

A force acting in the direction opposite to the displacement does negative work and hence decreases the kinetic energy: an example is a ball rising in a gravitational field. The work is negative. The kinetic energy decreases.

<u>Conservative and Dissipative Forces; Conservation of Mechanical Energy</u>

If the work done by a force on a body moving between two points does not depend on the path taken, the force is <u>conservative</u>; otherwise it is <u>dissipative</u>. When only conservative forces act, positive and negative parts of the work about a closed path cancel.

A familiar dissipative force is the force of sliding friction. To displace a box a distance s to the right one can take a straight path with work of friction $W_F = -\mathcal{F}_s s$, where the work is negative because the force is opposite to the displacement (See Fig. 7-2a); or one can displace the box a distance 2s to the right and then a distance s to the left arriving at the same endpoint. The work for the second displacement is

$$W_F' = -2\mathcal{F}_s s - \mathcal{F}_s s = -3\mathcal{F}_s s.$$

(See Fig. 7-2b) The work depends on the path.

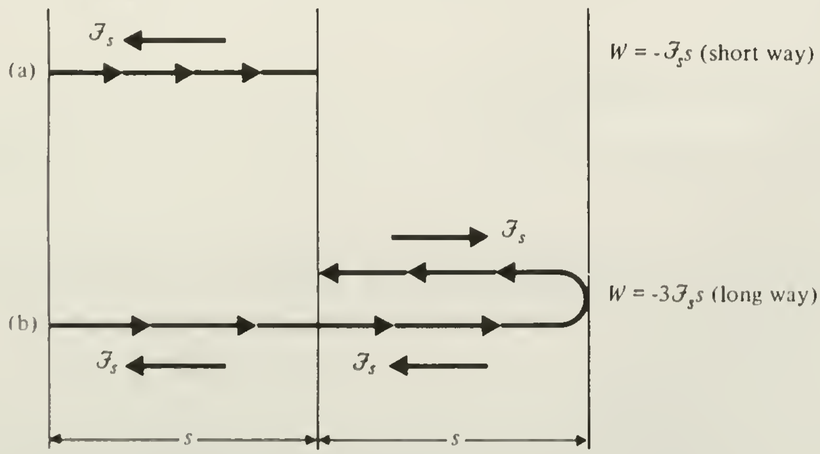


Figure 7-2

The gravitational force is a conservative force. To show this refer to Fig. 7-3 and consider the curved path connecting two points (x_1, y_1) and (x_2, y_2) in a constant gravitational field.

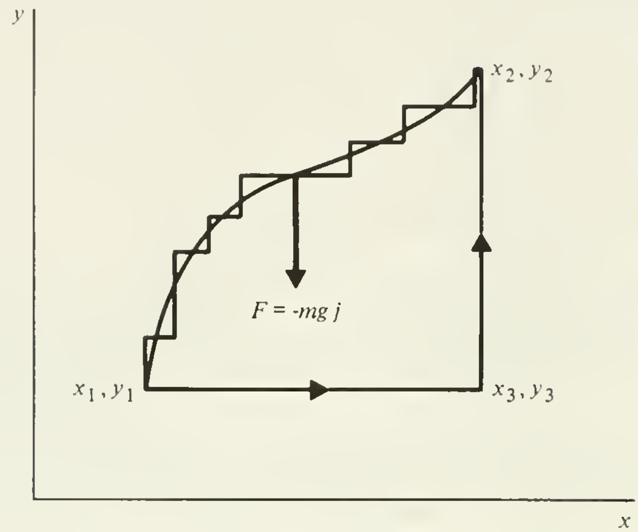


Figure 7-3

The path has been approximated by small horizontal and vertical staircase increments. The work of the weight is zero along the horizontal increments; no work is done in the horizontal displacements because

$$\mathbf{F} \cdot \mathbf{s} = 0$$

here. Along a vertical segment the work is $\Delta W_i = -F\Delta y_i = -mg\Delta y_i$, negative because the displacement and **F** are <u>anti</u>parallel. The total gravitational work is the sum of all these vertical increments

$$W_g = \sum_i (-mg\Delta y_i) = -mg\sum \Delta y_i = -mg(y_2 - y_1)$$

Any other path connecting the two end points, e.g. the 1 to 3 to 2 path of Fig. 7-3 has the same total ascent Δy and hence the same gravitational work. If the gravitational potential energy is defined as

$$U(y) = mgy$$

we find that the gravitational work is

$$W_g = mgy_1 - mgy_2 = U_1 - U_2$$

$$= -\Delta U$$

The sum K + U is the <u>total mechanical energy</u> E for this system, and changes according to the rule

$$W' = E_2 - E_1 = \Delta E$$

where W' is the work of all forces other than gravity. If W' is zero, the total mechanical energy is constant and is said to be conserved,

$$W' = 0 = E_1 - E_2; \quad E_1 = E_2;$$

$$\frac{1}{2} mv_1^2 + mgy_1 = \frac{1}{2} mv_2^2 + mgy_2$$

Elastic Forces

Any force which is conservative has a potential energy function. Its work can be grouped with the kinetic energy term, in an analogous way to the gravitational force considered in the last section. For example, consider the elastic force of a spring on a mass as shown in Fig. 7-4.

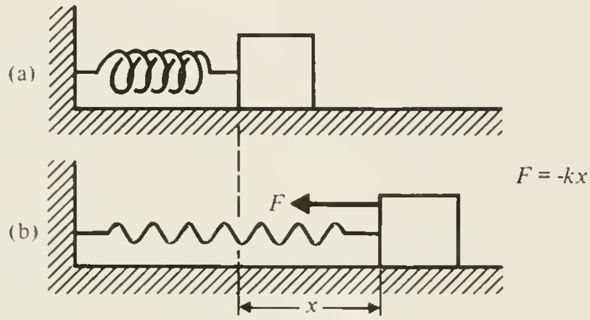


Figure 7-4

If the mass is displaced from x_1 to x_2, the work done by the elastic force is

$$W_e = \int_{x_1}^{x_2} \mathbf{F} \cdot d\mathbf{s} = \int_{x_1}^{x_2} F_x dx = -k \int_{x_1}^{x_2} x dx$$

$$= - \left. \frac{kx^2}{2} \right|_{x_1}^{x_2} = \frac{kx_1^2}{2} - \frac{kx_2^2}{2} = U_1 - U_2$$

Here the elastic potential energy has been defined as

$$U = \frac{kx^2}{2}$$

For this system the mechanical energy is conserved

$$E = K + U = \frac{1}{2} mv^2 + \frac{1}{2} kx^2,$$

provided the work of all the force, W', is zero.

Potential Energy and the Qualitative Description of Motion When Mechanical Energy is Conserved

In Fig. 7-5 the potential function U(x) is sketched for a mass m on a smooth horizontal surface attached to a wall by a spring of force constant k.

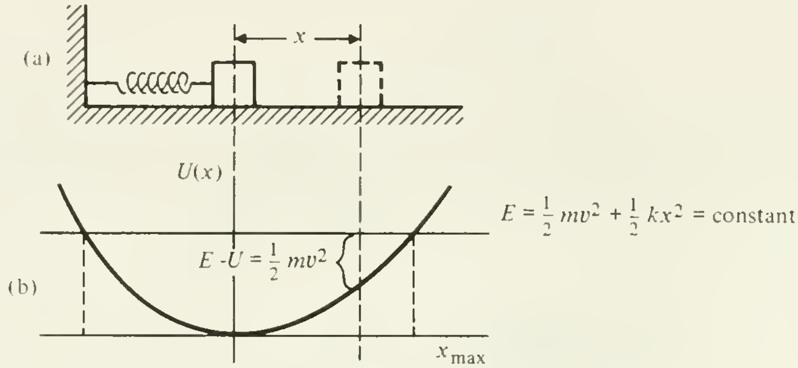


Figure 7-5

All other forces acting on the mass (e.g., gravity and the normal forces) do no work because they are perpendicular to the displacement, so W' = 0 and the mechanical energy E is conserved. This constant energy is indicated by the straight line E = constant in Fig. 7-5b. The difference of E and U is the kinetic energy

$$\frac{1}{2} mv^2,$$

also indicated in the figure. Since

$$K = \frac{1}{2} mv^2 = E - U > 0$$

motion is only allowed between the extreme coordinates $\pm x_{max}$ where K = 0 and

$$E = U(x_{max}) = \frac{1}{2} k(x_{max})^2$$

The maximum velocity and the maximum kinetic energy occur when U = 0 at x = 0, where

$$E = K\ (v_{max}) = \frac{1}{2} m(v_{max})^2$$

The total energy E is entirely potential energy at the endpoints and entirely kinetic energy at the equilibrium position x = 0. At other points the energy is split between the potential and kinetic parts, summing to the constant E.

If the spring is initially released from rest at x_{max}, the mass recoils toward $x = 0$, gaining maximum velocity at $x = 0$. As it overshoots $x = 0$, the spring pulls it back, decreasing its speed until it is zero again at $x = -x_{max}$, whereupon it reverses its direction, speeding up toward the origin, then slowing down as it approaches the initial point. The potential energy curve provides a quick qualitative description of the motion, its limits, and the velocity at each point x, if E is known.

The value of the constant E is determined by how the mass is set into motion, that is how much energy it is initially given. It may be evaluated at any point where x and v are both known.

Non-Uniform-Gravitational Field

When the displacement of a body in the earth's field is not small compared to the earth's radius, the field may not be approximated as constant; in calculating the work of the gravitational force the exact expression must be used,

$$F_g = G \frac{mm_E}{r^2}$$

This force, like that in a constant gravitational field, is also conservative, that is, path independent, as shown in Fig. 7-6;

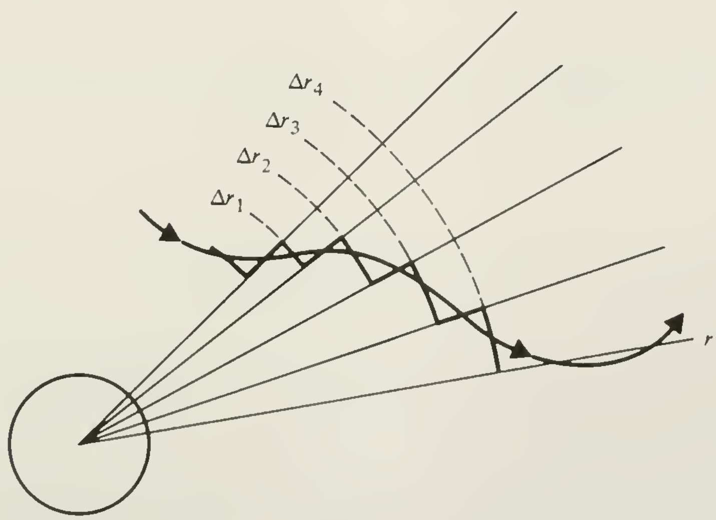


Figure 7-6

the path is divided into increments of a ray Δr_i and increments of a circular arc at constant radius. Since the force is radial the work along the arc is zero because **F** and Δ**s** are perpendicular. Thus the total gravitational work is

$$W_g \simeq \sum_i (-F_{g_i} \Delta r_i)$$

In the limit when all increments become very small

$$W_g = \mathrm{Lim}_{\Delta r_i \to o} \sum_i (- F_{gi} \Delta r_i)$$

$$= \int_{r_1}^{r_2} - \frac{Gmm_E}{r^2} dr$$

$$= - Gmm_E \left(- \frac{1}{r}\right) \Big|_{r_1}^{r_2} = Gmm_E \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

$$= - U_2 + U_1$$

where

$$U = -G \frac{mm_E}{r}$$

The mechanical energy for a particle of mass in the field of the earth is thus

$$E = K + U = \frac{1}{2} mv^2 - G \frac{mm_E}{r}$$

This energy is conserved if the work of all other forces is zero.

<u>Power</u>

The average power $\bar{P}$ of a force is defined as

$$\bar{P} = \frac{\Delta W}{\Delta t}$$

where ΔW is work done by the force in an interval Δt.

Instantaneous power is

$$P = \mathrm{Lim}_{\Delta t \to o} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

In SI, the units of P are joule$\cdot s^{-1}$ = watts. ($J \cdot s^{-1} = W$). In the British system the units are $ft \cdot lb \cdot s^{-1}$. A horsepower (hp) is

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb} \cdot s^{-1} = 746 \text{ W}$$

Another useful form is given by

$$\bar{P} = \frac{\Delta W}{\Delta t} = F(\cos\theta)\,\frac{\Delta s}{\Delta t} = F(\cos\theta)v$$

$$= F_t v = \mathbf{F}\cdot\mathbf{v}$$

As Δt approaches 0, P approaches $\bar{P}$ with

$$P = F\cos\theta\, v = F_t\, v = \mathbf{F}\cdot\mathbf{v}.$$

PROBLEM-SOLVING STRATEGIES

When applying energy conservation, make a list of the initial kinetic and potential energies (K_1, U_1) and the final kinetic and potential energies (K_2, U_2). Note the knowns and the unknowns. Solve for the unknowns.

Note the work, a scalar, may be positive or negative. A force acting perpendicular to the path of a particle (such as the tension of a pendulum bob) does no work. Such forces do not contribute to the work W or W′ in the work energy relation.

EXAMPLES AND SOLUTIONS

Example 1

A 10 kg box is pulled through a distance of 2 m by a horizontal force F of magnitude 20 N acting in the same direction as the displacement. Find the work done by F.

Solution:

$W = Fs\cos\theta$; $F = 20$ N, $s = 2$ m, $\cos\theta = 1$

$W = 20\text{ N}\cdot 2\text{ m}\cdot 1 = 40$ J

Note the work done by F doesn't depend on such things as the mass, velocity or acceleration of the box.

Example 2

The same box starts from rest on a smooth surface, acted upon by the same total force F = 20 N.

(a) Find the final kinetic energy after s = 2 m, using the methods of Ch. 2 and compare it to the work done by the total force F.

(b) Find the final velocity v.

Solution:

Because the force is constant, the acceleration is also constant, and we have

$$v^2 = v_0^2 + 2as \qquad a = F/m \qquad v_0 = 0$$

$$v^2 = 2\frac{Fs}{m}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{2Fs}{m} = Fs = W = 40\ \mathrm{J}$$

(b) $$v = \left(\frac{2Fs}{m}\right)^{1/2} = \left(\frac{2\cdot 20\ \mathrm{N}\cdot 2\ \mathrm{m}}{10\ \mathrm{kg}}\right)^{1/2} = 2.83\ \mathrm{m\cdot s^{-1}}$$

Example 3

The same box undergoes the same displacement on a rough floor at constant velocity. Find
(a) The work done by F;
(b) The coefficient of friction μ_k;
(c) The work done by the friction force;
(d) The work done by gravity;
(e) The work done by the normal force; and
(f) The work done by the total force acting on the body

Solution:

Refer to Fig. 7-7:

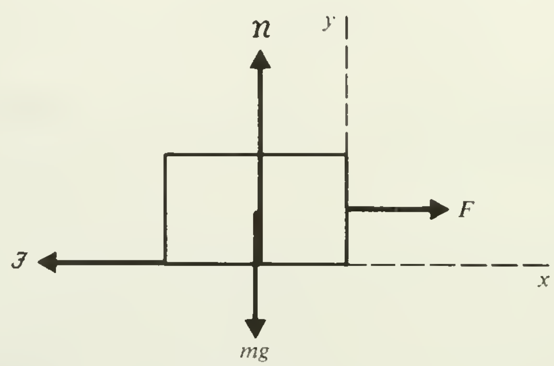


Figure 7-7

The body is in equilibrium because v = constant. Thus

$$\sum F_x = 0 = F - \mathcal{F} = F - \mu\mathcal{N} \qquad F = \mu\mathcal{N}$$

$$\sum F_y = 0 = \mathcal{N} - mg; \qquad \mathcal{N} = mg$$

and

$$\mu_k = \frac{F}{\mathcal{N}} = \frac{F}{mg} = \frac{20\ N}{10\ kg\cdot 9.8\ m\cdot s^{-2}} = 0.22$$

(a) $W_F = Fs\cos\theta = 20\ N\cdot 2\ m\cdot 1 = 40\ J$

(b) $\mu_k = 0.22$

(c) $W = \mathcal{F}s\cos\theta = 20\ N\cdot 2\ m\cdot(-1) = -40\ J$

The work done by the friction force is negative because ξ is in a direction opposite to the displacement, $\cos\theta = -1$.

(d) The work done by gravity, W_g, is zero because the displacement is perpendicular to the force, $\cos\theta = 0$.

(e) Similarly $W_\eta = 0$.

(f) The total force $\mathbf{F}_T$ is

$$\mathbf{F}_T = \mathbf{F} + \mathcal{F} + \mathcal{N} + \mathbf{w}\ (\text{weight}) = 0$$

and hence the total work W is

$$W = W_F + W_{\mathcal{F}} + W_\eta + W_g = 40\ J - 40\ J + 0 + 0$$

$$= 0$$

Example 4

The same box is pushed 2 m up a <u>smooth</u> inclined plane of angle $\theta = 30^\circ$ by a 100 N horizontal force.

(a) Find the work done on the box and

(b) the increase in its kinetic energy.

Solution:

Referring to Fig. 7-8,

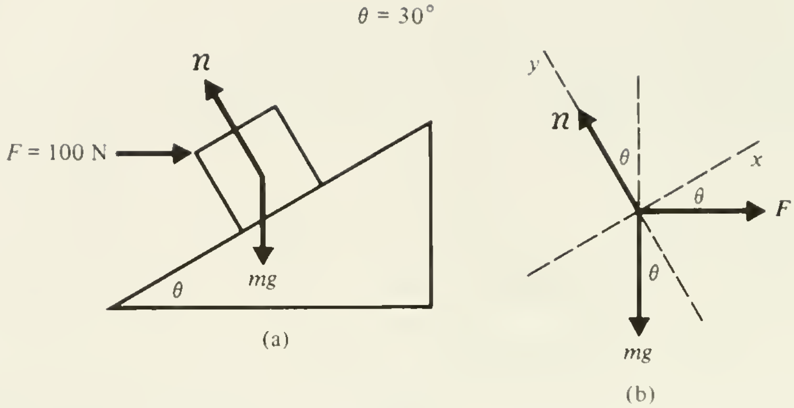


Figure 7-8

we have

$$\sum F_x = F\cos\theta - mg\sin\theta = ma_x$$

$$\sum F_y = n - mg\cos\theta - F\sin\theta$$

(a) $W_F = Fs\cos\theta = 100\ \mathrm{N}\cdot 2\ \mathrm{m}\cdot\cos 30^\circ$

$= 173\ \mathrm{J}$ = work of force F

$W_\eta = 0$ (Normal force does no work because $\mathbf{n}\cdot\mathbf{ds} = 0$)

$W_g = F_g\cos\theta\ s = -mg\sin\theta\ s$

$= -10\ \mathrm{kg}\cdot 9.8\ \mathrm{m}\cdot\mathrm{s}^{-2}\cdot\sin 30^\circ\cdot 2\ \mathrm{m}$

$= -98\ \mathrm{J}$ = negative work of gravitational force.

$W = W_F + W_g = 75\ \mathrm{J}$

(b) $a_x = \dfrac{1}{m}(F\cos\theta) - g\sin\theta$

$= 0.1\ \mathrm{kg}^{-1}\cdot 100\ \mathrm{N}\cdot\cos 30^\circ - 9.8\ \mathrm{m}\cdot\mathrm{s}^{-2}\sin 30^\circ$

$= 3.76\ \mathrm{m}\cdot\mathrm{s}^{-2}$

$v_x^2 = v_{ox}^2 + 2a_x x$, $v_{ox} = 0$

$v_x^2 = 2a_x x = 2(3.76\ \mathrm{m}\cdot\mathrm{s}^{-2})2\ \mathrm{m} = 15.0\ \mathrm{m}^2\cdot\mathrm{s}^{-2}$

$K = \dfrac{1}{2}mv_x^2 = \dfrac{1}{2}(10\ \mathrm{kg})(15.0\ \mathrm{m}^2\cdot\mathrm{s}^{-2}) = 75\ \mathrm{J}$

Verified here is the work-energy relation

$$W = \Delta K$$

Note W is the total work done by all the forces acting on the body.

Example 5

A body in a plane is acted upon a force

$$\mathbf{F} = -bx^2\mathbf{i} \qquad b = 20\ \mathrm{N\cdot m^{-2}}$$

$$F_x = -bx^2 \qquad F_y = 0$$

What work is done by F when a body is displaced along the paths $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$ of Fig. 7-9?

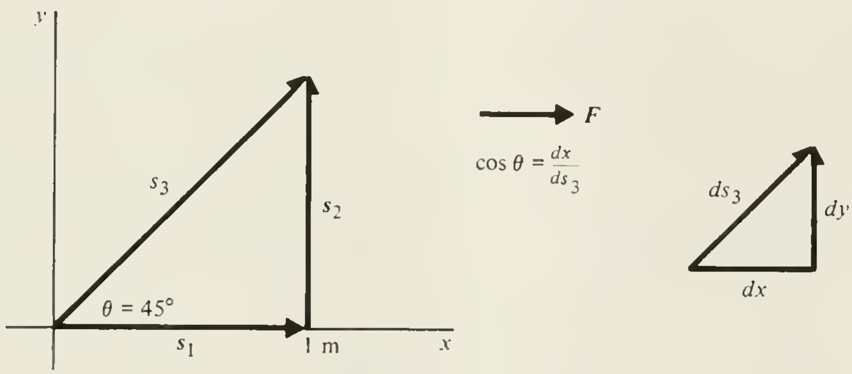


Figure 7-9

Solution:

Along $\mathbf{s}_1$, **F** and d**s** are parallel, $\cos\theta = 1$, and

$$W_1 = \int \mathbf{F}\cdot d\mathbf{s}_1 = \int F\cos\theta\, ds_1 = \int_0^{1\mathrm{m}} F_x dx$$

$$= -b\int_0^{1\mathrm{m}} x^2 dx = -20\ \mathrm{N\cdot m^{-2}}\ \frac{x^3}{3}\Big|_0^{1\mathrm{m}} = -6.7\ \mathrm{J}$$

Along $\mathbf{s}_2$, **F** is perpendicular to $d\mathbf{s}_2$ and

$$W_2 = \int \mathbf{F}\cdot d\mathbf{s}_2 = 0$$

Along $\mathbf{s}_3$, $\cos\theta = 45°$, $ds_3 = (\cos\theta)^{-1}\,dx$ and we have

$$W_3 = \int \mathbf{F}\cdot d\mathbf{s}_3 = \int F\cos\theta\, ds_3$$

$$= \int_{0}^{1m} F \cos\theta(\cos\theta)^{-1}\, dx$$

$$= \int_{0}^{1m} F dx = -6.7 \text{ J}$$

Note $W_3 = W_1 + W_2$, ie the work done is the same for each complete path connecting initial and final points. This is an example of a conservative force.

Example 6

A body sliding on a rough surface is given an initial velocity of 3 $m \cdot s^{-1}$ and comes to a stop in 1 m. Using the work energy relation, find the coefficient of friction.

Solution:

$$K_1 = \frac{1}{2}m(3m\cdot s^{-1})^2 \qquad K_2 = 0$$

$$W = \Delta K = K_2 - K_1; \qquad W = -\mathcal{F}x = -\mu\eta x$$

$$W = -\mu\eta x = -\mu mg\, x = 0 - \frac{1}{2}mv^2$$

$$\mu = \frac{v^2}{2\, gx} = \frac{(3\ m\cdot s^{-1})^2}{2\cdot 9.8\ m\cdot s^{-2}\cdot 1\ m} = 0.5$$

Note K_2 is the final kinetic energy, K_1 the initial so that ΔK is the change in kinetic energy, negative in this case; correspondingly the work is negative in this case because the force F is opposite to the displacement.

Example 7

A body is released from rest at the top of a rough inclined plane of height y and angle θ. The coefficient of friction is μ. Use the work-energy relation to find the velocity of the body at the bottom of the incline in terms of m, y, μ and θ.

Solution:

Referring to Fig. 7-10 we have

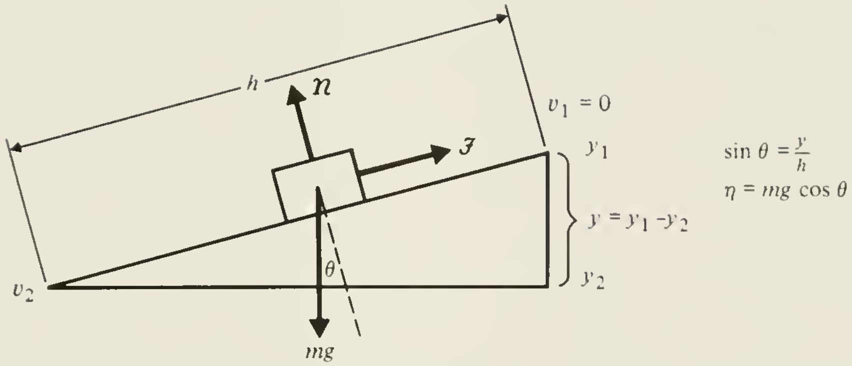


Figure 7-10

$$K_1 = 0 \qquad K_2 = \frac{1}{2} mv^2$$

$$W = W_g + W_{\mathcal{F}} + W_\eta = K_2 - K_1 = \frac{1}{2} mv_2{}^2 \; ; \quad K_1 = 0$$

Where W_g = work of gravity, $W_{\mathcal{F}}$ = work of friction, and W_η = work of the normal force.

$$W_g = mg \sin\theta \; h = mg \sin\theta \frac{y}{\sin\theta} = +mgy$$

$$W_{\mathcal{F}} = -\mathcal{F}h = -\mu\eta h = -\mu mg \cos\theta \; h$$

$$= -\mu mg \frac{\cos\theta}{\sin\theta} = -\frac{\mu mgy}{\tan\theta}$$

$$W = mgy \left(1 - \frac{\mu}{\tan\theta}\right) = \frac{1}{2} mv_2{}^2$$

(For slippage $\tan\theta > \mu_s$, where μ_s is the static coefficient; since $\mu_s > \mu$, the sliding coefficient of friction, $\tan\theta > \mu$ and $W > 0$ in the last expression.) Then we have

$$v_2 = \left[2gy\left(1 - \frac{\mu}{\tan\theta}\right)\right]^{1/2}.$$

Another approach is to handle the conservative part of the total force, the gravitation, through the potential energy $U = mgy$.

$$W' = \Delta E = E_2 - E_1 = (K_2 + U_2) - (K_1 + U_1)$$

$$= \left(\frac{1}{2} mv_2^2 + mgy_2 \right) - \left(\frac{1}{2} mv_1^2 + mgy_1 \right)$$

$$= \frac{1}{2} mv_2^2 - mg(y_1 - y_2)$$

$$= \frac{1}{2} mv_2^2 - mgy$$

where W′ is the work of all forces except the gravitational force. $W' = W_F = -\mu mgy/\tan\theta$. Thus

$$-\frac{\mu mgy}{\tan\theta} = \frac{1}{2} mv_2^2 - mgy$$

resulting in the same answer for v_2.

Example 8

A 2 kg mass attached by a string of length L = 1 m to the ceiling is released from rest at an angle of $\theta_0 = 60^\circ$ with the vertical. Find its maximum velocity.

Solution:

Refer to Fig. 7-11:

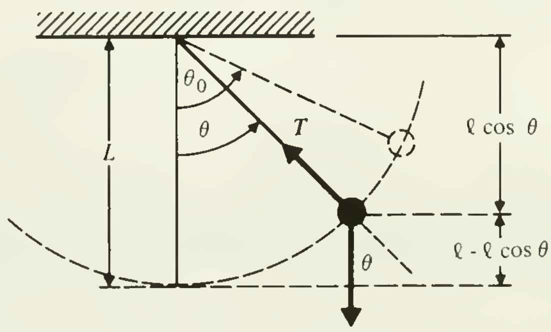


Figure 7-11

The only forces acting on the body are gravity and the tension T. T does no work (T and ds are perpendicular) so $W' = 0$ and mechanical energy is conserved,

$$W' = 0 = \Delta E = E_2 - E_1$$

The initial energy is

$$E_1 = K_1 + U_1 = 0 + mgy_1 = mgL(1 - \cos\theta_0)$$

The final energy is

$$E_2 = K_2 + U_2 = \frac{1}{2}\,mv_2{}^2 + mgy_2 = \frac{1}{2}\,mv_2{}^2 + mgL(1 - \cos\theta)$$

The conservation of energy yields

$$\frac{1}{2}\,mv_2{}^2 + mgL(1 - \cos\theta) = mgL(1 - \cos\theta_0)$$

$$v_2 = [2gL(\cos\theta - \cos\theta_o)]^{1/2}$$

$$= [2gL(\cos\theta - \frac{1}{2})]^{1/2}$$

The velocity is maximum at the bottom of the swing when $\theta = 0$ and

$$v_2 = (gL)^{1/2} = (9.8\ m\cdot s^{-2}\cdot 1\ m)^{1/2} = 3.1\ m\cdot s^{-1}$$

Example 9

In the last problem find the tension T in the string as a function of θ, and its value when T is maximum.

Solution:

$$\sum F = T - mg\cos\theta = \frac{mv^2}{R} = \frac{mv^2}{L}$$

$$T = mg\cos\theta + \frac{m}{L}\,2gL(\cos\theta - 1/2)$$

$$T = mg(3\cos\theta - 1)$$

T is maximum when $\theta = 0$ at the bottom of its swing, where

$$T = 2\ mg = 2.2\ kg\cdot 9.8\ m\cdot s^{-2} = 39.2\ N$$

Example 10

A mass of 1 kg resting on a smooth floor, attached to a wall by a spring of constant $k = 100\ N\cdot m^{-1}$, is pulled 0.5 m from its equilibrium position.

(a) What is its potential energy?

(b) If released from rest, what is its velocity when it passes through the equilibrium position?

(c) If it is given an initial velocity of 5 $m \cdot s^{-1}$ at $x = 0.5$ m, what is its velocity at $x = 0$?

Solution:

The system conserves total mechanical energy. Let x_1, v_1 be the initial position and velocity and x_2, v_2 the final position and velocity.

$$x_1 = 0.5 \text{ m} \qquad x_2 = 0 \qquad v_1 = 0 \qquad v_2 = ?$$

$$E = \frac{1}{2} mv^2 + \frac{1}{2} kx^2 = K + U$$

(a) $$U_1 = \frac{1}{2} kx_1^2 = \frac{1}{2}(100 \text{ N}\cdot\text{m}^{-1})(.5 \text{ m})^2 = 12.5 \text{ J}$$

(b) $$E_1 = K_1 + U_1 = 12.5 \text{ J} \qquad (K_1 = 0)$$

$$E_2 = K_2 + U_2 = \frac{1}{2} mv_2^2 + \frac{1}{2} kx_2^2 = \frac{1}{2} mv_2^2$$

$$E_1 = E_2$$

$$\frac{1}{2} mv_2^2 = 12.5 \text{ J}$$

$$v_2 = \left(\frac{2 \cdot 12.5 \text{ J}}{1 \text{ kg}} \right)^{1/2} = 5 \text{ m}\cdot\text{s}^{-1}$$

(c) $$E_1 = K_1 + U_1 = \frac{1}{2} mv_1^2 + \frac{1}{2} kx_1^2$$

$$= \frac{1}{2} (1 \text{ kg})(5 \text{ m}\cdot\text{s}^{-1}) + 12.5 \text{ J}$$

$$= 25 \text{ J}$$

$$E_1 = E_2 = \left[\frac{1}{2} mv_2^2 + \frac{1}{2} kx_2^2 \right] = \frac{1}{2} mv_2^2$$

$$v_2 = \left[\frac{2 \cdot 25 \text{ J}}{1 \text{ kg}} \right]^{1/2} = 7.1 \text{ m}\cdot\text{s}^{-1}$$

Note it does not matter whether the initial velocity is toward or away from the equilibrium position.

Example 11

A mass of 0.5 kg hangs from a spring whose unstretched length is 1 m = 2 L. It is stretched a length L = 0.5 m. Find the spring constant.

Solution:

Referring to Fig. 7-12b, when the mass is in equilibrium we have

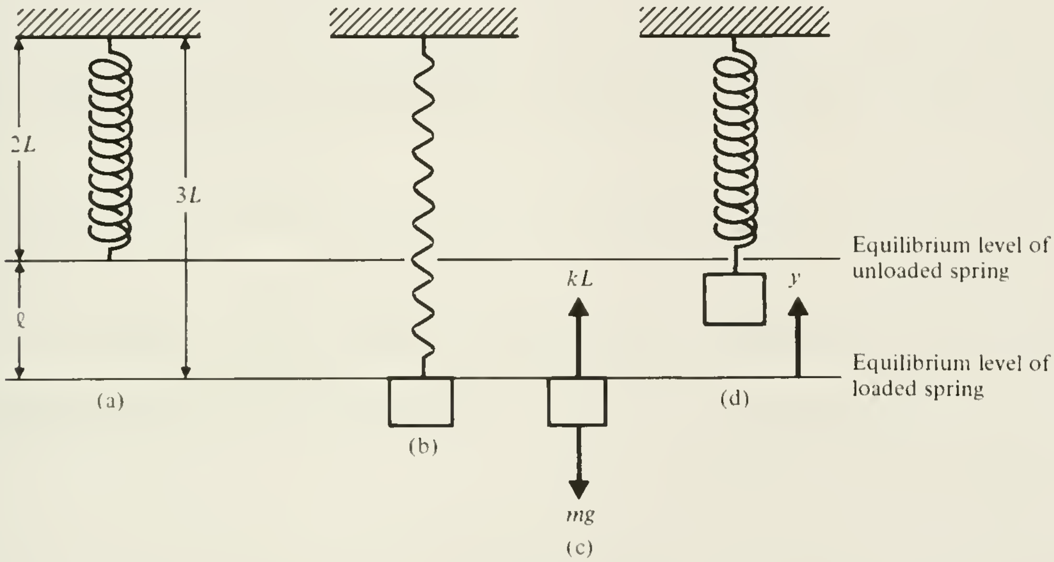


Figure 7-12

$$kL = mg \text{ so } \quad k = \frac{mg}{L} = \frac{0.5 \text{ kg}\cdot 9.8 \text{ m}\cdot\text{s}^{-2}}{0.5 \text{ m}}$$

$$k = 9.8 \text{ N}\cdot\text{m}^{-1}.$$

Example 12

The same mass is pulled down 0.5 m from its equilibrium position and released from rest. Find its maximum velocity and its maximum height.

Solution:

As shown in Fig. 7-12c, the coordinate y is the vertical distance from the equilibrium position. The system has gravitational potential energy U_g and elastic potential energy U_e. The elastic potential energy is

$$U_e = \frac{1}{2}k(L - y)^2 = \frac{1}{2}k(\frac{mg}{k} - y)^2$$

because L - y is the amount the spring is stretched. Thus

$$U_e = \frac{1}{2}kL^2 - kLy + \frac{1}{2}ky^2$$

$$= \frac{1}{2}kL^2 - mgy + \frac{1}{2}ky^2,$$

and

$$U_g = mgy.$$

The total potential energy is the sum of the elastic and gravitational potential energies

$$U = U_e + U_g = \frac{1}{2}kL^2 + \frac{1}{2}ky^2.$$

The constant term $\frac{1}{2}kL^2$ is the elastic potential energy when the mass is in its equilibrium condition. Since only potential energy differences enter the work-energy relation,

$$W' = E_2 - E_1 = (K_2 - K_1) + (U_2 - U_1)$$

this constant may be dropped without changing any physical result. Alternatively if the zero of gravitational potential energy is taken to be at L/2,

$$U_g = mg(y - \frac{L}{2}) = mgy - k\frac{L^2}{2},$$

and

$$U = U_g + U_e = \frac{1}{2}ky^2$$

<u>Thus when treating a spring in a gravitational field only the elastic potential energy appears in the potential function.</u>

Here we have $W' = 0 = E_2 - E_1$, with

$$E_1 = \frac{1}{2}mv_1^2 + U = 0 + \frac{1}{2}ky_1^2 \; ; \quad y_1 = -0.5 \text{ m}.$$

At an arbitrary final position y and velocity v

$$E_2 = \frac{1}{2} mv^2 + \frac{1}{2} ky^2 = E_1 = \frac{1}{2} ky_1^2$$

$$\frac{1}{2} mv^2 = \frac{1}{2} k(y_1^2 - y^2),$$

$$\frac{1}{2} mv_{max}^2 = \frac{1}{2} ky_1^2$$

The maximum velocity is when y = 0, where

$$v_{max} = [\frac{k}{m}]^{1/2} y_1 = [\frac{9.8\ m\cdot s^{-2}}{0.5\ kg}]^{1/2}\ 0.5\ m$$

$$= 2.2\ m\cdot s^{-1}$$

The height y as a function of v is

$$y = \pm(y_1^2 - \frac{m}{k} v^2)^{1/2}$$

and is maximum when $y = + |y_1| = 0.5$ m.

Example 13

For the frictionless system of Fig. 7-13a, find the velocity of the mass m when it hits the floor if it is released from rest a height L above the floor. $(m > m')$

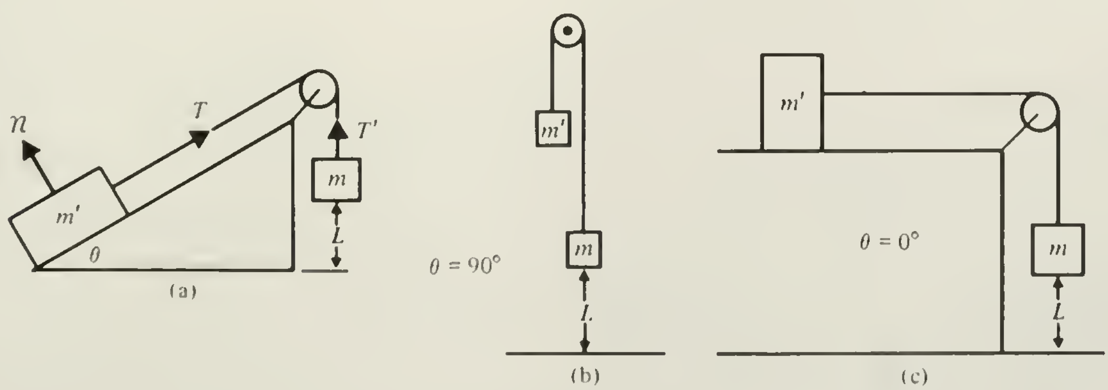


Figure 7-13

Solution:

Since η, the normal force of the plane on the mass, does no work, and the work of T is equal in magnitude but opposite in sign to the work of T', W' = 0 and the system is conservative, with mechanical energy

$$E = \frac{1}{2} mv^2 + \frac{1}{2} m'v^2 + mgy + mgy' = K + U$$

The initial values are

$$K_1 = 0, \; U_1 = mgL$$

$$E_1 = K_1 + U_1 = mgL.$$

If m falls a distance L, m' rises a distance L up the plane through a vertical height $y_2' = L \sin \theta$. Thus the final energy is

$$K_2 = \frac{1}{2} (m + m')v^2 \qquad U_2 = m'gL \sin \theta$$

$$E_2 = \frac{1}{2} (m + m')v_2{}^2 + m'gL \sin \theta$$

$$= E_1 = mgL$$

The final potential energy of m is zero becanse its final y coordinate is zero. Energy conservation $E_1 = E_2$ yields

$$mgl = \frac{1}{2} (m + m')v^2 + m'gL \sin \theta$$

The final velocity is thus

$$v_2 = [2 \frac{(m - m' \sin \theta)}{m + m'} gL]^{1/2}.$$

As θ approaches 90° we have an Atwood's machine (Fig. 7-11b) with

$$v_2 = [2 \frac{m - m'}{m + m'} gL]^{1/2}$$

As θ approaches 0 we have the table (Fig. 7-11c) with

$$v_2 = [\frac{2m}{m + m'} gL]^{1/2}.$$

Example 14

A 1 kg block attached to a spring and resting on a rough surface is given an initial velocity of $v = 10\ m\cdot s^{-1}$. The spring, initially unstretched, has a constant $k = 3\ N\cdot m^{-1}$. The maximum compression of the spring is $L = 4$ m. Find the coefficient of friction.

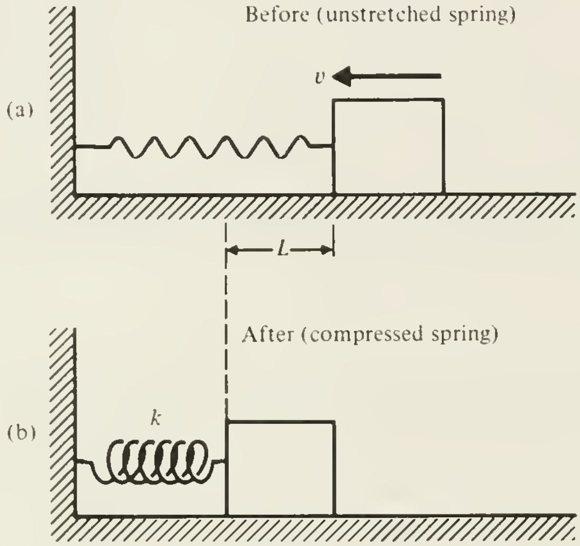


Figure 7-14

Solution:

Referring to Fig. 7-14, we have

$$W' = E_2 - E_1; \qquad W' = -\mathcal{F} L = -\mu mgL; \qquad E = \frac{1}{2} mv^2 + \frac{1}{2} kx^2$$

where W' is the work of the friction force. Evaluating the initial and final energies we have

$$E_2 = 0 + \frac{1}{2} kL^2 \qquad E_1 = \frac{1}{2} mv^2 + 0$$

Thus the work-energy relation yields

$$-\mu mgL = \frac{1}{2} kL^2 - \frac{1}{2} mv^2$$

$$\mu = \frac{mv^2/2 - kL^2/2}{mgL}$$

$$= \frac{(1/2)(1\ kg)(10\ m\cdot s^{-1})^2 - (1/2)(3\ N\cdot m^{-1})(4\ m)^2}{(1\ kg)(9.8\ m\cdot s^{-2})(4\ m)}$$

$$= .67$$

Example 15

A 0.5 kg mass is thrown vertically at 5 $m\cdot s^{-1}$ from the roof of a 100 m high building. Use the conservation of energy to find its velocity when it hits the ground.

Solution:

$$W' = 0 = E_2 - E_1 = \frac{1}{2} mv_2{}^2 + mgy_2 - \frac{1}{2} mv_1{}^2 - mgy_1$$

$$0 = \frac{1}{2} m(v_2{}^2 - v_1{}^2) - mg\ (y_1 - y_2)$$

$$v_2{}^2 = v_1{}^2 + 2g(y_1 - y_2)$$

$$v_1 = 5\ m\cdot s^{-1}$$

$$y_1 - y_2 = 100\ m$$

$$v_2 = [(5\ m\cdot s^{-1})^2 + 2(9.8\ m\cdot s^{-2})100\ m]^{1/2}$$

$$= 44.5\ m\cdot s^{-1}$$

Example 16

A mass slides in a frictionless vertical loop-the-loop, having been released from rest a distance L above the bottom of the loop.

(a) Find the velocity when the mass makes an angle θ with the vertical.

(b) Find the normal force of the loop on the mass.

Solution:

Referring to Fig. 7-15, we see that

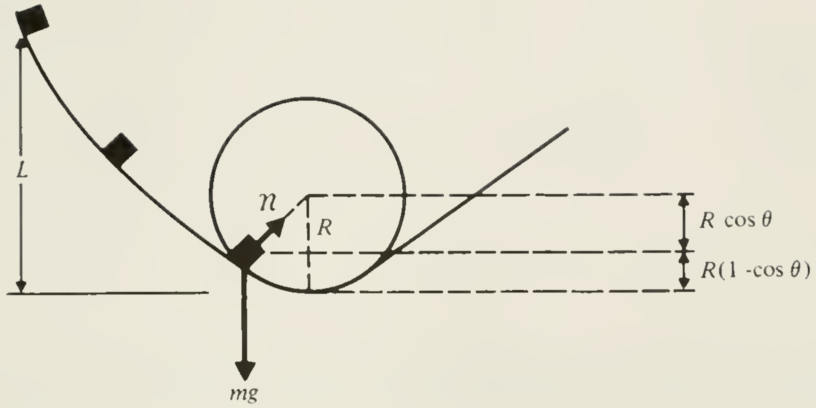


Figure 7-15

the kinetic plus the gravitational potential energy is conserved because the normal force does no work,

(a) $$W' = 0 = E_2 - E_1; \quad E = \frac{1}{2}mv^2 + mgy;$$

$$y_2 = R - R\cos\theta; \; y_1 = L$$

$$E_2 = \frac{1}{2}mv^2 + mg\,R(1 - \cos\theta)$$

$$E_1 = mgL$$

$$v^2 = 2gL - 2gR(1 - \cos\theta)$$

$$\sum F = n - mg\cos\theta = \frac{mv^2}{R}$$

$$n = mg\cos\theta + \frac{m}{R}[2gL - 2gR(1 - \cos\theta)]$$

$$= 3\,mg\cos\theta - 2\,mg + 2\,mg\frac{L}{R}$$

(b) The condition for the mass not to fall off the loop is $n > 0$. (If the mass were a bead on a wire, the wire could exert forces outward ($n < 0$) and the mass would never fall off.) At the top of the loop when $\theta = 180°$ this condition is

$$n = -5\,mg + 2\,mg\frac{L}{R} > 0$$

or

$$L > \frac{5}{2} R$$

Example 17

Suppose in the last problem the block is released from rest a distance $L = 2R$ above the bottom of the loop. When does the block fall off the loop?

Solution:

The block falls off when $\eta = 0$, i.e.

$$0 = 3 \text{ mg} \cos\theta - 2 \text{ mg} + 4 \text{ mg}$$

$$\cos\theta = -2/3 \qquad \theta = 132^\circ$$

Example 18

What velocity must a rocket have at the surface of the earth if it is to rise to a height equal to an earth radius before it begins to descend? Neglect air resistance.

Solution:

We have

$$W' = 0 = E_2 - E_1; \quad E = K + U; \quad U = -G\frac{mm_E}{r}$$

$$E_1 = \frac{1}{2} mv_1{}^2 - G\frac{mm_E}{R}$$

$$E_2 = 0 - G\frac{mm_E}{2R}$$

where R is the radius of the earth. Since

$$Gm_E = R^2 g$$

$E_1 = E_2$ implies

$$\frac{1}{2}mv_1^2 - mgR = -\frac{mgR}{2}; \quad \frac{1}{2}mv_1^2 = \frac{mgR}{2}$$

$$v_1 = (gR)^{1/2} = [(9.8\ m\cdot s^{-2})(6.37 \times 10^6\ m)]^{1/2}$$

$$= 7.9 \times 10^3\ m\cdot s^{-1}$$

Example 19

Discuss qualitatively the motion of a body whose potential energy is given in each part of Fig. 7-16.

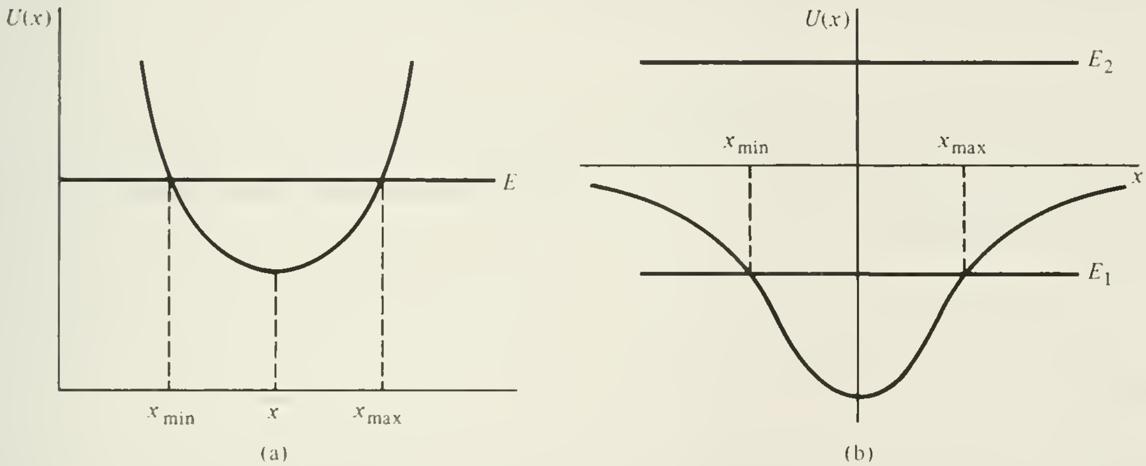


Figure 7-16

Solution:

(a) The particle is confined between x_{max} and x_{min} since $K = E - U > 0; E > U$. At these endpoints $K = 0$, $E = U$, and the velocity is zero. The body oscillates back and forth between x_{min} and x_{max}. The maximum velocity is at x where U is a minimum and K is a maximum.

(b) Case E_1 is similar to (a) above. Case E_2 has no endpoints; the kinetic energy $K = E_2 - U$ is always positive but greatest at $x = 0$. If the body comes in from the far left with a kinetic energy $K = E_2 - U \simeq E_2$, it speeds up as it approaches the origin, continuing to move in the positive direction, eventually far to the right with kinetic energy $K = E_2$. For $E > 0$ the body's motion is unbounded. For $E < 0$ the body's motion is bounded, that is has finite end points; it is confined to the region between x_{max} and x_{min}.

Example 20

What average power is required, in watts and hp, if an escalator is to lift twenty 100 kg people, 3 m high, in one minute? Does it matter whether they are lifted straight up or at an angle?

Solution:

$$P = Fv = \frac{\Delta W}{\Delta t}$$

$$\Delta W = mgh$$

$$P = \frac{(20)(100\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(3\ \text{m})}{60\ \text{s}} = 980\ \text{W}$$

$$= 0.98\ \text{kW} = \frac{(0.98\ \text{kW})(1\ \text{hp})}{0.746\ \text{kW}} = 1.31\ \text{hp}$$

The power is independent of the angle because gravitational work depends only on the vertical elevation.

Example 21

An automobile has a 150 hp engine. Its top speed is 100 mph. Assuming half the power is delivered by the engine to the tires on the road, find the drag of the air resistance and all other dissipative forces.

Solution:

$$P = Fv;\ \text{so}\ F = Pv^{-1}\ \ \text{and}$$

$$100\ \text{mph} = \frac{(100)(5280)}{3600}\ \text{ft}\cdot\text{s}^{-1} = 147\ \text{ft}\cdot\text{s}^{-1}$$

Thus

$$F = \frac{(75\ \text{hp})(550\ \text{ft}\cdot\text{lbs}\cdot\text{s}^{-1}\cdot\text{hp}^{-1})}{147\ \text{ft}\cdot\text{s}^{-1}}$$

$$= 281\ \text{lbs}$$

QUIZ

1. A 50 kg iceskater is coasting at 2 $m \cdot s^{-1}$. What constant force is necessary to stop the skater in a distance of 0.5 m?

Answer: 200 N

2. A spring attached to a 0.5 kg mass is compressed 5 cm and released from rest. When the mass is 2 cm from equilibrium it moves with a velocity of 0.3 $m \cdot s^{-1}$. Find the spring constant.

Answer: 21 $N \cdot m^{-1}$

3. A mass weighing 10 N slides without friction around a loop-the-loop of radius R. It is released from rest a distance 3R above the bottom of the loop. If θ is the angle the mass makes with the vertical, find the force of the loop against the mass at the top of the loop, at $\theta = 90^\circ$, and at the bottom of the loop.

Answer: 10 N, 40 N, 70 N.

4. 6 kg and 4 kg masses attached by a rope hang from a frictionless pulley. The 6 kg weight is 3 m above the ground, when the system is released from rest. Find the velocity of the system when the 6 kg mass hits the ground.

Answer: 3.4 $m \cdot s^{-1}$

8
IMPULSE AND MOMENTUM

OBJECTIVES

In this chapter the concepts of impulse, momentum and momentum conservation are developed. Your objectives are to:

Calculate the impulse of a force and relate it to the change of momentum of a body.

Distinguish between internal and external forces among interacting bodies.

Recognize when momentum is conserved.

Apply the impulse-momentum relation and the conservation of momentum to a variety of problems involving collisions(elastic and inelastic)and other kinds of interaction between bodies.

Calculate the coordinates of the center of mass of a body or a system of point masses.

REVIEW

The impulse J of a constant force acting on a single body between times t_1 and t_2 is

$$\mathbf{J} = \mathbf{F}\Delta t;$$

Note the direction of J is the direction of F and $\Delta t = t_2 - t_1$. If the force is not constant

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}\,dt; \qquad J_x = \int_{t_1}^{t_2} F_x dt, \text{ and similarly for } J_y \text{ and } J_x$$

If the average force is defined as

$$F_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} F \, dt$$

then

$$J = F_{av}(t_2 - t_1)$$

Newton's second law may be used to show that the impulse of a force gives the change in momentum

$$J = \int_{t_1}^{t_2} m \frac{dv}{dt} dt = \int_{v_1}^{v_2} m dv = mv_2 - mv_1 = p_2 - p_1 = \Delta p$$

where the momentum of a single body is defined as

$$p = mv.$$

Note the momentum is a vector quantity (unlike energy, which is a scalar.) The impulse J in a time interval is equal to the change in momentum of the body in that interval; if F or $F_{av} = 0$, $J = 0$ and the momentum does not change. If the force is constant and in the x direction,

$$J_x = \int_{t_1}^{t_2} F_x dt = F_x \int_{t_1}^{t_2} dt = F_x(t_2 - t_1) = m(v_{x2} - v_{x1}),$$

The last equality expresses the constant acceleration condition:

$$a_x = \frac{F_x}{m} = \frac{v_{x2} - v_{x1}}{t_2 - t_1}$$

Conservation of Momentum

When two bodies, for example two colliding billiard balls, interact with each other they exert forces on each other. These are internal forces. At the same time they may be acted upon by other forces, such as the friction on the table. These are external forces.

The total impulse depends only on external forces, because the internal forces cancel:

$$J = \int_{t_1}^{t_2} F^{ext} dt = P_2 - P_1$$

$$F^{ext} = F_A^{ext} + F_B^{ext}$$

If the total force F^{ext} or its integral in an interval vanishes then the total momentum is conserved:

$$\mathbf{P}_1 = \mathbf{P}_2; \quad \Delta\mathbf{P} = 0$$

The result may be generalized to any number of interacting bodies: the total momentum of a system is conserved if there is no net external force.

Collisions

When two bodies collide, internal forces are exerted during the short times of impact. These forces may be conservative or dissipative. If the forces are conservative, the collision is said to be elastic. In an elastic collision, the total energy and hence the total kinetic energy is conserved, since before and after the collision the potential energy is the same (generally zero.)

A collision is inelastic if total energy is not conserved.

An example of an approximately elastic collision is that between two billiard balls. No energy is lost or dissipated in permanent changes of the billiard balls; all energy is kinetic before and after the collision. During the collision kinetic energy may be transformed into elastic potential energy as the balls in contact compress each other; this potential energy is then, without loss, transformed back into kinetic energy of motion as the balls spring apart to their original shape and state.

An example of an inelastic collision is the impact of a bullet with a block of wood. Mechanical energy is not conserved; it is dissipated into heat and permanent changes of shape and state in the bullet and block.

Elastic Collision Between Two Bodies

If the initial velocities of two bodies (v_{A1} and v_{B1}) are known, then the two conditions of energy conservation and momentum conservation may be used to calculate their final velocities after a collision on a straight line. The general case is derived in the text. Note all velocities are referred to a single direction on a line (eq. positive when moving to the right, negative when moving to the left). A and B refer to the two bodies; 1 and 2 to before and after collisions. For an elastic collision between $m_A$and m_B with body B initial at rest the final velocities are

$$V_{A2} = \frac{m_A - m_B}{m_A + m_B} V_{A1} \qquad V_{B2} = \frac{2m_A}{m_A + m_B} V_{A1}$$

If $m_A \gg m_B$ (for example, baseball A hits ping-pong ball B at rest)

$$V_{A2} \simeq V_{A1}; \qquad V_{B2} \simeq 2V_{A1}.$$

The baseball hardly changes speed; the ping-pong ball flies off with twice the initial baseball velocity. Viewed from a frame of reference moving with the baseball, the ping-pong ball approaches with a speed V_{A1} and is reflected forward upon collision with an equal but opposite velocity. The <u>change</u> of velocity is $2V_{A1}$ in both frames.

If $m_A << m_B$ (for example, ping-pong ball A hits baseball B at rest), then

$$V_{A2} \simeq -V_{A1}; \qquad V_{B2} \simeq 0.$$

The baseball hardly moves and the ping-pong ball reverses its direction, changing its velocity by $2V_{A1}$, as in the last example. If

$$m_A = m_B, \; V_{A2} = 0 \text{ and } V_{B2} = V_{A1};$$

the moving ball stops upon collision, transferring all of its momentum to the ball originally at rest.

Center of Mass

The center of mass position vector R of a system of mass points m_1 at r_1, m_2 at r_2, etc., is given by

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3 + \ldots}{M} = \mathbf{R}\,(X, Y, Z)$$

The components of the center-of-mass are ($M = m_1 + m_2 + m_3 + \ldots$)

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{M}, \quad \text{and similarly for Y and Z}$$

The total momentum **P** of a system is the total mass times the velocity of the center of mass

$$M\mathbf{V} = \mathbf{P}$$

The rate of change of total momentum is

$$\frac{d\mathbf{P}}{dt} = M\frac{d\mathbf{V}}{dt} = M\mathbf{A} = \mathbf{F}^{\,ext}$$

when **A** is the acceleration of the center of mass. The center of mass moves as if it were a mass point at **R** acted upon by a total force $\mathbf{F}^{ext}$. If the total external force is zero, the total momentum is conserved, $\mathbf{A} = 0$, and the center of mass moves with uniform motion in a straight line.

PROBLEM-SOLVING STRATEGIES

Momentum is conserved in a system if no external forces act. Even if energy is not conserved, as in an inelastic collision, momentum is constant as long as the system is isolated from external forces.

Momentum is a vector; if P is conserved, the components P_x, P_y, and P_z are constant. If all motion is along a straight line, only that component of momentum need be considered.

As with energy conservation, list all your initial and final momenta, indicating knowns and unknowns.

Note that components of momentum may be positive or negative depending on direction of motion.

EXAMPLES AND SOLUTIONS

Example 1

A body falls for 5 s in a gravitational field. Its initial velocity is $v_o = 20$ $m \cdot s^{-1}$. Use the impulse-momentum relation to find the final velocity.

Solution:

$$J = \int_o^t F dt = \int_o^t (-mg) dt = -mgt.$$

$$\Delta p = mv - mv_o$$

Since $J = \Delta p \sim - mgt = mv - mv_o$, we may solve for v:

$$v = v_o - gt = 20\ m \cdot s^{-1} - (9.8\ m \cdot s^{-2}) 5\ s = -29\ m \cdot s^{-1}$$

Example 2

A force acts on a 0.5 kg body along a given direction with a magnitude given by the graph of Fig. 8-1. If the body is initially at rest find its velocity at t = 5 s.

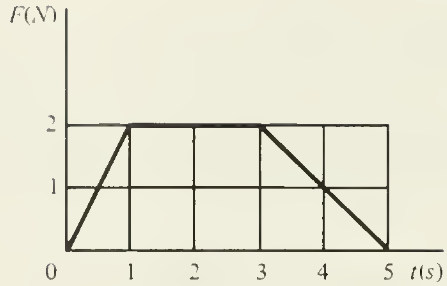


Figure 8-1

Solution:

First calculate the impulse in this interval:

$$J = \int_{0}^{5\ s} F dt = \text{area under the curve in Fig. 8-1}$$

$$= 7 \times \text{area of one box} \quad \text{(Count boxes!)}$$

$$= 7\ N{\cdot}s$$

Then equate that impulse to the change in momentum:

$$\Delta p = (0.5\ kg)v$$

$$7\ N{\cdot}s = (0.5\ kg)v$$

$$v = \frac{7\ N{\cdot}s}{0.5\ kg} = 14\ m{\cdot}s^{-1}$$

Example 3

A 0.15 kg baseball is pitched horizontally at 25 $m{\cdot}s^{-1}$. It is then batted at a velocity of 40 $m{\cdot}s^{-1}$ in the opposite direction. The ball is in contact with the bat for 0.004 s. Find the average force exerted by the bat.

Solution:

Since the impulse is the product of the average force with the time interval,

$$J = \int_{t_1}^{t_2} F dt = F_{av}\ \Delta t = mv_2 - mv_1$$

the average force is the momentum change divided by Δt:

$$F_{av} = \frac{m(v_2 - v_1)}{\Delta t} = \frac{0.15\ \text{kg}}{0.004\ \text{s}} [40\ \text{m}\cdot\text{s}^{-1} - (-25\ \text{m}\cdot\text{s}^{-1})]$$

$$= 2400\ \text{N}$$

Note v_1, the pitched ball velocity, is negative because the direction of the batted ball, v_2, has been taken to be positive.

Example 4

A box is kicked, and aquires a horizontal velocity $v_1 = 3\ \text{m}\cdot\text{s}^{-1}$. It slides for 2 s along a rough floor before coming to rest. Use the impulse momentum relation to find the coefficient of friction.

Solution:

$$J = \int_{t_1}^{t_2} F dt = \int_{t_1}^{t_2} (-\mu\eta) dt = \mu mg(t_2 - t_1) = -\mu mg\ \Delta t$$

$$\Delta p = m(v_2 - v_1) = -mv_1; \qquad v_2 = \text{final velocity} = 0$$

$$\mu = \frac{v_1}{g\Delta t} = \frac{3\ \text{m}\cdot\text{s}^{-1}}{9.8\ \text{m}\cdot\text{s}^{-1}\ 2\ \text{s}} = 0.15$$

Example 5

A 2000 kg car moving at 30 $\text{km}\cdot\text{hr}^{-1}$ strikes a stopped 1000 kg car and the two lock bumpers.

(a) What is their final common velocity just after the collision?

(b) What fraction of the initial energy is dissipated in the collision?

Solution:

Momentum is conserved because only internal forces act during the collision; energy is not because the two cars lock bumpers--the collision is inelastic.

Let

$$m_A = 2000\ \text{kg} \qquad v_{A1} = 30\ \text{km}\cdot\text{hr}^{-1}$$

$$m_B = 1000\ \text{kg} \qquad v_{B1} = 0$$

Then

$$m_A v_{A1} = (m_A + m_B)\, v_2$$

where v_2 is the common final velocity $v_{A2} = v_{B2} = v_2$.

$$v_2 = \frac{m_A}{m_A + m_B} v_{A1} = \frac{2000}{3000} 30\ \text{km}\cdot\text{hr}^{-1} = 20\ \text{km}\cdot\text{hr}^{-1}.$$

The final and initial mechanical energies are

$$E_2 = 1/2\ m_A v_2{}^2 + 1/2\ m_B v_2{}^2$$

$$= 1/2(m_A + m_B) v_2{}^2$$

$$E_1 = 1/2\ m_A v_{A1}{}^2$$

$$\frac{E_2}{E_1} = \frac{m_A + m_B}{m_A} \left[\frac{V_2}{V_{A1}} \right]^2 = \frac{m_A + m_B}{m_A} \left[\frac{m_A}{m_A + m_B} \right]^2$$

$$= \frac{m_A}{m_A + m_B} = \frac{2000}{3000} = \frac{2}{3}$$

2/3 of the initial energy remains kinetic; 1/3 is dissipated during the collision.

Example 6

Suppose in the last example the cars have spring loaded bumpers, so that the collision is perfectly elastic.

(a) What is the velocity of each car after the collision?

(b) What are the final momenta and energies?

Solution:

For this case (B initially at rest) the result of simultaneously solving the momentum conservation equation

$$m_A v_{A1} = m_A v_{A2} + m_B v_{B2}$$

and energy conservation equation

$$1/2\ m_A v_{A1}{}^2 = 1/2\ m_A v_{A2}{}^2 + 1/2\ m_B v_{B2}{}^2$$

is (see SZY)

$$v_{A2} = \frac{m_A - m_B}{m_A + m_B} v_{A1} = \frac{2000 - 1000}{3000} 30\ \text{km}\cdot\text{hr}^{-1} = 10\ \text{km}\cdot\ \text{hr}^{-1}$$

$$v_{B2} = \frac{2m_A}{m_A + m_B} v_{A1} = \frac{4000}{3000} 30 \text{ km hr}^{-1} = 40 \text{ km}\cdot\text{hr}^{-1}$$

The energies are

$$E_{A1} = 1/2\ m_A v_{A1}{}^2 = 1/2\ 2000 \text{ kg } \left(\frac{30 \times 10^3 \text{ m}}{3600 \text{ s}} \right)^2 = 70 \times 10^3 \text{J}$$

$$E_{A2} = 1/2\ m_A v_{A2}{}^2 = 1/2\ 2000 \text{ kg } \left(\frac{10 \times 10^3 \text{ m}}{3600 \text{ s}} \right)^2 = 8 \times 10^3 \text{J}$$

$$E_{B2} = 1/2\ m_B v_{B2}{}^2 = 1/2\ 1000 \text{ kg } \left(\frac{40 \times 10^3 \text{ m}}{3600 \text{ s}} \right)^2 = 62 \times 10^3 \text{J}$$

Note $E_{A1} = E_{A2} + E_{B2}$

The momenta are

$$p_{A1} = m_A v_{A1} = 2000 \text{ kg } \frac{30 \times 10^3 \text{ m}}{3600 \text{ s}} = 16.7 \times 10^3 \text{ kg}\cdot\text{m}\cdot\text{s}^{-1}$$

$$p_{A2} = m_A v_{A2} = 2000 \text{ kg } \frac{10 \times 10^3 \text{ m}}{3600 \text{ s}} = 5.6 \times 10^3 \text{ kg}\cdot\text{m}\cdot\text{s}^{-1}$$

$$p_{B2} = m_B v_{B2} = 1000 \text{ kg } \frac{40 \times 10^3 \text{ m}}{3600 \text{ s}} = 11.1 \times 10^3 \text{ kg}\cdot\text{m}\cdot\text{s}^{-1}$$

Note

$$p_{A1} = p_{A2} + p_{B2}$$

Example 7

A 2 kg block, moving to the right on a frictionless table at 3 $m\cdot s^{-1}$ makes a head-on collision with another 2 kg block moving 4 $m\cdot s^{-1}$ to the left.

(a) If the collision is completely elastic, find the final velocities of the blocks.

(b) If the collision is completely inelastic find the final velocity of the blocks.

(c) If half the initial energy is dissipated in the collision, find the final velocities of the blocks.

8-10

Solution:

Referring to Fig. 8-2, the initial velocities before collision (subscript '1') are

Figure 8-2

$$v_{A1} = 3\ \mathrm{m \cdot s^{-1}},\quad v_{B1} = -\ 4\ \mathrm{m \cdot s^{-1}}$$

and

$$m_A = m_B = m = 2\ \mathrm{kg}.$$

(a) The equations for conservation of momentum and energy are

$$mv_{A1} + mv_{B1} = m_A v_{A2} + mv_{B2} \quad \text{(momentum)}$$

$$v_{A1} + v_{B1} = \underline{v_{A2} + v_{B2} = 3\ \mathrm{m \cdot s^{-1}} - 4\ \mathrm{m \cdot s^{-1}} = -1\ \mathrm{m \cdot s^{-1}}}$$

$$1/2\ mv_{A1}^2 + 1/2\ mv_{B1}^2 = 1/2\ mv_{A2}^2 + 1/2\ mv_{B2}^2 \quad \text{(energy)}$$

$$1/2\ (3)^2\ \mathrm{m^2 s^{-2}} + 1/2\ (-4)^2\ \mathrm{m^2 s^{-2}} = 1/2(v_{A2}^2 + v_{B2}^2)$$

$$= 12.5\ \mathrm{m^2 \cdot s^{-2}}$$

$$\underline{v_{A2}^2 + v_{B2}^2 = 25\ \mathrm{m^2 \cdot s^{-2}}.}$$

The underlined equations may be solved simultaneously for v_{A2}, v_{B2} by solving the first equation for v_{B2} and substituting the result in the second equation

$$v_{A2}^2 + (-1 - v_{A2})^2 = 25$$

$$2v_{A2}^2 + 2v_{A2} - 24 = 0$$

$$(v_{A2} + 4)(v_{A2} - 3) = 0$$

$$v_{A2} = -4\ \mathrm{m \cdot s^{-1}},\ 3\ \mathrm{m \cdot s^{-1}}$$

The second root is extraneous, corresponding to the case where the bodies miss each other. Then, using the first boxed equation,

$$v_{B2} = -1 + 4 = 3\ \mathrm{m \cdot s^{-1}}.$$

(b) In this case $v_{A2} = v_{B2} = v_2$ and we may only use the first equation, since only momentum is conserved:

$$2v_2 = -1\ \mathrm{m \cdot s^{-1}} \qquad v_2 = -0.5\ \mathrm{m \cdot s^{-1}}$$

(c) Here we use conservation of momentum,

$$\underline{v_{A2} + v_{B2} = -1\ m\cdot s^{-1}}$$

and the fact that

$$E_{A2} + E_{B2} = 1/2(E_{A1} + E_{B1})$$

$$1/2\ mv_{A2}^2 + 1/2\ mv_{B2}^2 = 1/2(1/2\ mv_{A1}^2 + 1/2\ mv_{B1}^2)$$

$$\underline{v_{A2}^2 + v_{B2}^2 = 12.5\ m^2\cdot s^{-2}}$$

Solving the last two underlined equations simultaneously,

$$v_{A2}^2 + (-1 - v_{A2})^2 = 12.5$$

$$2v_{A2}^2 + 2v_{A2} + 1 - 12.5 = 0$$

$$v_{A2}^2 + v_{A2} - 5.75 = 0$$

$$v_{A2} = \frac{-1 \pm [1-4(1)(-5.75)]^{1/2}}{2}$$

$$= \frac{-1 \pm 4.9}{2} = \underline{1.95\ m\cdot s^{-1}},\ -2.95\ m\cdot s^{-1}$$

$$v_{B2} = -1 - v_{A2} = \underline{-2.95\ m\cdot s^{-1}},\ 1.95\ m\cdot s^{-1}$$

Here the underlined roots are physically relevant, the others corresponding to the case when the two bodies miss each other but a loss of energy is nonetheless suffered as a result of the 'collision'.

Example 8

A 200 g block moves to the right at a speed of 100 cm·s^{-1} and meets a 400 g block moving to the left with a speed of 80 cm·s^{-1}. Find the final velocity of each block if the collision is elastic.

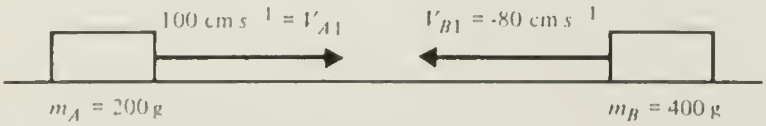


Figure 8-3

Solution:

Refer to Fig. 8-3 for the initial configuration.

$m_A = 200\ g;$ $\qquad 100\ cm\cdot s^{-1} = v_{A1}$

$v_{B1} = -80\ cm\cdot s^{-1}\ ;$ $\qquad m_B = 400\ g$

For an elastic collision, we have, from the text derivation,

$$v_{A2} = \frac{2m_Bv_{B1} + v_{A1}(m_A - m_B)}{m_A + m_B}$$

$$= \frac{2(400\ g)(-80\ cm\cdot s^{-1}) + 100\ cm\cdot s^{-1}\ (200\ g - 400\ g)}{200\ g + 400\ g}$$

$$= -140\ cm\cdot s^{-1}$$

$$v_{B2} = \frac{2m_Av_{A1} - v_{B1}\ (m_A - m_B)}{m_A + m_B}$$

$$= \frac{2(200\ g)(100\ cm\cdot s^{-1}) - (-80\ cm\cdot s^{-1})(200\ g - 400\ g)}{200\ g + 400\ g}$$

$$= 40\ cm\cdot s^{-1}$$

Example 9

A 200 g block (A) moving to the right at 100 $cm\cdot s^{-1}$ hits a 100 g block (B) moving to the right at 50 $cm\cdot s^{-1}$. Find each final velocity if the collision is elastic.

Solution:

Counting, as before, all velocities to the right as positive,

$m_A = 200\ g$ $\qquad v_{A1} = 100\ cm\cdot s^{-1}$

$m_B = 100\ g$ $\qquad v_{B1} = 50\ cm\cdot s^{-1}$

Thus we have

$$v_{A2} = \frac{2m_Bv_{B1} + v_{A1}\ (m_A - m_B)}{m_A + m_B}$$

$$= \frac{2(100\ g)\ 50\ cm\cdot s^{-1} + 100\ cm\cdot s^{-1}(200\ g - 100\ g)}{200\ g + 100\ g}$$

$$= 66.7\ cm\cdot s^{-1}$$

$$v_{B2} = \frac{2m_A v_{A1} - v_{B1}(m_A - m_B)}{m_A + m_B}$$

$$= \frac{2(200\ g)(100\ cm\cdot s^{-1}) - 50\ cm\cdot s^{-1}(200\ g - 100\ g)}{200\ g + 100\ g}$$

$$= 116\ cm\cdot s^{-1}$$

Example 10

A 200 g body traveling at 100 $cm\cdot s^{-1}$ hits a 500 g body at rest. Find the final velocities if the collision is elastic.

Solution

Referring to Fig. 8-4,

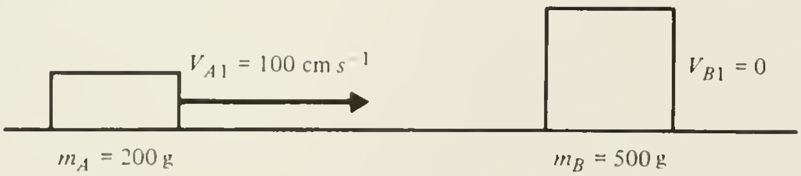


Figure 8-4

we have, for the case of $v_{B1} = 0$,

$$v_{A2} = v_{A1}\ \frac{m_A - m_B}{m_A + m_B} = 100\ cm\cdot s^{-1}\ \frac{200 - 500}{700}$$

$$= -\ 43\ cm\cdot s^{-1}$$

$$v_{B2} = v_{A1}\ \frac{2m_A}{m_A + m_B} = 100\ cm\cdot s^{-1}\ \frac{400}{700}$$

$$= 57\ cm\cdot s^{-1}$$

Example 11

Sand is dropped on a kitchen scale at the uniform rate of 100 $g \cdot s^{-1}$ from a height of $h = 1$ m. Find the force on the scale and its reading if it is calibrated in kg.

Solution:

In a time Δt a mass Δm of sand hitting the scale has its velocity changed from zero to v,

$$v^2 = v_0{}^2 - 2gh; \qquad v_0 = 0$$

$$v = (2gh)^{1/2} \quad \text{(= velocity gained in free fall through distance h)}$$

to $v = 0$ at impact. The change in momentum of the sand upon impact is

$$\Delta p = \Delta mv = \Delta m(2gh)^{1/2}$$

This change of momentum is produced by the impulse of the upward force of the scale on the sand,

$$J = \int Fdt = F\Delta t = \Delta p = \Delta m(2gh)^{1/2}$$

$$F = \frac{\Delta m}{\Delta t}\,(2gh)^{1/2} = (0.1\ kg \cdot s^{-1})(2 \cdot 9.8\ m \cdot s^{-1} \cdot 1\ m)^{1/2}$$

$$= 0.44\ N$$

The scale is calibrated so that a force F is registered as its corresponding mass:

$$F = mg; \qquad m = \frac{F}{g} = \frac{0.44\ N}{9.8\ m \cdot s^{-2}} = .045\ kg$$

$$= 45\ g \qquad \text{(45 grams)}$$

Example 12

Sand is dropped on a conveyor belt at the rate of 10 $kg \cdot s^{-1}$. What power is required to keep the belt moving at a constant velocity of $v = 1\ m \cdot s^{-1}$?

Solution:

In a time Δt an impulse $F\Delta t$ must be delivered to change the momentum of sand from $p_1 = 0$ in the belt direction to $p_2 = \Delta mv$ in the belt direction, that is

$$F\Delta t = \Delta p = p_2 - p_1 = \Delta mv$$

or

$$F = \frac{\Delta m}{\Delta t} v = (10 \text{ kg}\cdot s^{-1})(1 \text{ m}\cdot s^{-1})$$

$$= 10 \text{ N}$$

$$P = Fv = 10 \text{ N } (1 \text{ m}\cdot s^{-1}) = 10 \text{ watts}$$

To get the sand up to velocity, friction must act and energy is dissipated. The rate at which kinetic energy is added to the sand is

$$\frac{d}{dt}\left(\frac{1}{2} mv^2\right) = \frac{1}{2}\frac{dm}{dt} v^2 = \frac{1}{2}\left(\frac{dm}{dt} v\right) v = \frac{1}{2} Pv$$

The rest of the emergy is dissipated in friction.

Example 13

A 300 g ball is thrown a horizontal distance of 80 m as shown in Fig. 8-5. The initial projection angle is 45°. The thrower's hand is in contact with the ball for 0.3 s. Find the average force on the ball during contact with the thrower. Neglect air resistance.

Solution:

Referring to Fig. 8-5

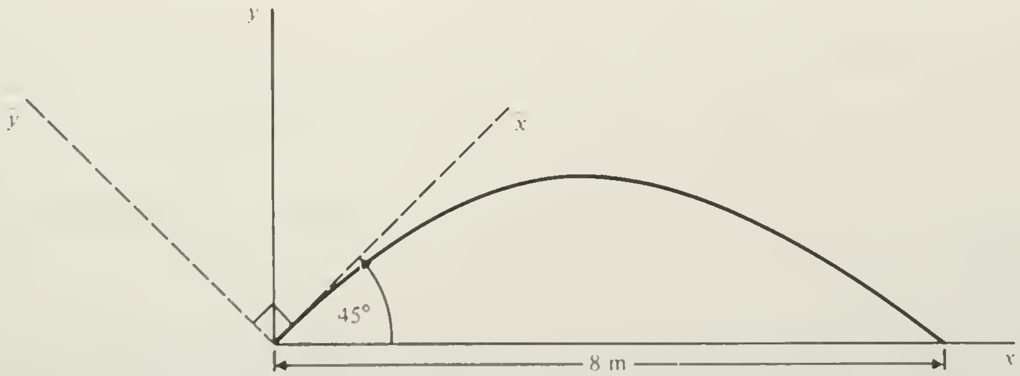


Figure 8-5

we have

$$x = v_{ox}t = \frac{v_o t}{(2)^{1/2}} \qquad v_{ox} = v_o \cos 45^\circ = \frac{v_o}{(2)^{1/2}}$$

$$y = v_{oy}t - 1/2\ gt^2 = \frac{v_o}{(2)^{1/2}} - 1/2\ gt^2$$

Setting y = 0 we can solve for the time t of impact,

$$t = \frac{2\ v_o}{(2)^{1/2} g}$$

yielding the range

$$x = \frac{v_o t}{(2)^{1/2}} = \frac{v_o}{(2)^{1/2}} \frac{2 v_o}{(2)^{1/2} g} = \frac{v_o^2}{g}$$

The initial velocity is thus

$$v_o = (gx)^{1/2}$$

We now change to the tilted $\bar{x}$, $\bar{y}$ coordinate system and consider the $J_{\bar{x}}$ impulse; the impulse necessary to give the ball this initial velocity (in the $\bar{x}$ direction) is

$$\int_o^b F_{\bar{x}} dt = F_{\bar{x},av}\ \Delta t = \Delta p = mv_o$$

$$F_{\bar{x},av} = \frac{mv_o}{\Delta t} = \frac{(0.3\ \text{kg})[9.8\ \text{m}\cdot\text{s}^{-1}\cdot\ 80\ \text{m}]^{1/2}}{0.3\ \text{s}}$$

$$= 28\ \text{N}$$

where Δt is time the thrower's hand is in contact with the ball.

Example 14

A 0.5 kg mass of putty is hurled at 1 $\text{m}\cdot\text{s}^{-1}$ against a 0.5 kg mass attached to a spring as shown in Fig. 8-6. The mass attached to the spring slides across the frictionless horizontal surface, depressing the spring a maximum distance of 10 cm. Find the spring constant if the putty sticks to the mass on the spring.

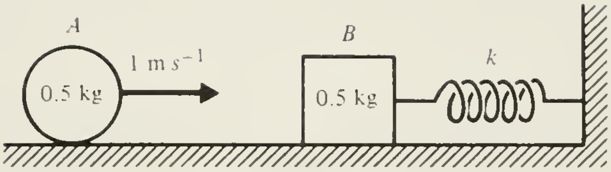


Figure 8-6

Solution:

The collision is inelastic but momentum is conserved during the interval from just before the collision to just after the collision, before the external force of the spring begins to act, i.e., before the spring is compressed appreciably. Letting

$v_{1A} = 1\ m \cdot s^{-1}$ $\qquad$ $v_{1B} = 0$

$m_A = 0.5\ kg = m_B$ $\qquad$ $v_{2A} = v_{2B} = v_2$

the conservation of momentum relation is

$$m_A v_{1A} = (m_A + m_B) v_2 = 2 m_A v_2$$

$$v_2 = 1/2\ v_{1A} = 0.5\ m \cdot s^{-1}$$

Just after the collision, mechanical energy is conserved, the dissipative forces between the putty and the mass B having ceased to act. The initial energy, just after the collision is

$$E_1 = 1/2(m_A + m_B) v_2^2$$

$$= 1/2(.5\ kg + .5\ kg)(0.5\ m \cdot s^{-1})^2$$

$$= 0.125\ J$$

When the velocity of the putty stuck to the mass becomes zero and the spring is maximally compressed,

$$E_2 = 1/2\ kx^2.$$

Conserving energy, we have

$$1/2\ kx^2 = 0.125\ J$$

$$k = \frac{2(0.125\ J)}{(0.1\ m)^2} = 25\ J \cdot m^{-2}$$

$$= 25\ N \cdot m^{-1}$$

Example 15

A 135 kg defensive lineman traveling 5 $m \cdot s^{-1}$ makes a completely inelastic collision with an 85 kg quarterback at rest. They are observed to slide 2 m on wet grass.

(a) What is the coefficient of friction between the grass and the players?
(b) What energy is dissipated in the collision?

Solution:

Letting

$$m_A = 135 \text{ kg} \qquad v_{A1} = 5 \text{ m}\cdot\text{s}^{-1} \qquad v_{A2} = v_{B2} = v_2$$

$$m_B = 85 \text{ kg} \qquad v_{B1} = 0$$

the momentum conservation condition is

$$m_A v_{A1} = (m_A + m_B)v_2$$

where v_2 is the velocity of the two players after the collision. Thus we have

$$v_2 = \frac{m_A}{m_A + m_B} v_{A1} = \frac{135}{135 + 85} 5 \text{ m}\cdot\text{s}^{-1}$$

$$= 3.07 \text{ m}\cdot\text{s}^{-1}$$

(Although the two players are not a completely isolated system, the external forces are so small compared to the forces that they exert on each other, that conservation of momentum is a reasonable approximation.)

From now on one can proceed in various ways to the coefficient of friction. We choose to use the work-energy relation, which here implies that the work of the frictional force, $-\mu\eta s$, is equal to the increase in mechanical energy from the moment just after collision to the end of the slide,

$$-\mu\eta s = -\mu(m_A + m_B)gs = 0 - 1/2(m_A + m_B)v_2^2$$

or

$$\mu = \frac{v_2^2}{2gs} = \frac{(3.07 \text{ m}\cdot\text{s}^{-1})^2}{(2)(9.8 \text{ m}\cdot\text{s}^{-2})(2 \text{ m})}$$

$$= .24$$

The energy before collision is

$$E_1 = 1/2\ m_A v_{A1}^2 = 1/2(135 \text{ kg})(5 \text{ m}\cdot\text{s}^{-1})^2 = 1687 \text{ J}$$

The energy just after collision is

$$E_2 = 1/2(m_A + m_B)v_2^2 = 1/2(135 \text{ kg} + 85 \text{ kg})(3.07 \text{ m}\cdot\text{s}^{-1})^2$$

$$= 1037 \text{ J}$$

The energy dissipated in the collision is thus

$$E_1 - E_2 = 1687 - 1037 = 650 \text{ J}$$

Example 16

A body of mass m = 100 g with a velocity 10 $\text{cm}\cdot\text{s}^{-1}$ hits another identical body at rest and the two recoil as shown in Fig. 8-7.

(a) Find the velocity of each mass after the collision.
(b) Is the collision elastic?
(c) Find the dissipated energy.

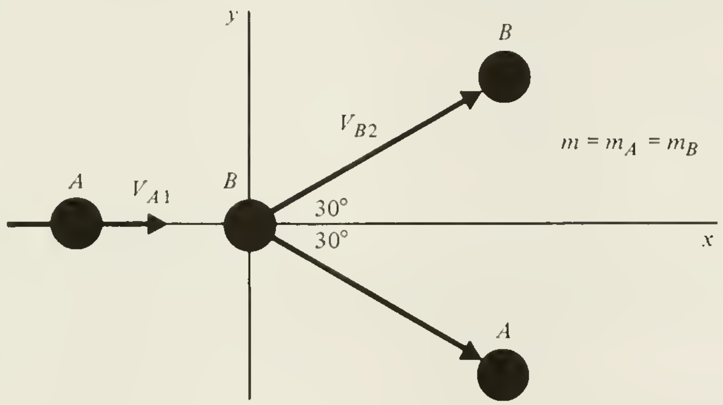


Figure 8-7

Solution:

This collision is in a plane rather than along a line as in the previous problems, so each component of momentum must be separately conserved.

The conservation of momentum yields $\mathbf{p}_1 = \mathbf{p}_2$, or in components,

(x comp) $\quad mv_{A1} = mv_{A2} \cos 30^\circ + mv_{B2} \cos 30^\circ$

(y comp) $\quad 0 = mv_{B2} \sin 30^\circ - mv_{A2} \sin 30^\circ$

The last relation implies $v_{B2} = v_{A2}$. The first then yields

$$v_{A1} = 2v_{A2} \cos 30^\circ$$

(a) $\quad v_{A2} = (2 \cos 30^\circ)^{-1} v_{A1} = .58\ v_{A1} = 5.8 \text{ cm}\cdot\text{s}^{-1}$

The initial energy is

$$E_1 = 1/2\ mv_{A1}^2$$

and the final energy is, using $v_{B2} = v_{A2}$,

$$E_2 = 1/2\ mv_{A2}^2 + 1/2\ mv_{B2}^2 = 2(1/2\ mv_{A2}^2)$$

$$= 1/2(2m)(.58v_{A1})^2$$

$$= .67(1/2\ mv_{A1}^2)$$

$$= .67\ E_1$$

Thus the collision is inelastic and the dissipated energy is

$$E_1 - E_2 = (1 - .67)1/2\ mv_{A1}^2$$

$$= (.33)(1/2)(0.1\ kg)(0.1\ m\cdot s^{-1})^2$$

$$= 1.65 \times 10^{-4}\ J$$

Example 17

A bullet weighing .02 lb is fired with a muzzle velocity $v = 2700$ ft s^{-1} into a ballistic pendulum weighing 20 lbs. Find the maximum height through which it rises.

Solution:

Referring to Fig. 8-8,

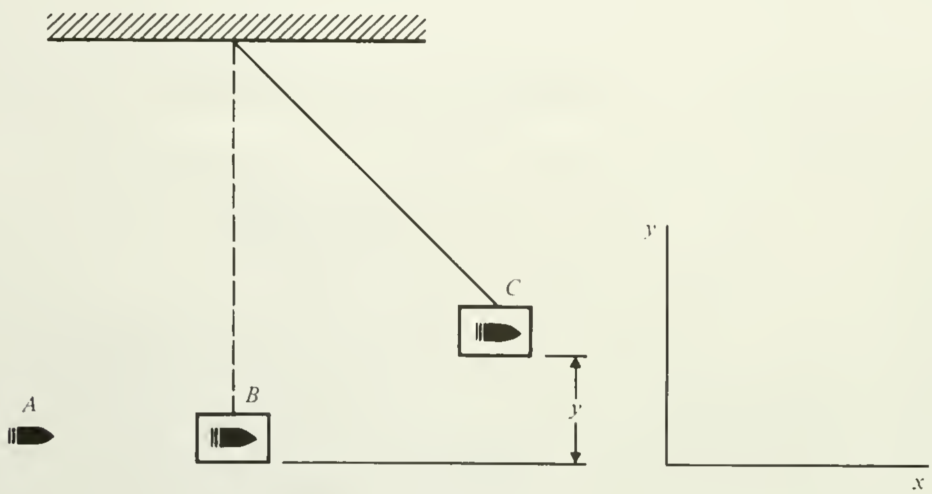


Figure 8-8

we let

m = bullet mass   M = block mass

v = velocity of bullet   V = velocity of bullet plus block just after collision

Consider the system of bullet plus block between time A when the bullet approaches the block to the time B just after collision when the system is about to start its upward swing. In this interval only internal forces act in the x direction and p_x is conserved

$$mv = (m + M)V; \qquad V = \frac{m}{m + M} v$$

In this interval mechanical energy is not conserved; the collision of bullet and block is inelastic, energy being dissipated as the bullet drills the block.

Now consider the interval between B and C. During this time all forces are conservative (mg) or do no work (T). Thus mechanical energy is conserved,

$$E_1 = 1/2(m + M)V^2 = (m + M)gy = E_2$$

Thus

$$y = \frac{V^2}{2g} = \frac{1}{2g}\left(\frac{m}{m + M}\right)^2 v^2$$

$$= \frac{1}{2 \cdot 32 \text{ ft}\cdot\text{s}^{-2}}\left(\frac{.02}{.02 + 20}\right)^2 (2700 \text{ ft}\cdot\text{s}^{-1})^2$$

$$= 0.11 \text{ ft.}$$

Example 18

A 75 kg man stands on ice and shoots a machine gun at 120 shots per minute for a 10 s burst. The mass of each bullet is 10 g and the muzzle velocity is 800 $\text{m}\cdot\text{s}^{-1}$. Find the average force on the man, his velocity after the burst, and how far he has recoiled from his initial position.

Solution:

The impulse delivered to the man is,

$$J = \int F dt = F_{av}\Delta t = \Delta p,$$

which is equal in magnitude to the impulse delivered to the bullets,

$$F_{av} = [\frac{\Delta p}{\Delta t}]_{man} = [\frac{\Delta p}{\Delta t}]_{bullets}.$$

In a time of 1 min/120 shots = 0.5 s a change $\Delta p = mv$ of momentum occurs where m is the bullet mass and v the muzzle velocity. This is approximately true when the man's velocity is small compared to the muzzle velocity. Thus

$$F_{av} = \frac{mv}{.5\ s} = (.010\ kg)(800\ m\cdot s^{-1})(.5\ s)^{-1} = 16\ N.$$

If M is the mass of the man, his acceleration is

$$a = \frac{F}{M} = \frac{16\ N}{75\ kg} = .21\ m\cdot s^{-2}.$$

His velocity is thus

$$v = at = 0.21\ m\cdot s^{-2}(10\ s) = 2.1\ m\cdot s^{-1}$$

and he recoils a distance

$$x = 1/2\ at^2 = 1/2(.21\ m\cdot s^{-1})(10\ s)^2$$

$$= 10.5\ m.$$

In the above solution certain approximations have been made. To make these explicit, consider the exact rocket equation

$$m\ \frac{dv}{dt} = v_r\ \frac{dm}{dt}$$

where in this case

m = mass of man + gun + unfired ammunition

$\frac{dm}{dt}$ = rate at which mass (bullets) are rejected

v_r = velocity of the bullets with respect to the gun

If m >> mass of the ammunition, $m \simeq M$, the mass of the man and gun, and

$$F = M\ \frac{dv}{dt} = v_r\ \frac{dm}{dt} = 800\ m\cdot s^{-1}\ \frac{.010\ kg}{0.5\ s} = 16\ N$$

as before.

Example 19

A 100 kg man begins to walk at 1 $m \cdot s^{-1}$ relative to the deck of a 200 kg boat. Their total momentum is zero. (Both are initially at rest.)

(a) Find the velocity of the boat relative to the water.

(b) Show that the center of mass of the man and the boat remains stationary.

Solution:

(a) Referring to Fig. 8-9, we see that

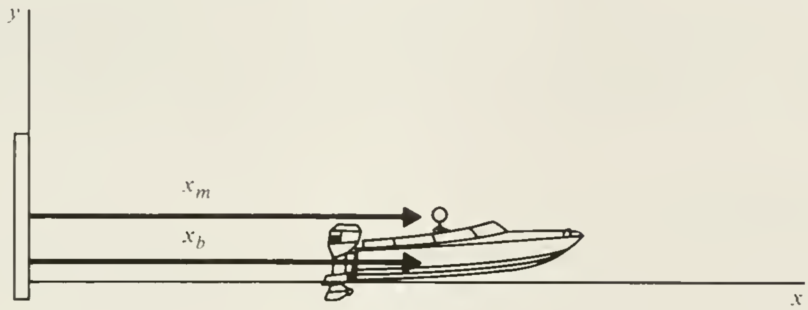


Figure 8-9

the momentum of the system man plus boat is conserved if the friction between the boat and the water is neglected. Thus

$$m_b v_b + m_m v_m = 0$$

where m_b = mass of boat and m_m = mass of man. The velocity of the man relative to the boat is ($v_{mb} = v_{me} + v_{eb} = v_{me} - v_{be}$ where e = earth:)

$$v_m - v_b = 1 \ m \cdot s^{-1}$$

Substituting the last relation into the first condition, we have

$$m_b v_b + m_m(1 \ m \cdot s^{-1} + v_b) = 0$$

or

$$v_b = \frac{-m_m}{m_b + m_m} 1 \ m \cdot s^{-1} = \frac{-100}{300} 1 \ m \cdot s^{-1}$$

$$= -0.33 \ m \cdot s^{-1}$$

If R is the center of the mass coordinate, we have

(b) $$m_m x_m + m_b x_b = R(m_b + m_m).$$

Solving for R and taking the time derivative we have

$$V = \frac{dR}{dt} = \frac{1}{m_b + m_b}(m_b v_b + m_m v_m) = 0$$

by the conservation of momentum condition.

Example 20

A 5 kg mass traveling 1 m·s^{-1} to the right makes a totally inelastic collision with a 3 kg mass traveling to the left at 5 m·s^{-1}. Find the velocity of the center of mass of the system before and after the collision.

Solution:

Refer to Fig. 8-10:

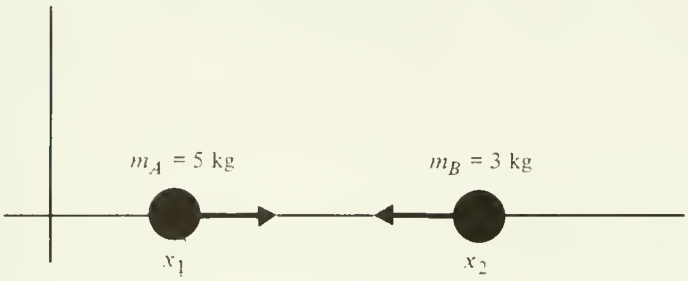


Figure 8-10

Before the collision, the center-of-mass position and velocity are

$$R_1 = \frac{1}{m_A + m_B}(m_A x_A + m_B x_B)$$

$$V_1 = \frac{dR}{dt} = \frac{1}{m_A + m_B}(m_A v_{A1} + m_B v_{B1})$$

$$= \frac{1}{8\ \text{kg}}[(5\ \text{kg})(1\ \text{m}\cdot\text{s}^{-1}) - (3\ \text{kg})(5\ \text{m}\cdot\text{s}^{-1})]$$

$$= -\ 1.25\ \text{m}\cdot\text{s}^{-1}$$

After the collision the masses stick together and their common velocity v_2 is given by $(m_A + m_B)v_2 = m_A v_{A1} + m_B v_{B1}$

$$v_2 = \frac{m_A v_{A1} + m_B v_{B1}}{m_A + m_B} = v_1 = -1.25 \text{ m s}^{-1}$$

The velocity of the center of mass remains the same because no external forces act on the system.

Example 21

Find the center of mass coordinates (x,y) of the three mass points
m_1 = 1 kg at (1 m, 2 m)
m_2 = 3 kg at (0, 1 m)
m_3 = 4 kg at (3 m, 1 m)
and plot the results.

Solution:

$$X = \frac{1}{m_1 + m_2 + m_3} (m_1 x_1 + m_2 x_2 + m_3 x_3)$$

$$= \frac{1}{8} (1(1) + 3(0) + 4(3)) = 1.63 \text{ m}$$

$$Y = \frac{1}{m_1 + m_2 + m_3} (m_1 y_1 + m_2 y_2 + m_3 y_3)$$

$$= \frac{1}{8} [1(2) + 3(1) + 4(1)] = 1.13 \text{ m}$$

(See Fig. 8-11, where CM marks the center of mass)

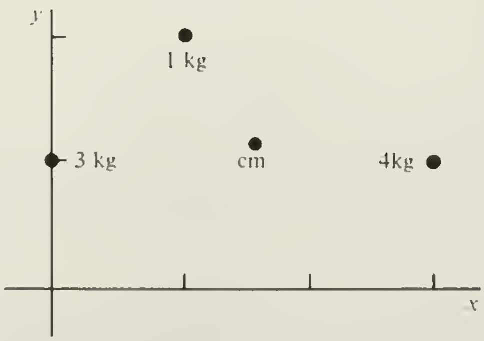


Figure 8-11

Example 22

Show that the coordinates of the center of mass of a composite object composed of parts A and B with masses m_A and m_B are given by

$$X_{AB} = \frac{m_A X_A + m_B X_B}{m_A + m_B}$$

$$Y_{AB} = \frac{m_A Y_A + m_B Y_B}{m_A + m_B}$$

Solution:

The sum of $m_i x_i$ over all the mass points composing the composite object may be split in the sum of $m_i x_i$ over the points in part A and the sum $m_i x_i$ over the points in part B,

$$X_{AB} = \frac{1}{M}\sum_{Ai} m_i x_i = \frac{1}{M}\sum_{Ai} m_i x_i + \frac{1}{M}\sum_{Bi} m_i x_i$$

$$= \frac{1}{M}(m_A X_A + m_B X_B) = \frac{m_A X_A + m_B X_B}{m_A + m_B}$$

with a similar derivation for Y_{AB}.

Example 23

The plate in Fig. 8-12a has a hole cut out of it at B. Find its center of mass.

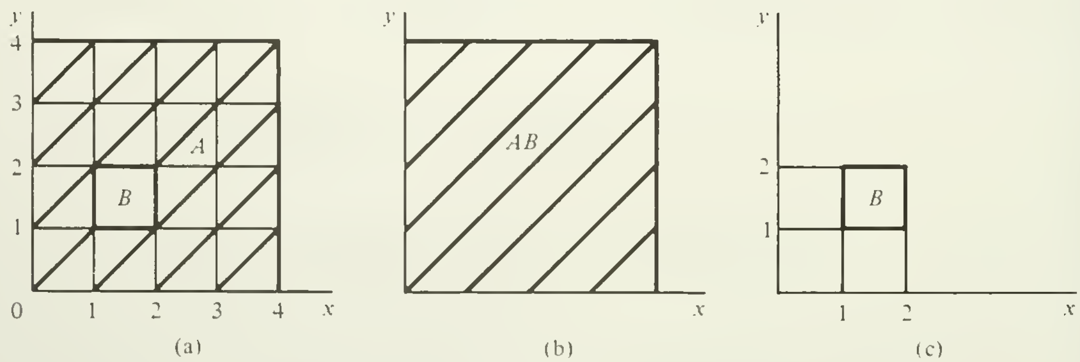


Figure 8-12

Solution:

The object A of Fig. 8-12a (slant shaded) may be considered a part of the composite object AB of Fig. 8-12b where B is the missing material of the square hole as shown in Fig. 8-12c.

$X_{AB} = 2 \qquad Y_{AB} = 2$

$X_B = 1.5 \qquad Y_B = 1.5$

From the previous problem, we find that

$$(M_A + M_B)X_{AB} = M_AX_A + M_BX_B$$

yielding

$$X_A = \frac{M_A + M_B}{M_A} X_{AB} - \frac{M_B}{M_A} X_B$$

$$= \frac{16}{15} - \frac{1}{15} 1.5 = 2.03$$

Similarly,

$$Y_B = 2.03$$

The center of mass of the plate is shifted slightly off center, away from the hole.

QUIZ

1. What is the momentum of a 1000 kg car travelling 60 $km \cdot hr^{-1}$?

Answer: 17,000 $kg \cdot m \cdot s^{-1}$

2. A 1000 kg automobile travelling 60 $km \cdot hr^{-1}$ brakes suddenly and the wheels lock as the car comes to a stop in 5 s.
   (a) What average force acted on the car?
   (b) What was the average acceleration?

Answer: (a) 3,300 N (b) 3.3 $m \cdot s^{-2}$

3. A 7500 kg truck moving at 50 $km \cdot hr^{-1}$ hits a stopped 3000 kg truck. They lock bumpers and skid to a stop.
   (a) What is the velocity of the two trucks just after the collision?
   (b) If the coefficient of friction is 0.25, how far do they skid?

Answer: 35 $km \cdot hr^{-1}$, 20 m

4. A bullet of mass 15 g strikes a ballistic pendulum of mass 3 kg. The center of mass of the pendulum rises a vertical distance of 10 cm. Find the velocity of the bullet.

Answer: 281 $m \cdot s^{-1}$

9
ROTATIONAL MOTION

OBJECTIVES

In this chapter you will learn the laws of motion for rigid bodies and apply them to bodies rotating about an axis of fixed direction. This chapter is a direct application to rotational motion of methods you developed in chapters 2 through 8. Your objectives are to:

> Master radian measure for the angle describing a body's orientation. Convert radians to revolutions.
>
> Describe rotational motion by use of angular velocity and angular acceleration. Convert rpm to $rev \cdot s^{-1}$ to $rad \cdot s^{-1}$.
>
> Solve problems involving constant angular acceleration. This will be closely analogous to material of chapter 2. The methods will be the same. The difference is that you will be dealing with the angular quantities θ, $d\theta/dt = \omega$, and $d\omega/dt = \alpha$ instead of the linear quantities x, $dx/dt = v$, and $dv/dt = a$.
>
> Relate linear quantities to angular quantities. Here you will apply some of the circular motion material of chapters 3 and 6.
>
> Solve problems involving rotational energy and the conservation of rotational energy plus translational energy for simple coupled systems.
>
> Calculate the moment of inertia I of a rigid body by use of integration and the parallel axis theorem, for simple geometries such as square plates and cylinders.
>
> Solve problems involving torques acting on bodies rotating about an axis of fixed direction.
>
> Solve problems involving angular impulse and angular momentum, using methods similar to those of chapter 8.

REVIEW

The orientation of a rigid body rotating about an axis of fixed direction is described by a single angle θ. It will be crucial in some of the equations of this chapter that θ be measured in radians (rad),

$$\theta = \frac{s}{r}$$

where s is the arc subtended by θ on a circle of radius r. A complete revolution (rev) is given by an arc of length $2\pi r$. Thus

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}$$

$$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ = (2\pi)^{-1} \text{ rev}$$

The angular velocity is

$$\omega = \frac{d\theta}{dt}$$

and may be given in $\text{rad}\cdot s^{-1}$, $\text{deg}\cdot s^{-1}$, $\text{rev}\cdot s^{-1}$, or rpm (= $\text{rev}\cdot\text{min}^{-1}$). Useful conversions are

$$1 \text{ rev}\cdot s^{-1} = 2\pi \text{ rad}\cdot s^{-1}$$

$$1 \text{ rpm} = \frac{\text{rev}}{60 \text{ s}} = \frac{2\pi \text{ rad}\cdot s^{-1}}{60}$$

The angular acceleration is given by

$$\alpha = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} = \frac{d^2\theta}{dt^2}$$

and has units $\text{rev}\cdot s^{-2} = 2\pi \text{ rad}\cdot s^{-2}$, etc.

If the <u>acceleration α is constant</u>, the basic equations of rotational motion are

$$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2 \qquad \theta(o) = \theta_o$$

$$\omega = \omega_o + \alpha t \qquad \omega(o) = \omega_o$$

$$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$$

The relation between linear and angular quantities is

$$s = r\theta \qquad \underline{(\theta \text{ in rad})}$$

where s is the arc subtended by a point in a rigid body a distance r from the axis as the body rotates through θ. This point has speed

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \qquad (\omega \text{ in rad}\cdot s^{-1})$$

Its component of acceleration, a_t, along the arc is

$$a_t = \frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = r\alpha \qquad (\alpha \text{ in rad}\cdot s^{-2})$$

Note radian measure must be used in the last three formulas. From the results of chapter 5, the centripetal acceleration toward the axis is

$$a_n = \frac{v^2}{r} = \omega^2 r$$

The rotational counterpart to mass is the moment of inertia I,

$$I = \sum mr^2 = \int (r^2)dm = \int (r^2)\rho dV$$

where the sum is over all points in the body and r is the perpendicular distance to the axis of rotation. I depends on the axis of rotation. If the body is continuous, I must be evaluated by integration. If the body has uniform density,

$$I = \rho\int(r^2)dV = \frac{M}{V}\int(r^2)dV.$$

Moments of inertia of various simple solids are given in the text.

The parallel axis theorem helps in calculating moments of inertia. If I_{cm} is the moment of inertia about the cm and I_p is the moment about a parallel axis a distance d from the cm,

$$I_p = I_{cm} + Md^2$$

where M is the body's mass.

As shown in Fig. 9-1, a force F on a body has a <u>point of application</u> and a <u>line of action.</u> The <u>moment</u> or <u>torque</u> of F about an arbitrary point 0 is

$$\Gamma_o = FL = Fr\sin\theta = F_n r$$

where F_n is the projection of F perpendicular to r (see Fig. 9-1).

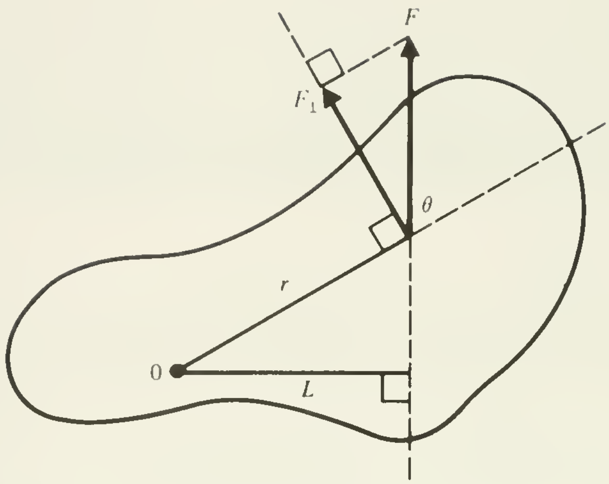


Figure 9-1

L is called the lever or moment arm of F. The torque Γ_o is counted positive if it tends to make the body rotate counterclockwise, negative otherwise. The torque of F in Fig. 9-1 is positive.

A more general definition of a moment is given by the vector

$$\Gamma_o = \mathbf{r} \times \mathbf{F}$$

whose magnitude is

$$\Gamma_o = rF \sin \theta = F_n r$$

The direction of Γ_o is perpendicular to $\mathbf{r}$ and $\mathbf{F}$. By the right hand rule, it points out of the paper for the example in Fig. 9-1. (Curl the fingers of your right hand in the direction from $\mathbf{r}$ to $\mathbf{F}$, the shorter way; your thumb points in the direction of $\mathbf{r} \times \mathbf{F} = \Gamma_o$.)

The work done by a torque Γ moving a body through an angle $\Delta\theta$ is

$$W = \Gamma\Delta\theta$$

and the power is

$$P = \Gamma\omega$$

Newton's Laws applied to rigid body motion about a fixed axis imply that

$$\sum \Gamma = I\alpha$$

where the sum over Γ is the total torque about the fixed axis, I is the moment of inertia, and α the angular acceleration.

The angular momentum L of a point mass moving in a circle of radius r with velocity v is

$$L = mvr = m\omega r^2.$$

If these are added up for the points in a rotating rigid body, we find the total angular momentum is

$$L = I\omega$$

The angular impulse of a torque is

$$J_\theta = \int_{t_1}^{t_2} \Gamma dt = \int_{t_1}^{t_2} I\alpha dt = \int_{t_1}^{t_2} I \frac{d\omega}{dt} dt = \int_{\omega_1}^{\omega_2} I d\omega$$

$$= I\omega_2 - I\omega_1 = L_2 - L_1 = \Delta L$$

and yields the change in angular momentum. In differential form we find

$$\frac{dL}{dt} = I\alpha$$

or

$$\sum \Gamma = I\alpha = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

If the total external torque on a system is zero, $\sum \Gamma = 0$, the total angular momentum is conserved, $\Delta L = 0$.

PROBLEM-SOLVING STRATEGIES

The problems of this chapter are very similar to those of chapters 2-8. The difference is that they involve angular rather than linear quantities. As before, decide on a positive direction and stick to it. List all pertinent known and unknown quantities. Typical problems encountered involve:

Radian measure, conversions, and the relation between angular and linear quantities. See examples 1 and 2 below.

Equations of constant angular acceleration. See examples 3 and 4 below.

Finding the radial and tangential acceleration of a point in a rigid body. See examples 5 and 6.

Calculation of moments of inertia. See examples 7 through 9 below.

Rotational work and power, and the conservation of energy, including rotational energy for conservative systems. See examples 10 and 11.

Finding torques of forces. See examples 12 and 13.

Applications of $\Gamma = I\alpha$ to simple and coupled systems. Conservation of energy is also useful here. In the coupled systems there is often a relation between the linear coordinate of one component and the angular coordinate of the other component. See Examples 14 through 19 below.

Application of the parallel axis theorem. See Example 20.

Angular impulse and the conservation of angular momentum. See Examples 21 through 23.

EXAMPLES AND SOLUTIONS

Example 1

Referring to Fig. 9-2, find the angle θ in degrees and radians, and the arc length s'.

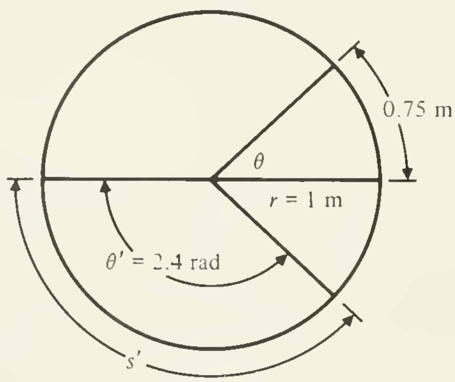


Figure 9-2

Solution:

$$\theta = \frac{s}{r} = \frac{0.75\ \text{m}}{1\ \text{m}} = 0.75\ \text{rad}$$

$$= 0.75\ \text{rad}\ \frac{360^\circ}{2\pi\text{rad}} = 43^\circ$$

$$s' = \theta' r = 2.4(1\ \text{m}) = 2.4\ \text{m}$$

The equation $s = \theta r$ is true only when θ is measured in radians.

Example 2

A phonograph record rotates at 33(1/3) rpm (revolutions per minute). Find its angular velocity in $rad \cdot s^{-1}$ and the speed of a point 5 cm from the center.

Solution:

$$1 \text{ rev} = 2\pi \text{ rad}$$

$$33(1/3) \text{ rpm} = \omega = 33(1/3) \text{ rev}(60 \text{ s})^{-1} \frac{2\pi \text{rad}}{\text{rev}}$$

$$\omega = 3.5 \text{ rad} \cdot s^{-1}$$

$$= 3.5 \ s^{-1}$$

$$v = \omega r = 3.5 \ s^{-1} \ (5 \text{ cm}) = 17.5 \text{ cm} \cdot s^{-1}.$$

Example 3

The record in the previous problem starts from rest and accelerates with constant angular acceleration to 33(1/3) rpm in 2 s. Find the angular acceleration and the angle turned through, in degrees and revs.

Solution:

For constant angular acceleration we have:

$$\theta = \theta_o + \omega_o t + \frac{1}{2} \alpha t^2$$

$$\omega = \omega_o + \alpha t$$

In this problem $\theta = 0$ and $\omega = 0$. Thus we have

$$\alpha = \frac{\omega - \omega_o}{t} = \frac{3.5 \text{ rad} \cdot s^{-1}}{2 \text{ s}} = 1.75 \text{ rad} \cdot s^{-2}$$

$$\theta = \frac{1}{2} \alpha t^2 = \frac{1}{2} 1.75 \text{ rad} \cdot s^{-2} \ (2s)^2 = 3.5 \text{ rad}$$

$$= 3.5 \text{ rad} \ \frac{360^\circ}{2\pi \text{ rad}} = 200^\circ$$

$$= 3.5 \text{ rad } \frac{1 \text{ rev}}{2\pi \text{ rad}} = 0.56 \text{ rev}$$

Example 4

The record player in the previous problem is shut off and the record decelerates uniformly from 33(1/3) rpm to a stop in 2 complete revolutions. Find the angular acceleration and the time it takes to come to a stop.

Solution:

$$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2 \qquad \theta_o = 0$$

$$\omega = \omega_o + \alpha t \qquad \omega_o = 33\frac{1}{3} \text{ rpm}$$

$$\omega^2 = \omega_o{}^2 + 2\alpha(\theta - \theta_o) \qquad \omega = 0 \text{ at } \theta = 2 \text{ rev}$$

The last equation is most convenient to use because all terms but α are known:

$$\alpha = \frac{\omega_2 - \omega_o{}^2}{2(\theta - \theta_o)} = \frac{-\omega_o{}^2}{2\theta} = \frac{-[33(1/3) \text{ rpm}]^2}{2(2 \text{ rev})}$$

$$= -278 \text{ rev}\cdot\text{min}^{-2} = -278 \text{ rev}\cdot\text{min}^{-2} \frac{2\pi\text{rad}}{\text{rev}} \left(\frac{\text{min}}{60 \text{ s}}\right)^2$$

$$= -0.485 \text{ rad}\cdot\text{s}^{-2}$$

$$= -0.077 \text{ rev}\cdot\text{s}^{-2}$$

From the middle equation

$$t = \frac{\omega - \omega_o}{\alpha} = \frac{-\omega_o}{\alpha} = \frac{-33(1/3)\text{rpm}}{-278 \text{ rev}\cdot\text{min}^{-2}}$$

$$= 0.12 \text{ min} = 7.2 \text{ s}$$

Example 5

Find, in Example 3, the components of the linear acceleration of a point on the edge of a 12 inch record

(a) 1 s after the turntable is turned on, and

(b) 3 s after the turntable is turned on. ($\alpha = 0$ after $t = 2s$)

Solution:

(a) $\omega = \omega_o + \alpha t$ $\qquad \omega_o = 0$

$a_t = \alpha r$ $\qquad \alpha = 1.75 \text{ rad}\cdot\text{s}^{-2}$

$a_n = \dfrac{v^2}{r}$ $\qquad v^2 = (\omega r)^2 = (\alpha t r)^2$

$r = 6$ in

Thus

$$a_t = 1.75 \text{ rad}\cdot\text{s}^{-2} \cdot 6 \text{ in} = 10.5 \text{ in}\cdot\text{s}^{-1}.$$

$$a_n = \frac{\alpha^2 t^2 r^2}{r} = (1.75 \text{ rad}\cdot\text{s}^{-2})^2 (1 \text{ s})^2\, 6 \text{ in}$$

$$= 18.4 \text{ in}\cdot\text{s}^{-2}$$

(b) For $t > 2$ s, the angular acceleration is zero, $a_t = 0$, and

$$a_n = \frac{v^2}{r}$$

The angular velocity at 3 s is the same as the velocity at t = 2 s when 33(1/3) rpm was reached,

$$\omega = \alpha t = 1.75 \text{ rad}\cdot\text{s}^{-2}(2 \text{ s})$$

$$= 3.5 \text{ rad}\cdot\text{s}^{-1}$$

$$v = r\omega = (6 \text{ in})(3.5 \text{ rad}\cdot\text{s}^{-1}) = 21 \text{ in}\cdot\text{s}^{-1}$$

$$a_n = \frac{(21 \text{ in}\cdot\text{s}^{-1})^2}{6 \text{ in}} = 73 \text{ in}\cdot\text{s}^{-2}$$

Example 6

A flywheel 1.0 m in diameter rotating with an initial velocity of 500 rpm is rotating at 1000 rpm after 20 s. Assuming constant acceleration find

(a) the angular acceleration,

(b) the angle through which the flywheel rotates between 500 and 1000 rpm, and

(c) the acceleration of a point on the rim.

Solution:

$$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2 \qquad \omega_o = 500 \text{ rpm}$$

$$\omega = \omega_o + \alpha t \qquad \omega = 1000 \text{ rpm}$$

$$\omega^2 = \omega_o{}^2 + 2\alpha(\theta - \theta_o)$$

(a) From the middle equation, we find

$$\alpha = \frac{\omega - \omega_o}{t} = \frac{1000 - 500}{20 \text{ s}} \frac{\text{rpm}}{\text{s}}$$

$$= \frac{500}{20} \cdot \frac{\text{rev}}{60 \text{ s}^2} = 0.42 \text{ rev}\cdot\text{s}^{-2}$$

$$= 0.42 \text{ rev } \left(\frac{1 \text{ min}}{60} \right)^{-2}$$

$$= 1{,}510 \text{ rev}\cdot\text{min}^{-2}$$

(b) Using this last result, we find

$$\theta - \theta_o = \frac{\omega^2 - \omega_o{}^2}{2\alpha} = \frac{(1000 \text{ rev}\cdot\text{min}^{-1})^2 - (500 \text{ rev}\cdot\text{min}^{-1})^2}{2(1{,}510 \text{ rev}\cdot\text{min}^{-2})}$$

$$= 248 \text{ rev}$$

(c) $a = \alpha r = (0.42)(2\pi)\text{rad}\cdot\text{s}^{-2}(0.5 \text{ m})$

$$= 1.32 \text{ m}\cdot\text{s}^{-2}.$$

Note in the forms $a = \alpha r$ and $v = \omega r$, α must be in $\text{rad}\cdot\text{s}^{-2}$ and ω must be in $\text{rad}\cdot\text{s}^{-1}$.

Example 7

Four small bodies of mass 0.5 kg are connected by light rods and form the shape of a square 1 m on a side. Find the moment of inertia of the system about an axis

(a) Perpendicular to the plane of the square and passing through one mass.
(b) Passing through two adjacent masses.
(c) Passing through two opposite masses.

Solution:

Referring to Fig. 9-3,

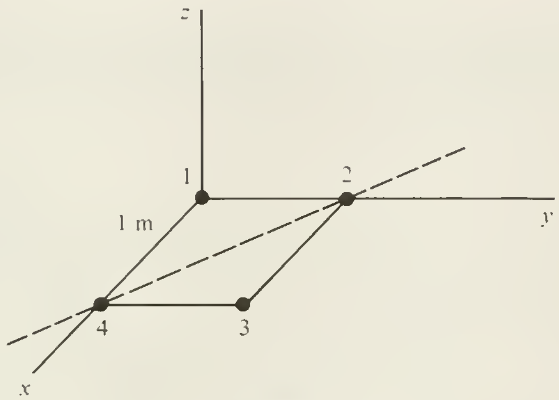


Figure 9-3

we first consider case (a) and take the axis to be the z-axis, finding

$$I_a = \sum mr^2 = m(r_1{}^2 + r_2{}^2 + r_3{}^2 + r_4{}^2)$$

where r is the <u>perpendicular distance</u> of each mass to the z-axis:

$r_1 = 0;$ $\quad r_2 = \text{side} = 1\ \text{m};$

$r = \text{diagonal} = 1.41\ \text{m};$ $\quad r_4 = \text{side} = 1\ \text{m}.$

Thus we have

$$I_a = (0.5\ \text{kg})[(1\ \text{m})^2 + (1.4\ \text{m})^2 + (1\ \text{m})^2]$$

$$= 2\ \text{kg}\cdot\text{m}^2.$$

For case (b) we take the axis to be the x-axis; then

$r_1 = 0;$ $\quad r_2 = 1\ \text{m};$ $\quad r_3 = 1\ \text{m};$ $\quad r_4 = 0.$

$$I_b = (0.5\ \text{kg})[(1\ \text{m})^2 + (1\ \text{m})^2]$$

$$= 1\ \text{kg}\cdot\text{m}^2.$$

For case (c) the axis is the dashed line in Fig. 9-3:

$$r_1 = (1/2)\text{diagonal} = 0.71 \text{ m} = r_3$$

$$r_2 = r_4 = 0$$

$$I_c = (0.5 \text{ kg})[(0.71 \text{ m})^2 + (0.71 \text{ m})^2]$$

$$= 0.5 \text{ kg}\cdot\text{m}^2.$$

Example 8

Find the moment of inertia of a thin rectangular uniform sheet of metal, of mass M and dimension L = length by w = width, about the x axis in Fig. 9-4.

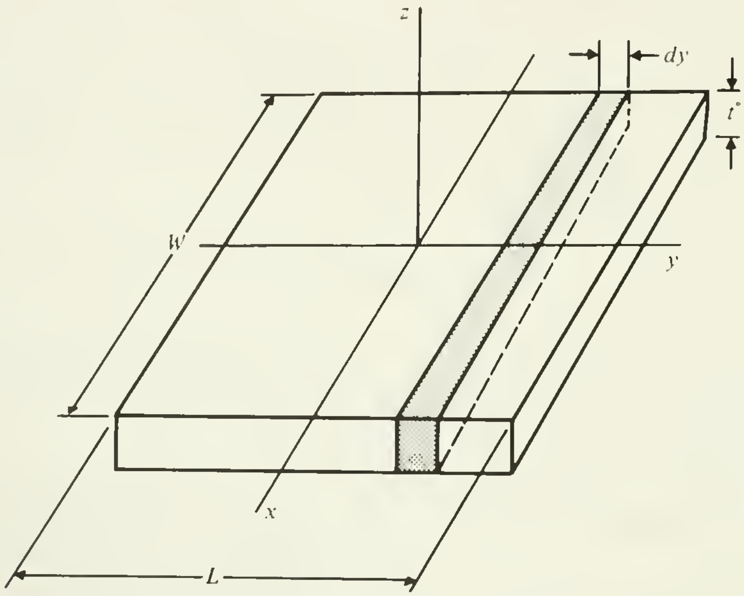


Figure 9-4

Solution:

Referring to Fig. 9-4, we sub-divide the plate into thin strips all parts of which have the same perpendicular distance r = y to the x axis. If the plate has thickness t,

$$\rho = \frac{M}{wLt} = \text{mass per unit volume}$$

$$dV = \text{volume of strips} = wt\ dy$$

$$I = \int \rho\ r^2\ dV = \rho \int y^2\ wt\ dy$$

$$= \frac{M}{wLt}\, wt \int_{-L/2}^{L/2} y^2 dy = \frac{M}{L} \frac{y^3}{3} \Big|_{-L/2}^{L/2} = \frac{ML^2}{12}$$

Example 9

A rod of mass m = 1 kg and length L = 1 m connects 2 small 1 kg masses. Find the moment of inertia of the composite system about an axis through the center, and perpendicular to the rod.

Solution:

According to the text the moment of inertia of the <u>rod</u> is

$$I_{rod} = \frac{1}{12} mL^2$$

(This result could have been inferred from Example 8.) To find the total moment we must add the moment of inertia of the masses at the ends,

$$I_{masses} = m(\frac{L}{2})^2 + m(\frac{L}{2})^2 = 2m(\frac{L}{2})^2$$

$$I = \frac{1}{12} mL^2 + 2m(\frac{L}{2})^2$$

$$= 0.58\ mL^2 = 0.58(1\ kg)(1\ m)^2$$

$$= 0.58\ kg{\cdot}m^2$$

Example 10

How much energy is dissipated when a 2 kg grinding wheel of radius 0.1 m is brought to rest from an initial velocity of 3000 rpm?

Solution:

The energy dissipated is the kinetic energy of the wheel,

$$K = \frac{1}{2} I\omega^2 \qquad\qquad I = \frac{1}{2} MR^2 \text{ (solid disk)}$$

$$\omega = 3000 \text{ rpm} = 3000\left(\frac{2\ \pi \text{ rad}}{60 \text{ s}}\right); \ M = 2 \text{ kg}; \text{ and } R = 0.1 \text{ m}.$$

$$K = \frac{1}{2}\left[\frac{1}{2}\, 2 \text{ kg } (0.1 \text{ m})^2\right]\left(\frac{3000 \cdot 2\ \pi \text{ rad}}{60 \text{ s}}\right)^2$$

$$= 493 \text{ J}.$$

Example 11

What is the average power dissipated in the last example if the grinding wheel is brought to rest in 10 rev? Assume constant angular acceleration.

Solution:

The average power dissipated is

$$\bar{P} = \frac{\Delta E}{\Delta t} = \frac{\Delta K}{\Delta t}$$

and we must calculate the time it takes the wheel to stop. First, from the initial data we calculate the acceleration:

$$\omega^2 = \omega_o^2 + 2(\theta - \theta_o)\alpha; \qquad \omega = 0$$

$$\theta - \theta_o = 10 \text{ rev}; \qquad \omega_o = 3000 \text{ rpm}$$

$$\alpha = \frac{-\omega_o^2}{2(\theta - \theta_o)} = \frac{-(3000 \text{ rpm})^2}{2(10 \text{ rev})} = -\ 4.5 \text{ x } 10^5 \text{ rev}\cdot\text{min}^{-2}.$$

Then, knowing the acceleration and the initial and final velocities, we calculate the stopping time,

$$\omega = \omega_o + \alpha t$$

$$t = \frac{\omega - \omega_o}{\alpha} = \frac{-\omega_o}{\alpha} = \frac{-3000 \text{ rpm}}{-4.5 \text{ x } 105 \text{ rev}\cdot\text{min}^{-2}}$$

$$= 6.7 \text{ x } 10^{-3} \text{ min} = 0.40 \text{ s}$$

The average power dissipated is

$$\frac{\Delta K}{\Delta t} = \frac{493 \text{ J}}{0.40 \text{ s}} = 1.23 \text{ kW}$$

Example 12

Find the sum of the moments, $\sum \Gamma_0$, of the forces **F** and **F′** shown in Fig. 9-5a about the point 0, in terms of the lever arms h and h′.

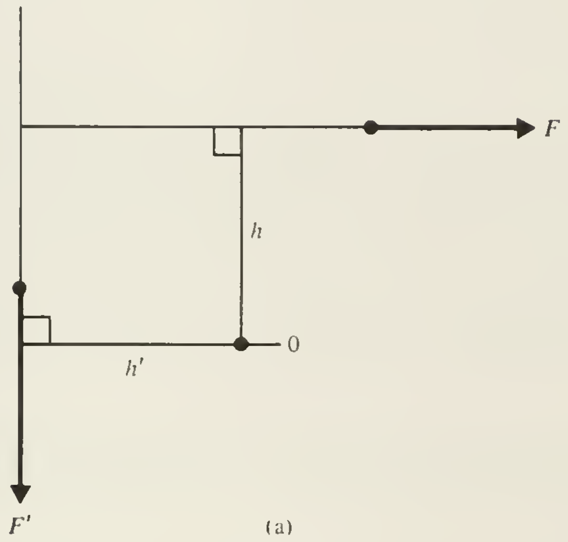


Figure 9-5a

Solution:

$$\sum \Gamma_0 = -Fh + h'F'$$

The moment of F about 0 is negative because F tends to produce a clockwise rotation about 0; correspondingly the moment of F′ about 0 is positive because it tends to cause a counterclockwise rotation about 0. In this case, it was most convenient to calculate the moments in terms of the lever arms h and h′ because they were directly given.

Example 13

Find the sum of the moments about 0 of the forces **F** and **F′** shown in Fig. 9-5b.

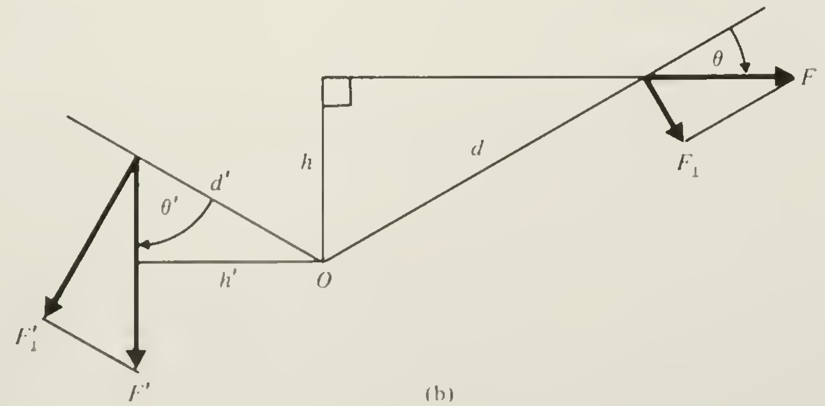


Figure 9-5b

Solution:

$$\sum \Gamma_o = -F \sin\theta\, d + F' \sin\theta' d'$$

$$= -F_n d + F_n' d'$$

$$= -Fh + F'h'$$

In this example the same forces as in Example 12 were oriented by giving the distance between the points of application and the origin, and the angle between this line and the force. The convenient way to calculate the moment is to find the component of each force perpendicular to the line d' and d. Since $h = d \sin\theta$ and $h' = d' \sin\theta'$, the result is the same.

Example 14

In example 11, what torque is necessary to produce the calculated acceleration of the grinding wheel?

Solution:

$$\Gamma = I\alpha; \qquad I = \frac{1}{2} MR^2$$

From Example 11,

$$\alpha = -\ 4.5 \times 10^5 \text{ rev}\cdot\text{min}^{-2} = -\ 4.5 \times 10^5 \times 2\pi \text{ rad} \times (60 \text{ s})^{-2}$$

$$= -\ 785 \text{ rad}\cdot\text{s}^{-1}$$

$$\Gamma = (\frac{1}{2} MR^2)\alpha = -\ \frac{1}{2}\, 2 \text{ kg } (0.1 \text{ m})^2 \times 785 \text{ s}^{-1}$$

$$= -\ 7.85 \text{ N}\cdot\text{m}.$$

The negative sign indicates that the torque opposes the flywheel's motion, tending to slow it down.

Example 15

In the last example, what work is done by the torque Γ as the grinding wheel is brought to rest?

Solution:

$$\theta = 10 \text{ rev} = 10(2\pi) \text{ rad}$$

$$W = \Gamma\theta = -\ 7.85 \text{ N}\cdot\text{m } (10 \cdot 2\pi \text{ rad})$$

$$= -\ 493 \text{ J.}$$

The work of Γ is negative because the angular displacement is opposite to the direction of Γ. The work done by Γ is equal to the <u>increase</u> in kinetic energy of the wheel; since this energy is decreased, W is negative.

Example 16

In the last example, what is the instantaneous power dissipated at t = 0 as the grinding wheel starts to slow down?

Solution:

The initial velocity is 3000 rpm. Thus

$$P = \Gamma\omega = \Gamma\omega_0 = -7.85 \text{ N}\cdot\text{m}\left[\frac{3000(2\ \pi)\text{rad}}{60 \text{ s}}\right]$$

$$= -2.46 \text{ kW}$$

The initial power <u>dissipated</u> is 2.46 kW, twice the average power dissipated as calculated in Example 11.

Example 17

In the last example, what tangential force must be applied to the rim of the grindstone to yield the torque of Example 14?

Solution:

$$\Gamma = Fr$$

$$F = \frac{\Gamma}{r} = \frac{-7.85 \text{ N}\cdot\text{m}}{0.1 \text{ m}} = -\ 78.5 \text{ N}$$

Example 18

A rod pivoted at an end has mass M = 1 kg and length 1 = 0.5 m. Neglect the friction at the pivot. It is released from rest in a horizontal position. What is its angular velocity when it is vertical?

Solution:

The system is conservative. The potential and kinetic energies are, referring to Fig. 9-6, and taking U = 0 at the initial horizontal position,

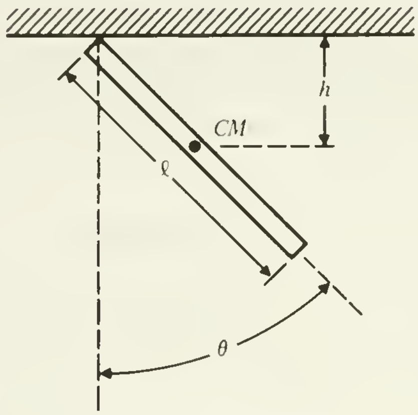


Figure 9-6

$$U = Mgh = -Mg\ \frac{L}{2}\ \cos\theta = \text{potential energy of CM.}$$

$$K = \frac{1}{2}\ I\omega^2 \qquad\qquad I = \frac{1}{3}\ ML^2$$

The initial energy E_i is

$$E_i = K_i + U_i = 0 \qquad\qquad (\theta_i = \frac{\pi}{2}\ \ \omega_i = 0)$$

and the final energy E_f is

$$E_f = K_f + V_f = \frac{1}{2}\ \omega_f{}^2 - Mg(L/2) = E_i = 0$$

The final angular velocity is thus

$$\omega_f = \left(\frac{MgL}{I} \right)^{1/2} = \left(\frac{3g}{L} \right)^{1/2} = 7.67 \text{ rad}\cdot\text{s}^{-1}$$

Example 19

A cord is wrapped around the rim of a uniform flywheel of radius R = 0.2 m and mass M = 10 kg. A mass of m = 10 kg is suspended from the cord 10 m above the floor.

(a) How much time does it take the mass to hit the floor?
(b) What is the velocity when the mass hits the floor?
(c) What is the tension in the rope?

Solution:

Referring to Fig. 9-7, we first solve the problem for the general case $m \neq M$.

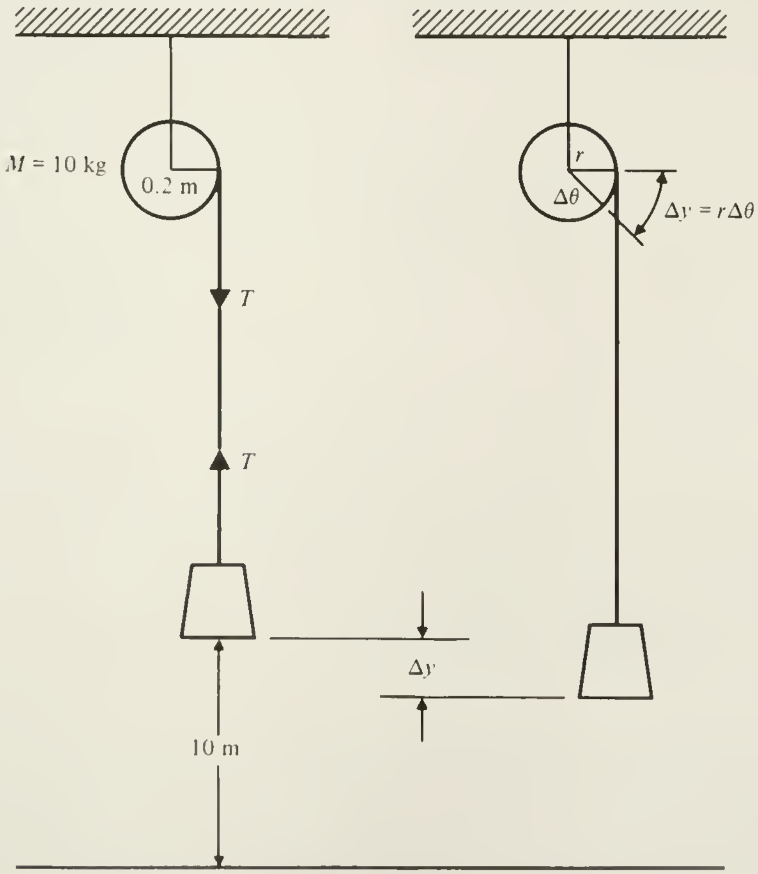


Figure 9-7

The equation of motion of the flywheel is

$$\Gamma = I\alpha; \qquad I = \frac{1}{2} MR^2 \qquad \Gamma = TR$$

and the equation of motion of the mass m is

$$mg - T = ma$$

where a is the downward acceleration of the mass. (Note $T \neq mg$. The equality is true only in equilibrium.)

The key to solving this problem is to recognize that there is a relation between the linear acceleration a of the mass and the angular acceleration α of the flywheel. If a length Δy of cord is pulled from the flywheel it moves through an angle

$$\Delta\theta = \frac{\Delta y}{R}$$

while the mass drops a distance Δy. Thus

$$\frac{\Delta r}{\Delta t} = \frac{1}{R}\,\frac{\Delta y}{\Delta t} \qquad \text{or} \qquad \omega = \frac{v}{R}$$

implying

$$\Delta\omega = \frac{\Delta v}{R} \qquad \text{or} \qquad \frac{\Delta 1}{\Delta t} = \frac{1}{R}\,\frac{\Delta v}{\Delta t} = \alpha$$

yielding

$$\alpha = \frac{a}{R}$$

The equation of motion for the flywheel tells us

$$\alpha = \frac{\tau}{I} = \frac{(TR)}{1/2(MR^2)} = \frac{2T}{MR} = \frac{a}{R}$$

or

$$T = \frac{Ma}{2}$$

The equation of motion for the weight then may be solved for a by substituting T,

$$mg - \frac{Ma}{2} = ma$$

$$a = \frac{g}{1 + \frac{M}{2m}}$$

When m = M. a = 2/3 g. In this case we have for the falling mass,

$$y = y_o + v_o t + \frac{1}{2} at^2 \qquad y_o = 0 \qquad v_o = 0$$

$$y = \frac{1}{2} at^2$$

$$v = \frac{dy}{dt} = at$$

$$v^2 - v_o^2 = 2ay \qquad v_o = 0$$

From the last equation we have

$$v = (2ay)^{1/2} = (2 \ \frac{2}{3} \ 9.8 \ m\cdot s^{-2} \cdot 10 \ m)^{1/2}$$

$$= 11.43 \ m\cdot s^{-1}$$

The time to hit the floor is

$$t = \frac{v}{a} = \frac{v}{2/3g} = \frac{3(11.4 \ m\cdot s^{-1})}{2(9.8 \ m\cdot s^{-2})} = 1.74 \ s.$$

Finally, the tension in the rope is

$$T = \frac{Ma}{2} = \frac{M}{2} \ \frac{2}{3} \ g = \frac{Mg}{3} = \frac{(10 \ kg) 9.8 \ m\cdot s^{-2}}{3}$$

$$= 32.7 \ N$$

Another approach to this problem is through energy conservation. The initial energy, E_i, is

$$E_i = K_i + V_i$$

$$= \frac{1}{2} I\omega_i^2 + \frac{1}{2} Mv_i^2 + Mgh_i$$

= rotational plus translational energy

$$= Mgh_i \;; \qquad h_i = 10 \text{ m}, \qquad v_i = 0 = \omega_i$$

The final energy is

$$E_f = \frac{1}{2} I\omega_f^2 + \frac{1}{2} Mv_f^2 + mgh_f$$

$$= \frac{1}{2} I\left(\frac{v}{R}\right)^2 + \frac{1}{2} Mv^2 + Mgh_f; \; h_f = 0$$

$$= \frac{1}{2}\left(\frac{1}{2} MR^2\right)\frac{v^2}{R^2} + \frac{1}{2} Mv^2$$

$$= \frac{3}{4} Mv^2 = E_i = Mgh_i$$

Thus we have

$$v = \left(\frac{4\,gh_i}{3}\right)^{1/2} = \left[\frac{4(9.8 \text{ m}\cdot\text{s}^{-2})10 \text{ m}}{3}\right]^{1/2}$$

$$= 11.4 \text{ m}\cdot\text{s}^{-1}$$

Example 20

The moment of inertia of a uniform circular disk about an axis through its center and perpendicular to the plane of disk is $I = 1/2(MR^2)$, where M is the mass and R the radius of the disk. Find the moment of inertia of the disk about an axis perpendicular to the disk through an edge of the disk (a distance R from the center of the disk.)

Solution:

By the parallel axis theorem, we have

$$I_p = I_{cm} + Md^2$$

where in this case $d = R$. Thus

$$I_p = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$$

Example 21

A baggage carousel has a mass of $M = 500$ kg and is approximately a disk of radius $2\ m = R$. It is freely rotating at an angular velocity $\omega = 1\ rad\cdot s^{-1}$ when 10 pieces of baggage with masses of $M' = 20$ kg a piece are dropped on the carousel a distance $R = 2$ m from the axis of rotation.

(a) Assuming no external torques act on the system of carousel plus baggage, what is the final angular velocity?

(b) What is the energy of system before and after the 10 pieces of baggage are added?

Solution:

Angular momentum is conserved because only internal forces act. The initial angular momentum is

$$L_i = I_c\omega_{ci} = (\frac{1}{2}MR^2)\omega_{ci}$$

where M is the carousel mass I_c its moment of inertia, and ω_{ci} the initial carousel velocity.

The final angular momentum is

$$L_f = I_c\omega_{cf} + 10\ M'(R)^2\omega_{cf}$$

Note this is the angular momentum of carousel plus baggage where ω_{cf} is the final angular velocity.

By the conservation of angular momentum,

$$L_i = L_f$$

$$I_c\omega_{ci} = I_c\omega_{cf} + 10M'(R)^2\omega_{cf}$$

$$\frac{1}{2}MR^2\ \omega_{ci} = \frac{1}{2}MR^2\ \omega_{cf} + 10\ M'(R)^2\omega_{cf}$$

The final angular velocity is

$$\omega_{cf}[\frac{MR^2}{2} + 10M'(R)^2] = \omega_{ci}\frac{MR^2}{2}$$

$$\omega_{cf} = (\frac{1}{1 + 20M'/M})\ \omega_{ci}$$

$$= (\frac{1}{1 + 20(20/500)})\omega_{ci} = 0.56(1\ rad\cdot s^{-1})$$

(b) The initial energy is

$$E_i = K_i = (1/2)I_c\omega_{ci}^2 = 1/2(1/2\ M\ R^2)\omega_{ci}^2$$

$$= (1/4)MR^2\omega_{ci}^2 = (1/4)500\ \text{kg}(2\ \text{m})^2(1\ \text{rad}\cdot\text{s}^{-1})^2 = 500\ \text{J}$$

The final energy is

$$E_f = K_f = \frac{1}{2}(I_c\omega_{cf}^2) + \frac{1}{2}(I_b\omega_{cf}^2)$$

with I_b the moment of inertia of the baggage,

$$I_b = 10\ M'R^2$$

Thus

$$E_f = \frac{1}{2} I_c\omega_{cf}^2 + \frac{1}{2}(10\ M'R^2)\omega_{cf}^2$$

$$= \frac{1}{2}\left(\frac{1}{2} MR^2\right)\omega_{cf}^2 + \frac{1}{2}(10\ M'R^2)\omega_{cf}^2$$

$$= \left[\frac{1}{4} 500\ \text{kg}(2\text{m})^2 + \frac{1}{2} 10(20\ \text{kg})(2\ \text{m})^2\right](0.56\ \text{rad}\cdot\text{s}^{-1})^2$$

$$E_f = 282\ \text{J}$$

The collision did not conserve energy; as the baggage dropped on the carousel, dissipation occurred.

Example 22

A block of mass 0.1 kg is attached to a cord passing through a hole in a horizontal frictionless surface. The block is originally rotating in a circle at a distance 0.2 m from the hole with an angular velocity 7 rad·s^{-1}. A force P pulls on the cord and it is shortened to 0.1 m.

(a) What is the new angular velocity?

(b) What is the work done by the force P?

Solution:

(a) Since there is no external torque (P has no moment about the hole) angular momentum is conserved. The initial angular momentum is

$$L_i = I_i\omega_i = MR_i^2\omega_i$$

The final angular momentum is

$$L_f = I_f\omega_f = MR_f^2\omega_f$$

By conservation of momentum,

$$MR_i^2\omega_i = MR_f^2\omega_f$$

$$\omega_f = \left(\frac{R_i}{R_f}\right)^2 \omega_i = \left(\frac{0.2\ \text{m}}{0.1\ \text{m}}\right)^2 7\ \text{rad}\cdot\text{s}^{-1}$$

$$= 28\ \text{rad}\cdot\text{s}^{-1}$$

(b) The work W′ done by the external force, P acting through the distance s, is equal to the increase in kinetic energy,

$$W' = K_f - K_i\ ; \qquad s = 0.1\ \text{m}$$

$$= \frac{1}{2} I_f\omega_f^2 - \frac{1}{2} I_i\omega_i^2$$

$$= \frac{1}{2} MR_f^2\omega_f^2 - \frac{1}{2} MR_i^2\omega_i^2$$

Note however that

$$R_f^2 = R_i^2 \frac{\omega_i}{\omega_f}$$

and thus

$$W' = \frac{1}{2} MR_i^2\omega_f^2\left(\frac{\omega_i}{\omega_f}\right) - \frac{1}{2}MR_i^2\omega_i^2$$

$$= \frac{1}{2} R_i^2 \left(\frac{\omega_f}{\omega_i} - 1\right)\omega_i^2$$

$$= \frac{1}{2} (0.1\ \text{kg})(0.2\ \text{m})^2((28/7) - 1)\ (7\ \text{rad}\cdot\text{s}^{-1})^2$$

$$= 0.29\ \text{J}$$

Example 23

A turntable with moment of inertia $I = 2000\ kg\cdot m^2$ makes one revolution every 5 s. A man of mass $M = 100$ kg standing at the center of the turntable runs out along a radius fixed in the turntable.

(a) What is the angular velocity when he is $R = 3$ m from the center?

(b) How much work does he do during the run?

Solution:

Since there are no external torques, total angular momentum is conserved. The initial angular momentum is ($\omega_i = 2\pi$ rad / 5 s)

$$L_i = I_i\omega_i = 2000\ kg\cdot m^2 \left(\frac{2\pi\ rad}{5\ s} \right)$$

$$= 2513\ kg\cdot m^2\ s^{-1}$$

The final angular momentum is

$$L_f = (I_i + MR^2)\omega_f = [2000\ kg\cdot m^2 + 100\ kg\ (3\ m)^2]\omega_f$$

$$= 2900\ kg\cdot m^2\ \omega_f$$

The conservation of angular momentum yields, $L_i = L_f$, or

$$\omega_f = \frac{2513\ kg\cdot m^2\ s^{-1}}{2900\ kg\cdot m^2} = 0.87\ s^{-1}$$

$$= 0.87\ \frac{1}{2\pi}\ rev\cdot s^{-1} = 0.14\ rev\cdot s^{-1}$$

$$= 1 \text{ rev every } 7.25\ s.$$

The work done <u>by</u> the man, W', is equal to the <u>increase</u> in kinetic energy,

$$W' = K_f - K_i = \frac{1}{2} I_f\omega_f^2 - \frac{1}{2} I_i\omega_i^2$$

$$= 1/2[2000\ kg\cdot m^2 + 100\ kg\ (3m)^2](.87\ s^{-1})^2$$

$$- 1/2\ 2000\ km^2 \left(\frac{2\pi}{5\ s} \right)^2$$

$$= 1097\ J - 1579\ J$$

$$= -482\ J$$

Since the work is negative, work is done <u>on</u> the man as he runs out; if he tries to run in he will have to do work to get closer to the center.

QUIZ

1. A 33 and 1/3 rpm phonograph record takes 3 s to get up to speed. Find the average angular acceleration in rad·s^{-2}.

Answer: 1.16 rad·s^{-2}

2. A flywheel in the shape of a solid disk has a mass of 5 kg and a radius of 5 cm. Find its kinetic energy when it rotates at 100 rpm.

Answer: 0.34 J

3. A mass of 1 kg hangs by a rope wrapped around a pulley of moment of inertia 2 kg·m^2 and radius 50 cm. If the mass starts from rest, find its acceleration and how far it falls after a time of 1 s.

Answer: 1.1 m·s^{-2}, 0.54 m

4. A block of mass 0.5 kg is attached to a cord passing through a hole in a horizontal frictionless surface. The block originally rotates in a circle of radius 0.4 m with an angular velocity of 14 rad·s^{-1}. The cord is then pulled so that the new radius is 0.2 m.
   (a) What is the new angular velocity?
   (b) What work is done when the cord is pulled?

Answer: 56 rad·s^{-1}, 23 J

10
EQUILIBRIUM OF A RIGID BODY

OBJECTIVES

The first condition of equilibrium of a rigid body is that the total force acting on the body is zero:

$$\sum \mathbf{F} = 0.$$

This guarantees translational equilibrium. The second condition of equilibrium guarantees that there is no tendency to rotate. This is guaranteed by the torque condition (see below.) Your objectives in this chapter are to:

Continue to gain proficiency at calculating the moment or torque of a force.

Apply the second condition of equilibrium to a variety of problems involving static structures and bodies in rotational equilibrium, such as see-saws or teeter-totters, booms supported by guy wires, and ladders against walls.

Calculate the position of the center-of-gravity of a rigid body.

REVIEW

Second Condition of Equilibrium

To insure static equilibrium of a rigid body, the total torque must vanish about any axis: 0, 0′, etc.,

$$\sum \Gamma_0 = 0, \qquad \sum \Gamma_{0'} = 0, \text{ etc.}$$

This is a good time to review how to calculate a torque: see Examples 12 and 13 of Chapter 9.

Center-of-Gravity

The moment of the gravitational force is the sum of the moments of gravity acting on the individual mass points,

$$\Gamma = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{g}$$

where $\mathbf{g}$ is a vector pointing down, of magnitude g.

Thus

$$\Gamma = (\sum_i m_i \mathbf{r}_i) \times \mathbf{g} = M\mathbf{R} \times \mathbf{g} = \mathbf{R} \times M\mathbf{g}$$

$$= \mathbf{R} \times \mathbf{W}$$

where R is the center-of-mass position vector previously defined, and

$$\mathbf{W} = \sum_i \mathbf{w}_i = M\mathbf{g}$$

is the total weight of the body.

PROBLEM-SOLVING STRATEGIES

The equilibrium problems of Chapter 5 included extended bodies but were chosen so that the moment condition was automatically satisfied. Referring to the problem-solving strategy section of the study guide, add the condition that the total moment about any point must vanish. After writing down the first conditions, write down the second condition about as many points as you wish until you have as many independent equations as there are unknowns.

As illustrated below, the equilibrium equations are not always independent; there will often be a choice as to what combinations of the two components of the first equilibrium condition and the various second equilibrium conditions you use to find a solution.

As always, painstaking care must be exercised with respect to the signs of the moments of the forces: In Fig. 10-1, for illustration, forces A and B have positive moments about 0; C and D have negative moments about 0. What are the signs of the moments of the forces A, B, C, D about 0'? (Negative, positive, positive, positive, respectively.)

A choice of coordinate system and a free-body diagram are always useful. If a force is resolved into components, then each component may have a torque about a given point.

A convenient choice of the point 0 about which moments are taken will often simplify the moment condition equation.

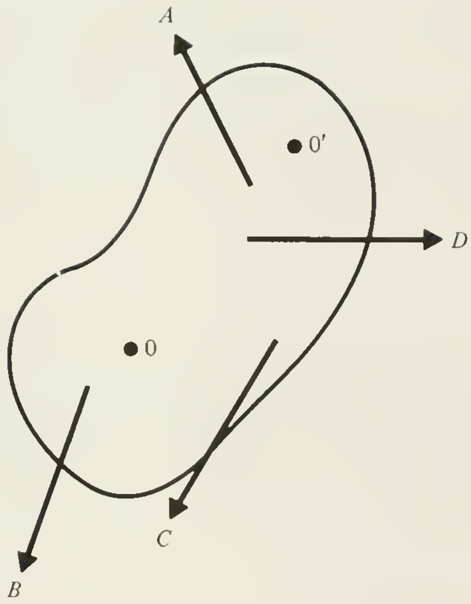


Figure 10-1

EXAMPLES AND SOLUTIONS

Example 1

Show that if a body is in equilibrium under the action of three coplanar forces, no two of which are parallel, their lines of action must meet at a point.

Solution:

Referring to Fig. 10-2, we write the first condition of equilibrium as **A** + **B** + **C** = 0. The lines of action of any two of the three forces, say **B** and **C**, must meet at a point 0, as shown in the Fig. 10-2:

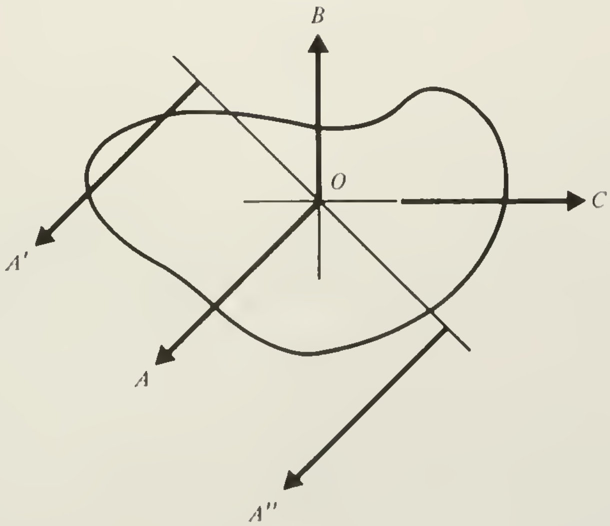


Figure 10-2

For translational equilibrium, the vector **A** must be equal to $-$ **B** $-$ **C** but its line of action is as yet unknown, as indicated by the parallel vectors of equal magnitude, **A**, **A'**, **A''**. The second condition requires that all moments vanish, in particular those about 0, $\sum \Gamma_0 = 0$. B and C have no moment about 0; thus the moment of A about 0 must vanish. This is possible only if its lever arm vanishes, i.e., if its line of action passes through 0.

Example 2

The case excluded above, when the forces are parallel is illustrated in the classic playground question: A 20 kg boy sits on one end of a 3 m see-saw (teeter-totter). His 30 kg sister wishes to be in balance with him. Where should she adjust the pivot?

Solution:

The first step is to draw a free-body diagram showing all the forces acting on the see-saw. Referring to Fig. 10-3, we see that

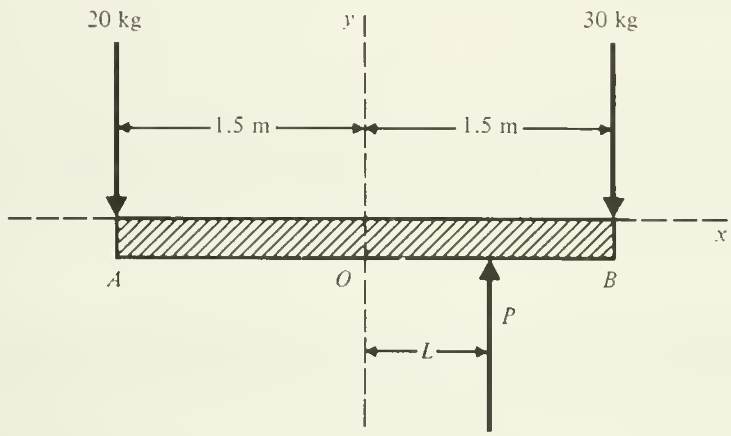


Figure 10-3

the first condition, $\sum F_y = 0$, implies

$$P = A + B$$

The second condition takes various forms, according to the point about which moments are calculated.

About A, we have

$$0 = \sum \Gamma_A = (1.5 \text{ m} + L)P = (3 \text{ m})B.$$

About 0, we have

$$0 = \sum \Gamma_o = 1.5 \text{ m } A + LP - 1.5 \text{ m } B.$$

About P, we have

$$0 = \sum \Gamma_P = (1.5 \text{ m} + L)A - (1.5 \text{ m} - L)B.$$

About B, we have

$$0 = \sum \Gamma_B = 3 \text{ m } A - (1.5 \text{ m} - L)P.$$

Note that A, P and B were convenient points because only two of the three forces have moments there. Any two of the moment equations may be used to solve for the unknowns P and 1. For example, the third yields

$$\frac{1.5 \text{ m} + L}{1.5\text{m} - L} = \frac{B}{A} = \frac{3}{2}\,; \qquad L = 0.3 \text{ m}$$

Then the first yields

$$P = \frac{3 \text{ m } B}{1.5 \text{ m} + L} = \frac{3}{1.8}\, 30 \text{ kg} = 50 \text{ kg}$$

and the second yields

$$P = \frac{1.5 \text{ m } B - 1.5 \text{ m } A}{L} = \frac{1.5 \text{ m}(30 \text{ kg}) - 1.5 \text{ m}(20 \text{ kg})}{.3\text{m}}$$

$$= 50 \text{ kg}$$

Thus of the four moment equations plus the $\sum F_y = 0$ condition, only two are independent. One could as well choose any other independent two to find the unknowns p and L.

Example 3

A 150 kg uniform diving board 4 m long is mounted as in Fig. 10-4a. Find the forces acting on the board when a 100 kg man is standing on its end.

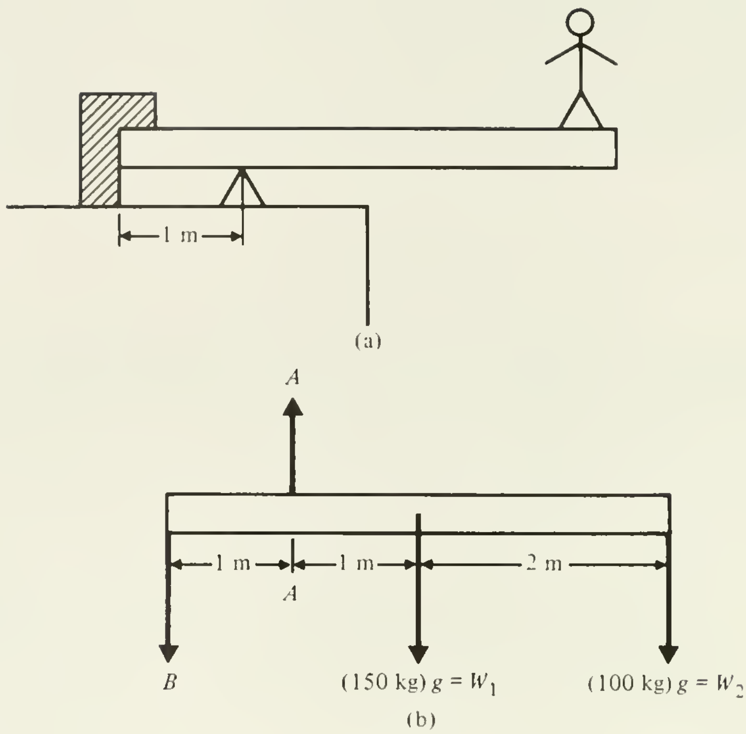


Figure 10-4

Solution

Referring to the free-body diagram of Fig. 10-4b, we see that the first condition of equilibrium yields

$$0 = \sum F_y = A - B - W_1 - W_2$$

Taking moments about B, we have

$$0 = \sum \Gamma_B = (1\ \mathrm{m})A - (2\ \mathrm{m})W_1 - (4\ \mathrm{m})W_2$$

yielding

$$A = 2W_1 + 4W_2 = 2(150\ \mathrm{kg})(9.8\ \mathrm{m \cdot s^{-2}}) + 4(100\ \mathrm{kg})(9.8\ \mathrm{m \cdot s^{-2}})$$

$$= 6860\ \mathrm{N}$$

Combining this result with the first condition yields

$$B = A - W_1 - W_2$$

$$= 6860\ \mathrm{N} - (150\ \mathrm{kg})(9.8\ \mathrm{m \cdot s^{-2}}) - (100\ \mathrm{kg})(9.8\ \mathrm{m \cdot s^{-2}})$$

$$= 4410\ \mathrm{N}$$

The result can be checked by calculating the total moment about A,

$$\sum \Gamma_A = (1\ \mathrm{m})B - (1\ \mathrm{m})W_1 - (3\ \mathrm{m})W_2$$

$$= (1\ \mathrm{m})4410\ \mathrm{N} - (1\ \mathrm{m})(150\ \mathrm{kg})(9.8\ \mathrm{m\cdot s^{-2}}) - (3\ \mathrm{m})(100\ \mathrm{kg})(8.8\ \mathrm{m\cdot s^{-2}})$$

$$= (4410 - 1470 - 2940)\mathrm{N\cdot m}$$

$$= 0,$$

verifying that it vanishes.

Example 4

The weightless strut in Fig. 10-5a supports a 300 N weight as shown. Find the tension in the supporting cable and the force of the wall against the strut.

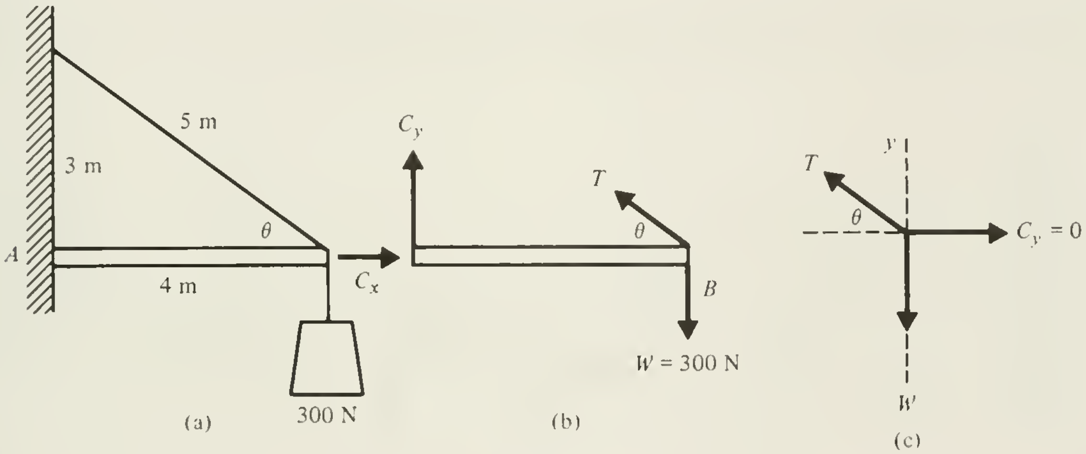


Figure 10-5

Solution:

The first step is to draw a free-body diagram showing all forces acting on the strut as illustrated in Fig. 10-5b. C_x and C_y are the rectangular components of the force <u>of</u> the wall <u>on</u> the strut. Since only three forces act on the strut (C, T, W) all their lines of action must meet at a single point, as proved in Example 1. Thus the line of action of C passes through the end of the strut where T and W act, implying $C_y = 0$. Alternatively, taking moments about B, we have

$$0 = \sum \Gamma_B = -C_y \cdot 4\mathrm{m} \qquad C_y = 0$$

leading to the same result. The first condition of equilibrium then yields (see Fig. 10-5c)

$$0 = \sum F_x = C_x - T\cos\theta = C_x - \frac{4}{5}T$$

$$0 = \sum F_y = T\sin\theta - W = \frac{3}{5}T - W,$$

with the solutions,

$$T = \frac{5}{3}W = \frac{5}{3}(300\ \text{N}) = 500\ \text{N}$$

$$C_x = \frac{4}{5}T = \frac{4}{5}(500\ \text{N}) = 400\ \text{N}$$

Example 5

Find the magnitude and direction of the force exerted by the wall on the uniform strut in Fig. 10-6a if the strut weighs 100 N.

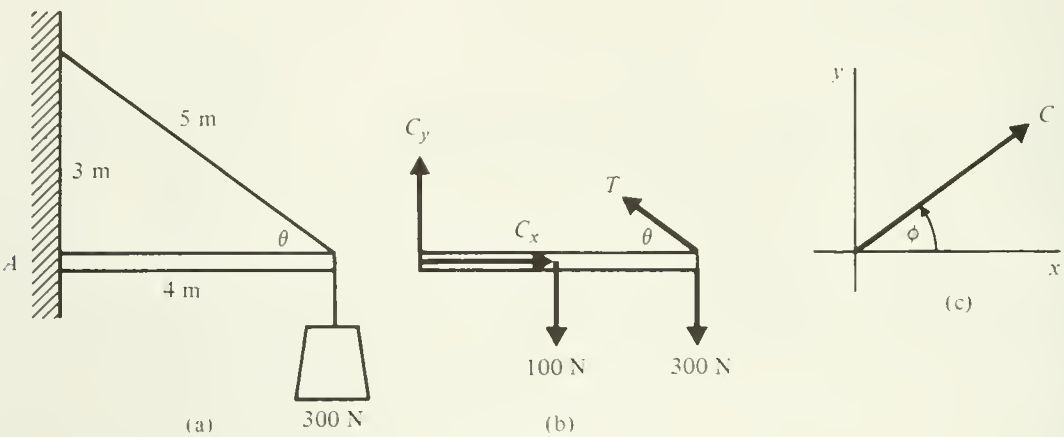


Figure 10-6

Solution:

In this case there are more than three forces acting on the strut and they need not meet at a single point. After drawing a free body diagram and coordinate axes (see Fig. 10-6b) we write down the first conditions of equilibrium,

$$0 = \sum F_x = C_x - T\cos\theta = C_x - \frac{4}{5}T$$

$$0 = \sum F_y = C_y + T\sin\theta - 300\ \text{N} - 100\ \text{N}$$

$$= C_y + \frac{3}{5} T - 400 \text{ N}$$

A convenient point to take moments about for the second condition of equilibrium is A because only one unknown, T, will be involved,

$$0 = \sum \Gamma_A = 4 \text{ m} \cdot T \sin \theta - 2 \text{ m} \cdot 100 \text{ N} - 4 \text{ m} \cdot 300 \text{ N}$$

yielding ($\sin \theta = 3/5$)

$$T = \frac{200 \text{ N}\cdot\text{m} + 1200 \text{ N}\cdot\text{m}}{4 \text{ m}(3/5)} = 583 \text{ N}$$

The first conditions then yield

$$C_x = \frac{4}{5} T = \frac{4}{5} (583 \text{ N}) = 467 \text{ N}$$

$$C_y = 400 \text{ N} - \frac{3}{5} T = 400 \text{ N} - \frac{3}{5} (583 \text{ N}) = 50 \text{ N}$$

In the free-body sketch of Fig. 10-6b we evidently guessed correctly the directions of C_x, C_y. Had we reversed these directions in Fig. 10-6b and calculated the equilibrium conditions accordingly, our answers would have come out negative for C_x and C_y, signaling that their directions are opposite to those originally assumed.

It is often wise to check your results by taking moments about another point: about B we have

$$\sum \Gamma_B = -4 \text{ m} \cdot C_y + 2 \text{ m} \cdot (100 \text{ N})$$

$$= -4 \text{ m} \cdot 50 \text{ N} + 2 \text{ m} \cdot (100 \text{ N}) = 0$$

Note C, T and the two weights are not concurrent. The angle ϕ that C makes with the horizontal is

$$\phi = \tan^{-1} \frac{C_y}{C_x} = \tan^{-1} \frac{50}{467} = 6.1°$$

Example 6

A ladder of mass $M = 25$ kg rests against a frictionless wall as shown in Fig. 10-7a. Find all the forces acting on the ladder.

Solution:

The first step is to draw the free-body diagram, as shown in Fig. 10-7b.

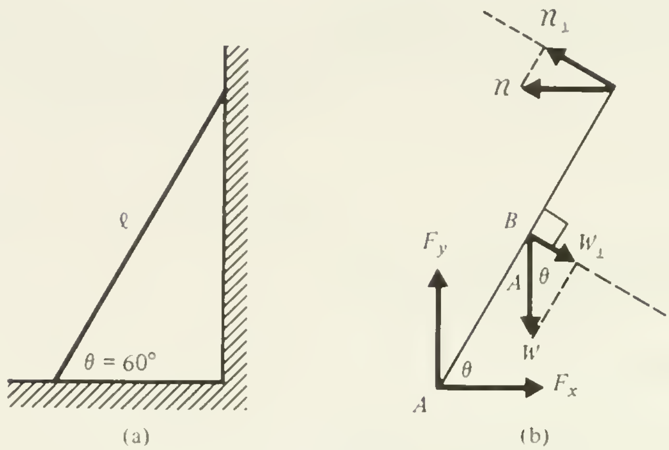


Figure 10-7

The unknown force F of the ground against the foot of the ladder has been resolved into its horizontal and vertical components. Since the wall is frictionless, it only exerts a force on the ladder perpendicular to the wall, n. Also indicated on Fig. 10-7b is W, the projection of W_n perpendicular to the lever arm or ladder. Note W_n and n_n are not extra forces; they are components of W and n useful for calculating moments.

The next step is to write down the conditions of equilibrium. The first conditions are

$$0 = \sum F_x = F_x - n$$

$$0 = \sum F_y = F_y - W$$

About the point A, the second condition of equilibrium is

$$0 = \sum \Gamma A = (-\frac{L}{2} W_n) + Ln_n$$

$$= - \frac{L}{2} W \cos\theta + Ln \sin\theta$$

Regarding F_x, F_y, and n as unknown in the three equations above, we solve the second one for F_y,

$$F_y = W = Mg = 25 \text{ kg}\cdot 9.8 \text{ m}\cdot\text{s}^{-1} = 245 \text{ N},$$

the last one for n,

$$\mathcal{N} = \frac{W \cos\theta}{2 \sin\theta} = \frac{245\ N}{2 \cdot 1.73} = 71\ N$$

and finally the first for F_x,

$$F_x = 71\ N$$

To check these results, calculate the moments about C,

$$\sum \Gamma_c = F_x(\sin\theta)L - F_y(\cos\theta)L + W(\cos\theta)(\frac{L}{2})$$

$$= L[71\ N(0.87) - 245\ N(0.50) + 245(0.500)(\ \frac{1}{2}\ )] = 0$$

Note the length scale L is irrelevant in this problem.

Example 7

In the previous problem, what is the minimum coefficient of friction between the ladder and the ground which allows the ladder to stand without slipping?

Solution:

The force of friction is $\mathcal{F} = F_x$ and the normal force at the ground is $\mathcal{N} = F_y$. The condition

$$\mathcal{F} \leq \mu_s \mathcal{N}$$

is thus

$$F_x \leq \mu_s F_y$$

or

$$\mu_s \geq \frac{F_x}{F_y} = \frac{71\ N}{245\ N} = 0.29$$

Example 8

A boom of length L supported by a guy wire supports a weight of W = 200 N as shown in Fig. 10-8a. The boom also weighs W. Find the tension in the guy wire and the force exerted by the ground on the foot of the boom.

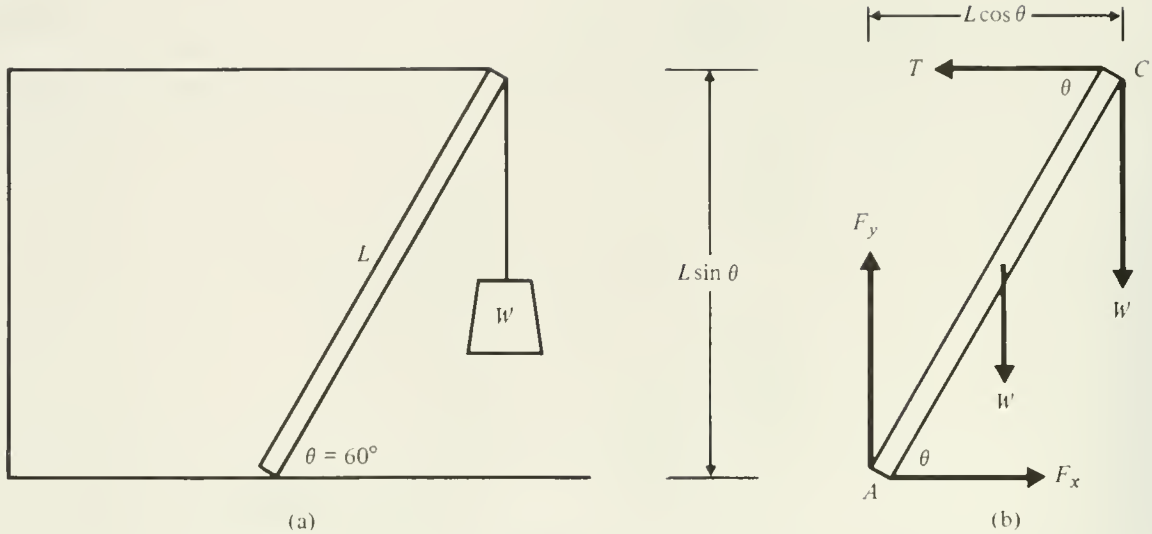


Figure 10-8a,b

Solution:

After sketching the free body diagram of Fig. 10-8b, accounting for all forces acting on the boom, we write down the first conditions of equilibrium,

$$0 = \sum F_x = F_x - T$$

$$0 = \sum F_y = F_y - W - W = F_y - 2W$$

and the second condition of equilibrium, taking moments about A,

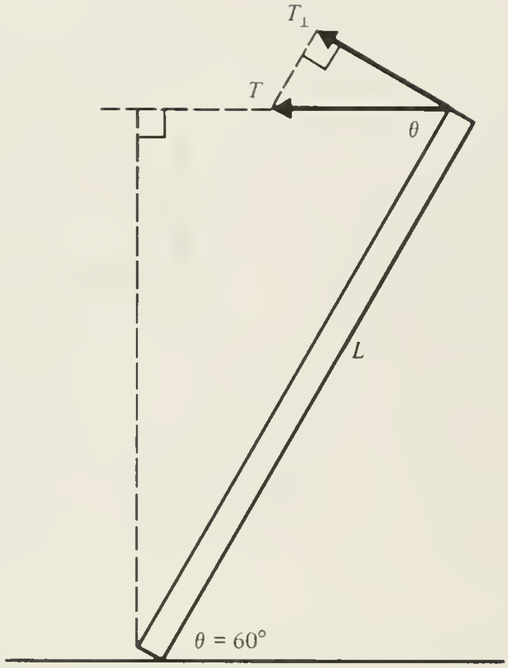
$$0 = \sum \Gamma_A = LT\sin\theta - LW\cos\theta - \frac{1}{2}\cos\theta\, LW$$

$$= L(T\sin\theta - \frac{3}{2} W\cos\theta)$$

At this point it is perhaps well to pause, and review techniques for calculating moments. Indicated on Fig. 10-8b are the components of the ladder in the vertical and horizontal direction. Review Example 12 and Example 13 of Study Guide Chapter 9 for the two methods of handling the geometry. The moment of T is positive because it tends to make the boom rotate counterclockwise about A; its magnitude is given by

$$T_nL = (T\sin\theta)L = T(L\sin\theta) = TL_n$$

as indicated in Fig. 10-8c



(c)

Figure 10-8c

Returning to the moment condition we have

$$T = \frac{3}{2} \frac{W}{\tan \theta} = \frac{3}{2} \frac{200 \text{ N}}{1.73} = 173 \text{ N}$$

The first conditions then imply

$$F_x = 173 \text{ N}$$

$$F_y = 2 W = 400 \text{ N}$$

To check that the solution is correct, verify that $0 = \sum \Gamma_B$.

Example 9

Find the minimum coefficient of friction between the weightless strut and the wall of the system in Fig. 10-9, if the strut is not to slip.

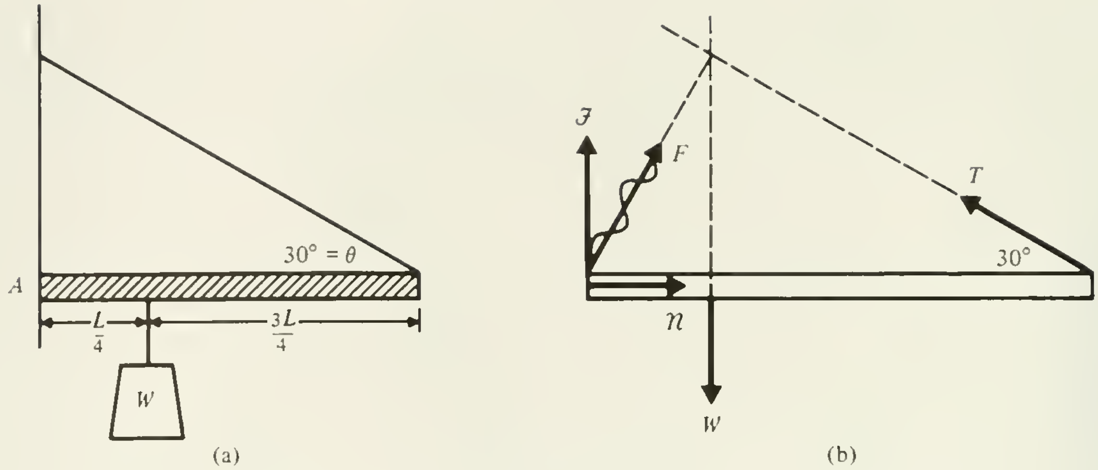


Figure 10-9

Solution:

$\mathcal{F}$ and $\mathcal{N}$, as indicated on the free-body diagram of Fig. 10-9b, are the components of the force **F** of the wall on the strut. Since there are only three coplanar forces acting on the strut (**F**, **W**, **T**) they must meet at a point as shown. We will not explicitly use that fact in the solution below, but the observation is useful in indicating the correct directions of **F** and $\mathcal{N}$

The first conditions of equilibrium are

$$0 = \sum F_x = \mathcal{N} - T \cos 30^\circ$$

$$0 = \sum F_y = \mathcal{F} + T \sin 30^\circ - W$$

The second condition, taking moments about A, is

$$0 = \sum \Gamma_A = T \sin 30^\circ\ L - \frac{1}{4} LW$$

$$T = \frac{1}{4} \frac{W}{\sin 30^\circ}$$

From the first condition we then have

$$\mathcal{N} = T \cos 30^\circ = \frac{W}{4 \tan 30^\circ}$$

$$\mathcal{F} = W - T \sin 30^\circ = W - \frac{1}{4} W = 3/4\ W$$

For no slipping we require

$$\mathcal{F} < \mu_s\, n$$

$$\frac{3}{4}\ W < \mu_s\ \frac{W}{4\ \tan 30^\circ}$$

$$3\ \tan 30^\circ < \mu_s$$

or

$$\mu_s > 1.73$$

Solve this problem when W hangs a distance λ from the wall, with $0 < \lambda < 1$. The answer is

$$\mu_s > \frac{1 - \lambda}{\lambda}\ \tan 30^\circ$$

Example 10

The guy wire on the hinged door of Fig. 10-10 is adjusted so that the top hinge exerts only a vertical force on the door. The hinges are set so that they share the weight of the 300 N door equally. Find the force exerted by each hinge. The hinges are 0.5 m from the top and bottom of the 2 m high door.

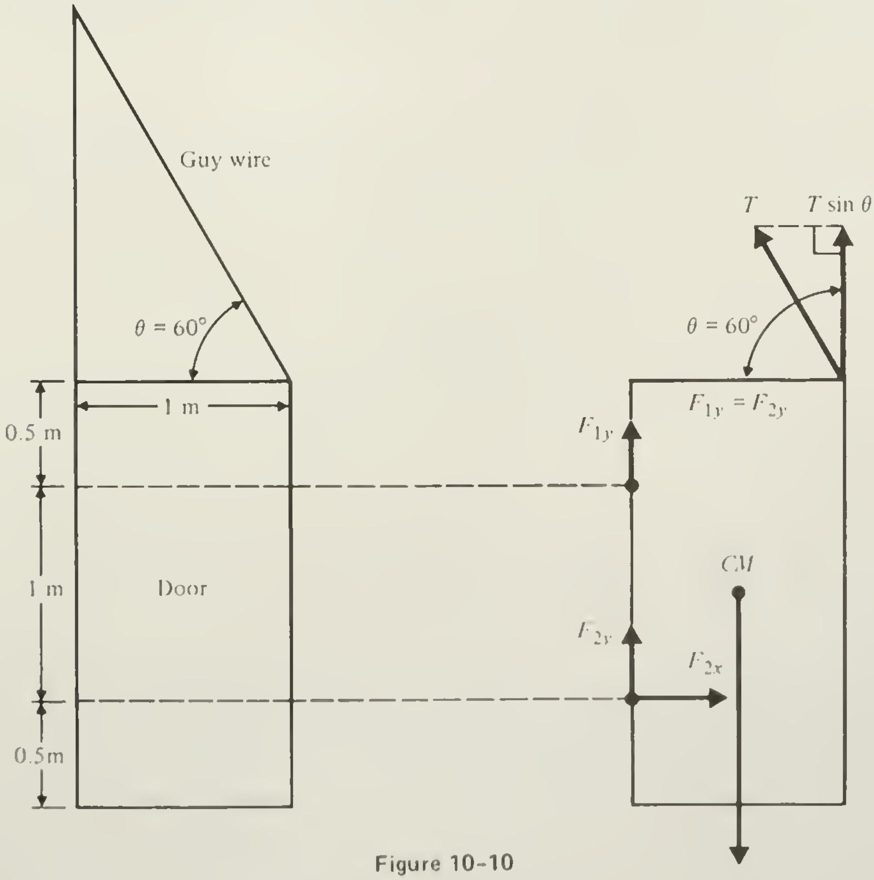


Figure 10-10

Solution:

Indicated on the free body diagram of Fig. 10-10b are the problem conditions that $F_{1y} = F_{2y}$ (equal sharing of weight by the hinges) and $F_{1x} = 0$ (no horizontal component of upper hinge.)

The first conditions of equilibrium are

$$\underline{0 = \sum F_x = F_{2x} - T\cos\theta}$$

$$0 = \sum F_y = T\sin\theta + F_{1y} + F_{2y} - W$$

$$\underline{0 = T\sin\theta + 2F_{2y} - W}$$

The second condition is, taking moments about the center-of-mass (CM),

$$0 = \sum \Gamma_{CM}$$

$$= T\sin\theta(0.5\text{ m}) + T\cos\theta(1\text{ m}) - (F_{1y} + F_{2y})(0.5\text{ m}) + F_{2x}(0.5\text{ m})$$

or since $F_{1y} = F_{2y}$,

$$\underline{0 = T\sin\theta + 2T\cos\theta - 2F_{2y} + F_{2x}}$$

The underlined equations are three simultaneous equations in the unknowns T, F_{2x}, and F_{2y}. Solving the first for T and substituting the result in the other two yields $[T = F_{2x}(\cos\theta)^{-1}]$

$$0 = F_{2x}\tan\theta + 2F_{2y} - W$$

$$0 = F_{2x}\tan\theta + 2F_{2x} - 2F_{2y} + F_{2x} = F_{2x}\tan\theta - 2F_{2y} + 3F_{2x}$$

These are two simultaneous linear equations in the unknowns F_{2x}, F_{2y}. Adding them yields

$$F_{2x} = \frac{W}{2\tan\theta + 3} = \frac{300\text{ N}}{2\tan 60^\circ + 3} = 46.4\text{ N}$$

Subtracting them yields

$$0 = 4F_{2y} - W - 3F_{2x}$$

$$F_{2y} = \frac{W + 3F_{2x}}{4} = \frac{300\text{ N} + 3(46.4\text{ N})}{4}$$

$$= 110\text{ N}$$

Finally, the first underlined equation yields

$$T = \frac{F_{2x}}{\cos\theta} = \frac{46.4\ N}{\cos 60^\circ} = 93\ N$$

As a check, calculate the moment about the upper left corner and verify that it vanishes.

Example 11

Find the center-of-gravity of the uniform boomerang-like object in Fig. 10-11.

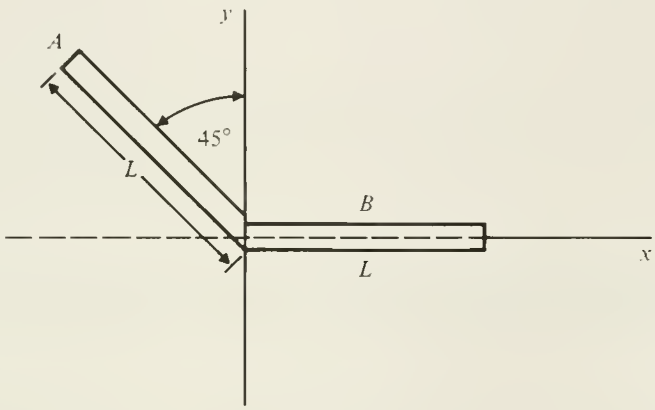


Figure 10-11

Solution:

Dividing the object into the pieces A and B, we have

$$M_A = M_B$$

$$X_A = -\frac{L}{2}\cos 45^\circ = -\frac{L}{2(2)^{1/2}}$$

$$Y_A = \frac{L}{2}\cos 45^\circ = \frac{L}{2(2)^{1/2}}$$

$$X_B = \frac{L}{2}$$

$$Y_B = 0$$

where (X_A, Y_A) and (X_B, Y_B) are the coordinates of the center-of-gravity of the two pieces. The coordinates (X,Y) of the whole object is

$$X = \frac{M_A X_A + M_B Y_B}{M_A + M_B} = \frac{M_A[-L/2(2)^{1/2} + L/2]}{2M_A}$$

$$= \frac{L}{4}\,[1 - \frac{1}{(2)^{1/2}}\,] = 0.07\ L$$

$$Y = \frac{M_A Y_A + M_B Y_B}{M_A + M_B} = \frac{M_A[L/2(2)^{1/2} + 0]}{2M_A}$$

$$= \frac{L}{4}\ \frac{1}{(2)^{1/2}} = 0.18\ L$$

Example 12

A cubic box is dragged along a floor at constant velocity as shown in Fig. 10-12. The coefficient of friction is $\mu = 0.25$. Find the line of action of the normal force of the floor on the box.

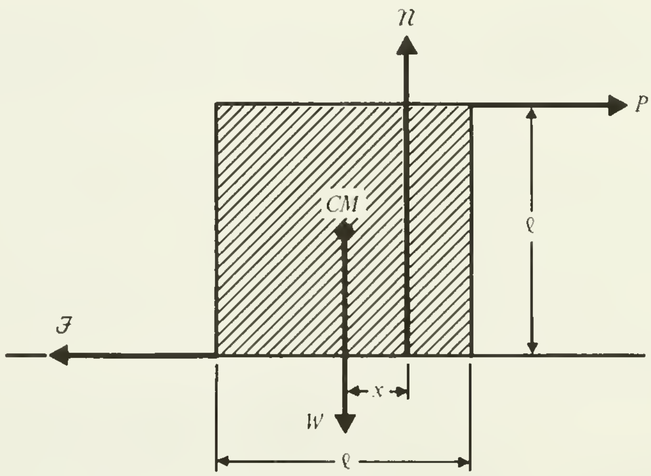


Figure 10-12

Solution:

From the free body diagram of Fig. 10-12, the first conditions of equilibrium are

$$0 = \sum F_x = P - \mathcal{F} = P - \mu \mathcal{N}$$

$$0 = \sum F_y = \mathcal{N} - W$$

Combining these we find

$$P = \mu W$$

Taking moments about the CM, we have

$$0 = \sum \Gamma_{CM} = x\mathcal{N} - \frac{L}{2} P - \frac{L}{2} \mathcal{F}$$

$$0 = xW - \frac{1}{2}\mu W - \frac{1}{2}\mu W$$

or

$$x = L\mu = .25\ L$$

If $\mu > \frac{1}{2}$ the point of application would need to be beyond the edge of the cube and it would tip.

QUIZ

1. A 18 kg child sits 1.5 m from the pivot of a weightless teeter-totter (see-saw). How far away on the opposite side should his 40 kg mother sit if she is to balance him?

Answer: 0.7 m

2. A 90 kg painter is halfway up a 8 m ladder which makes an angle of 30° with a vertical wall. What is the torque of his weight about the foot of the ladder?

Answer: 1800 N·m

3. A post weighing 500 N rests on a rough horizontal surface with a static coefficient of friction $\mu_s = 0.3$. The upper end is held by a rope fastened to the surface and making an angle of 37° with the post. A horizontal force acts at the midpoint of the post. What is the largest value this force may have if the post is not to slip?

Answer: 500 N

4. A uniform ladder 5 m long and weighing 400 N rests against a frictionless wall with its lower end 3 m from the wall. The coefficient of static friction of the ladder with the ground is 0.40. A 800 N man stands on the ladder at its midpoint. Find the frictional force.

Answer: 450 N

11
PERIODIC MOTION

OBJECTIVES

In this chapter you will study an oscillatory motion along a straight line called simple harmonic motion (SHM). It occurs when a body is attracted to an equilibrium position by a force proportional to its displacement from the equilibrium position. An example is a body on the end of a spring. Since the acceleration in this motion is not constant, the kinematical equations of Chapter 2 are not applicable. Your objectives are to:

Obtain, solve, and apply the equations of motion of SHM.

Define the amplitude, frequency, and period for SHM.

Find the position and velocity of a body in SHM, given the initial position and velocity.

Solve problems involving SHM, using the conservation of energy.

Apply the equations of SHM to a body attached to a spring and to a simple and physical pendulum.

Treat damped and forced oscillations.

REVIEW

A body of mass m attached to the origin by a spring of force constant k has the equation of motion

$$F = -kx = ma, \qquad a = \frac{-k}{m}x,$$

where x is the displacement from the origin. Since the spring force is conservative, the total energy is conserved (constant),

$$E = \frac{mv^2}{2} + \frac{kx^2}{2} = \text{constant} = K + U_e$$

where the elastic potential energy is

$$U_e = \frac{kx^2}{2} .$$

The maximum displacement, or <u>amplitude</u> $A = x_{max}$, occurs when $v = 0$, yielding

$$E = \frac{mv^2}{2} + \frac{kx^2}{2} = \frac{kA^2}{2}$$

Solving this equation for v, we have

$$v = \pm \left[\frac{k}{m} (A^2 - x^2)\right]^{1/2}$$

$$v_{max} = \left[\frac{k}{m} \right]^{1/2} A$$

$$(1/2)mv_{max}^2 = (1/2)kA^2 = E.$$

The general solution to the equation of motion is

$$x = A \cos \left(\left[\frac{k}{m}\right]^{1/2} t + \theta_o\right)$$

$$= A \cos (\omega t + \theta_o)$$

$$= A \cos (2\pi f t + \theta_o)$$

$$= A \cos \left(\frac{2\pi t}{\tau} + \theta_o\right)$$

where A is the amplitude and the other constants are related according to

$$\omega = \left[\frac{k}{m}\right]^{1/2} = 2\pi f = \frac{2\pi}{\tau}$$

The motion repeats itself in the period

$$\tau = 2\pi \left[\frac{m}{k}\right]^{1/2},$$

which is the time for one complete <u>cycle</u>. The frequency, or number of cycles per unit time, is

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

with units

one cycle$\cdot s^{-1}$ = one hertz = 1 hz.

The velocity of the body is the time rate of change of the displacement,

$$v = \frac{dx}{dt} = -[\frac{k}{m}]^{1/2} A \sin ([\frac{k}{m}]^{1/2} t + \theta_o) = - \omega A \sin (\omega + \theta_o)t$$

The constants A (= amplitude) and θ_o (= phase) in x(t) and v(t) must be determined by the initial conditions of the motion.

If x = 0 at t = 0, we have

$$x = \pm A \sin \omega t$$

$$v = \pm A\omega \cos \omega t$$

where the sign is chosen according to the initial direction of the velocity.

If v = 0 at t = 0, we have

$$x = \pm A \cos \omega t$$

$$v = \mp \omega A \sin \omega t$$

where, again, the sign is chosen according to the initial conditions.

If neither x nor v is zero at t = 0, then, from the general form for x and v evaluated at t = 0, we have

$$x_o = A \cos \theta_o$$

$$v_o = - \omega A \sin \theta_o,$$

with the solution

$$\theta_o = \text{arc tan } (- \frac{v_o}{\omega_o})$$

$$A^2 = x_o^2 + \frac{v_o^2}{\omega^2}$$

A <u>simple pendulum</u> (point mass suspended by a light string) undergoes SHM if its displacement is small, with

$$\omega = [\frac{g}{L}]^{1/2} = 2\pi f = \frac{2\pi}{\tau}$$

where L is the length of the string. In a simple pendulum all the mass is concentrated at a single point.

An extended object pivoted to swing about a fixed axis is a physical pendulum. If its moment of inertia is I and the angular displacement from equilibrium is small, the body undergoes SHM with

$$\omega = [\frac{mgh}{I}]^{1/2} = 2\pi f = \frac{2\pi}{\tau}$$

where h is the distance from the pivot to the center-of-gravity.

Hints and Problem-Solving Strategies

Choose the form $x = \pm A \sin \omega t$ when the SHM motion starts at the origin. Choose the form $x = \pm A \cos \omega t$ when the motion starts at maximum or minimum displacement. Otherwise the general form $x = A \sin (\omega t + \theta)$ or $A \cos(\omega t + \theta_0)$ must be used.

Commit to memory: $\omega = 2\pi f = 2\pi/\tau$.

When the restoring force is provided by a spring of constant k, $\omega = (k/m)^{1/2}$. In a simple pendulum $\omega = (g/L)^{1/2}$. In a physical pendulum $\omega = (mgh/I)^{1/2}$.

Examples and Solutions

Example 1

A body of mass 0.1 kg is attached to a wall by a spring of constant k = 15 $N \cdot m^{-1}$ (see Fig. 11-1a). It is initially pulled a distance x = 0.2 m from its equilibrium position and released from rest (see Fig. 11-1b).

(a) What is its initial potential energy?
(b) What is its initial kinetic energy?
(c) What is its initial total energy?
(d) What is its velocity when it passes through the equilibrium position?

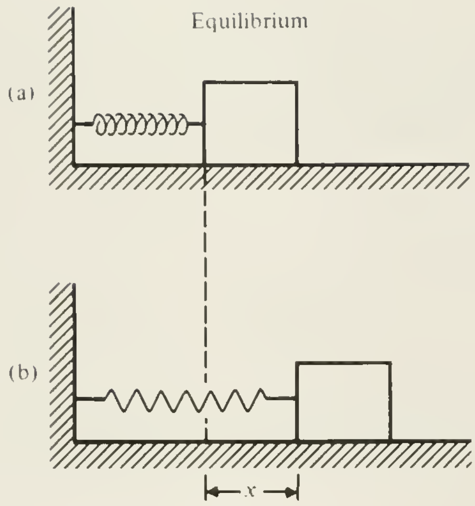


Figure 11-1

Solution:

(a) The initial potential energy is

$$U_e = \frac{kx_i^2}{2} = \frac{(15\ N\cdot m^{-1})(0.2\ m)^2}{2} = 0.30\ J.$$

(b) The initial kinetic energy is

$$K = \frac{mv_i^2}{2} = 0.$$

(c) The total energy is

$$E = K + U = 0.30\ J,$$

and is conserved.

(d) When the body passes through its equilibrium position ($x = 0$ where $U_e = 0$) its energy is

$$E = \frac{mv^2}{2} + 0 = 0.30\ J$$

$$v = [\frac{2E}{m}]^{1/2} = \frac{2(0.30)J}{0.1\ kg} = 2.5\ m\cdot s^{-1}.$$

Example 2

In the last example, what are the amplitude, frequency and period of the motion?

Solution:

The amplitude is the maximum displacement. Since the body is released from rest its amplitude is its initial displacement,

$$A = 0.2 \text{ m}.$$

The frequency of the motion is

$$f = \frac{1}{2\pi}[\frac{k}{m}]^{1/2} = \frac{1}{2\pi}[\frac{15 \text{ N m}^{-1}}{0.1 \text{ kg}}]^{1/2} = 1.95 \text{ s}^{-1}$$

$$= 1.95 \text{ cycles per second} = 1.95 \text{ Hz}.$$

The period of the motion is

$$\tau = f^{-1} = (1.95 \text{ s}^{-1})^{-1} = 0.51 \text{ s}.$$

The angular freqnency ω is

$$\omega = 2\pi f = 2\pi(1.95 \text{ s}^{-1}) = 12.25 \text{ s}^{-1}.$$

Example 3

In the last example, find the displacement x and the velocity v for t = 0.1 s after release.

Solution:

Since the displacement x is maximum at t = 0 the appropriate choice among the forms

$$x = A \sin \omega t$$

$$x = A \cos \omega t$$

$$x = A \cos (\omega t + \theta_o)$$

is

$$x = A \cos \omega t \quad [x_{max} = x(o) = A]$$

with

$$\omega = 12.25 \text{ s}^{-1}, \text{ and } A = 0.2 \text{ m}.$$

(If the spring had been compressed in Fig. 11-1b then, with x defined in Fig. 11-1b, the correct form would have been $x = -A \cos\omega t$. Choose the sign according to the initial conditions and your sign convention for positive x.)

At time t = 0.1 s,

$$x(0.1\ s) = (0.2\ m)\cos[(12.25\ s^{-1})(0.1\ s)]$$

$$= 0.07\ m$$

$$v(0.1\ s) = \frac{dx}{dt} = -A\omega \sin \omega t$$

$$= -(0.2\ m)(12.25\ s^{-1}) \sin [(12.25\ s^{-1})(0.1\ s)]$$

$$= -2.3\ m\cdot s^{-1}.$$

Note that in the above formula, ωt, the argument of the trigonometric functions, is measured in radians. Your calculator must be informed accordingly when evaluating $\sin \omega t$ or $\cos \omega t$.

Example 4

In the examples 1 through 3, find x and v at $t = n\tau/8$, when $n = 1,2,3,\ldots,8$ and plot the results.

Solution:

$$x = A\cos\omega t = (0.2\ m)\cos(12.25\ s^{-1}t) \qquad \tau = 0.51\ s$$

$$x(o) = (0.2\ m)\cos(0) = 0.2\ m$$

$$x\left(\frac{\tau}{8}\right) = (0.2\ m)\cos\left[(12.25\ s^{-1})\left(\frac{0.51\ s}{8}\right)\right] = 0.14\ m$$

$$x\left(\frac{2\tau}{8}\right) = (0.2\ m)\cos\left[(12.25\ s^{-1})\left(\frac{0.51\ s}{4}\right)\right] = 0.00\ m$$

$$x\left(\frac{3\tau}{8}\right) = -0.14\ m$$

$$x\left(\frac{4\tau}{8}\right) = -0.20\ m$$

$$x\left(\frac{5\tau}{8}\right) = -0.14\ m$$

$$x\left(\frac{6\tau}{8}\right) = 0.00\ m$$

$$x\left(\frac{7\tau}{8}\right) = 0.14 \text{ m}$$

$$x\left(\frac{8\tau}{8}\right) = 0.20 \text{ m}$$

$$v = \frac{dx}{dt} = -\omega A \sin \omega t = -(12.25 \text{ s}^{-1})(0.2 \text{ m}) \sin (12.25 \text{ s}^{-1}t)$$

$$= -(2.45 \text{ m}\cdot\text{s}^{-1}) \sin (12.25 \text{ s}^{-1}t)$$

$$v(o) = 0$$

$$v\left(\frac{\tau}{8}\right) = -(2.45 \text{ m s}^{-1}) \sin [(12.25 \text{ s}^{-1})(0.51 \text{ s}/8)]$$

$$= -1.72 \text{ m}\cdot\text{s}^{-1}$$

$$v\left(\frac{2\tau}{8}\right) = -2.45 \text{ m}\cdot\text{s}^{-1}$$

$$v\left(\frac{3\tau}{8}\right) = -1.72 \text{ m}\cdot\text{s}^{-1}$$

$$v\left(\frac{4\tau}{8}\right) = 0$$

$$v\left(\frac{5\tau}{8}\right) = 1.72 \text{ m}\cdot\text{s}^{-1}$$

$$v\left(\frac{6\tau}{8}\right) = 2.45 \text{ m}\cdot\text{s}^{-1}$$

$$v\left(\frac{7\tau}{8}\right) = 1.72 \text{ m}\cdot\text{s}^{-1}$$

$$v\left(\frac{8\tau}{8}\right) = 0$$

The results are plotted in Fig. 11-2.

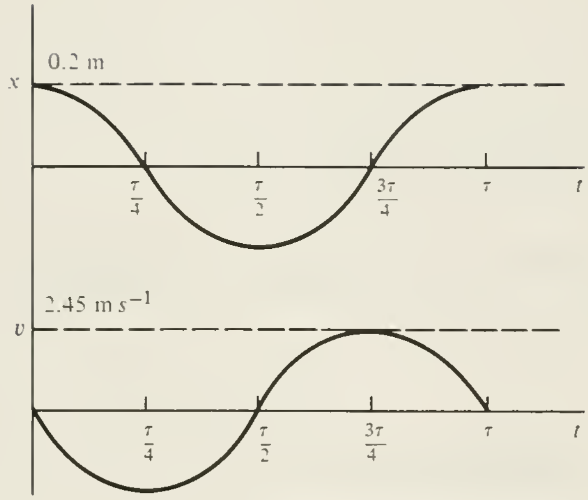


Figure 11-2

Example 5

A body of mass 0.5 kg is attached, as in Fig. 11-1a, to a wall by a spring of constant $k = 100\ N\cdot m^{-1}$. It is given an initial velocity, at $x = 0$, of $5\ m\cdot s^{-1}$.

(a) Find the total energy of the body.

(b) Find the amplitude of oscillation.

(c) Find the velocity when the displacement is half the amplitude.

(d) Find the displacement when the velocity is half the initial velocity.

(e) Find the displacement when the kinetic and potential energy are the same.

(f) Find the frequency and the period of the motion.

(g) Plot the displacement, velocity, and acceleration of the motion against time for one period.

Solution:

The total conserved energy is

$$E = \frac{mv^2}{2} + \frac{kx^2}{2} .$$

(a) Evaluating the energy at the initial time, when $x = 0$,

$$E = \frac{mv^2}{2} = \frac{(0.5\ kg)(5\ m\cdot s^{-1})^2}{2} = 6.25\ J.$$

(b) The maximum displacement, or amplitude is achieved when $v = 0$, when

$$E = \frac{kx^2{}_{max}}{2} = \frac{kA^2}{2} = 6.25\ J.$$

$$A = [\frac{2E}{k}]^{1/2} = [\frac{2(6.25\ J)}{100\ N\cdot m^{-1}}]^{1/2} = 0.35\ m.$$

(c) When x = A/2,

$$E = 6.25\ \text{J} = \frac{mv^2}{2} + \frac{kx^2}{2} = \frac{mv^2}{2} + \frac{1}{2}k\left(\frac{A}{2}\right)^2$$

$$v = \left[\frac{2E - k(A/2)^2}{m}\right]^{1/2} = \left[\frac{2(6.25\ \text{J}) - (100\ \text{N}\cdot\text{m}^{-1})(0.35\ \text{m}/2)^2}{0.5\ \text{kg}}\right]^{1/2}$$

$$= \pm\ 4.34\ \text{m}\cdot\text{s}^{-1}.$$

(d) When $v = \frac{v_{max}}{2} = \frac{5\ \text{m}\cdot\text{s}^{-1}}{2} = 2.5\ \text{m}\cdot\text{s}^{-1}$,

$$E = 6.25\ \text{J} = \frac{mv^2}{2} + \frac{kx^2}{2}\ .$$

$$x = \left[\frac{2E - mv^2}{k}\right]^{1/2} = \left[\frac{2(6.25\ \text{J}) - 0.5\ \text{kg}(2.5\ \text{m}\cdot\text{s}^{-1})^2}{100\ \text{N}\cdot\text{m}^{-1}}\right]^{1/2}$$

$$= \pm\ 0.31\ \text{m}$$

(e) When the kinetic energy and potential are the same,

$$\frac{1}{2}\,mv^2 = \frac{1}{2}\,kx^2$$

$$E = \frac{mv^2}{2} + \frac{kx^2}{2} = 2\left(\frac{kx^2}{2}\right) = kx^2$$

$$x = \left[\frac{E}{k}\right]^{1/2} = \left[\frac{6.25\ \text{J}}{100\ \text{N}\cdot\text{m}^{-1}}\right]^{1/2} = 0.25\ \text{m}$$

(f) The frequency is

$$f = \frac{1}{2\pi}\left[\frac{k}{m}\right]^{1/2} = \frac{1}{2\pi}\left[\frac{100\ \text{N}\cdot\text{m}^{-1}}{0.5\ \text{kg}}\right]^{1/2} = 2.25\ \text{Hz}.$$

The period is

$$\tau = f^{-1} = (2.25\ \text{Hz})^{-1} = 0.44\ \text{s}$$

$$\omega = 2\pi f = 14.14\ \text{s}^{-1}.$$

(g) Since the minimum displacement and maximum velocity are at t = 0, the appropriate forms for x and v are

$$x = A\sin\omega t \qquad\qquad x(o) = 0$$

$$v = \frac{dx}{dt} = \omega A \cos \omega t \qquad v(o) = \omega A.$$

The acceleration a is

$$a = \frac{dv}{dt} = - \omega^2 A \sin \omega t = - \omega^2 x = - \frac{k}{m} x.$$

The constants are (see (b) and (f) above)

$$A = 0.35 \text{ m} \qquad \omega = 14.14 \text{ s}^{-1}$$

$$a_{max} = \omega^2 A = (14.14 \text{ s}^{-1})^2 (0.35 \text{ m}) = 70 \text{ m} \cdot \text{s}^{-2}.$$

For a plot of x, v and a, see Fig. 11-3.

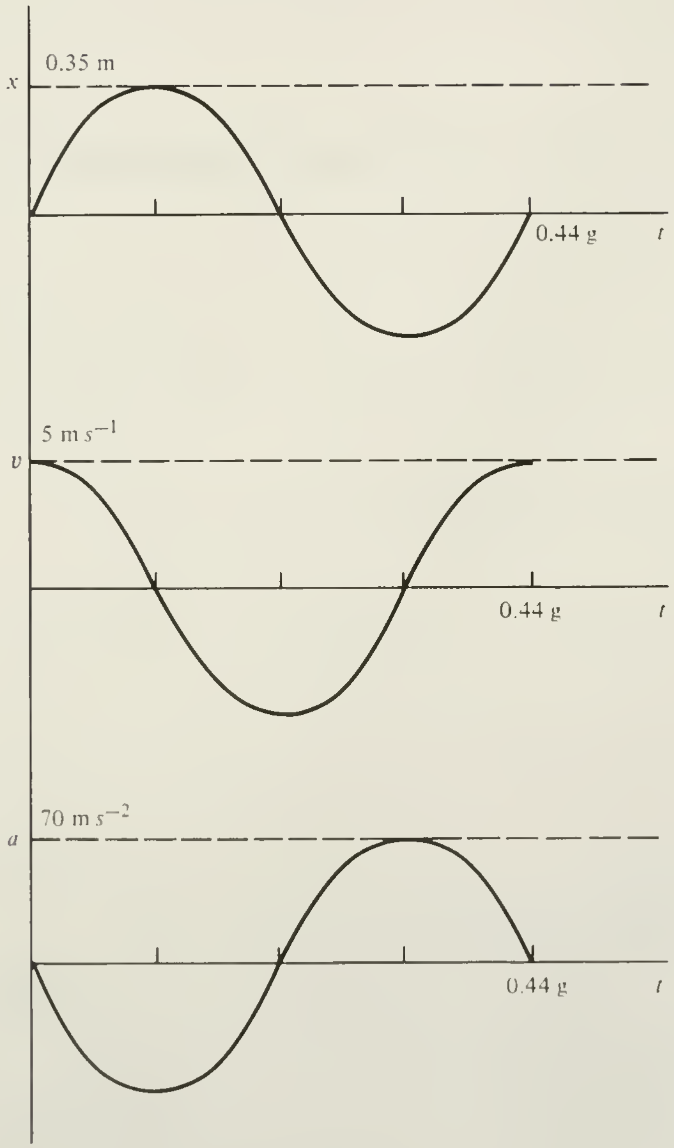


Figure 11-3

Example 6

A 200 g mass vibrates in SHM with amplitude 0.5 m and frequency 5 Hz. Find
(a) the energy of the motion.
(b) the maximum values of the velocity and acceleration.
(c) the time it takes to move from 25 cm below to 25 cm above the equilibrium position.

Solution:

(a) The total energy is

$$E = \frac{mv^2}{2} + \frac{kx^2}{2} .$$

At maximum displacement, x = A = 0.5 m and v = 0, so

$$E = \frac{kA^2}{2} .$$

The force constant may be evaluated from the known frequency, since

$$f = \frac{1}{2\pi}\left[\frac{k}{m}\right]^{1/2} \quad \text{or} \quad (2\pi f)^2 = \frac{k}{m} ,$$

$$k = m(2\pi f)^2 = (0.2 \text{ kg})(2\pi\cdot 5 \text{ s}^{-1})^2 = 197 \text{ N}\cdot\text{m}^{-1}.$$

Thus

$$E = \frac{kA^2}{2} = \frac{(197 \text{ N}\cdot\text{m}^{-1})(0.5 \text{ m})^2}{2} = 24.7 \text{ J}.$$

(b) The maximum value of the velocity may be evaluated in two ways. First, from the energy expression,

$$E = \frac{mv^2}{2} + \frac{kx^2}{2} = \frac{mv^2{}_{max}}{2} ,$$

because v is maximum at the origin where x = 0:

$$v_{max} = \left[\frac{2E}{m}\right]^{1/2} = \left[\frac{2(24.7)\text{J}}{0.2 \text{ kg}}\right]^{1/2} = 15.7 \text{ m}\cdot\text{s}^{-1}.$$

Another way is to use the general form

$$x = A\cos(\omega t + \theta_0)$$

$$v = \frac{dx}{dt} = -\ \omega A \sin(\omega t + \theta_o)$$

$$v_{max} = \omega A = 2\pi f A = 2\pi(5\ s^{-1})(0.5\ m) = 15.7\ m\cdot s^{-1}.$$

The acceleration is

$$a = \frac{dv}{dt} = -\ \omega^2 A \cos(\omega t + \theta_o) = -\ \omega^2 x$$

$$a_{max} = \omega^2 A = (2\pi f)^2 A$$

$$= [2\pi(5\ s^{-1})]^2(0.5\ m) = 493\ m\cdot s^{-2}.$$

(c) Here we must choose a convenient form for x(t) among those listed in Example 3. If we start the clock (take t = 0) as the body passes through x = 0 with positive velocity,

$$x = A \sin \omega t.$$

Referring to Fig. 11-4, we wish to find the times t_+ and t_- when the body has displacement of ± 25 cm. Since $x = A \sin \omega t$

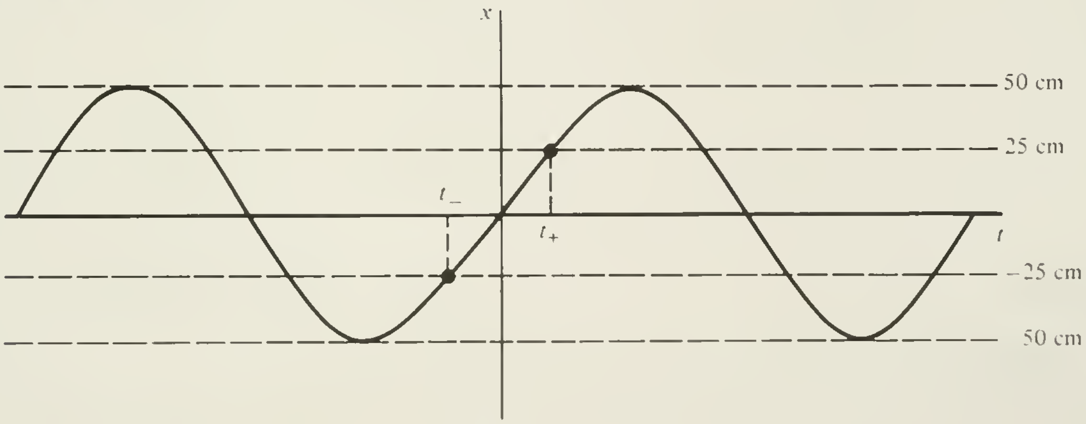


Figure 11-4

$$t = \frac{1}{\omega} \text{ arc sin } \frac{x}{A} = \frac{1}{2\pi f} \text{ arc sin } \frac{x}{A}$$

$$t_+ = \frac{1}{2\pi(5\ s^{-1})} \text{ arc sin } \frac{25}{50} = 0.017\ s$$

$$t_- = \frac{1}{2\pi(5\ s^{-1})} \text{ arc sin } (-\ \frac{25}{50}\ ) = -0.017\ s.$$

The total time is

$$t_+ - t_- = 0.033\ s.$$

Example 7

A body of mass 200 g stretches a spring 10 cm when suspended from it in a gravitational field. What is the period of oscillation if a 500 g mass is suspended from the spring?

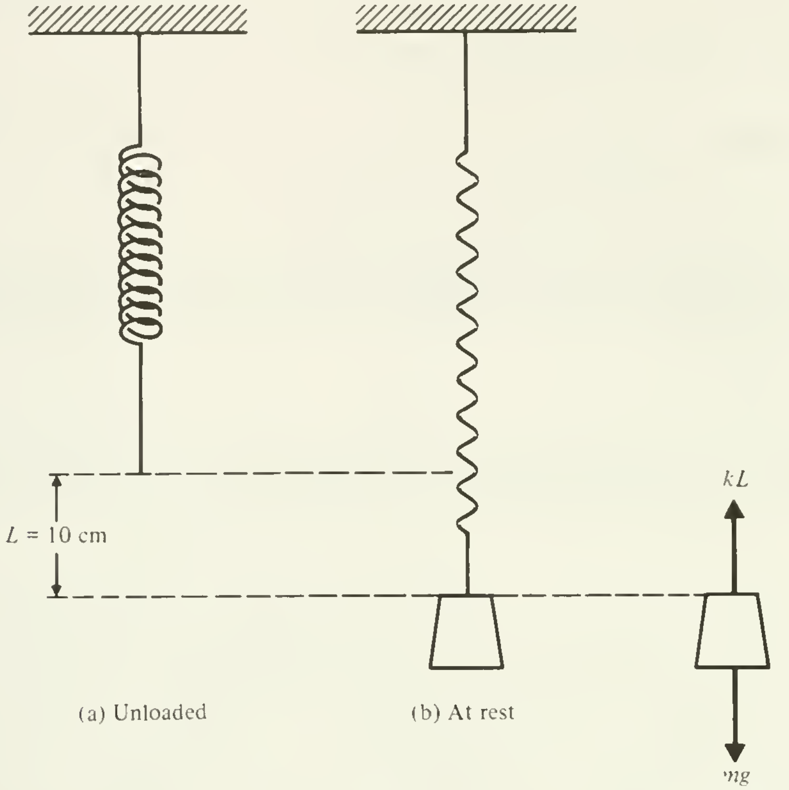


Figure 11-5

Solution:

Referring to Fig. 11-5, we see that when the mass is at rest it is in equilibrium:

$$kL = mg$$

$$k = \frac{mg}{L} = \frac{0.2\ \text{kg}\ (9.8\ \text{m}\cdot\text{s}^{-2})}{0.1\ \text{m}} = 19.6\ \text{N}\cdot\text{m}^{-1}.$$

If now a mass m' = 500 g is suspended from the spring of force constant k,

$$f = \frac{1}{2\pi}\left[\frac{k}{m'}\right]^{1/2} = \frac{1}{2\pi}\left[\frac{19.6\ \text{N}\cdot\text{m}^{-1}}{0.5\ \text{kg}}\right]^{1/2} = 0.99\ \text{s}^{-1}.$$

$$\tau = f^{-1} = 1\ \text{s}$$

Example 8

A spring stretches 10 cm when its tension is 50 N. A body of mass 5 kg is hung from the spring. When at rest the body is given an initial upward velocity of 1 $m \cdot s^{-1}$.

(a) Find the amplitude and the frequency of the motion.

(b) Find the acceleration of the mass when it is 5 cm above its equilibrium position.

(c) Find the force of tension in the spring at this point.

(d) What is the displacement and velocity 0.4 s after the initial time?

Solution:

(a) First we find the spring constant,

$$k = \frac{F}{x} = \frac{50\ N}{0.1\ m} = 500\ N \cdot m^{-1},$$

from which we determine the frequencies

$$\omega = 2\pi f = [\frac{k}{m}]^{1/2} = [\frac{500\ N \cdot m^{-1}}{5\ kg}]^{1/2} = 10\ s^{-1}$$

$$f = \frac{10\ s^{-1}}{2\pi} = 1.6\ Hz.$$

To find the amplitude, use the energy method:

$$E = \frac{mv^2}{2} + \frac{kx^2}{2} = \frac{kA^2}{2}\ .$$

In this case the initial energy is $\frac{mv_i^2}{2}$ ($x_i = 0$) so that

$$\frac{mv_i^2}{2} = \frac{kA^2}{2}$$

$$A = [\frac{m}{k}]^{1/2}\ v_i = \frac{v_i}{\omega} = \frac{1\ m \cdot s^{-1}}{10\ s^{-1}} = 0.1\ m = 10\ cm.$$

(b)

$$a = -\frac{k}{m}\ x = -\ \omega^2 x = -(10\ s^{-1})^2(0.05\ m) = -\ 5\ m \cdot s^{-2}.$$

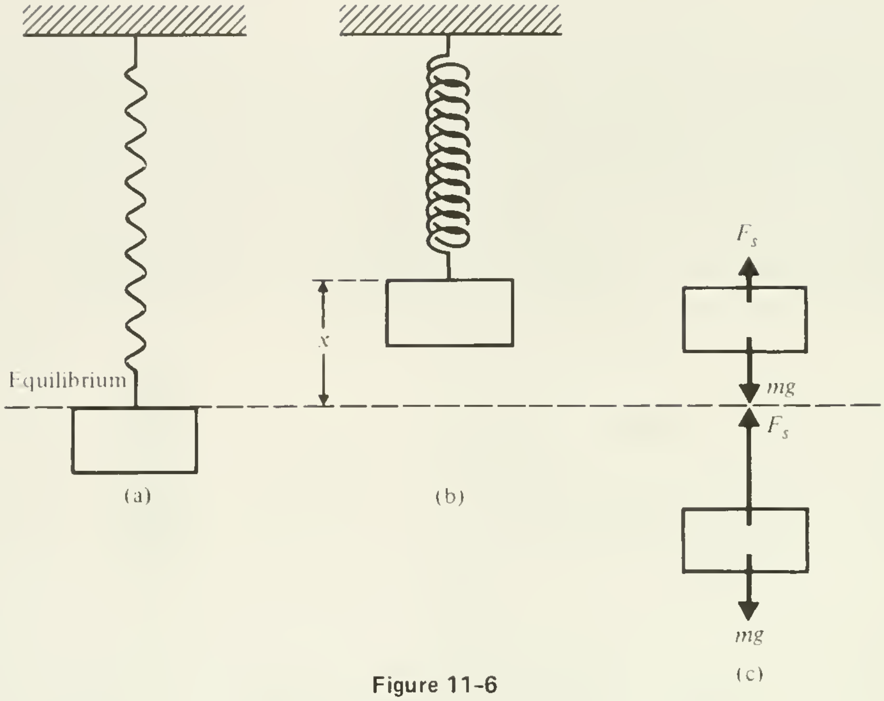


Figure 11-6

(c) Referring to Fig. 11-6, which includes free body diagrams for the cases in which the mass is above and below its equilibrium position, we have, for the total force F in terms of the spring force F_s,

$$F = -kx = F_s - mg$$

$$F_s = mg - kx$$

$$= (5\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2}) - (500\ \text{N}\cdot\text{m}^{-1})(0.05\ \text{m})$$

$$= 24\ \text{N}$$

(d) Since the initial displacement is zero, the forms

$$x = A \sin \omega t; \quad x(o) = 0; \quad A = 0.1\ \text{m (from part a)}$$

$$v = \omega A \cos \omega t; \quad v(o) = \omega A = (10\ \text{s}^{-1})(0.1\ \text{m}) = 1\ \text{m}\cdot\text{s}^{-1}$$

obey the initial conditions. Thus we have

$$x = (0.10\ \text{m}) \sin (10\ \text{s}^{-1}t)$$

$$x(0.4\ \text{s}) = (0.10\ \text{m}) \sin (10\ \text{s}^{-1} \times 0.4\ \text{s}) = -\ 0.076\ \text{m} = -\ 7.6\ \text{cm}$$

$$v = (1\ \text{m}\cdot\text{s}^{-1}) \cos (10\ \text{s}^{-1}t)$$

$$v(0.4\ \text{s}) = (1\ \text{m}\cdot\text{s}^{-1}) \cos (10\ \text{s}^{-1}\cdot 0.4\ \text{s})$$

$$= -0.65\ \text{m}\cdot\text{s}^{-1} = -65\ \text{cm}\cdot\text{s}^{-1}.$$

The block is 7.6 cm below its equilibrium position moving downward with speed 65 $\text{cm}\cdot\text{s}^{-1}$.

Example 9

A body in SHM with angular frequency 0.5 s^{-1} is initially 10 cm from its equilibrium position and moving back toward its equilibrium position with a velocity 5 $cm \cdot s^{-1}$, as shown in Fig 11-7.

(a) Find the period of the motion.

(b) Find the coordinate and velocity of the body as a function of time.

(c) How long does it take the body to return to its equilibrium position?

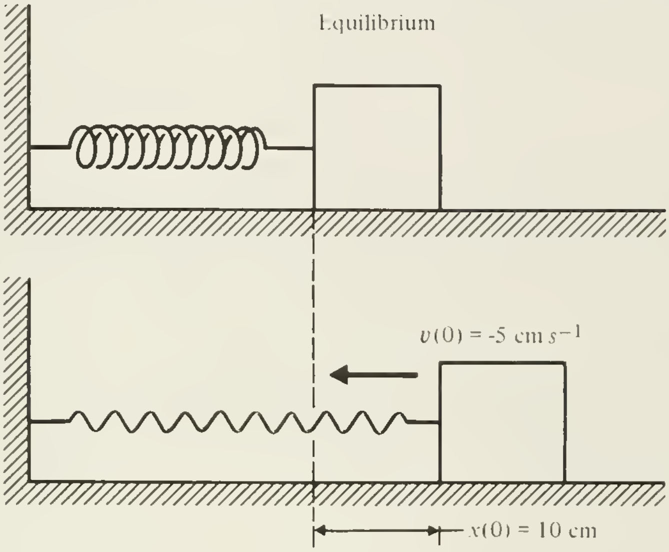


Figure 11-7

Solution:

(a) The period is

$$\tau = f^{-1} = \frac{2\pi}{\omega} = \frac{2\pi}{0.5\ s^{-1}} = 12.6\ s.$$

(b) Since both initial displacement and initial velocity are non-zero, we must use the general forms

$$x = A \cos (\omega t + \theta_0); \qquad \omega = 0.5\ s^{-1}$$

$$v = \frac{dx}{dt} = - \omega A \sin (\omega t + \theta_0)$$

and evaluate the unknown constants A and θ_0 from the initial conditions

$$x(o) = 10 \text{ cm and } v(o) = - 5 \text{ cm } s^{-1}.$$

(The initial displacement has been taken as positive. This amounts to a choice of coordinate system. See Fig. 11-7. The initial velocity is then negative because the body is moving back toward the origin from a positive coordinate.)

Thus we have, at t = 0

$$10\text{ cm} = A\cos\theta_o \qquad \text{(displacement condition)}$$

$$-5\text{ cm}\cdot\text{s}^{-1} = -\omega A\sin\theta_o \qquad \text{(velocity condition)}$$

yielding

$$\tan\theta_o = \omega^{-1}\frac{5\text{ cm}\cdot\text{s}^{-1}}{10\text{ cm}} = (0.5\text{ s}^{-1})^{-1}(0.5\text{ s}^{-1}) = 1$$

$$\theta_o = \text{arc tan}(1) = 0.79\text{ rad}$$

$$A = 10\text{ cm}(\cos\theta_o)^{-1} = 14.1\text{ cm}.$$

Thus

$$x = (14.1\text{ cm})\cos(0.5\text{ s}^{-1}t + 0.79)$$

$$v = -(0.5\text{ s}^{-1})(14.1\text{ cm})\sin(0.5\text{ s}^{-1}t + 0.79).$$

(c) Solving the first equation for t when $x = 0$,

$$0 = (14.1\text{ cm})\cos(0.5\text{ s}^{-1}t + 0.79)$$

For positive t starting at $t = 0$, the cosine function vanishes when its argument passes through $\pi/2$:

$$0.5\text{ s}^{-1}t + 0.79 = \frac{\pi}{2}$$

$$t = \left(\frac{\pi}{2} - 0.79\right)(0.5\text{ s}^{-1})^{-1}$$

$$= 1.56\text{ s}.$$

Example 10

A clock pendulum is approximately a simple pendulum. How long should the suspension be if its period is to be 2 s and its mass is 5 kg?

Solution

For a simple pendulum

$$\omega = 2\pi f = \frac{2\pi}{\tau} = \left[\frac{g}{L}\right]^{1/2}$$

$$\left(\frac{2\pi}{\tau}\right)^2 = \frac{g}{L}$$

Solving for the length, L,

$$L = \left(\frac{\tau}{2\pi}\right)^2 g = \left(\frac{2\ s}{2\pi}\right)^2 (9.8\ m\cdot s^{-2}) = 0.99\ m$$

Note the mass is irrelevant in the small oscillation approximation.

Example 11

A simple pendulum consists of a 5.0000 kg mass suspended by a 2.0000 m wire. It is pulled 10.0000 cm from the vertical and released. Find the maximum velocity to 5 significant figures, using $g = 9.8000\ m\cdot s^{-2}$:

(a) exactly, and

(b) making the small oscillation approximation.

Solution

Refer to Fig. 11-8:

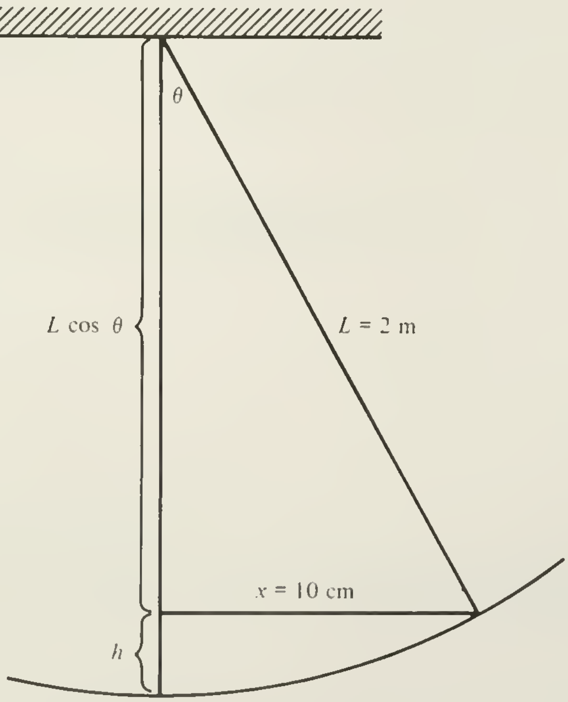


Figure 11-8

(a) The initial energy is

$$E = mgh = mgL(1 - \cos\theta)$$

and the energy at the bottom of the swing with v maximum is

$$E = \frac{1}{2} mv^2_{max}$$

By energy conservation

$$\frac{1}{2}\, mv^2{}_{max} = mgL(1 - \cos\theta)$$

$$v_{max} = [2gL(1 - \cos\theta)]^{1/2}$$

$$\sin\theta = \frac{10.000\ \text{cm}}{2.0000\ \text{m}} = 5.0000 \times 10^{-2}$$

$$\cos\theta = .99875$$

$$v_{max} = [2(9.8000\ \text{m}\cdot\text{s}^{-2})(2.0000\ \text{m})(1.2508 \times 10^{-3})]^{1/2}$$

$$= 0.22142\ \text{m}\cdot\text{s}^{-1}$$

(b) Using the small oscillation approximation

$$x = A \sin \omega t$$

$$v = \omega A \cos \omega t$$

$$v_{max} = \omega A \text{ with } \omega = [\frac{g}{L}]^{1/2}$$

$$v_{max} = (0.10000\ \text{m})[\frac{9.8000\ \text{m}\cdot\text{s}^{-2}}{2.000\ \text{m}}]^{1/2}$$

$$= 0.22136\ \text{m}\cdot\text{s}^{-1}$$

Example 12

A thin uniform rod is pivoted at a point one quarter of its length from one end. The period is measured. Then it is pivoted at its end and the period is again measured. Find the ratio of the initial to the final period.

Solution

The moment of inertia of a rod about its center of mass is

$$I_{cm} = \frac{1}{12}\, mL^2 .$$

By the parallel axis theorem

$$I_x = I_{cm} + mx^2$$

where x is the distance from the 'cm' axis to the parallel axis.

Initially we have

$$I_i = I_{cm} + m(\frac{L}{4})^2 = \frac{1}{12} mL^2 + \frac{1}{16} mL^2$$

$$= \frac{7}{48} mL^2$$

and finally

$$I_f = I_{cm} + m(\frac{L}{2})^2 = \frac{1}{12} mL^2 + \frac{1}{4} mL^2$$

$$= \frac{1}{3} mL^2$$

The period of a physical pendulum is

$$\tau = 2\pi \left[\frac{I}{mgh}\right]^{1/2},$$

where h is the distance from the CM to the pivot point.

Thus

$$\frac{\tau_i}{\tau_f} = 2\pi\left[\frac{I_i}{mgh_i}\right]^{1/2} \div 2\pi\left[\frac{I_f}{mgh_f}\right]^{1/2}$$

$$= \left[\frac{I_i h_f}{I_f h_i}\right]^{1/2} = \left[\frac{(7/48)mL^2 \cdot (L/2)}{(1/3)mL^2 \cdot (L/4)}\right]^{1/2}$$

$$= \left[\frac{7 \cdot 3 \cdot 4}{48 \cdot 2}\right]^{1/2} = 0.94$$

Example 13

A body of mass 2 kg is suspended at a point 3 cm from its center-of-mass and observed to oscillate with a period of 2 s. Find its moment of inertia.

Solution

$$\tau = 2\pi \left[\frac{I}{mgh}\right]^{1/2}$$

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{I}{mgh} \quad \text{or} \quad I = mgh\left(\frac{\tau}{2\pi}\right)^2$$

$$I = (2\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(0.03\ \text{m})\left(\frac{2\ \text{s}}{2\pi}\right)^2$$

$$= 5.96 \times 10^{-2}\ \text{kg}\cdot\text{m}^2$$

Example 14

A body of mass 0.3 kg hangs by a spring of force constant 50 $N\cdot m^{-1}$. By what factor is the frequency of oscillation reduced if the oscillation is damped and reaches $e^{-1} = 0.37$ of its original amplitude in 100 oscillations?

Solution:

The fractional frequency shift is

$$\frac{\Delta w}{w} = \frac{w' - w}{w} = \frac{[(k/m) - (b^2/4m^2)]^{1/2} - [k/m]^{1/2}}{[k/m]^{1/2}}$$

$$= \left[1 - \frac{m}{k}\frac{b^2}{4m^2}\right]^{1/2} - 1$$

For small damping we make the approximation

$$\frac{\Delta w}{w} = -\left(\frac{m}{2k}\right)\left(\frac{b^2}{4m^2}\right)$$

The amplitude is diminished by the factor $e^{-bt/2m}$. This factor is e^{-1} when $t = 2m/b = 100$ periods $= 100(2\pi)[m/k]^{1/2}$. Thus we can evaluate the factor $b/2m = [k/m]^{1/2}[1/(100)(2\pi)]$ and determine the frequency shift

$$\frac{\Delta w}{w} = -\frac{m}{2R}\frac{k}{m}\left[\frac{1}{100(2\pi)}\right]^2$$

$$= -1.3 \times 10^{-6}$$

QUIZ

1. A body of mass 1 kg rests on a frictionless table and is attached to the wall with a spring of spring constant 10 $N \cdot m^{-1}$. It is pulled 10 cm from its equilibrium position and released from rest.

   (a) Find the period of the motion.

   (b) Find the velocity when the displacement is 5 cm.

Answer: 1.99 s, $\pm$ 0.27 $m \cdot s^{-1}$

2. A simple pendulum consists of a 10 kg mass suspended by an 8 m long wire. It is pulled 40 cm from the vertical and released. Find the maximum velocity of the pendulum bob.

Answer: 0.44 $m \cdot s^{-1}$

3. A thin solid uniform disk of mass 0.8 kg and radius 4 cm is pivoted at a point on its edge so that it can swing freely in its plane. Find the period of oscillation.

Answer: 0.5 s

4. An automobile spring has an undamped period of 1 s and supports one quarter of the weight of a 1000 kg car. What is the damping constant b of the damping force F = - bv supplied by the car's shock absorber if the system is critically damped?

Answer: 3.1×10^3 $kg \cdot s^{-1}$

12
ELASTICITY

OBJECTIVES

In this short chapter you will learn a few facts about how the size and shape of materials change when forces are applied to them. Your objectives are to:

Define stress and strain and evaluate their magnitudes for various applied force.

Define pressure.

Recognize elastic behavior and define the elastic moduli associated with elastic behavior.

Solve problems involving Young's modulus Y, Poisson's ratio σ, the bulk modulus B, and the shear modulus S.

REVIEW

If a body is considered as composed of two parts, a stress is the force per unit area of one part of the body on the other part, at the boundary layer, a plane. The force may be normal to the plane, in which case the stress is

$$\text{Normal stress} = \frac{F_n}{A} ,$$

or parallel to the plane, in which case the stress is

$$\text{Shear stress} = \frac{F_t}{A} .$$

If the body is being squeezed together at the plane, the stress is compressive. If the body is being pulled apart, the stress is tensile. The units of stress are the units of pressure, which we discuss next: force per unit area.

A body at rest and submerged in a fluid at rest feels a normal stress of compression, independent of direction, called hydrostatic pressure,

$$p = \frac{F}{A},$$

measured in units of $N \cdot m^{-2} = Pa$. The average pressure of the earth's atmosphere at sea level is equal to one atmosphere,

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} = 14.7 \text{ lb} \cdot \text{in}^{-2}.$$

The strain of a body is the relative change of the dimensions of the body when subjected to stress. It is always a dimensionless quantity.

If ΔL is the change of length of a wire of initial length L_o subjected to tensile stress, the strain is

$$\text{Tensile strain} = \frac{\Delta L}{L_o}.$$

If a body is subjected to shear stress, it tends to change its shape by altering an angle by a small amount ϕ, and the strain is

$$\text{Shear strain} = \phi \text{ (radians)}.$$

The forces responsible for hydrostatic pressure produce a volume change ΔV of a submerged object of volume V,

$$\text{Volume strain} = \frac{\Delta V}{V}.$$

If stress (force per unit area producing strain) is proportional to strain (deformation) we are in the <u>elastic</u> regime where Hooke's law is valid and <u>elastic moduli</u> are useful:

$$\text{Young's modulus} = Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{\text{compressive stress}}{\text{compressive strain}} = \frac{F_n/A}{\Delta L/L_o}$$

When a body stretches it also becomes smaller in width w. The fractional change in width is proportional to the fractional change in length:

$$\frac{\Delta w}{w} = -\sigma \frac{\Delta L}{L_o},$$

where σ is called Poisson's ratio.

$$\text{Shear modulus} = S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F_t/A}{\phi}$$

$$\text{Bulk modulus} = B = \frac{-\Delta p}{\Delta V/V_o} = -V_o \frac{\Delta p}{\Delta V} .$$

The compressibility is the inverse bulk modulus, $k = B^{-1}$.

In the elastic range the force necessary to stretch a spring a distance x is $F = kx$. The equal and opposite force of the spring is $F = -kx$. k is the spring constant.

PROBLEM-SOLVING STRATEGIES

Refer to SZY, Tables 12-1 and 12-2, and 12-3 for useful elastic constants.

EXAMPLES AND SOLUTIONS

Example 1

A 10 kg weight is hung from a steel wire of unstretched length 1 m and diameter 2 mm. How much does the wire stretch, and what is its final diameter?

Solution:

We know the force (weight), the area and the unstretched length, so by looking up Young's modulus, Y, for steel, we can find the change in length:

$$Y = \frac{F_n/A}{\Delta L/L_o} , \text{ so}$$

$$\Delta L = \frac{F_n L_o}{AY} = \frac{mgL_o}{AY} = \frac{(10\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})(1\text{ m})}{\pi(10^{-3}\text{ m})^2(2 \times 10^{11}\text{ N}\cdot\text{m}^{-2})}$$

$$= 1.56 \times 10^{-4}\text{ m} = 0.156\text{ mm}.$$

The change in the width Δw is

$$\Delta w = -\sigma \frac{\Delta L}{L_o} w = -(0.19)\frac{1.56 \times 10^{-4}\text{ m}}{1\text{ m}} 2\text{ mm} = -5.9 \times 10^{-5}$$

$$w = w + \Delta w = 2\text{ mm} - 5.9 \times 10^{-5}\text{ mm}.$$

Example 2

What is the Young's modulus of a sample of brass wire 0.5 m long and 2mm in diameter which stretches 0.15 mm when stressed by a 10 kg weight?

Solution:

$$Y = \frac{F_n /A}{\Delta L/L_o} = \frac{F_n L_o}{\Delta A} = \frac{mgL_o}{\Delta LA} = \frac{(10\ kg)(9.8\ m\cdot s^{-2})(0.5\ m)}{(0.15\ x\ 10^{-3}\ m)\pi(10^{-3} m)^2}$$

$$= 1.04\ x\ 10^{11}\ N\cdot m^{-2} = 1.04\ x\ 10^{11}\ Pa$$

Example 3

A steel wire has a breaking stress of 7.2 x 10^8 Pa. Its cross section is 0.06 cm^2 and its length is 3 m. What is the maximum load it will support?

Solution:

$$\text{Tensile stress} = \frac{F_n}{A}$$

$$\text{Maximum stress} = 7.2\ x\ 10^8\ Pa = \frac{(F_n)_{max}}{0.06\ cm^2}$$

$$(F_n)_{max} = 0.06(10^{-2}\ m)^2(7.2\ x\ 10^8\ N\cdot m^{-2})$$

$$= 4{,}300\ N.$$

Example 4

A copper wire of cross-sectional area 0.05 cm^2 and length 5 m is attached end to end to a steel wire of length 3 m and cross-sectional area 0.02 cm^2. The wires are stretched to a tension of 200 N. Find the stress in each wire and the length of the combination.

Solution:

(s = steel, c = copper)

$$(\text{Tensile stress})_c = F_n/A_c = \frac{200\ N}{0.05(10^{-2}\ m)^2} = 4.0\ x\ 10^7\ Pa$$

$$(\text{Tensile stress})_s = F_n/A_s = \frac{200\ \text{N}}{0.02(10^{-2}\ \text{m})^2} = 1.0 \times 10^8\ \text{Pa}$$

$$(\Delta L)_s = \frac{(F_n/A)_s L_s}{Y_s} = \frac{(1.0 \times 10^8\ \text{Pa})(3\ \text{m})}{2 \times 10^{11}\ \text{Pa}} = 1.5 \times 10^{-3}\text{m}$$

$$(\Delta L)_c = \frac{(F_n\ /A)_c L_c}{Y_c} = \frac{(4.0 \times 10^7\ \text{Pa})(5\ \text{m})}{1.1 \times 10^{11}\ \text{Pa}} = 1.8 \times 10^{-3}\text{m}$$

$$\Delta L = (\Delta L)_s + (\Delta L)_c = 3.3 \times 10^{-3}\ \text{m}.$$

Example 5

A steel elevator cable of breaking stress 7.2×10^8 Pa is to support a 2000 kg elevator whose maximum upward acceleration is 2 m s^{-2}. What diameter cable should be used if the maximum stress is one-quarter the breaking stress?

Solution:

If F is the upward force of the cable on the elevator, and a is its upward acceleration,

$$F - Mg = Ma$$

where $M = 2000$ kg.

Thus

$$\text{stress} = \frac{F}{A} = \frac{M(a + g)}{A} = (1/4)(\text{breaking stress})$$

$$= (1/4)(7.2 \times 10^8\ \text{Pa}) = 1.8 \times 10^8\ \text{Pa}$$

$$A = \frac{M(a + g)}{1.8 \times 10^8\ \text{Pa}} = \frac{(2000\ \text{kg})(9.8 + 2)\text{m}\cdot\text{s}^{-2}}{1.8 \times 10^8\ \text{N}\cdot\text{m}^{-3}}$$

$$= 1.3 \times 10^{-4}\ \text{m}^2 = \pi(\ \frac{D}{2}\ ),$$

where D is the cable diameter. Solving for D:

$$D = 2[\ \frac{1.3 \times 10^{-4}\ \text{m}^2}{\pi}\ ]^{1/2} = 1.29 \times 10^{-2}\ \text{m} = 1.29\ \text{cm}.$$

Example 6

What is the change in volume of a cube of lead, one inch on a side, originally at atmospheric pressure, subjected to a hydrostatic pressure of 21 atmospheres? What is its new edge length L?

Solution:

$$B = -V\frac{\Delta p}{\Delta V}\text{, with } \Delta p = (21 - 1)\text{atm or } (20)(14.7 \text{ lbs}\cdot\text{in}^{-2})$$

$$\Delta V = \frac{-V\Delta p}{B} = \frac{-(1 \text{ in}^3)(20)(14.7 \text{ lbs}\cdot\text{in}^{-2})}{1.1 \times 10^6 \text{ lb}\cdot\text{in}^{-2}} = -2.67 \times 10^{-4}\text{in}^3$$

$$L^3 = V + \Delta V = (1 - 2.67 \times 10^4)\text{in}^3$$

$$L = (1 - 2.67 \times 10^{-4})^{1/3} \simeq (1 - 0.89 \times 10^{-4})\text{in.}$$

To four significant figures, L is equal to 1 in.

Example 7

Find the compressibility of a 0.1 m^3 sample of oil whose decrease in volume is 2.04×10^{-4} m^3 when subjected to a pressure increase of 1.02×10^7 Pa. Express your result in atmospheres of pressure.

Solution:

$$B = \frac{-V\Delta p}{\Delta V} = \frac{(0.1 \text{ m}^3)(1.02 \times 10^7 \text{ Pa})}{2.04 \times 10^{-4} \text{ m}^3}$$

$$= 5.0 \times 10^9 \text{ Pa} = \frac{5.0 \times 10^9 \text{ Pa}}{1.013 \times 10^5 \text{ Pa}\cdot\text{atm}^{-1}}$$

$$= 4.9 \times 10^4 \text{ atm}$$

$$k = \frac{1}{B} = 2.0 \times 10^{-5} \text{ atm}^{-1}.$$

Example 8

What is the shear stress at each of the glued wooden joints under the tensions of Fig. 12-1 if each glued area is 10 cm^2?

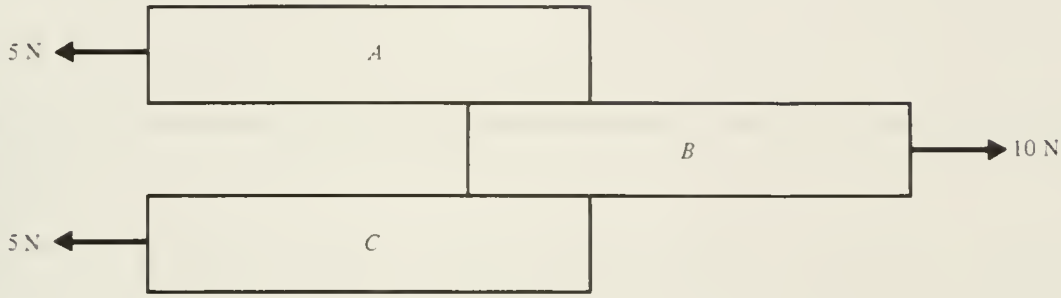


Figure 12-1

Solution:

The force of A on B is 5 N and is tangential to the joint.

$$\text{shear stress} = \frac{F_t}{A} = \frac{5\ N}{10\ cm^2} = 5.0 \times 10^3\ N\cdot m^{-2}$$

QUIZ

1. A steel rod 8 m long and 0.5 cm^2 in cross section is found to stretch 0.4 cm under a tension of 12,000 N. What is Young's modulus for this steel?

Answer: $4.8 \times 10^{11}\ N\cdot m^{-2}$

2. An aluminum wire, originally 5 m long and 0.1 cm in diameter, stretches 5.71 mm when subjected to a tension. Find the tension and its change in width.

Answer: $8 \times 10^7\ N\cdot m^{-2}$, 1.8×10^{-5} cm

3. A cube of lead one centimeter on a side is subjected to a pressure increase of 14 atmospheres. Find the fractional change in length of one of its edges.

Answer: $\Delta L/L = 6 \times 10^{-5}$

13
FLUID MECHANICS

OBJECTIVES

In this chapter you will learn about density, pressure, bouyancy and Archimedes' Principle, surface tension, streamline flow of an ideal fluid, Bernoulli's equation, viscosity, turbulence, and Reynolds number. Your objectives are to:

Define <u>density</u> and <u>pressure</u>.

<u>Convert</u> the <u>pressure units</u> of atmospheres, mm of mercury and $N \cdot m^{-2}$, to each other.

Solve problems involving the <u>variation</u> of <u>pressure</u> with <u>depth</u>.

Apply <u>Archimedes' Principle</u> to bodies floating in liquids and determine their apparent weight.

Calculate the <u>forces</u> on the <u>walls</u> of a vessel containing a fluid.

Derive and apply the <u>equation of continuity</u> and <u>Bernoulli's equation</u> to the flow of fluids out of tanks and through tubes of varying diameter.

Define the coefficient of viscosity; develop <u>Poiseuille's law</u> for the flow of a viscuos fluid through a circular pipe. Apply <u>Stokes' law</u> to a sphere falling in a viscous fluid.

Define <u>Reynolds number</u> and use it to determine when flow is <u>turbulent</u> and when it is <u>laminar</u>.

REVIEW

<u>Pressure and Density</u>

The density of a substance is its mass per unit volume

$$\rho = \frac{m}{V}$$

The SI units of density are $kg \cdot m^{-3} = 10^3\ g \cdot cm^{-3}$. In the British system, since weight $w = mg$, mass has units of $w \cdot g^{-1}$ or

$$\text{British units of mass} = lb \cdot g^{-1} = lb\ (32\ ft\ s^{-2})^{-1} = slug$$

Thus in the British system of units, $\rho g = mg/V$ = weight per unit volume has units $lb \cdot ft^{-3}$.

The hydrostatic pressure in a fluid (liquid or gas) is, as we learned in the last chapter, the normal force per unit area against a surface ΔA within the fluid,

$$p = \frac{\Delta F}{\Delta A}$$

It is independent of the direction of the area but increases with depth. If the density ρ is a constant,

$$p = p_a + \rho g h,$$

where p is the pressure a depth h below a level where the pressure is atmospheric pressure. More generally the pressure difference between two levels at elevation y above a reference level is

$$p_2 - p_1 = -\rho g (y_2 - y_1).$$

A pressure gauge usually measures the difference between absolute pressure p and atmospheric pressure p_a,

$$\text{gauge pressure} = p - p_a = \rho g h.$$

Given a fluid of density ρ, the gauge pressure is proportional to the height h and may be quoted in units of h referred to the specific substance, such as 'millimeters of mercury'. Common pressure units are

$$1\ N \cdot m^{-2} = 1\ Pa = 10^{-5}\ bar$$

$$\text{one Torr} = 1\ \text{mm mercury}.$$

A pressure of one Torr corresponds to

$$p = \rho g h = (13.6 \times 10^3\ kg \cdot m^{-3})(9.8\ m\ s^{-2})(10^{-3}\ m)$$

$$= 133\ N \cdot m^{-2} = 133\ Pa$$

where ρ is the density of mercury. A pressure of 30 in = 0.76 m mercury is an average atmospheric pressure, p_a,

$$p_a = 760\ Torr = 1.013 \times 10^5\ Pa = 14.7\ lb \cdot in^{-2}.$$

Archimedes' Principle

When a body is immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid that is displaced by the body. If the fluid has density ρ and the body a volume V, the upward bouyant force is

$$F_B = mg = \rho Vg.$$

Surface Tension

A surface tension is a force on the surface of a liquid tending to minimize its area. It may be measured by the thin film experiment of SZY, Fig. 13-8, with

$$\gamma = \frac{F}{2L}$$

in units $N\cdot m^{-1} = 1000\ dyne\cdot cm^{-1}$.

For a soap bubble of radius R the difference between the interior air pressure p and the exterior pressure p_a is

$$p - p_a = \frac{4\gamma}{R}\ .$$

For a liquid drop

$$p - p_a = \frac{2\gamma}{R}$$

where p is the pressure within the fluid.

An ideal fluid is incompressible and has no viscosity. In steady or stationary flow the velocity at each point in space is constant. A streamline is a curve along which a particle would flow if the flow were stationary. The tangent to a streamline at any point is the direction of the fluid velocity at that point. The streamlines passing through the boundary of an area enclose a tube of flow. In a streamline or laminar flow adjacent layers of fluid slide smoothly past each other. A pattern of streamlines is steady or changes smoothly with time. In turbulent flow the streamline pattern is always changing in a seemingly random way.

An incompressible fluid obeys the equation of continuity,

$$A_1v_1 = A_2v_2,$$

where A_1 and A_2 are two cross-section areas of a flow tube where the corresponding velocities of flow are v_1 and v_2. The volume V of fluid passing through the surfaces per unit time, or discharge rate is:

$$\text{Discharge rate} = Av = \frac{dV}{dt} .$$

<u>Bernoulli's equation</u> is a relation among the pressure, elevation and velocity at two points in a tube of streamline flow of an ideal fluid,

$$p_1 + \rho g y_1 + \frac{\rho v_1{}^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2{}^2}{2} ,$$

where ρ is the density, y is the elevation and v the velocity.

In a viscous fluid, adjacent layers of flowing fluid exert shear stress on each other. If a layer a perpendicular distance L from a stationary layer has velocity v, the coefficient of viscosity is

$$\eta = \frac{F/A}{v/L}$$

where F is the shear force and A the area of the layer. (See SZY, Fig. 13-24).

In steady viscous flow a pressure difference must be maintained to balance the viscous forces on a tube of flow. The volume rate of flow in a pipe of radius R is given by <u>Poiseuille's law</u>

$$\frac{dV}{dt} = \frac{\pi}{8} \frac{R^4}{\eta} \frac{p_1 - p_2}{L}$$

where L is the length of the pipe.

A sphere falling in a viscous medium where r is the radius and v the velocity (Stokes' Law) experiences a retarding force $F = 6\pi\eta r v$. A raindrop in the atmosphere eventually reaches a terminal velocity v where the downward weight is balanced by the upward viscous force and bouyant force. If r is its radius, ρ the density of the sphere, and ρ' the density of the fluid, then

$$v = \frac{2}{9} \frac{r^2 g}{\eta} (\rho - \rho').$$

The Reynolds number is:

$$N_R = \frac{\rho v D}{\eta},$$

where ρ = density, v = velocity, D = the tube diameter and η the viscosity. N_R characterizes the nature of the flow:

$N_R \lesssim 2000$, laminar flow;

$N_R \gtrsim 3000$, turbulent flow.

The ~ symbol means 'approximately'. Between 2000 and 3000 the flow may be of either kind and may change from one kind to the other.

HINTS AND PROBLEM-SOLVING STRATEGIES

The density of water is $\rho_w = 10^3\ \text{kg}\cdot\text{m}^{-3} = 1.9\ \text{slug}\cdot\text{ft}^{-3}$.

Add $14.7\ \text{lb}\cdot\text{in}^{-2} = 1.013 \times 10^5$ Pa to gauge pressure to get absolute pressure.

EXAMPLES AND SOLUTIONS

Example 1

32 grams of a gas occupy a volume of 22 liters. What is the density of the gas in $\text{kg}\cdot\text{m}^{-3}$?

Solution:

$$1 \text{ liter} = 1000 \text{ cc} = 1000(10^{-2}\ \text{m})^3$$

$$\rho = \frac{m}{V} = \frac{32\ \text{g}}{22\ \text{l}} = \frac{32 \times 10^{-3}\ \text{kg}}{(22)(1000)(10^{-2}\ \text{m})^3}$$

$$= 1.45\ \text{kg}\cdot\text{m}^{-3}$$

Example 2

Referring to Table 13-1, SZY, what is the density of mercury in $\text{kg}\cdot\text{m}^{-3}$? Of water in $\text{kg}\cdot\text{m}^{-3}$?

Solution:

$$\rho_{Hg} = 13.6\ \text{g}\cdot\text{cm}^{-3} = (13.6 \times 10^{-3}\ \text{kg})(10^{-2}\ \text{m})^{-3}$$

$$= 13.6 \times 10^3\ \text{kg}\cdot\text{m}^{-3}$$

$$\rho_w = 1\ \text{g}\cdot\text{cm}^{-3} = (10^{-3}\ \text{kg})(10^{-2}\text{m})^{-3}$$

$$\rho_w = 10^3\ \text{kg}\cdot\text{m}^{-3}.$$

Example 3

A bicycle pump has a piston of diameter one inch. What force on the piston is necessary to add air to a tire at gauge pressure 60 $lb \cdot in^{-2}$?

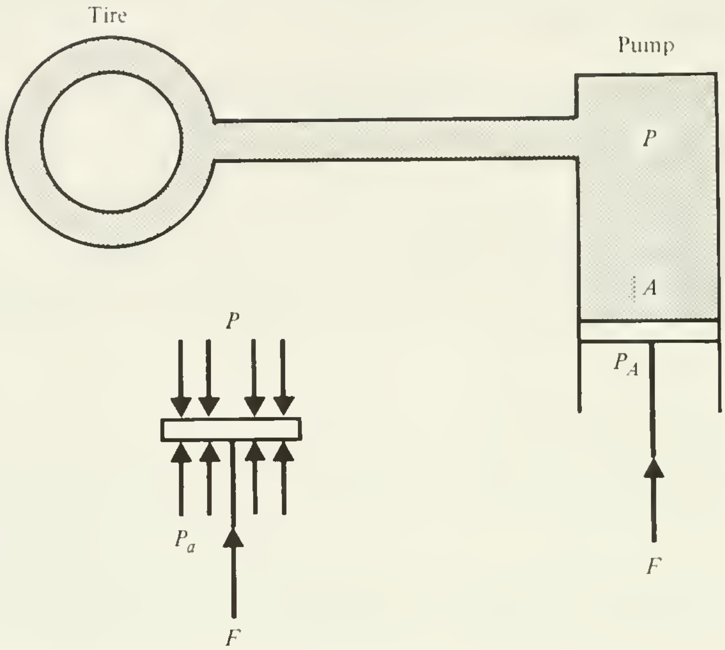


Figure 13-1

Solution:

Referring to Fig. 13-1, we see that the net upward force on the piston is

$$F + p_a A - pA.$$

When this force is positive, air enters the tire,

$$F + p_a A - pA > 0$$

$$F > (p - p_a)A = \text{gauge pressure x A}$$

$$= (60\ lb \cdot in^{-2})\pi\left(\frac{1\ in}{2}\right)^2$$

$$= 47\ lb.$$

Example 4

Water pressure in a town is maintained by a water tower 100 ft high, open to the atmosphere at the top.

(a) What is the gauge pressure at ground level, in atmospheres?

(b) A garden hose in this town, of diameter 5/8 inch, is open at an end and spilling water. What force at the end of the hose is necessary to close the leak?

Solution:

(a) $$p - p_a = \rho gh = \text{gauge pressure}$$

at ground level, where the density of water is

$$\rho = 1\ \text{g}\cdot\text{cm}^{-3} = 10^3\ \text{kg}\cdot\text{m}^{-3}.$$

In the British system the weight per unit volume is

$$\frac{w}{V} = \frac{mg}{V} = 62\ \text{lb}\cdot\text{ft}^{-3}$$

so that the mass density is

$$\rho = \frac{m}{V} = \frac{62\ \text{lb}\cdot\text{ft}^{-3}}{g}$$

Thus

$$p - p_a = \rho gh = \frac{62\ \text{lb}\cdot\text{ft}^{-3}}{g}\ g\ (100\ \text{ft})$$

$$= 6200\ \text{lb}\cdot\text{ft}^{-2}$$

$$= 6200\ \text{lb}\ (12\ \text{in})^{-2} = 43\ \text{lb}\cdot\text{in}^{-2}.$$

$$= \frac{43\ \text{lb}\cdot\text{in}^{-2}}{14.7\ \text{lb}\cdot\text{in}^{-2}(\text{atm})^{-1}} = 2.93\ \text{atm}.$$

(b) The force needed to contain the water, referring to Fig. 13-2, is

$$pA = p_aA + F$$

$$F = (p - p_a)A = (43\ \text{lb}\cdot\text{in}^{-2})\pi(5/16\ \text{in})^2$$

$$= 13.2\ \text{lb}.$$

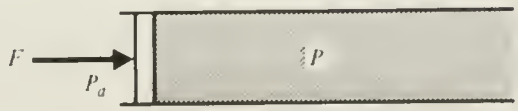


Figure 13-2

Example 5

(a) What is the pressure 100 ft below the ocean surface if sea water weighs 64 $lb \cdot ft^{-3}$?
(b) A Dutch submarine descends to this depth and springs a leak with a hole of diameter one inch. A sea scout plugs the hole with his thumb. What force must he apply?

Solution:

(a) $$p = p_a + \rho g h$$

$$\rho = \frac{m}{V} = \frac{w}{gV} = \frac{64\ lb \cdot ft^{-3}}{g}$$

$$p - p_a = \rho g h = \frac{64\ lb \cdot ft^{-3}}{g}\ g\ (100\ ft)$$

$$= 6400\ lb \cdot ft^{-2} \quad \text{(about 3 atm)}$$

(b) Assuming the interior of the submarine is at atmospheric pressure, a force pA of water pushes in and p_aA of air pushes out. Thus the force F necessary to keep the sea out is

$$F = (p - p_a)A \text{ with } A = \pi\left(\frac{D}{2}\right)^2 = \pi\left(\frac{1\ in}{2}\right)^2$$

$$F = (6400\ lb \cdot ft^{-2})\ \pi(1/24\ ft)^2$$

$$F = 35\ lb.$$

Example 6

A water barometer consists of a column of water below a vacuum as shown in Fig. 13-3.
(a) What is the height in feet of the column?
(b) What is the maximum height that a suction pump can raise water?

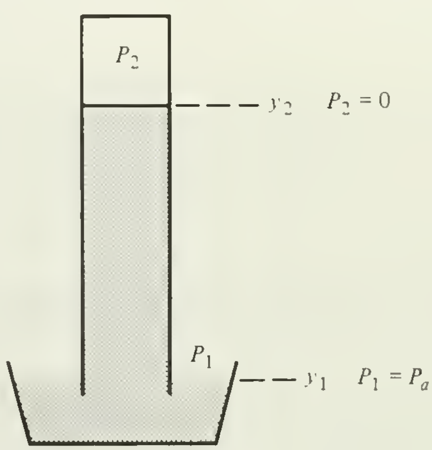


Figure 13-3

Solution:

The pressure difference between the two points is ($p_2 = 0$ and $p_1 = p_a$)

$$p_2 - p_1 = -\rho g(y_2 - y_1) = -\rho gh = -p_a.$$

Thus

$$h = \frac{p_a}{\rho g} = \frac{14.7\ \mathrm{lb \cdot in^{-2}}}{\left[\frac{62\ \mathrm{lb \cdot ft^{-3}}}{g}\right] g}$$

$$= \frac{14.7\ \mathrm{lb}\ (1/12\ \mathrm{ft})^{-2}}{62\ \mathrm{lb \cdot ft^{-3}}} = 34\ \mathrm{ft}.$$

(b) The answer is 34 ft. The best the suction pump can do is create a perfect vacuum, as in case (a).

Example 7

What fraction of an ice-cube is submerged when floating in glycerin?

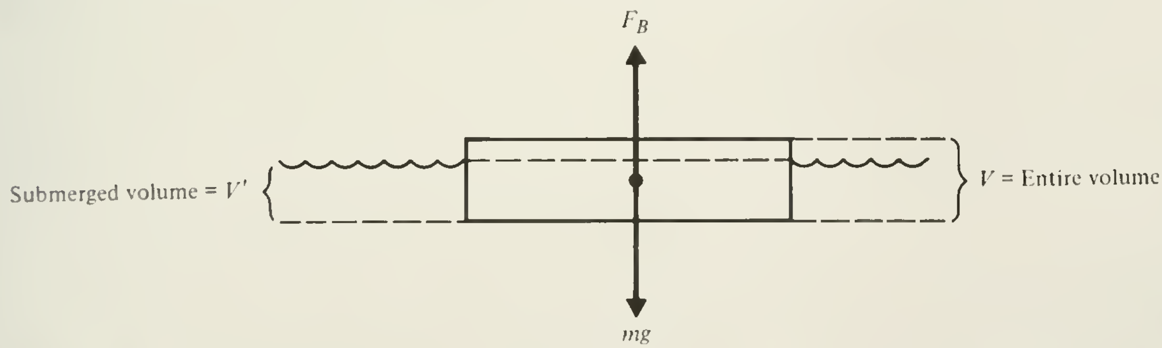


Figure 13-4

Solution:

Referring to Fig. 13-4, the bouyant force is

$$F_B = \rho_G V' g$$

where ρ_G is the density of the glycerin and V' is the submerged volume, equal to the volume of displaced glycerin. The weight mg of the cube is

$$mg = \rho_I V g$$

where ρ_I is the density of ice and V the volume of the ice-cube. Thus

$$\rho_I V g = \rho_G V' g$$

and

$$\frac{V'}{V} = \frac{\rho_I}{\rho_G} = \frac{0.92}{1.26} = 0.73.$$

Example 8

A dumpling floats two-thirds submerged in water. What is its density?

Solution:

The bouyant upward force is

$$F_B = \rho_W V' g$$

where ρ_W = density of water and V' is the submerged volume of the dumpling. The weight of the dumpling is

$$w = \rho_O V g$$

where ρ_O is the dumpling density and V its volume. Thus

$$\rho_W V' g = \rho_O V g$$

$$\rho_O = \rho_W \frac{V'}{V} = \frac{2}{3} \rho_W = \frac{2}{3} \quad 1 \text{ g}\cdot\text{cm}^{-3}$$

$$= 0.66 \text{ g}\cdot\text{cm}^{-3}.$$

Example 9

A sphere of volume 10 cm^3 'weighs' 50 g in water. (The spring balance reads 50 g.)

(a) What does it 'weigh' in air?
(b) What is its density?

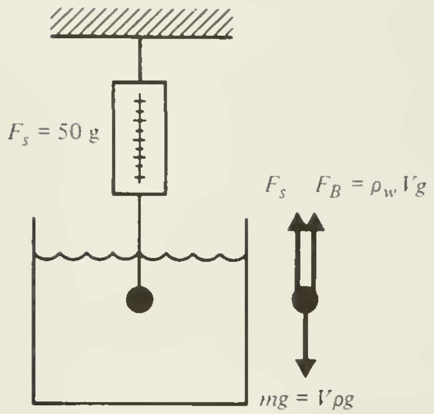


Figure 13-5

Solution:

Referring to Fig. 13-5, the equilibrium condition is

$$F_s + F_B = mg$$

$$F_s + \rho_w Vg = \rho Vg$$

$$\rho = \frac{F_s + \rho_w Vg}{Vg} = \frac{F_s}{Vg} + \rho_w$$

$$= \frac{mg}{Vg} + \rho_w = \frac{50\ \text{g}}{10\ \text{cm}^3} + 1\ \text{g}\cdot\text{cm}^{-3}$$

$$= 6\ \text{g}\cdot\text{cm}^{-3}.$$

The 'weight' in air is ρV or 60 g as measured by the scale. Its weight is of course

$$(60\ \text{g})(980\ \text{cm}\cdot\text{s}^{-2}) = 58{,}800\ \text{dynes}.$$

Example 10

A block of mass m = 15 kg and volume 0.01 m^3 hangs by a cord from a spring balance and is submerged in an unknown liquid as shown in Fig. 13-6. The spring scale reads 50 N.

(a) Draw a diagram indicating all forces acting on the mass m.

(b) Find the density of the liquid.

(c) If the combined weight of the liquid and its container is W = 500 N, find the force of the table against the container.

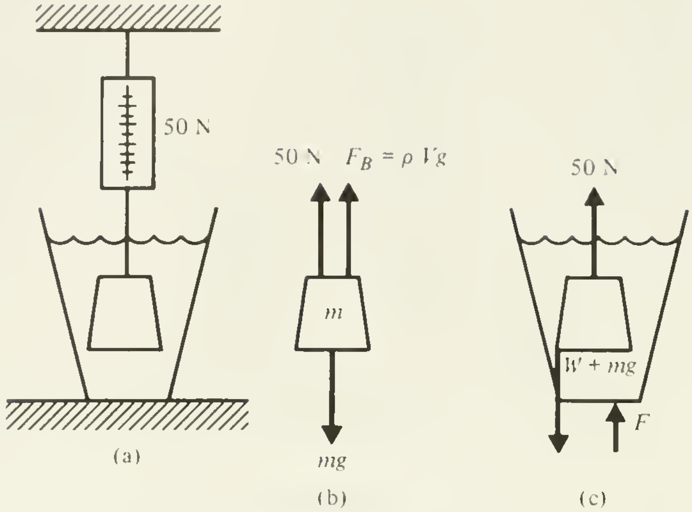


Figure 13-6

Solution:

(a) See Fig. 13-6b.

(b) Since the block is in equilibrium,

$$50\ \text{N} + \rho V g = mg$$

$$\rho = \frac{mg - 50\ \text{N}}{Vg} = \frac{(15\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2}) - 50\ \text{N}}{(0.01\ \text{m}^3)(9.8\ \text{m}\cdot\text{s}^{-2})} = 990\ \text{kg}\cdot\text{m}^{-3}.$$

(c) We consider all forces acting on the system composed of the liquid, its container, and the mass m. These are indicated in Fig. 13-6c. Thus we have

$$F + 50\ \text{N} = W + mg$$

$$F = W + mg - 50\ \text{N} = 500\ \text{N} + (15\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2}) - 50\ \text{N}$$

$$= 597\ \text{N}.$$

Example 11

An aluminum block floats at the interface between water and mercury. What fraction of the block is submerged in the mercury?

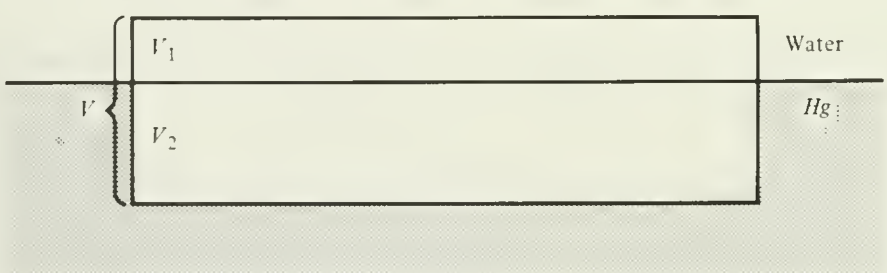


Figure 13-7

Solution:

Referring to Fig. 13-7, we see that the weight of displaced water is $\rho_w V_1 g$ and the weight of displaced mercury is $\rho_{Hg} V_2 g$. Their sum is the net bouyant force and is equal to the weight, $mg = \rho_{Al} V g$:

$$\rho_{Al} V g = \rho_w V_1 g + \rho_{Hg} V_2 g$$

$$= \rho_w (V - V_2) g + \rho_{Hg} V_2 g$$

$$(\rho_{Al} - \rho_w) V = (\rho_{Hg} - \rho_w) V_2$$

$$\frac{V_2}{V} = \frac{\rho_{Al} - \rho_w}{\rho_{Hg} - \rho_w} = \frac{2.7 - 1.0}{13.6 - 1.0} = 0.13$$

Example 12

The vessel shown in Fig. 13-8 is filled with water. Find the forces of the walls against the water contained in the vessel. Neglect atmospheric pressure.

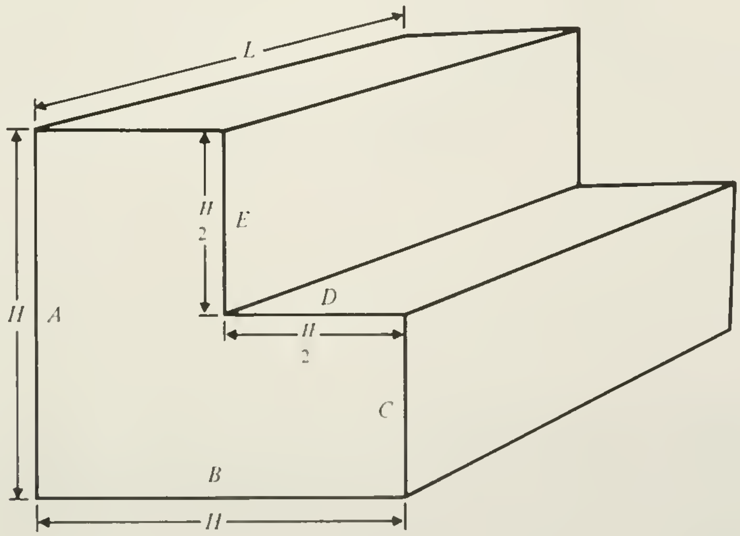


Figure 13-8

Solution:

The forces on the walls A, C and E are found by integrating $dF = \rho dA = \rho L dy$:

$$F_A = \int_0^H \rho g L(H - y)\,dy = \rho g L \frac{H^2}{2}$$

$$F_C = \int_0^{H/2} \rho g L(H - y)\,dy = \rho g L \frac{3H^2}{8}$$

$$F_E = \int_{H/2}^{H} \rho g L(H - y)\,dy = \rho g L \frac{H^2}{8}$$

Note $F_A = F_C + F_E$ so that the vessel does not spontaneously move sideways.

The vertical forces are

$$F_B = p_B HL \quad \text{(up)}$$

$$F_D = p_D HL/2 \quad \text{(down)}$$

The net upward force is

$$F = F_B - F_D = p_B HL - p_D \frac{H}{2} L.$$

Neglecting the atmospheric pressure,

$$p_B = \rho g H$$

$$p_D = \rho g \frac{H}{2}$$

Thus

$$F = \rho g H(HL) - \rho g \frac{H}{2} \left(\frac{H}{2} L\right)$$

$$= \rho g [H^2 L - \left(\frac{H}{2} \right)^2 L] = \rho g V$$

where V is the volume of water. The net upward force is equal to the weight of the water.

Example 13

Find the gauge pressure inside a raindrop of radius 1 mm at 0°C.

Solution:

$$p - p_a = \frac{2\gamma}{R} \qquad (2\gamma \text{ because the raindrop is solid})$$

$$= \frac{2(72.8)\ \text{dyne}\cdot\text{cm}^{-1}}{10^{-1}\ \text{cm}}$$

$$= 1456\ \text{dyne}\cdot\text{cm}^{-2}.$$

Example 14

What is the gauge pressure, in atmospheres, necessary to blow a soap bubble of radius 5 cm?

Solution:

$$p - p_a = \frac{4\gamma}{R} = \frac{(4)(25\ \text{dyne}\cdot\text{cm}^{-1})}{5\ \text{cm}} \qquad (4\gamma \text{ because the bubble is hollow})$$

$$= 20\ \text{dyne}\cdot\text{cm}^{-2} = (20 \times 10^{-5}\ \text{N})(10^4\ \text{m}^{-2})$$

$$= 2\ \text{N}\cdot\text{m}^{-2} = 2\ \text{N}\cdot\text{m}^{-2}\ \left(\frac{1\ \text{atm}}{10^5\ \text{N}\cdot\text{m}^{-2}}\right)$$

$$= 2 \times 10^{-5}\ \text{atm}.$$

Example 15

A 30 ft high cylindrical tank of area 1 ft^2 is filled with water.

(a) Find the velocity of discharge as a function of the height of water remaining in the tank when a hole of area 0.5 ft^2 is opened in the bottom of the tank.

(b) Find the initial discharge velocity.

(c) Find the initial volume rate of discharge.

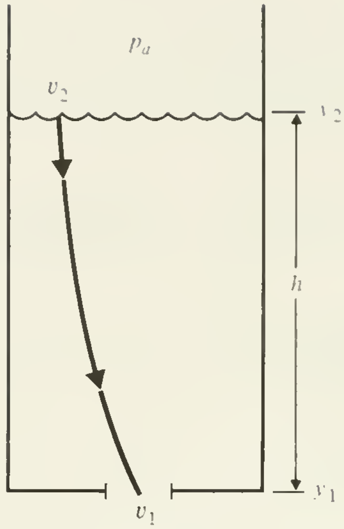


Figure 13-9

Solution:

(a) Referring to Fig. 13-9, we apply Bernoulli's equation to a flow tube which starts at the upper surface and ends just outside the hole:

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}$$

$$p_1 = p_2 = p_a$$

$$\rho g(y_2 - y_1) = \frac{\rho(v_1^2 - v_2^2)}{2}$$

$$v_1^2 - v_2^2 = 2gh.$$

We now use the equation of continuity to determine a relation between v_1 and v_2

$$v_2 A_2 = v_1 A_1 \text{ or } v_2 = \frac{A_1}{A_2} v_1$$

yielding

$$v_1^2 -(\frac{A_1}{A_2})^2 v_1^2 = 2gh$$

$$v_1 = (2gh)^{1/2}[1 - (A_1/A_2)^2]^{-1/2}.$$

Often in problems of this type $A_1 \ll A_2$ and the last factor may be neglected. This is not now the case:

$$[1 - (A_1/A_2)^2]^{-1/2} = [1 - (0.5/1)^2]^{-1/2} = 1.15,$$

and thus

$$v_1 = 1.15\ [2(32\ \text{ft}\cdot\text{s}^{-2})h]^{1/2} = 9.24\ [h\ \text{ft}\cdot\text{s}^{-2}]^{1/2}.$$

The velocity is greatest when the 'head' h is the greatest. At the beginning

(b) $$v_1 = 9.24\ [30\ \text{ft}^2\cdot\text{s}^{-2}]^{1/2} = 51\ \text{ft}\cdot\text{s}^{-1}.$$

(c) The discharge rate at the beginning is:

$$A_1v_1 = (0.5\ \text{ft}^2)(51\ \text{ft}\cdot\text{s}^{-1}) = 25\ \text{ft}^3\cdot\text{s}^{-1}.$$

Example 16

Water inside an enclosed tank is subjected to a pressure of two atmospheres at the top of the tank. What is the velocity of discharge from a small hole 3 m below the top surface of the water?

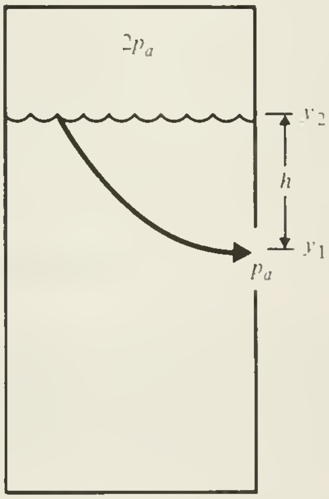


Figure 13-10

Solution:

Referring to Fig. 13-10, we apply Bernoulli's equation to the flow tube from the top of the water to the hole,

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}$$

$$p_a + \rho g y_1 + \frac{\rho v_1^2}{2} = 2p_a + \rho g y_2$$

where we neglect the velocity v_2,

$$v_2 = \frac{A_1}{A_2} v_1 \cong 0$$

because the hole is small relative to the cross-section of the tank,

$$\frac{A_1}{A_2} << 1.$$

Thus we find

$$\frac{\rho v_1{}^2}{2} = p_a + \rho g(y_2 - y_1) = p_a + \rho gh$$

$$v_1{}^2 = \frac{2}{\rho}(p_a + \rho gh) = \frac{2p_a}{\rho} + 2gh$$

The density of water is $\rho = 1 \cdot g\ cm^{-3} = 10^3\ kg \cdot m^{-3}$ and atmospheric pressure is $p_a = 1.013 \times 10^5$ pa. Thus we find

$$v_1 = [\frac{2 \times 1.01 \times 10^5\ Pa}{10^3\ kg \cdot m^{-3}} + 2\ (9.8\ m \cdot s^{-2})(3m)]^{1/2}$$

$$v_1 = [203\ m^2 \cdot s^{-2} + 59\ m^2 \cdot s^{-2}]^{1/2} = 16\ m \cdot s^{-1}$$

The pressure term is approximately three times the gravity term. This is reasonable because an atmosphere of pressure corresponds to a water height of 34 ft (about 10 m), approximately three times the gravity head of 3 m in this problem.

Example 17

Water discharged from a hose reaches a maximum height of 10 m. What is the gauge pressure in the water system at the hose nozzle level?

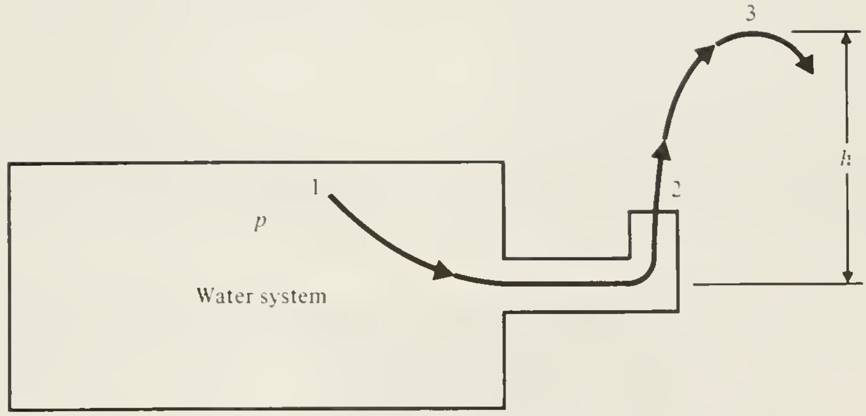


Figure 13-11

Solution:

First apply Bernoulli's equation to a flow line from somewhere inside the water system at the same height as the nozzle to the point of discharge. Referring to Fig. 13-11, we find

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}$$

$$p_1 = p, \qquad p_2 = p_a, \qquad y_1 = y_2, \qquad \text{and } v_1 = 0$$

$$p - p_a = \frac{\rho v_2^2}{2} = \text{gauge pressure.}$$

To rise to a height h above the nozzle the water must have an initial velocity

$$v_2^2 = 2gh.$$

Thus

$$p - p_a = \frac{\rho 2gh}{2} = \rho gh.$$

Alternatively apply Bernoulli's equation to the tube of flow 1 to 2 to 3,

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_3 + \rho g y_3 + \frac{\rho v_3^2}{2}$$

$$p_1 = p; \qquad p_3 = p_a; \qquad y_3 - y_1 = h; \qquad v_1 - v_3 = 0$$

$$p - p_a = \rho g(y_3 - y_1) = \rho gh$$

$$= (10^3\ \text{kg}\cdot\text{m}^{-3})(9.8\ \text{m}\cdot\text{s}^{-2})(10\ \text{m}) = 9.8 \times 10^4\ \text{Pa}.$$

Example 18

A toy rocket of diameter 2 in. consists of water under the pressure of compressed air created by pumping up a nose chamber. When the gauge air pressure is 60 $lbs \cdot in^{-2}$ the water is ejected through a hole of diameter 0.2 in. Find the propelling force or 'thrust' of the rocket.

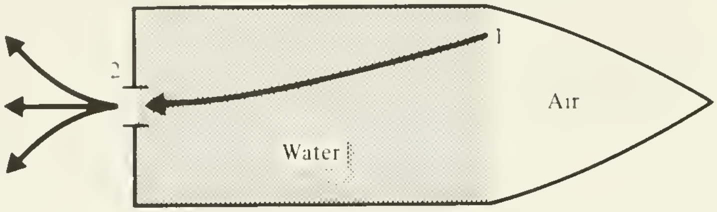


Figure 13-12

Solution:

Applying Bernoulli's equation between points 1 and 2 of the flow line in Fig. 13-12,

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}$$

$$p_2 = p_a; \qquad y_1 = y_2; \qquad \text{and } v_1 = \frac{A_2}{A_1} v_2 = 0.$$

$$\frac{\rho v_2^2}{2} = p_1 - p_2 = \text{gauge pressure inside rocket} = p - p_a.$$

The thrust is

$$F = v_2 \frac{dm_2}{dt} = v_2 \rho \frac{dV_2}{dt} = v_2 \rho A_2 v_2$$

$$= \rho A_2 v_2^2,$$

where $\dfrac{dV_2}{dt}$ = volume discharge rate = $v_2 A_2$.

$$F = \rho A_2 v_2^2 = 2A_2 \frac{\rho v_2^2}{2} = 2A_2(p - p_a)$$

$$= 2\pi(0.1 \text{ in})^2(60 \text{ lbs}\cdot\text{in}^{-2})$$

$$= 3.8 \text{ lbs}.$$

Example 19

Water at gauge pressure of an atmosphere is flowing in a pipe of 2 cm diameter at a velocity of 6 cm s^{-1}. Find the pressure drop when it meets an obstruction of diameter 1 cm.

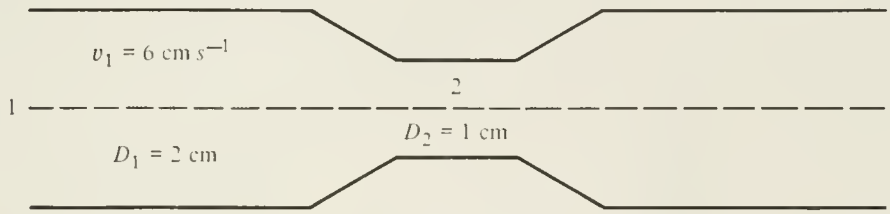


Figure 13-13

Solution:

Referring to Fig. 13-13, we apply Bernoulli's equation along the flow line from point 1 to 2,

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}$$

$$y_1 = y_2$$

$$p_1 + \frac{\rho v_1^2}{2} = p_2 + \frac{\rho v_2^2}{2} \; .$$

The velocities v_1 and v_2 are related by the continuity equation

$$v_1 A_1 = v_2 A_2 \quad \text{so } v_2 = \frac{A_1}{A_2} V_1$$

so that

$$p_1 + \frac{\rho v_1^2}{2} = p_2 + \frac{\rho (A_1/A_2)^2}{2} v_1^2$$

$$p_2 = p_1 + (\rho v_1^2/2)[1 - (A_1/A_2)^2].$$

The pressure drop is

$$p_1 - p_2 = (\rho v_1^2/2)[(A_1/A_2)^2 - 1]$$

$$= (1/2)(10^3\ \text{kg}\cdot\text{m}^{-3})(0.06\ \text{m}\cdot\text{s}^{-1})^2[(2/1)^2 - 1]$$

$$= 5.4\ \text{Pa}.$$

Example 20

Referring to Fig. 13-14, find the manometer height difference $h_1 - h_2$ when water of velocity $v_1 = 15\ \text{cm}\cdot\text{s}^{-1}$ enters a tube of area $A_1 = 2 \times 10^{-4}\ \text{m}^2$ and then meets a constriction of area $A_2 = 1 \times 10^{-4}\text{m}^2$.

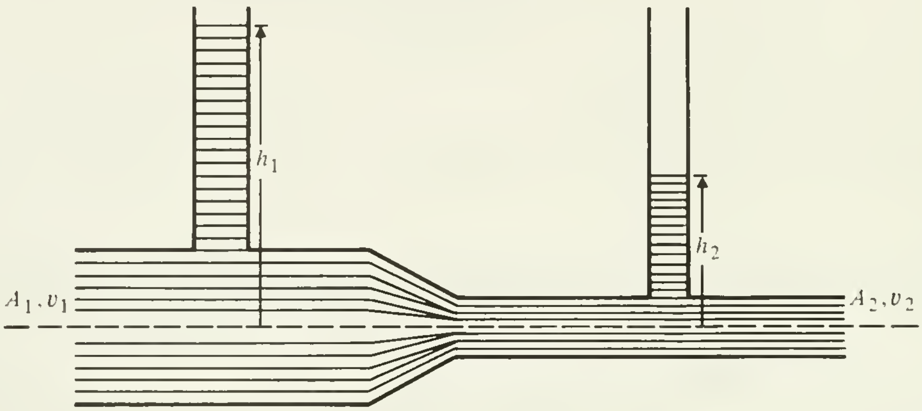


Figure 13-14

Solution:

First we check that the flow is laminar so that we may apply Bernoulli's equation. The test is that the Reynolds number $N_R < 2000$,

$$N_R = \frac{\rho v D}{\eta}$$

where in this case the parameters are:

$$\rho = 10^3\ \text{kg}\cdot\text{m}^{-3}, \quad v = 15\ \text{cm}\cdot\text{s}^{-1}, \quad A = \pi r^2 = \pi(\frac{D}{2})^2$$

$$D = 2[\ \frac{A}{\pi}\ ]^{1/2}, \qquad \text{and } \eta = 1.0005 \times 10^{-3}\ \text{N}\cdot\text{s}\cdot\text{m}^2.$$

Thus the product $\rho v D$ is

$$\rho vD = 2(10^3\ \mathrm{kg \cdot m^{-3}})(0.15\ \mathrm{m \cdot s^{-1}})(2 \times 10^{-4}\ \mathrm{m^2}/\pi)^{1/2}.$$

Dividing by the viscosity η gives for N_R

$$N_R = 2390.$$

The flow is in the transition region where it may be laminar or turbulent. To proceed we assume laminar flow and apply Bernoulli's equation on a streamline:

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}$$

Since $y_1 = y_2$,

$$p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2).$$

By the equation of continuity,

$$v_2 A_2 = v_1 A_1 \text{ so } v_2/v_1 = A_1/A_2$$

and

$$p_1 - p_2 = (\rho v_1^2/2)[(A_1/A_2)^2 - 1].$$

However, in the hydrostatic columns

$$p_1 - p_a = \rho g h_1$$

$$p_2 - p_a = \rho g h_2,$$

so that

$$p_1 - p_2 = \rho g(h_1 - h_2) = (\rho v_1^2/2)[(A_1/A_2)^2 - 1]$$

or

$$h_1 - h_2 = (v_1^2/2g)[(A_1/A_2)^2 - 1]$$

$$= \frac{(0.15\ \mathrm{m \cdot s^{-1}})^2}{2(9.8\ \mathrm{m \cdot s^{-2}})}(2^2 - 1)$$

$$= 3.4 \times 10^{-3}\ \mathrm{m} = 0.34\ \mathrm{cm}.$$

Example 21

Castor oil at 20°C flows through a pipe of radius 3 cm and length 1 m. The flow velocity at the center of the pipe is 5 cm s^{-1}. Find the pressure drop along the pipe.

Solution:

The velocity at a point r from the center of a pipe of radius R is

$$v = \frac{p_1 - p_2}{4\eta L}(R^2 - r^2)$$

where in this case

$$R = 0.03\ m, \quad L = 1\ m, \quad r = 0, \quad v = 0.05\ m \cdot s^{-1},$$

and

$$\eta = 9.86 \text{ poise} = 9.86 \times 10^{-1}\ N \cdot s \cdot m^{-2}$$

Thus

$$p_1 - p_2 = \frac{4\eta L v}{R^2}$$

$$= \frac{(4)(9.86 \times 10^{-1}\ N \cdot s \cdot m^{-2})(1\ m)(0.05\ m \cdot s^{-1})}{(0.03\ m)^2}$$

$$= 219\ N \cdot m^{-2}.$$

Example 22

A steel sphere of radius 0.5 cm falls through castor oil at 20° C. Find its terminal velocity.

Solution:

The terminal velocity is given by

$$v_T = \frac{2}{9}\ \frac{r^2 g}{\eta}\ (\rho - \rho')$$

where in this case

$$r = 0.005\ m, \quad \eta = 0.986\ N \cdot s \cdot m^{-2},$$

$$\rho = \rho_s = 7.8\ \mathrm{g \cdot cm^{-3}}, \quad \text{and } \rho' = \rho_{air} << \rho_s.$$

Neglecting the density of air compared to steel,

$$v_T = \frac{2(0.005\ \mathrm{m})^2\ 9.8\ \mathrm{m \cdot s^{-2}}(7.8 \times 10^3\ \mathrm{kg \cdot m^{-3}})}{9(0.986\ \mathrm{N \cdot s \cdot m^{-1}})}$$

$$= 0.43\ \mathrm{m \cdot s^{-1}}.$$

Example 23

Water flows at 20°C through a pipe of 2 cm diameter.
(a) Find the velocity of flow above which turbulence occurs.
(b) Find the velocity of flow below which the flow is laminar. Calculate the discharge rate in each case.

Solution:

For the Reynolds number

$$N_R = \frac{\rho v D}{\eta} > 3000$$

the flow is turbulent, i.e. for

$$v > \frac{3000\ \eta}{\rho D}$$

where in this case

$$\eta = 1.005 \text{ centipoise} = 1.005 \times 10^{-2} \text{ poise}$$

$$= 1.005 \times 10^{-1}\ \mathrm{N \cdot s \cdot m^{-2}}$$

$$\rho = 10^3\ \mathrm{kg \cdot m^{-3}} \quad \text{and } D = 0.02\ \mathrm{m}.$$

Thus

$$v > \frac{(3000)(1.005 \times 10^{-3}\ \mathrm{N \cdot s \cdot m^{-2}})}{(10^3\ \mathrm{kg \cdot m^{-2}})(0.02\ \mathrm{m})}$$

$$v > 0.15\ \mathrm{m \cdot s^{-1}}.$$

The discharge rate is ($r = 1$ cm $= 10^{-2}$ m)

$$vA = (0.15\ \mathrm{m\ s^{-1}})\pi(0.01\ \mathrm{m})^2 = 4.71 \times 10^{-5}\ \mathrm{m^3 \cdot s^{-1}}.$$

The flow is laminar for

$$N_R = \frac{\rho v D}{\eta} < 2000$$

$$v < \frac{2000\ \eta}{\rho D} < 0.10\ \mathrm{m \cdot s^{-1}}$$

with discharge rate

$$vA = (0.10\ \mathrm{m \cdot s^{-1}})\pi(0.01\ \mathrm{m})^2 = 3.14 \times 10^{-5}\ \mathrm{m^3 \cdot s^{-1}}.$$

Example 24

Calculate the Reynolds number for the flow in Example 19. Was it valid to assume laminar flow?

Solution:

$$N_R = \frac{\rho v D}{\eta}$$

where the density of water is $10^3\ \mathrm{kg \cdot m^{-3}}$, $v = 6\ \mathrm{cm \cdot s^{-1}}$, $D = 2$ cm, and the coefficient of viscosity is $\eta = 1.005 \times 10^{-3}\ \mathrm{N \cdot s \cdot m^2}$, yielding

$$N_R = \frac{(10^3\ \mathrm{kg \cdot m^{-3}})(0.06\ \mathrm{m \cdot s^{-1}})(0.02\ \mathrm{m})}{1.005 \times 10^{-3}\ \mathrm{N \cdot s \cdot m^{-2}}} = 1200.$$

The flow is laminar.

QUIZ

1. A body whose density is three times that of water is released at rest at the surface of a pond. Find its downward acceleration.

Answer: $6.5\ \mathrm{m \cdot s^{-1}}$

2. A 10 m high standpipe, open to the atmosphere, is filled with water. Find the pressure at the bottom of the standpipe.

Answer: $1.99 \times 10^5\ \mathrm{N \cdot m^{-2}}$

3. A 10 m high standpipe, filled with water and open to the atmosphere at its top, has a small hole at its base. Find the velocity of discharge.

Answer: 14 $m \cdot s^{-1}$

4. Water flows in a pipe of 4 cm diameter at a velocity of 12 $cm \cdot s^{-1}$. Is the flow turbulent or laminar?

Answer: Turbulent

14
TEMPERATURE AND EXPANSION

OBJECTIVES

In this chapter you begin the study of the equilibrium properties of large numbers of particles that constitute macroscopic systems. The system's temperature is of central importance to this discussion. Your objectives are to:

Apply the different numerical values of the fixed temperature points to establish the four most common temperature scales.

Familiarize yourself with a few common thermometers.

Calculate thermal expansion of common materials.

Calculate stresses that can be set up by temperature changes.

REVIEW

Three undefined physical quantities (mass, length, and time) were introduced when we studied the mechanics of a particle. We were assumed to be familiar with these quantities (i.e. fast, slow, long, short, heavy, light being frequently used comparisons) and were given only operational definitions (told how to measure these quantities in terms of an agreed upon standard). A fourth such undefined basic quantity is temperature.

Two macroscopic bodies A and B placed in contact with each other in such a way that they can exchange energy (diathermal wall) eventually reach a state where the variables describing the two bodies stop changing with time. The two bodies are said to be in thermal equilibrium. The temperature of body A is now equal to that of body B. If A and B are now isolated from each other and a third body C is put in contact with A, A and C will be in thermal equilibrium if the temperature of body C is equal to that of body A. Thus there exists a scalar quantity, temperature, that is a property of all macroscopic systems in equilibrium states such that temperature equality is a necessary and sufficient condition for thermal equilibrium. Alternatively: two systems each in thermal equilibrium with a third system are in thermal equilibrium with each other. This is a statement of the zeroth law of thermodynamics.

A thermometer can be any instrument or device capable of assigning a numerical value to the temperature of a body which it is in contact with. Various physical properties can be measured in order to infer temperature: length of a column of liquid, pressure of a gas, resistivity of a metal or semi-conductor, magnetic properties of certain materials, etc., so long as they give adequate sensitivity, accuracy, reproducibility, speed of measurement, and ease of interpolation. Concerning this last point, thermometers are usually calibrated at a few fixed temperature points and other values of the temperature are either interpolated or extrapolated. It is desirable that the material property to be measured, the state coordinate, change linearly with temperature if the interpolation is to be reliable. This is not always the case.

Various temperature scales are in use: the Fahrenheit scale, the Celsius scale, the Rankine scale, and the Kelvin scale. Thus a particular temperature could have four different numerical values on the four different scales. These temperature scales are established by assuming the temperature (t) is linearly proportional to the physical parameter (X) being measured by the thermometer,

$$t(X) = AX + B$$

Measurement of X at two known temperatures (X_1 at t_1 and X_2 at t_2) determines both A and B and fixes the temperature scale:

$$t_1 = AX_1 + B$$

$$t_2 = AX_2 + B$$

so that A and B are now known functions of t_1, t_2, X_1, and X_2.

For the Celsius scale, the temperature of the melting point of ice is assigned the value 0°C and the temperature of the boiling point of water is assigned the value 100°C. For the Fahrenheit scale, these same two fixed points are assigned the values 32°F and 212°F respectively. Thus both the size of the degree and the zero of the scale are different for the two scales. From just the information given here, it is possible to derive the conversion equation (see Example 1 for one method):

$$t_c = \frac{5}{9}(t_f - 32^\circ F)$$

The Kelvin and Rankine scales are both absolute scales established such that the zero of the temperature scale coincides with zero value of the thermometric property, X, being measured (think of X as the gas pressure of a constant volume thermometer rather than the length of a mercury column). This is equivalent to setting B equal to zero in our previous linear relationship between t and X. Now only one other fixed point is necessary to establish the temperature scale. That fixed point is chosen by international agreement to be the triple point of water ($t_c = 0.01^\circ C$).

The Kelvin scale uses the same size of temperature unit (degree) as the Celsius scale. The Rankine scale uses the same size degree as the Fahrenheit scale. Thus $A_K = A_c$ and $A_R = A_f$. Capital T's are used to denote these absolute temperatures.

$$T_K(X) = A_K X = A_c X$$

$$T_R(X) = A_R X = A_f X$$

Extrapolation of these linear relationships (T verses X) using gas-law thermometry yields the result that the triple point of water (0.01°C) corresponds to a temperature of 273.16 K on the Kelvin scale. This establishes the conversion formula:

$$t_c + 273.15\text{ K} = T_K.$$

With very few exceptions (water below 4°C happens to be one), liquids and solids expand when heated. If the temperature change ΔT is not too large, the increase in a typical linear dimension of a homogeneous material, ΔL, is found to be proportional to the mean length and the temperature change. The proportionality constant (coefficient of linear expansion α) is material dependent and tabulated in reference books over wide ranges of temperature for most materials:

$$\Delta L = \alpha L \Delta T$$

For most practical problems, it does not matter significantly whether the value of L in the above equation is taken to be the initial value, the final value, or the mean value (see Example 4) since α is extremely small (~ 10^{-5} per C°) and not known to many significant figures. When considering expansions (or contractions) of areas or volumes of complicated objects, it is useful to imagine the expansions (or contractions) as photographic enlargements (or reductions). Thus in a donut shaped object, when heated, both the inner and outer radius have the same fractional increase ($\Delta R/ R = \alpha \Delta T$).

Since different materials have different thermal expansion coefficients, stresses can be set up as a result of temperature changes in heterogeneous bodies. These stresses arise from either real elongations or compressions of the materials or length changes that would occur if not otherwise prevented. To estimate the magnitude of such stresses, one usually finds the fractional change in length, $\Delta l/l = \alpha \Delta T$, from the temperature change and then uses the definition of Young's modulus to find the force per unit area (stress). This is illustrated in Examples 6 and 7.

PROBLEM-SOLVING STRATEGIES

Values of the necessary elastic constants such as Young's modulus are given in Chapter 12. To calculate the thermal stress in a given piece of material, it is frequently useful to calculate the length change that would occur if the motion was unrestricted and then compute the force per unit area needed to compress or stretch the material from that hypothetical length to the real length.

EXAMPLES AND SOLUTIONS

Example 1

Given the melting point of ice, $t_c = 0°C$ and $t_f = 32°F$ and the boiling point of water, $t_c = 100°C$ and $t_f = 212°F$, derive a formula that converts temperature on the Celsius scale into temperature on the Fahrenheit scale.

Solution:

Writing generally that $t_c = A_cX + B_c$ and $t_f = A_fX + B_f$, we can eliminate the common state coordinate to obtain a linear relationship between t_f and t_c.

$$X = \frac{t_c - B_c}{A_c}$$

so that

$$t_f = A_f\left(\frac{t_c - B_c}{A_c}\right) + B_f = C_1 t_c + C_2$$

where C_1 and C_2 are constants. Using the given numerical values, we have

$$212°F = C_1(100°C) + C_2$$

$$32°F = C_1(0°C) + C_2$$

The second equation yields

$$C_2 = 32°F.$$

Subtracting the second equation from the first gives $180\ F° = C_1(100\ C°)$, or

$$C_1 = 1.8\ F°(C°)^{-1}.$$

Rewriting the linear relationship, we have

$$t_f - 32°F = (1.8\ F°)(C°)^{-1}t_c$$

or

$$t_f = \frac{9}{5}\,t_c + 32°F$$

Example 2

At what temperature, t, do the two scales of the last example give an identical reading?

Solution:

In the previous formula, set $t_c = t_f = t$ to give:

$$t = \frac{5}{9}(t - 32) \text{ or}$$

$$\frac{4}{9} t = -\frac{5}{9} 32$$

$$t = -40.$$

Thus when $t_c = -40°C$, $t_f = -40°F$, and the temperatures are identical. Above this temperature, a given number of degrees Celsius (say 20) is warmer than the same number of degrees Fahrenheit whereas −50°C is colder than −50°F.

Example 3

What temperature does 0 K correspond to on the Celsius scale and on the Fahrenheit scale ?

Solution:

Since $T_K = t_C + 273.15$ K, then if $T_K = 0$ K, one has $t_C = -273.15$ K.

Using $t_C = \dfrac{5°C}{9°F}(t_F - 32°F)$ and then substituting $t_C = -273.15$ K,

we can solve for t_F:

$$t_F = [\frac{9}{5}(-273.15) + 32]\ °F$$

$$= -459.67°F.$$

Example 4

Compute the length change for a brass bar ($\alpha = 2 \times 10^{-5}$ per C^{0}) originally 1 meter long for a $100^{\circ}C$ temperature rise from the expression $\Delta L/L = \alpha\Delta T$ using for L the initial length, the final length, and the mean length.

Solution:

(a) $$L - L_0 = \Delta L = \alpha L_0 \Delta T$$

$L = L_0(1 + \alpha\Delta T)$, this is the form nearly always used.

We have

$$L = 1\ \mathrm{m}[1 + (2 \times 10^{-5})(10^2)]$$

or

$$L - 1\ \mathrm{m} = 2 \times 10^{-3}\ \mathrm{m}.$$

(b) $$L - L_0 = \Delta L = \alpha L \Delta T$$

$$L = L_0/(1 - \alpha\Delta T)$$

Numerically we now have

$$L = 1\ \mathrm{m}/(1 - 2 \times 10^{-3}) = 1.002004\ \mathrm{m}$$

but the last 3 digits are not significant, so $L - 1\ \mathrm{m} = 2 \times 10^{-3}$ m as before.

(c) $$L - L_0 = \Delta L = \alpha\left(\frac{L + L_0}{2}\right)\Delta T$$

$$L = L_0 \frac{(1 + \alpha\Delta/2)}{(1 - \alpha\Delta/2)}$$

Numerically we have

$$L = 1\ \mathrm{m}\ (1 + 10^{-3})/(1 - 10^{-3}) = 1.002002\ \mathrm{m}\ \text{or}$$

$$\Delta L = L - 1\ \mathrm{m} = 2 \times 10^{-3}\ \mathrm{m}.$$

Thus in the formula $\Delta L/L = \alpha\Delta T$, for most problems, it won't matter which value of L is used. The form seen in part (a) where L is taken to be L_0 is most frequently encountered.

Example 5

A flat sheet of material with coefficient of linear expansion $\alpha = 3 \times 10^{-5}$ per C° of original dimensions L_o, w_o, at temperature T_o is heated to temperature T. Calculate the new area if $L_o = 0.80$ m, $w_o = 0.60$ m, $T_o = 20^\circ$C, and $T = 120^\circ$C.

Solution:

Let A(T) be the area at temperature T. Then A(T) = L(T)w(T).

$$\begin{aligned} A(T) &= L_o[1 + \alpha(T - T_o)]w_o[1 + \alpha(T - T_o)] \\ &= L_o w_o[1 + \alpha(T - T_o)]^2 \\ &= L_o w_o[1 + 2\,\alpha(T - T_o) + \alpha^2(T - T_o)^2]. \end{aligned}$$

We will evaluate this square bracket numerically. In order, the three factors are: 1, 6×10^{-3}, and 9×10^{-6}. The sum of these terms is: 1.006009. Thus to the desired accuracy of this problem, $(1 + \alpha\Delta T)^2 \simeq 1 + 2\alpha\Delta T$.

$$A(120^\circ C) = 0.48\ m^2\ (1.006) = 0.483\ m^2.$$

This order of approximation leads to a simple interpretation if we write:

$$A(T) - A(T_o) = A(T_o)(2\alpha)\Delta T.$$

Thus 2α (twice the coefficient of linear expansion) plays the role of the coefficient of area expansion. In the text, these considerations are carried over into three dimensions where it is shown, with the same approximations, that the coefficient of volume expansion is equal to three times the coefficient of linear expansion.

Example 6

In Fig. 14-1, consider an aluminum wire stretched across a steel yoke. Assume there are no stresses in the wire at 20°C and that the whole system is now cooled by 50°C ($\Delta T = -50^\circ$C). If the area of contact of the wire to the yoke is $9 \times 10^{-6}\ m^2$, what force is exerted on the wire?

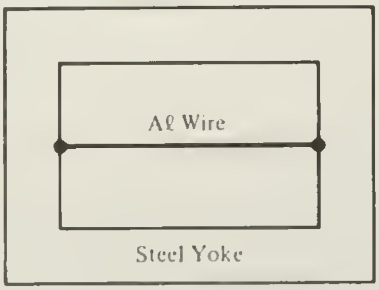


Figure 14-1

Solution:

For the wire,

$$\Delta L/L = \alpha_{Al}\Delta T$$

$$= -\ 2.4 \times 10^{-5} \times 0.5 \times 10^{2}$$

$$= -\ 1.2 \times 10^{-3}.$$

For the yoke,

$$\Delta L/L = \alpha_{s}\Delta T$$

$$= -\ 1.2 \times 10^{-5} \times 0.5 \times 10^{2}$$

$$= -\ 0.6 \times 10^{-3}.$$

The thin wire will contract by the amount dictated by the massive yoke. Thus $\Delta L/L$ will be $-\ 0.60 \times 10^{-3}$ so tensile stresses are set up to take the length from the value it would like to have ($\Delta L/L = -1.2 \times 10^{-3}$) to the length of the yoke ($\Delta L/L = -0.60 \times 10^{-3}$).

Since

$$Y = \frac{F/A}{\Delta L/L}$$

we have

$$F/A = \frac{Y_{Al}\Delta L}{L}$$

$$= (0.7 \times 10^{11}\ \mathrm{N \cdot m^{-2}})(-\ 1.2 \times 10^{-3} + 0.6 \times 10^{-3}).$$

After multiplying by the given area, the force is found to be:

$F = 3.8 \times 10^{2}$ N, tending to stretch the wire.

Example 7

For a piece made as shown in Fig. 14-2 where the aluminum bar is so large that its effect on the steel member cannot be ignored, calculate the forces on the Al for a temperature change of $-50\ C^{o}$ and then of $+50\ C^{o}$. Take the area of contact (of each end) to be $9 \times 10^{-4}\ m^2$ and assume the pieces are welded at the ends and don't break loose. Take $Y_{Al} = 0.7 \times 10^{11}\ N\cdot m^{-2}$ and $Y_s = 2 \times 10^{11}\ N\cdot m^{-2}$.

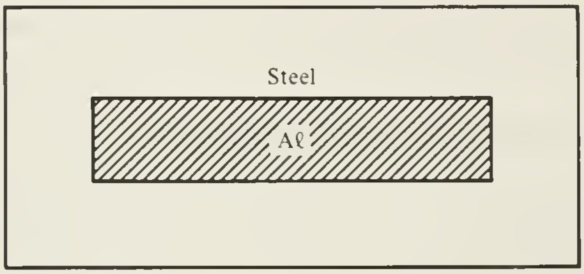


Figure 14-2

Solution:

Let the actual fractional change in length be $= \Delta L/L$. Let the fractional change in length of an isolated Al bar $= \Delta L_{Al}/L$. Let the fractional change in length of an isolated steel bar $= \Delta L_s/L$.

Recalling that Y = stress/ strain, we have

$$\text{strain in the Al} = \frac{\Delta L_{Al} - \Delta L}{L} = \frac{1}{Y_{Al}}\ (F/A).$$

$$\text{strain in the steel} = \frac{\Delta L - \Delta L_s}{L} = \frac{1}{Y_s}\ (F/A).$$

Since the force on the aluminum due to the steel is equal and opposite to the force on the steel due to the aluminum, we have

$$FL/A = Y_s(\Delta L - \Delta L_s) = Y_{Al}(\Delta L_{Al} - \Delta L).$$

We can solve for ΔL giving:

$$\Delta L = \left(\frac{Y_{Al}\Delta L_{Al} + Y_s \Delta L_s}{Y_s + Y_{Al}} \right)$$

Dividing by L and writing $\Delta L_{Al}/L = \alpha_{Al}\ \Delta T$ and $\Delta L_s/L = \alpha_s \Delta T$, we have:

$$\Delta L/L = \left(\frac{Y_{Al}\alpha_{Al} + Y_s \alpha_s}{Y_s + Y_{Al}} \right)\Delta T$$

$$= 1.51 \times 10^{-5}\ \Delta T.$$

The strain in the aluminum for a 50 C° temperature change = 4.4×10^{-4}. The stress in the aluminum for a 50 C° temperature change = 3.1×10^{7} N $\cdot$m^{-2}. The force is F = 2.8×10^{4} N with the sign (compression or extension) depending on the sign (+ or −) of the temperature change. The treatment used in Example 6 amounted to assuming that Young's modulus for steel was much larger than that for aluminum, which is in fact not such a good approximation.

QUIZ

1. Calculate the temperature for which the Fahrenheit and Kelvin scales coincide.

Answer: t = 574.6 K (or F)

2. Suppose an absolute temperature scale was created with the single fixed point (except absolute zero), the boiling point of water (100 °C), being assigned the value 1000 °A. Find the values of the following temperatures on this scale: (a) the freezing point of water, 0 °C, and (b) normal body temperature, 37 °C.

Answer: (a) 732 °A, (b) 831.2 °A.

3. Sections for a railroad track are laid on a day when the temperature is 41 °F and their length is 10.000 m. If the expansion coefficient of the metal is α = 1.2×10^{-5} (C°)$^{-1}$, what must be the spacing between adjacent sections so that they will just touch on a day when the temperature is 104 °F?

Answer: 4.2×10^{-3} m.

4. At 30°C a steel rod and an aluminum rod (of equal cross section) with equal initial lengths of 0.5000 m are placed end-to-end between totally rigid supports. There is no stress initially. (a) Calculate the length change of the aluminum rod if the system is heated to 100°C and (b) the pressure developed at the interface.

Answer: (a) − 9.33×10^{-5} m
(b) 1.31×10^{8} Pa

15
QUANTITY OF HEAT

OBJECTIVES

In this chapter you will calculate the quantity of heat transferred in several specific processes and relate it to temperature changes and phase changes in the system under consideration. Your objectives are to:

Relate the temperature change of an object to the quantity of heat given to or taken from the object.

Calculate the quantity of heat required to convert a pure substance from one phase (e.g. solid) to another (e.g. liquid).

Calculate the temperature changes and/or phase changes of pure substances resulting from the conversion of some other form of energy (mechanical, electrical, etc.) into thermal energy.

REVIEW

Heat is introduced in this chapter as another form of energy. A heat flow is an energy transfer that takes place exclusively due to a temperature difference. When two objects at different temperatures are placed in contact, the final temperature is between the two initial temperatures. The conclusion is that heat energy flowed from the hotter body to the colder body.

Not all energy transfers involve a heat flow. Heat flow and performance of work are equivalent in the sense that an energy increase (decrease) occurs and no later experiment can tell how this change took place.

The ratio of the temperature change (dT) of a given object to the input of a specified infinitesimal quantity of heat energy, dQ, depends on the mass of the object and a material dependent property, the specific heat capacity, c. An essential definition is:

$$c = \frac{1}{m}\frac{dQ}{dT}$$

As pointed out in the text, c is generally a function of temperature and this equation must be integrated to find the quantity of heat Q involved in a given finite temperature change. Problems 15-27 and 15-28 of SZY treat specific heat capacities that are temperature dependent. For simplicity we usually assume that c is independent of T over the temperature range of interest.

If c is independent of T, then we can write:

$$\Delta Q = mc\Delta T$$

where ΔQ is the heat energy added to (or taken from) the system and ΔT is the temperature change.

Since heat is a form of energy and the joule (J) is the S.I. unit of energy, the proper units for c are $J \cdot kg^{-1}(C^{\circ})^{-1}$ as seen from either of the above equations. Two other units of heat energy are still used frequently. Both units were originally introduced to make the specific heat capacity of water numerically equal to 1.00. The calorie (cal) is the amount of heat energy required to raise the temperature of 1 g of water 1 C°. Thus c for water is 1 $cal \cdot g^{-1}(C^{\circ})^{-1}$. One calorie is found experimentally to be the equilavent of 4.186 J so for water c = 4186 $J \cdot kg^{-1}(C^{\circ})^{-1}$ in the units of Tables 15-1 and 15-2 of the text. The kilocalorie (kcal) is 10^3 calories, so for water c = 1 $kcal \cdot kg^{-1}(C^{\circ})^{-1}$.

The Btu is defined such that the specific heat capacity of unit weight (1 pound) of water is equal to unity. Thus 1 Btu heat energy input will raise the temperature of 1 pound of water by 1 F°. Conversions between the Btu and cal units are illustrated in Example 1 but joules are used in all other examples.

Change of phase is still a topic of considerable interest in current research. Melting of a pure substance is a complicated physical phenomenon and not completely understood (at least in detail) on the microscopic level. Here we concentrate on the gross aspects of this problem which are well understood.

Melting of ice and the boiling of water are two common examples of a change of phase. In this context, we identify three phases of matter: solid, liquid and gaseous. There are other phases at higher temperatures (energies), such as plasmas and nuclear matter, but we will focus on these three.

The remarkable fact is that the phase changes, melting, freezing, vaporization, or liquefaction take place at <u>fixed temperature</u>. For instance, in melting as more heat is supplied, more of the solid is converted into the liquid phase but no temperature change occurs until all the solid disappears. This provides an excellent fixed temperature point for calibration of thermometers. The quantity of heat energy required for a given phase change, such as fusion (or melting), depends on the mass of material present and a material dependent property of the substance called the heat of fusion (L_F. If the phase change being made involves vaporization (or boiling), the heat of vaporization (L_V) is used. In either case, the heat required is:

$$Q = mL.$$

Phase change salts are now of great practical importance as they may be useful in storing energy conveniently for solar houses. Problem 15-31 in the text is a good illustration of this point.

PROBLEM-SOLVING STRATEGIES

When only thermal energy transfer occurs, the sum of the heat energy increases for all the bodies is zero. Be careful with signs; $Q>0$ means in our sign convention that heat is added to a body, $Q<0$ that heat is taken away from it. For many of the problems the specific heat must be expressed in $J\cdot kg^{-1}\cdot C^{o-1}$.

EXAMPLES AND SOLUTIONS

The problems in this chapter are based on energy conservation and occasionally involve the conversion of energy from one form to another. Hence once the important definitions $\Delta Q = mc\Delta T$ and $Q = mL$ are memorized, the problems become basically accounting problems. More so than most sets of problems met in this course, these problems require early substitution of numerical values since the number of physical processes that must be taken into account depend on how much heat energy is available. Examples 5 and 6 illustrate this point.

Example 1

Derive a formula for the conversion from Btu's to calories.

Solution:

The respective units Btu and calorie were chosen so that C_w (heat capacity per unit weight) or C_m (heat capacity per unit mass) will be equal to unity for water (see section 15.2 of text). Thus since $C_w = 1\ Btu\cdot lb^{-1}\cdot F^{o-1}$ and $C_m = 1\ cal\cdot g^{-1}\cdot C^{o-1}$ are the specific heat capacities of the same substance, we can find the conversion formula by writing:

$$1\ Btu\cdot lb^{-1}\cdot F^{o-1} = 1\ cal\cdot g^{-1}\cdot C^{o-1}.$$

Now we use the conversions, 1 lb = 454 g (i.e. a quatity of water with a weight of 1 lb has a mass of 454 g.) and 1 F^o= 5/9 C^o, so that

$$1\ Btu\ (454\ g)^{-1}\left(\frac{5}{9}\ {}^{o}C\right)^{-1} = 1\ cal\cdot g^{-1}\cdot C^{o-1}.$$

Thus we have

$$1\ Btu = 252\ cal.$$

Example 2

Water flowing at a speed of 5 $m\cdot s^{-1}$ falls over a 50 m high waterfall (approximate height of Niagara Falls) into a still pool below. Calculate the approximate rise in water temperature due to the conversion of mechanical energy into thermal energy.

Solution:

Let m be the mass of water and equate the loss of mechanical energy to the increase in thermal energy.

$$\frac{1}{2}\,mv^2 + mgh = mc\,\Delta T.$$

Note that the mass is unimportant so we can solve for T to obtain:

$$\Delta T = \frac{1}{c}\,(v^2/2 + gh).$$

Numerically:

$$\Delta T = \frac{1/2\ 25(m\cdot s^{-1})^2 + 9.8\ m\cdot s^{-2}\ (50\ m)}{4186\ J\cdot kg^{-1}\cdot C^{o-1}}\ \text{in S.I. units.}$$

Since $1\ J = 1\ kg\cdot m^2\cdot s^{-2}$, we have $\Delta T = 0.12°C$.

As a historical note, Joule actually tried to measure this temperature increase but since it is so small, he was unsuccessful.

Example 3

A silver bullet with speed of 500 $m\cdot s^{-1}$ initially at a temperature of 20°C stops suddenly and all its mechanical energy is converted into thermal energy. What is its temperature rise?

Solution:

From Table 15-1 of the text we find $c = 234\ J\cdot kg^{-1}\cdot C^{o-1}$. Equating the loss of mechanical energy to the increase in thermal energy, we have:

$$\frac{1}{2}\,mv^2 = mc\,\Delta T$$

The mass cancels out so $\Delta T = v^2/2c$.

Numerically, we have

$$\Delta T = \frac{(5 \times 10^2\ m\cdot s^{-1})^2}{2[2.34 \times 10^2\ J\cdot kg^{-1}\cdot(C^o)^{-1}]}$$

$$\Delta T = 5.34 \times 10^2 \ \frac{(m\cdot s^{-1})^2\ C^o}{(m\cdot s^{-1})^2}$$

$$\Delta T = 534\ C^o.$$

This is a very large temperature increase but the melting point of silver is 960°C, (see Table 15-2) so the only error we have made is in assuming that the specific heat is constant over this very large temperature interval. Had the bullet been a lead bullet (lead melts at 327°C), it would have melted. This might be the reason that vampires can only be taken out with silver bullets.

Example 4

Suppose in Example 2 the temperature of the flowing water and the still pool below the falls is about 0°C. If a block of ice flowing with the water goes over the falls, what fraction of the ice melts?

Solution:

Again we equate the mechanical energy loss to the increase in thermal energy. In this case however no temperature increase occurs but some fraction, f, of the ice melts. Let L be the latent heat for ice. (The value of L 3.35 x 10^5 $J\cdot kg^{-1}$ from Table 15-2.)

$$\frac{1}{2}\ mv^2 + mgh = fmL$$

Again m cancels out on both sides of the equation and the fraction, f, is:

$$f = \frac{(1/2\ v^2 + gh)}{L}$$

Numerically

$$f = \frac{1/2\ 25(m\cdot s^{-1})^2 + 9.8\ m\cdot s^{-2}(50\ m)}{3.35 \times 10^5\ J\cdot kg^{-1}}$$

$$f = 1.5 \times 10^{-3} \quad \text{or } 0.15\% \text{ melts.}$$

Example 5

A copper calorimeter of mass 2 kg initially contains 1.5 kg of ice at −10°C. How much heat energy must be added to convert all of the ice to water and then half of the water into steam?

Solution:

The heat added increases the temperature of the ice until the melting point of ice is reached, melts the ice, raises the temperature of the water until it reaches its vaporization point, and then vaporizes half the water. For the calorimeter, no phase change occurs so heat added only results in a temperature increase. We will calculate the heat required in five steps.

(a) Heat the ice to $0^\circ C$, $C_I = 2300\ J\cdot kg^{-1}\ C^{\circ -1}$ (see Section 15-6).

$$\Delta Q_a = mC_I\Delta T = (1.5\ kg)(2300\ J\cdot kg^{-1}\cdot C^{\circ -1})(10^\circ C)$$

$$\Delta Q_a = 3.45 \times 10^4\ J.$$

(b) Melt the ice at fixed temperature (0°). $L_F = 3.33 \times 10^5\ J\cdot kg^{-1}$ (Table 15-2).

$$\Delta Q_b = mL_F = (1.5\ kg)(3.33 \times 10^5\ J\cdot kg^{-1})$$

$$\Delta Q_b = 5.00 \times 10^5\ J$$

(c) Heat the water to $100^\circ C$. $C_w = 4190\ J\cdot kg^{-1}\cdot C^{\circ -1}$.

$$\Delta Q_c = mC_w\Delta T = (1.5\ kg)(4190\ J\cdot kg^{-1}C^{\circ -1})(100^\circ\ C)$$

$$\Delta Q_c = 6.28 \times 10^5\ J.$$

(d) Vaporize half the water at fixed temperature $(100^\circ C)$, $L_v = 2.256 \times 10^6\ J\cdot kg^{-1}$ (Table 15-2).

$$\Delta Q_d = (1/2)mL_v \text{ (since only 1/2 of the total mass is being vaporized)}$$

$$\Delta Q_d = (1/2)(1.5\ kg)(2.256 \times 10^6\ J\cdot kg^{-1})$$

$$\Delta Q_d = 1.69 \times 10^6\ J$$

(e) Heat the calorimeter from $-10^\circ C$ to $100^\circ C$. $C_c = 390\ J\cdot kg^{-1}C^{\circ -1}$ (Table 15-1).

$$\Delta Q_e = M_c\ C_c\ \Delta T = (2\ kg)(390\ J\cdot kg^{-1}C^{\circ -1})(110^\circ\ C)$$

$$\Delta Q_e = 8.59 \times 10^4\ J.$$

The total heat energy required is the sum of these five contributions.

$$\Delta Q_T = 2.94 \times 10^6\ J.$$

If we had ignored the contribution of the copper calorimeter altogether, we would have made an error of ξ°∝

Example 6

A copper calorimeter of mass 2 kg contains 1.5 kg of ice at $-10^\circ C$.
(a) What will the final temperature be if the heat added, ΔQ, is equal to 5×10^5 J?
(b) If $\Delta Q = 10^6$ J what will the final temperature be?

Solution:

(a) The answer to the first part is trivial if we look at the numerical solution to Example 5. There we found that 5×10^5 J were required to melt all the ice but 3.45×10^4 J would be needed to heat the ice to $0^\circ C$. Thus the final temperature is $0^\circ C$ with nearly all the ice melted. Without reference to the previous problem we could make an error if we assumed that the heat added was more than enough to cause a phase change and that an additional temperature rise occurred. To illustrate this difficulty we write:

$$\Delta Q = m_I C_I(0^\circ C + 10^\circ C) + m_I L_F + m_I C_W(T_f - 0^\circ C) + m_c C_c(T_f - 10^\circ C)$$

If we now substitute numerically we find that $T_f = -5.99^\circ C$! This nonphysical result came from introducing too many processes for the amount of heat energy available.

(b) If $\Delta Q = 10^6$ J, then we have enough heat energy to melt the ice so an additional temperature rise occurs. If this temperature rise exceeds $100^\circ C$, we will have to reformulate the problem to include vaporization of the water as well.

$$\Delta Q = m_I C_I(10^\circ C) + m_I L_F + m_I C_W(T_f - 0^\circ C)$$

$$+ m_c C_c(T_f + 10^\circ C).$$

Numerically:

$$10^6 \text{ J} = 3.45 \times 10^4 \text{ J} + 5 \times 10^5 \text{ J}$$

$$+ (1.5 \text{ kg})(4190 \text{ J}\cdot\text{kg}^{-1}\cdot\text{C}^{\circ -1})T_f$$

$$+ 7.8 \times 10^3 \text{ J}$$

$$+ (2 \text{ kg})(390 \text{ J}\cdot\text{kg}^{-1}\cdot\text{C}^{\circ -1})T_f.$$

Solving for the unknown T_f;

$$T_f = 64.8^\circ C.$$

QUIZ

1. A copper calorimeter of mass 1 kg contains 300 g of water initially at 10°C. Another 200 g of water at 70°C is poured in. What is the final equilibrium temperature, T_f?

Answer: $T_f = 30.2°C$

2. A metal calorimeter of mass 0.05 kg contains 0.300 kg of water at 20 °C. A second piece of the <u>same</u> metal of mass 0.20 kg is heated to 75 °C and then dropped into the calorimeter plus water. At equilibrium, the final temperature is 35 °C. Calculate the specific heat of the unknown metal from this data.

Answer: $c = 2598\ J \cdot kg^{-1}$.

3. A lead bullet initially at 20°C traveling with speed v is stopped suddenly with all of its kinetic energy converted into heat energy of the bullet. The bullet just melts without further temperature rise. Calculate v using data from Tables 15-1 and 15-2.

Answer: $v = 359\ m \cdot s^{-1}$. The heat of fusion and the specific heat must be expressed in S.I. units.

4. An aluminum pail ($c_{Al} = 910\ J \cdot kg^{-1}$, $m_{Al} = 1.5$ kg) contains 1 kg of water and 2 kg of ice at 0 °C. If 3 kg of water at 70 °C is poured into this pail, calculate the final temperature of the pail plus water.

Answer: $T_f = 7.92$ °C.

16
MECHANISMS OF HEAT TRANSFER

OBJECTIVES

In this chapter you will be introduced to the three processes by which heat energy can be transferred from one body to another. These three processes are conduction, convection, and radiation. Your objectives are to:

Calculate the heat energy per unit time (the heat current) transferred by conduction due to a temperature difference in a given material.

Estimate the heat current due to a convective process.

Calculate the heat current radiated from a given surface at temperature T.

Calculate the power absorbed by an object from specific environments.

REVIEW

Conduction of heat energy is a transport of energy (without a net motion of matter) from a higher temperature to a lower temperature. This energy transport requires matter (some solid, liquid, gas or a combination of these) to be present. For a single substance, the heat energy transported per unit time (H) by conduction depends on the cross-sectional area, A, the temperature gradient (dT/dx), and a material dependent property, the thermal conductivity, k:

$$H = -kA\left(\frac{dT}{dx} \right).$$

Since heat is transported from 'hot' to 'cold', the origin of coordinates can be placed at the high temperature end of the material so dT/dx is negative. If H, k, and A in this equation are constants, then T changes linearly with x as shown for the steady-state temperature in Fig. 16-3 in the text. If the area is not constant, as would be the case for radial heat flow in a cylinder or sphere, the temperature distribution will depend on the particular geometry. This point is illustrated in Example 7.

Conduction of heat through various combinations of objects is treated by extending the basic equation of conduction. If two objects are joined at a common interface (a series connection), the steady-state heat current is constant and the same through both objects. Example 3 treats this case. If two objects are connected side-by-side (a parallel combination) between two temperature reservoirs, the total heat current is the sum of the individual heat currents.

Heat transfer by convection involves the actual motion of matter and hence is of no practical importance in solids but is important in both liquids and gases. One important example of heat transport by convection is the warming of the Scandinavian Peninsula by the Gulf Stream. Convection is much more complicated than conduction (and hence less well understood) because the heat transport depends on details of the matter transport such as whether the flow is laminar or turbulent. For simple situations nevertheless, we can write an equation not too different from the heat conduction equation:

$$H = hA(\Delta T),$$

where H, the heat current carried by convection, is proportional to the area (A) and the temperature difference (ΔT) between the surface and the main body of the fluid. The lack of precision in this last statement is an indication that the convection equation is at best an approximation. In fact, h, the convection coefficient is a weak function of the temperature difference between the surface in question and the main body of the fluid. The text develops an approximate method of treating a practical convection problem. The validity of that approximation is examined in Example 4 where it is concluded that the answer given in the text is in excellent agreement with an exact numerical calculation.

Heat transfer by radiation (meaning electromagnetic radiation or waves) is different from conduction or convection in that no intervening material is required--it works perfectly well in a vacuum. Thus in outer space where conduction and convection are negligible (since the necessary matter is not there), radiation becomes the dominant heat transfer mechanism. The power radiated by an object has been found to depend on the fourth power of its absolute temperature (T^4), its surface area, A, and specific properties of its surface through a factor e (called the emissivity). The proportionality constant here is denoted σ and called the Stefan-Boltzmann constant. Summarizing this:

$$H = Ae\sigma T^4 \quad \text{(Stefan Law)}$$

If a body is a perfect radiator (e = 1) it is called a 'blackbody'. A perfect reflector would have e = 0. All real surfaces lie somewhere in between these two idealizations. The power absorbed by a body depends on its particular environment. In general a good radiator is a good absorber also. Conversely a body that reflects most of the radiation incident on it is a poor absorber and a poor radiator.

When the heat current is radiated isotropically (uniformly in all directions), the energy per unit area per unit time (power per unit area) falls off as r^{-2} where r is the distance from the source. This is necessary to keep the power crossing an imaginary surface of area $4\pi r^2$ independent of r and equal to the power being radiated.

PROBLEM-SOLVING STRATEGIES

The quantity (H) used here to symbolize a heat current is related to the quantity of heat (ΔQ) calculated in Chapter 15 by $H = (\Delta Q/\Delta t)$.

The problems in this chapter are further examples of energy conservation and conversion. When the temperature at each point is no longer changing with time, the heat current into an element is equal to the heat current out of the same element. Problems involving conduction or convection are usually straightforward to set up. Occasionally the problems involving convection are computationally messy.

The problems involving radiation are based on the Stefan law. Typical errors that arise involve not using an absolute (Kelvin) temperature or in not using the correct area in the formula. Frequently the effective area for absorption is <u>different</u> from that used for radiation.

EXAMPLES AND SOLUTIONS

Example 1

A brass rod of length 0.15 m, and cross-sectional area 10^{-4} m^2 has one end at 0°C and the other end at $+5^\circ$C. How large is the heat current through the rod ?

Solution:

We use the defining equation for the conduction heat current, H, Eq. 16-1 of the text and the data from Table 16-1. $k = 109\ J\cdot(s\cdot m\cdot C^\circ)^{-1}$, $A = 10^{-4} m^2$, $L = 0.15$ m, and $\Delta T = 5\ C^\circ$.

$$H = \left(\frac{kA}{L}\Delta T\right)$$

$$= \frac{109\ J\cdot(s\cdot m\cdot C^\circ)^{-1}(10^{-4}\ m^2)(5-0)C^\circ}{0.15\ m}$$

$$= 0.363\ J\cdot s^{-1} = 0.363\ \text{Watts}$$

Example 2

Suppose the heat current is not steady but corresponds to the situation in Fig. 16-1 of the text. Find the expression for the rate of change of temperature at an arbitrary point in the rod in terms of the heat current into the element (H_{in}) and the heat current out of the element (H_{out}). Refer to Fig. 16-1.

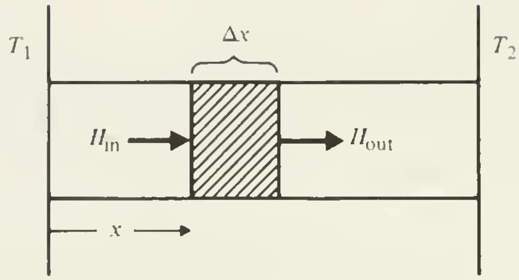


Figure 16-1

Solution:

For the shaded element between x and x + Δx, we show a heat current H_{in}, going into the element and a different current H_{out} coming out the other side. If $H_{in} > H_{out}$, the matter inside this element heats up since energy is conserved. If the material in the shaded region has mass ΔM, specific heat capacity c, and temperature T, the heat influx causes the temperature to change with time (i.e. dT/dt = 0).

$$\Delta M = \rho A \Delta x$$

where ρ is the mass density of the material. The heat energy per unit time entering and staying in the region is the difference between H_{in} and H_{out} so:

$$H_{in} - H_{out} = \Delta M \; c \; \frac{dT}{dt}$$

Thus for steady flow, dT/dt = 0, and $H_{in} = H_{out}$.

Example 3

Two rods of different materials but each of uniform cross-section area, A, are joined at the interface of area A, as shown in Fig. 16-2. The lengths of the rods are l_1 and l_2 and the respective thermal conductivities are k_1 and k_2. The ends are maintained at temperatures T_1 and T_2 where $T_1 > T_2$. Derive a general expression for the interface temperature T assuming all the heat conducted through rod 1 passes through rod 2.

Solution:

The heat current through rod 1 , $H_1 = k_1A(T_1 - T)/l_1$, is equal to the heat current through rod 2, $H_2 = k_2A(T - T_2)/l_2$. Equating the two expressions, we have:

$$\frac{k_2A}{l_2}(T - T_2) = \frac{k_1A}{l_1}(T_1 - T).$$

We solve this equation for the unknown interface temperature T to obtain:

$$T = \frac{(k_1/l_1)T_1 + (k_2/l_2)T_2}{(k_1/l_1) + (k_2/l_2)}$$

(A cancels out.)

To illustrate this formula, let rod 1 be the brass rod of the previous example and rod 2 be a steel rod [$k = 50.2\ J(s{\cdot}m{\cdot}C^o)^{-1}$] of length 0.2 m and the same area as 1. If $T_1 - T_2 = 5^oC$ with $T_2 = 0^oC$, then the interface temperature is equal to 3.72^oC. For a steady heat current, the temperature gradient (dT/dx) will be equal to $-H/k_1A$ in rod 1 and equal to $-H/k_2A$ in rod 2. Thus the temperature will decrease linearly with x in both rods but with different slopes.

Example 4

Consider the air inside a room to be at 25^oC while the air outside is at -15^oC. Calculate the heat transferred per unit area of glass for a windowpane of thermal conductivity $1.05\ J{\cdot}(s{\cdot}m{\cdot}C^o)^{-1}$. Note that this is the same example that is used in the text in section 16-2 where an approximate solution was obtained. We will attempt an exact solution to test the validity of the approximate one. Refer to Fig. 16-3.

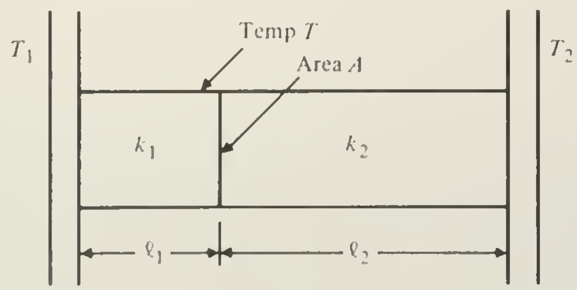


Figure 16-2

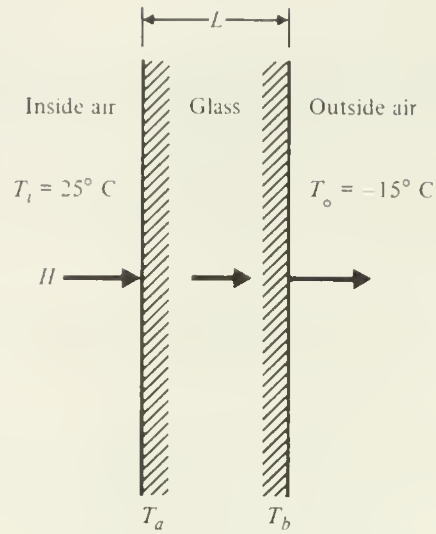


Figure 16-3

Solution:

Let T_a and T_b be the inside and outside temperatures of the glass plate. Heat is transferred by convection in the air, both inside and outside, and by conduction through the glass plate. The heat currents per unit area must be equal for steady-state heat flow. Thus we write:

$$H/A = h_i(T_i - T_a) = \frac{k}{L}(T_a - T_b)$$

$$h_o(T_b - T_o) = \frac{k}{L}(T_a - T_b)$$

Using

$$h = 1.77(\Delta T)^{1/4}\ \mathrm{J \cdot s^{-1} \cdot m^{-2} \cdot C^{o-1}}$$

from Table 16-2, we obtain that

$$(T_i - T_a)^{5/4} = (T_b - T_o)^{5/4}$$

or

$$T_i - T_a = T_b - T_o.$$

Thus we find that:

$$T_i + T_o = T_a + T_b.$$

We can use this equation to eliminate T_b in the first of the above equations and solve numerically for T_a. The equation to be solved is:

$$(25 - T_a)^{5/4} = 5.896 \times 10^2(T_a - 5)$$

$$T_a = 5.071^\circ C.$$

Since

$$T_a + T_b = T_i + T_o = 10^\circ C, \text{ we have } T_b = 4.929^\circ C.$$

The heat current per unit area is then:

$$H/A = \frac{k}{L}(T_a - T_b) = 74.8\ J \cdot s^{-1} \cdot m^{-2}.$$

Comparing these values with the text's approximate solution yields the fact that the heat currents differ only in round-off error and the glass face temperatures are also in near perfect agreement. The key to this good agreement is choosing the initial average temperature of the glass to be equal to the average of the inside and outside air temperatures.

Example 5

The average radiated power from the sun reaching the earth's surface is given by the 'solar constant', $S = 1.395\ kw \cdot m^{-2}$. Assuming the sun radiates isotropically as an ideal blackbody, calculate the absolute temperature of the sun's surface. Refer to Fig. 16-4.

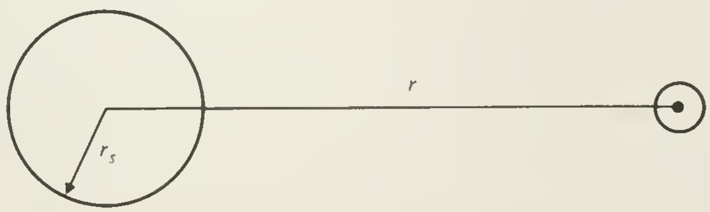


Figure 16-4

Solution:

Take for the distance between the earth and sun $r = 1.5 \times 10^{11}$ m and the radius of the sun, $r_s = 7 \times 10^8$ m. If we draw a series of spheres concentric with the sun, we see that the energy per unit area per unit time (the solar constant for that sphere) multiplied by the surface area of the sphere $(4\pi r^2)$ is a constant and equal to the power radiated by the sun. Using Eq. 16-6, where H is constant, we write (with e = 1 for the sun):

$$(4\pi r_s{}^2)\sigma T_s{}^4 = (4\pi r^2)S$$

Numerically:

$$T_s = [(\frac{r}{r_s})^2 \frac{S}{\sigma}]^{1/4}$$

$$T_s = 5.8 \times 10^3 \text{ K.}$$

In this problem we have used a measured value of the radiated power that falls on a unit area on the earth's surface (S) and two known lengths to deduce the temperature at the Sun's surface from Stefan's law.

Example 6

Assuming the moon is a perfect blackbody and absorbs all the radiation falling on it from the sun, estimate the temperature of the moon's surface.

Solution:

Since only an estimate is called for, we assume the distance between the moon and the sun is on average the same as the average distance between the earth and the sun. This means we can use the same solar constant, S, for the moon as we used for the earth. Let r_m be the radius of the moon.

$$H_{in} = \pi r_m^2 \, S.$$

Since πr_m^2 is the area of a circle the size of the moon and is the effective area for absorption of energy from the sun.

$$H_{out} = 4\pi r_m^2 \, \sigma T_m^4 \text{ (the whole area radiates).}$$

If the moon is in thermal equilibrium, then $H_{in} = H_{out}$, giving:

$$T_m = (S/4\sigma)^{1/4}$$

$$= \left(\frac{1.4 \times 10^3 \text{ W}\cdot\text{m}^{-2}}{4 \times 5.67 \times 10^{-8} \text{ W}\cdot\text{m}^{-2}\cdot\text{K}^{-4}}\right)^{1/4}$$

$$= 280 \text{ K.}$$

Because of the assumptions made, this would apply to the earth as well. The model used is vastly oversimplified. The observed mean temperature is 287 K presumably due to the core and mantle being at elevated temperatures and cooling, so the assumption of thermal equilibrium was not justified for the present.

Example 7

A piece of copper with constant thickness, t, has a rectangular cross- section of variable area as shown in Fig. 16-5. At $x = 0$, $y = y_0$ and at $x = L$, $y = y_0/2$, changing linearly in between. The temperature is equal to T_1 at the origin and equal to T_2 at $x = L$ with $T_1 > T_2$. Find the temperature as a function of x.

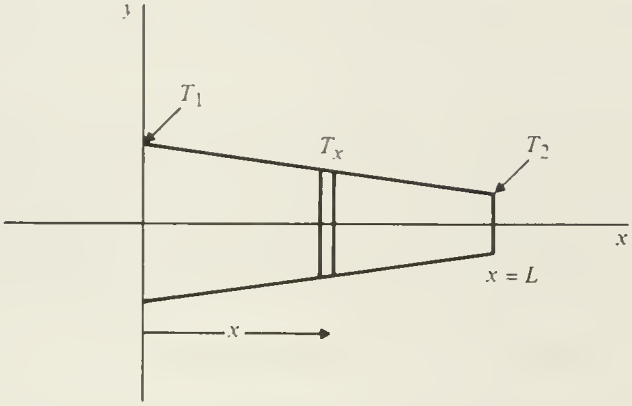


Figure 16-5

Solution:

The equation for the straight line $y = y(x)$ is

$$y = y_0[1 - (x/2L)].$$

Thus the area, $A(x) = 2yt$ is given by:

$$A(x) = A_o(1 - \frac{x}{2L})$$

where

$$A_o = 2y_o t.$$

Here it is essential to recognize that the heat is constant through the sample and given by:

$$H = \frac{-kA(x)}{x}[T(x) - T_1].$$

In particular at $x = L$, the heat current is: $H = \dfrac{kA_o}{2L}(T_1 - T_2)$.

Equating these two expressions for H and solving for T(x), we obtain:

$$T(x) = T_1 - \frac{\frac{x}{2L}\,(T_1 - T_2)}{1 - \frac{x}{2L}} .$$

QUIZ

1. Two pieces of metal with the same cross-sectional area of 0.02 m^2 are welded together. One piece is stainless steel and of length 0.3 m (K_{ss} = 50.2 $W \cdot m^{-1} \cdot C^{o-1}$). The second piece is of length 0.2 m but its thermal conductivity is unknown. When the stainless steel end is maintained at a temperature of 50 °C and the other end maintained at 20 °C, the temperature of their common junction is 27 °C. (a) Calculate the heat current in the stainless steel piece. (b) Calculate the thermal conductivity of the unknown piece.

Answer: (a) H = 77.0 Watts, (b) K_u = 110 $W \cdot m^{-1} \cdot C^{o-1}$.

2. A flat horizontal plate with face area of 0.64 m^2 is maintained by an electrical heater at a temperature of 30°C in a room where the air temperature is 20°C. Assuming the only heat loss of the plate results from convection from both the upward facing surface and the downward facing surface, calculate the power delivered by the heater. Use Table 16-2.

Answer: 43.1 $J \cdot s^{-1}$ = 43.1 watts

3. Calculate the steady state temperature a thin blackened disc of cross-sectional area A would reach on earth when directly exposed to sunlight if no heat is lost via conduction or convection.

Answer: Since only one side is illuminated but both sides radiate, the correct answer is 333 K.

4. Taking the mean distance from the earth to the sun to be 1.49 x 10^{11} m and that from the sun to Venus to be 1.08 x 10^{11} m, (a) use the known solar constant on earth, S = 1.4 $kW \cdot m^{-2}$, to calculate the solar constant on Venus. (b) Assuming Venus absorbs and radiates like a blackbody, calculate the equilibrium temperature on that planet.

Answer: (a) S = 2.66 $kW \cdot m^{-2}$, (b) T = 329 K (or 56 °C).

17
THERMAL PROPERTIES OF MATTER

OBJECTIVES

In this chapter, the concept of an equation of state is introduced. The ideal gas is used to illustrate this concept. Your objectives are to:

Apply the ideal gas equation of state to calculations of the pressure, volume, temperature, and number of moles.

Recognize the differences in behavior between the ideal gas and real substances.

Construct p-T diagrams of real substances showing the triple point and critical point and the solid, liquid, and vapor regions.

Calculate relative humidity from the partial pressure of water vapor and the vapor pressure.

REVIEW

An equation of state connects the variables that characterize the equilibrium states of the system. For the next few chapters we will be considering gases so that the variables most frequently used are the volume (V), pressure (p), temperature (T) and the mass m. Since the above variables are not independent, we can express one of them as a function of the remaining variables:

$$V = f(p,T,m)$$

Other possibilities will be discussed later.

One useful idealization is the ideal gas in which p,V,T, and m are connected by the equation:

$$pV = nRT$$

where the mass, m, is contained in the factor n that gives the number of moles of gas in the volume V. The constant R is called the universal (same for all) gas constant. The various numerical values and different units are examined in

Example 1.

If the number of moles of an ideal gas is constant, then the equation of state predicts that pV/T is a constant. This equation in three variables (only two of which are independent) defines a surface. Projection of this three dimensional surface onto two dimensions yields p vs T, V vs T, and p vs V diagrams. We will have use for such diagrams in the next few chapters.

Real substances have substantially different p,V,T surfaces from that of an ideal gas. This is because the real substances can undergo phase transformations such as liquefaction and solidification. For a real substance, although some functional relationships give good indications of much of the behavior (see Example 4), we should regard the p,V,T surfaces as graphical displays of experimental data.

Isobaric processes are those occurring at constant pressure while isothermal processes occur at constant temperature.

The two dimensional projection p vs T is called the phase diagram. As a rule, a point on this diagram gives the coordinates of a single phase but there are lines on this diagram where two phases (such as vapor and liquid) coexist. In Fig. 17-4 (SZY), the positive slope of dp/dT indicates that the volume (for fixed mass) contracts as you pass from vapor to liquid and again when you pass from liquid to solid. A negative slope of the fusion curve, dp/dT, usually is a signal that the substance expands upon freezing (as water does) but the phase boundary between liquid and solid helium is an exception that will not be treated here.

The fusion curve and the vaporization curve intersect at a point called the triple point where liquid, solid, and vapor can all coexist at a unique pressure and temperature. The vaporization curve terminates in a point called the critical point. If, at a temperature below the critical temperature, the pressure is increased from a point in the vapor phase to a point in the liquid phase, a discontinuous change in specific volume (as well as a latent heat) is found in crossing the vaporization curve. You can tell that a phase transformation has occurred since the two phases had different specific volumes at the same p and T. If, on the other hand, T is greater than the critical temperature and the pressure is increased, no discontinuous change in specific volume occurs so we can't tell the difference between vapor and liquid.

Along the vaporization curve, the pressure (called the vapor pressure) is a function of the temperature alone and does not depend on the volume of the container. For this reason it is easy to reduce the temperature of a liquid-vapor system in equilibrium simply by using a vacuum pump to reduce the pressure. The relationship between p and T along the vaporization curve is so reproducible that a measurement of the pressure accurately yields the temperature. The helium vapor pressure-temperature curve forms the basis of the provisional international temperature scale below 4.2 K.

PROBLEM-SOLVING STRATEGIES

Express all temperatures in Kelvin degrees. Frequently numerical substitution for R can be avoided by remembering that

$$R = \frac{p_s V_s}{T_s}$$

where p_s, V_s, and T_s are the standard pressure, volume, and temperature for one mole. This relationship is useful when two of the known quantities are easily expressed in terms of these standard values yielding the ratio of the unknown quantity to its standard value.

EXAMPLES AND SOLUTIONS

Example 1

Show that the values of

(a) $R = 8.314\ J\cdot mol^{-1}\cdot K^{-1}$

(b) $R = 8.314 \times 10^7\ ergs\cdot mol^{-1}\cdot K^{-1}$

(c) $R = 1.99\ cal\cdot mol^{-1}\cdot K^{-1}$

(d) $R = 8.20 \times 10^{-2}\ atm\cdot mol^{-1}\cdot K^{-1}$

are all equivalent.

Solution:

On occasion it is difficult to remember the value of R, the molar gas constant, or to recall the value of R in the units most appropriate to the problem under consideration. If you can remember that 1 mole of any ideal gas occupies 22.4 liters at standard temperature and pressure, you can overcome either difficulty.

(a) Using the ideal gas law

$$pV = nRT \quad \text{or} \quad R = pv/nT,$$

then standard pressure is 1 atm (1.013×10^5 Pa) and standard temperature is $0^\circ C$ (273 K) so for $n = 1$,

$$R = \frac{(1.013 \times 10^5\ N\cdot m^{-2})(22.4 \times 10^{-3}\ m^3)}{(273\ K)(1\ mole)}$$

$$= 8.31\ J\cdot mol^{-1}\cdot K^{-1}$$

(b) Since $1\ J = 10^7$ ergs, the conversion is trivial.

(c) Similarly, since 4.19 J = 1 cal, dividing the result (a) by this factor produces

$$R = 1.99 \text{ cal}\cdot\text{mol}^{-1}\cdot\text{K}^{-1}$$

(d) This result merely summarizes the starting data. Since standard pressure is 1 atm and standard temperature is 273 K we have:

$$R = \frac{(1 \text{ atm})(22.41)}{(1 \text{ mole})(273\text{K})}$$

$$R = 8.20 \times 10^{-2} \text{ atm}\cdot\text{l}\cdot\text{mol}^{-1}\cdot\text{K}^{-1}$$

Example 2

Helium gas is admitted to a volume of 200 cm^3 at a temperature of 77°K until the pressure is equal to 1 atm.

(a) If the temperature of this container is raised to 20°C, what will the pressure be?

(b) If the system has a pressure relief valve that will not permit the pressure to exceed 1 atm, what fraction of the gas remains at 20°C?

Solution:

(a) For this closed system, the number of moles, n, and the volume are both constant so:

$$\frac{P_i}{T_i} = \frac{P_f}{T_f}$$

Thus

$$\frac{P_f}{1 \text{ atm}} = \frac{273 + 20}{77}$$

$$P_f = 3.8 \text{ atm.}$$

(b) In this situation, both the pressure and volume are constant but the number of moles is not constant since gas can leak out the pressure relief valve. Therefore:

$$n_f(RT_f) = n_i(RT_i)$$

The fraction of gas left is:

$$n_f/n_i = T_i/T_f = 77/293$$

Thus

$$n_f/n_i = 0.263$$

Example 3

The molar volume of liquid helium is approximately 28 $cm^3 \cdot mol^{-1}$. A high pressure (100 atm) helium cylinder has a volume of 0.05 m^3. How many liters of liquid helium can be produced from the gas in the cylinder?

Solution:

Since one liter contains 1000 cm^3, one liter of liquid helium represents 35.7 moles. To calculate the number of moles stored in the high pressure cylinder we use the ideal gas law:

$$pV = nRT$$

Let

$$p = 100 \text{ atm} = 10^7 \text{ Pa}$$

$$V = 5 \times 10^{-2} \text{ m}^3$$

$$R = 8.31 \text{ J}\cdot\text{K}^{-1}\cdot\text{mol}^{-1}.$$

and take room temperature for T(293°K). Then n is:

$$n = \frac{10^7(5 \times 10^{-2})}{8.31(293)} = 2.05 \times 10^2 \text{ moles}$$

Thus dividing this by 35.7 moles per liter, we find that we can make 5.8 liters of liquid from one high pressure cylinder of gas.

Example 4

Treating N_2 as though it is an ideal gas, (a) calculate the volume occupied by a mole of this gas at the critical temperature and pressure. Express your answer as a ratio of the volume to the known critical volume ($V_c = 90.1 \times 10^{-6}$ m^3). (b) Calculate the pressure at V_c, T_c.

Solution:

(a) Using $pV = nRT$

$$V = \frac{nRT}{p}$$

$$= \frac{(1\ \text{mole})(8.314\ \text{J}\cdot\text{mol}^{-1}\cdot\text{K}^{-1})(126.2\ \text{K})}{33.9 \times 10^{5}\ \text{N}\cdot\text{m}^{-2}}$$

$$= 3.095 \times 10^{-4}\ \text{m}^3$$

Since $V_c = 90.1 \times 10^{-6}\ \text{m}^3$, then

$$\frac{V}{V_c} = 3.44$$

(b) Solving the ideal gas law for p,

$$p = \frac{(1\ \text{mole})(8.314\ \text{J}\cdot\text{mol}^{-1}\cdot\text{K}^{-1})(126.2\ \text{K})}{90.1 \times 10^{-6}\ \text{m}^3}$$

$$= 1.16 \times 10^{7}\ \text{Pa}.$$

The critical pressure is $p_c = 33.9 \times 10^5$ Pa so for the ratio we have

$$\frac{p}{p_c} = 3.44.$$

Example 5

Assume the temperature and pressure in the lower part of the atmosphere are given by

$$T = T_o - \alpha y$$

$$\ln\left(\frac{p_o}{p}\right) = \left(\frac{Mg}{R\alpha}\right)\ln\left(\frac{T_o}{T_o - \alpha y}\right)$$

where T_o and p_o are the temperature and pressure at the surface, M the molecular mass, and $\alpha = 6 \times 10^{-3}\ \text{C}^\circ\cdot\text{m}^{-1}$. y is the height above the surface. Calculate the temperature and pressure at a height of 1 mi(1.6 km). Take M = 29 g as the mean molecular mass of air.

Solution:

We will assume T_o = 293 K (or 20 °C). The temperature 1 mi above the earth's surface will be

$$T = T_o - \alpha y = 293\ \text{K} - (6 \times 10^{-3}\ \text{K}\cdot\text{m}^{-1})(1.6 \times 10^3\ \text{m})$$

$$= 293\ \text{K} - 9.6\ \text{K} = 283.4\ \text{K}.$$

To calculate the pressure, first evaluate the factor

$$\frac{Mg}{R\alpha} = \frac{(29 \times 10^{-3}\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})}{(8.314\ \text{J}\cdot\text{mol}^{-1}\cdot\text{K}^{-1})(6 \times 10^{-3}\ \text{K}\cdot\text{m}^{-1})}$$

$$= 5.70$$

This yields

$$\ln\left(\frac{p_o}{p}\right) = 5.70\ \ln(1.034)$$

or

$$p = 0.827\ p_o.$$

Example 6

Along the phase equilibrium line between liquid and gas, the slope is given by:

$$\frac{dp}{dT} = \frac{L_v}{T(V_g - V_l)}$$

where L_v is the heat of vaporization, V_g the volume of a mole of gas, and V_l the volume of a mole of liquid. Assuming the gas obeys the ideal gas law and that $V_g \gg V_l$, obtain the vapor pressure as a function of temperature.

Solution :

Making the approximation $V_g \gg V_l$, we have

$$\frac{dp}{dT} = \frac{L_v}{TV_g}.$$

Replacing the gas volume, $V_g = RT/p$ in the denominator and rearranging,

$$\frac{dp}{p} = \frac{L_v}{R}\left(\frac{dT}{T^2}\right).$$

Integrating this expression yields

$$\ln (p/p_o) = -\frac{L_v}{RT},$$

where p_o is the pressure at infinitely high temperature. Solving for p,

$$p = p_o \exp\left(-\frac{L_v}{RT}\right).$$

This shows that the equilibrium vapor pressure decreases (but not linearly) with decreasing temperature.

QUIZ

1. A 50 liter tank at 293 K contains N_2 gas at a pressure of 0.6 atm.
(a) How many moles of O_2 must be added to make a mixture that is 80% N_2 and 20% O_2?
(b) What will the final pressure be?

Answer: (a) 0.312 moles of O_2, (b) $p_f = 0.75$ atm.

2. On a really hot day (temperature 40 °C, relative humidity 75 %), the partial pressure of water vapor in the air is $p_w = 5.50 \times 10^3$ Pa. Assuming the water vapor behaves as an ideal gas, calculate the mass of water that can be removed from one cubic meter of air if all the water vapor is liquefied.

Answer: m = 38.1 grams (or 38.1 cc).

3. A 50 liter tank contains 3 moles of gas, the composition of the gas being 20 % oxygen and 80 % nitrogen. If the temperature of the tank is 293 K, (a) calculate the pressure in the tank and (b) the pressure that would be obtained if all the oxygen molecules were removed leaving only nitrogen.

Answer: (a) $p = 1.46 \times 10^5$ Pa, (b) $p_N = 1.17 \times 10^5$ Pa.

4. A cylinder of volume 0.5 liter fitted with a piston contains an ideal gas at 400 K with a pressure of 6 atm. The piston is pushed in at constant pressure until the volume is halved, the temperature increasing to 450 K. (a) Calculate the number of moles in the system initially. (b) How many moles remain at the end?

Answer: (a) $n_i = 9.14 \times 10^{-2}$ moles, (b) $n_f = 4.06 \times 10^{-2}$ moles.

18
THE FIRST LAW OF THERMODYNAMICS

OBJECTIVES

In this chapter the first law of thermodynamics is stated. It is basically a restatement of energy conservation. Your objectives are to:

> Calculate the heat taken in by the system of interest for various processes (constant volume, constant pressure, or constant temperature).
>
> Calculate the work done by the system for the same processes.
>
> Apply the first law of thermodynamics to the ideal gas.

REVIEW

Thermodynamics enables you to analyse physical systems without reference to their microscopic properties. Usually the universe is divided into two parts, a system of interest and the rest of the universe, which makes up the environment for the 'system'. The energy of the universe is taken as constant, but the system can change its energy by transfer of heat, the performance of work, or both, by interacting with the rest of the universe.

The work done <u>by</u> a fluid, W, as it changes from a state labeled 1 to a state labeled 2 is equal to the area under a pressure (p) versus volume (V) curve connecting the points (p_1, V_1) and (p_2, V_2).

$$W = \int_1^2 p\,dV$$

Since an infinite number of curves (leading to an infinite number of values of W) can connect the points 1 and 2, the work done <u>by</u> the system in general <u>depends on the path taken</u> between points 1 and 2. This is illustrated in Example 1. Since the pressure is always greater than or equal to zero, the work done by the system in an expansion (the final volume is larger than the initial volume) is positive and the work done by the system in a contraction (final volume less than initial volume) is negative.

Like work, the heat taken in or rejected by a system depends on the path taken between its initial and final thermal equilibrium states. Thus the 'heat content' of a body is not a well defined concept, although heat transfer is. Heat taken in by the system will be considered positive while heat rejected will be negative.

While neither the heat taken in nor the work done by a system are functions of just the coordinates of the initial and final equilibrium states and depend on the path taken between these states, their difference $\Delta Q - \Delta W$ is independent of the path and depends only on the coordinates of the initial and final states. This 'state function', $U_2 - U_1 = \Delta U$, is called the internal energy. The first law of thermodynamics can be stated in the following way: there exists a function, ΔU, of the thermodynamic coordinates (p,V, and T for an ideal gas) of the initial and final equilibrium states of the system that is equal to the difference of the heat taken in by the system and the work done by the system.

$$\Delta U = \Delta Q - \Delta W$$

It might be easier to remember this equation if it is rewritten in the form:

$$\Delta Q = \Delta U + \Delta W$$

Here the heat taken in is equal to the increase in internal energy plus the work done by the system. The first law of thermodynamics recognizes heat as a form of energy and basically is a restatement of energy conservation.

Five specific thermodynamic processes are given names in the text:

(1) Adiabatic Process: no heat energy enters or leaves the system so $\Delta Q = 0$. If $\Delta Q = 0$, the first law implies that $\Delta U = -\Delta W$ so that if one quantity, ΔU or ΔW, is known, the other can be found.

(2) Isochoric Process: there is no volume change, $\Delta V = 0$, so no work is done and $\Delta Q = \Delta U$.

(3) Isothermal Process: there is no temperature change, $\Delta T = 0$.

(4) Isobaric Process: there is no pressure change, $\Delta p = 0$.

(5) Throttling Process: this is a specific adiabatic process in which the sum of the internal energy (U) and the product of pressure and volume is held constant (i.e. $pV + U$ = constant).

The first law of thermodynamics can also be stated in terms of infinitesimal processes so that it appears in differential form:

$$dQ = dU + dW$$

Again, for emphasis, we note that to calculate ΔQ or ΔW from dQ or dW, the path must be specified but to obtain ΔU we only need to know the initial and final equilibrium states.

Many of these general (and very powerful) ideas can be applied profitably to the ideal gas. For the ideal gas, it is found that the internal energy depends only on the temperature (T) and not on the pressure or volume. This is a very important conclusion and we will use it frequently although in this chapter it is simply asserted without proof. (See Example 2).

The two specific heat capacities that are important for the ideal gas are c_p, the specific heat at constant pressure and the specific heat at constant volume, c_v. C_p and C_v are the heat capacities of 1 mole of ideal gas.

When the heat taken in (dQ) by the system (containing n moles), at constant volume, results in a temperature increase (dT), the heat capacity per mole, C_v, is equal to:

$$nC_v = \frac{dQ}{dT}$$

Since dV = 0 here (isochoric process), no work is done and the first law gives dQ = dU. We can conclude that $dU = nC_v\ dT$. The change in U is always independent of the path and since U is a function of T only for an ideal gas, we can now calculate ΔU for <u>any</u> process for an ideal gas:

$$\Delta U_{ideal} = nC_v\Delta T$$

If heat is added at constant pressure, then the molar heat capacity, C_p, is defined by:

$$nC_p = \frac{dQ}{dT}$$

For dQ we can use the first law and write dQ = dU + pdV. We know that $dU = nC_v dT$ and since p is constant, pdV = nRdT. Substituting these results into the equation for C_p yields:

$$C_p = C_v + R.$$

The ratio of C_p/C_v is denoted γ and will be regarded as an experimental quantity for our purposes although it can be calculated by appealing to microscopic models of gases. See Examples 3 and 4.

When an ideal gas undergoes an adiabatic (dQ = 0) expansion or compression, the values of the thermodynamic variables are constrained by the two equations:

$$TV^{\gamma-1} = \text{constant} \quad \text{and} \quad pV^{\gamma} = \text{constant}.$$

Since dQ = 0, the work done by the gas is most simply obtained by first calculating the internal energy change from the temperature change and then writing that

$$\Delta W = -\Delta U = -\ nC_v\Delta T.$$

This procedure is illustrated in Example 5.

PROBLEM-SOLVING STRATEGIES

Nearly all calculations in this chapter involve the ideal gas. The internal energy of an ideal gas depends only on the temperature and is usually the easiest quantity to calculate in a problem as $\Delta U = nC_V\Delta T$. For many of the processes considered here, either ΔW or ΔQ can be calculated without undo difficulty. By using the first law of thermodynamics, the third or unknown quantity can be found.

Sign conventions will be important here. ΔQ is positive if heat is taken in by the system, negative otherwise. The work done by the system is positive when the final volume is larger than the initial volume.

EXAMPLES AND SOLUTIONS

Example 1

Consider the path shown in the Fig. 18-1 from the initial state i to the final state f, involving two constant volume processes and one constant pressure process. Calculate the heat taken in by the system and the work done by the system in terms of p, V_i, V_f, the number of moles n, C_V, T_f and T_i.

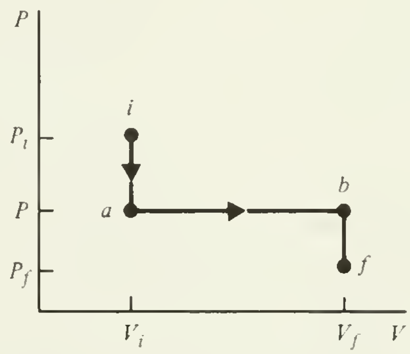


Figure 18-1

Solution:

The work done in the processes from i to a and from b to f is zero since no volume change occurs in either case. Thus the total work done occurs between a and b where the pressure is p and the volume change is $V_f - V_i$.

$$\Delta W = p(V_f - V_i)$$

The volume change is determined by the coordinates of the initial (point a) and final (point b) states and the arbitrary pressure, p. Therefore the work done depends on the particular value of p chosen. The heat taken in along the constant volume segment i to a is:

$$\Delta Q = nC_V(T_a - T_i).$$

The heat taken in along the constant pressure segment a to b is:

$$\Delta Q = nC_p(T_b - T_a).$$

The heat taken in along the constant volume segment b to f is:

$$\Delta Q = nC_V(T_f - T_b)$$

Summing these contributions:

$$\Delta Q_T = nC_V(T_f - T_b + Ta - T_i) + nC_p(T_b - T_a)$$

Rearranging this:

$$\Delta Q_T = nC_V(T_f - T_i) + n(C_p - C_V)(T_b - T_a)$$

However

$$C_p - C_V = R \text{ and } nRT_b = pV_f \text{ while } nRT_a = pV_i$$

Rewriting:

$$\Delta Q_T = nC_V(T_f - T_i) + p(V_f - V_i)$$

$$= nC_V(T_f - T_i) + \Delta W_T.$$

Here we should notice that ΔQ_T also depends on the arbitrary pressure p as well as the coordinates of the initial and final states so ΔQ_T must depend on the path chosen. The difference between ΔQ_T and ΔW_T does not depend on the arbitrary pressure, p, but only on the coordinates of the initial and final states.

Example 2

Referring to Fig. 18-1, let the path from i to f be one where there is an expansion at constant pressure, p_i, from V_i to V_f followed by a reduction in pressure at constant volume from p_i to p_f. Calculate the heat taken in and the work done by the system.

Solution:

The change in internal energy is the same as in Example 1 since the initial and final points are the same.

$$\Delta U = nC_V(T_f - T_i)$$

The work done is the area of the rectangle

$$\Delta W = p_i(V_f - V_i)$$

The net heat taken in is equal to

$$\Delta Q_T = \Delta W + \Delta U$$

$$= p_i(V_f - V_i) + nC_V(T_f - T_i)$$

The heat taken in along this path is different than that computed in Example 1 but the difference between the heat taken in and the work done by the gas is the same (independent of path).

Example 3

An ideal monatomic gas (n moles) is taken from point 1 on the T_1 isotherm to point 3 on the T_2 isotherm along the path 1-2-3 shown in Fig. 18-2.

(a) In terms of the parameters shown, calculate the change in internal energy of the gas and the heat that must be added to it in this process.

(b) Suppose the path 1-4-3 is followed instead. Calculate the heat added along this path.

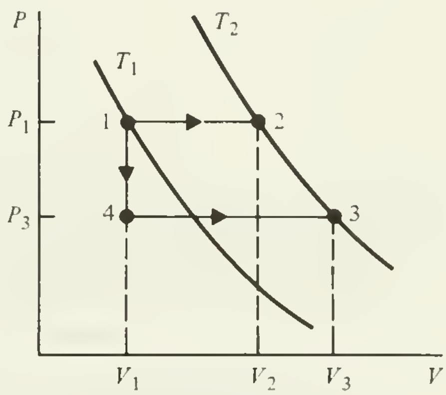


Figure 18-2

Solution:

For this gas assume C_v and C_p are known and constant.

(a) In going from 1 to 3, the internal energy change depends on the temperature change (but not on the path taken).

$$\Delta U = nC_v(T_3 - T_1)$$

The heat taken in along 1 to 2 can be written as:

$$\Delta Q = n\,C_p(T_2 - T_1)$$

or the first law can be used to give:

$$\Delta Q = \Delta U + \Delta W$$

$$\Delta Q = nC_v(T_2 - T_1) + p_1(V_2 - V_1)$$

Along the path 2 to 3, the internal energy change is zero (because the temperature is constant) so the heat taken in can be calculated from the work done:

$$\Delta Q = \Delta W = \int_{V_2}^{V_3} p dV = nRT_2 \int_{V_2}^{V_3} \frac{dV}{V}$$

$$= nRT_2 \ln\left(\frac{V_3}{V_2} \right)$$

Therefore along the path 1-2-3,

$$\Delta Q = n\, C_p(T_2 - T_1) + nRT_2 \ln\left(\frac{V_3}{V_2} \right)$$

(b) Since the work done in the process from 1 to 4 is zero, the total work done is:

$$\Delta W = P_3(V_3 - V_1).$$

The change in internal energy is the same, $nC_v(T_2 - T_1)$, so:

$$\Delta Q = nC_v(T_2 - T_1) + P_3(V_3 - V_1)$$

If we subtract the heat taken in along 1-2-3 from the heat taken in along 1-4-3, the difference is equal to:

$$(P_1 - P_3)V_1 - nRT_2 \ln (V_3/V_2)$$

We can conclude that for different paths, ΔQ is different.

Example 4

One mole of an ideal monatomic gas starts at point A in Fig. 18-3 ($T_A = 273$ K, $P_A = 1$ atm.) and undergoes an adiabatic expansion to point B where $V_B = 2V_A$. This is followed by an isothermal compression to the original volume at C ($V_c = V_A$) and a pressure increase at constant volume back to the original pressure. Take $\gamma = 5/3$.

(a) Compute the temperature at point B.
(b) Compute the pressure at point C.
(c) Compute the total work done for this cycle.

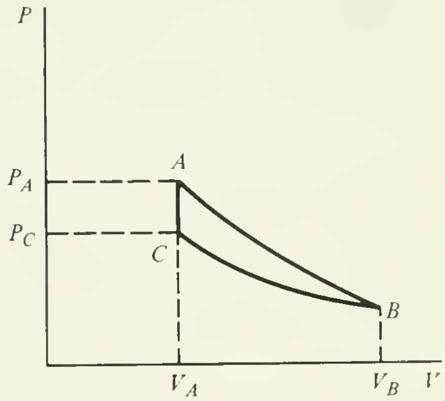


Figure 18-3

Solution:

(a)
$$T_B V_B^{\gamma-1} = T_A V_A^{\gamma-1}$$

so

$$T_B = (273\text{K})\left(\frac{V_A}{V_B}\right)^{\gamma-1}$$

$$= (273\text{ K})\left(\frac{V_A}{2V_A}\right)^{5/3-1}$$

$$= \frac{273\text{ K}}{2^{2/3}}$$

$$= 172\text{ K}$$

(b) The temperature at C is the same as at B and $V_c = V_A$, so the ratio

$$\frac{P_A V_A}{T_A} = \frac{P_c V_c}{T_c}$$

simplifies to:

$$P_c = P_A\left(\frac{T_c}{T_A}\right)$$

$$= (1\ \text{atm})\left(\frac{173\ \text{K}}{273\ \text{K}}\right) = 0.630\ \text{atm}.$$

(c) The total work done by the gas in the cycle is equal to the area ABC enclosed by the cycle in the P-V plane.

$$\Delta W_{AB} = -\ \Delta U_{AB} = -\ C_V(T_B - T_A) \quad (\text{since } \Delta Q = 0)$$

$$\Delta W_{BC} = RT_B[\ln(V_c/V_B)] \qquad (\text{since T is constant})$$

and the work done is zero from C to A since no volume change occurs. Summing these contributions:

$$W_{NET} = C_v(T_A - T_B) - RT_B\ (\ln 2)$$

$$= \frac{3}{2}\ R(T_A - T_B) - RT_B\ (\ln 2)$$

Using $C_v = 3R/2$ for an ideal monatomic gas, we have

$$W_{NET} = 268\ \text{J}.$$

Example 5

For the thermodynamic cycle shown in Fig. 18-4 use the equation $TV^{\gamma-1}$= constant and the ideal gas law to show that:

$$\frac{V_2}{V_1} = \frac{V_3}{V_4}$$

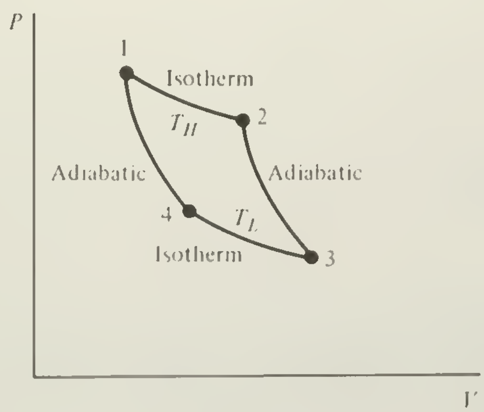


Figure 18-4

Solution:

For the adiabatic connecting points 1 and 4,

$$T_H V_1^{\gamma-1} = T_L V_4^{\gamma-1}$$

and for the adiabataic connecting 2 and 3,

$$T_H V_2^{\gamma-1} = T_L V_3^{\gamma-1}$$

Dividing the first equation by the second to eliminate the temperatures,

$$\left(\frac{V_1}{V_2} \right)^{\gamma-1} = \left(\frac{V_4}{V_3} \right)^{\gamma-1}$$

Since both sides are raised to the same power, we can conclude that

$$V_2/V_1 = V_3/V_4.$$

Example 6

An ideal diatomic gas initially at a pressure of 1 atm, a temperature of 300 K, and a volume of 0.05 m^3 is compressed adiabatically to a final volume of 0.03 m^3. Calculate the final temperature and pressure and the work done on the gas.

Solution:

For an adiabatic process, $TV^{\gamma-1}$ is constant as is pV^{γ}. First we find the pressure, p_f,

$$p_f V_f^{\gamma} = p_i V_i^{\gamma}$$

or

$$\frac{p_f}{p_i} = \left(\frac{V_i}{V_f} \right)^{\gamma}$$

Substituting, using $\gamma = 1.40$ for a diatomic gas, we have

$$\frac{p_f}{1\ \text{atm}} = \left(\frac{0.05\ m^3}{0.03\ m^3} \right)^{1.40} = 2.04$$

$$p_f = 2.04\ \text{atm}.$$

We can find T_f by using either the ideal gas law, since p_f and V_f are now known, or the adiabatic relations

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

$$\frac{T_f}{T_i} = \left(\frac{V_i}{zf} \right)^{\gamma-1}$$

Using the last formula we have

$$\frac{T_f}{300\ K} = \left(\frac{0.05\ m^3}{0.03\ m^3} \right)^{1.40-1} = 1.23$$

Thus

$$T_f = 368\ K.$$

To calculate the work done by the gas, we calculate the integral

$$\Delta W = \int_{V_i}^{V_f} p dV$$

where for p we substitute

$$p = \frac{K}{V^\gamma}$$

where the constant K in this relationship is equal to pV^γ at <u>any particular point</u>. Completing this integral gives:

$$\Delta W = \frac{K}{1 - \gamma} [V_f^{1-\gamma} - V_i^{\delta-\gamma}]$$

This can be put in a more recognizable form by noting that

(a) $KV_f^{1-\gamma} = (p_f\ V_f^\gamma)V_f^{1-\gamma} = p_f\ V_f = nRT_f$

(b) $KV_i^{1-\gamma} = (p_i\ V_i^\gamma)V_i^{1-\gamma} = p_i\ V_i = nRT_i$

and

(c) $1 - \gamma = 1 - (C_p/C_v) = (C_v - C_p)/C_v = - R/C_v$

Making all these substitutions, we get for the work done by the gas,

$$\Delta W = - nC_v(T_f - T_i)$$

We note that this equals $- \Delta U$, a result dictated by the first law of thermodynamics and the fact that $\Delta Q = 0$ (so that $\Delta W = - \Delta U$) for an adiabatic process. In order to avoid the above integration and tricky substitutions, we can obtain the work done on the gas (the negative of the work done by the gas) by calculating ΔU directly. To finish this calculation we use the specific heat value for H_2 from Table 18-1,

$$C_v = 20.42 \ \mathrm{J\cdot mol^{-1}\cdot K^{-1}}$$

and calculate the number of moles present. At standard temperature and pressure one mole of an ideal gas occupies 0.0224 m^3. Thus the number of moles is

$$n = \frac{(1 \text{ mole})(1 \text{ atm})(0.05 \text{ m}^3)(273 \text{ K})}{(1 \text{ atm})(.0224 \text{ m}^3)(300 \text{ K})} = 2.03 \text{ moles}$$

Therefore the work done on the gas:

$$\Delta W_{on} = - \Delta W_{by} = (2.03)(20.42 \text{ J})(68) = 2821 \text{ J}$$

QUIZ

δ∝ two moles of argon gas ($C_v = 12.47 \ \mathrm{J\cdot mol^{-1}\cdot K^{-1}}$) are compressed at a constant pressure of 2 atm from a volume of 5 liters to a volume of 2 liters. During this process, the absolute temperature increases by 20%. How much heat is taken in by the system?

Answer: Approximately 300 J of heat is rejected by the system in this process.

2. Calculate the work done by 1.5 moles of an ideal gas ($C_v = 20.76 \ \mathrm{J\cdot mol^{-1}\cdot K^{-1}}$) in going along a straight line path in the p-V plane from the point (p_1,V_1) to the point (p_2,V_2) where p_1 = 3 atm, V_1 = 1 liter, p_2 = 1 atm, and V_2 = 2 liters. This problem can be done without integration.

Answer: 2 atm·liter or about 200 J.

3. An ideal gas with $C_v = (5/2)R$ and $\gamma = 1.4$ expands adiabatically from an initial pressure and volume (p_i = 20 atm, V_i = 2 liters) to a final pressure of 1 atm. Assuming there are 1.5 moles of gas present, calculate (a) the initial temperature, (b) the final volume, (c) the final temperature, and (d) the work done by the gas.

Answer: (a) T_i = 325.6 K, (b) V_f = 17.0 liters, (c) T_f = 138.3 K, (d) W = 5838 J.

4. An ideal gas with $\gamma = 1.50$ is initially at a pressure of 1 atm and a temperature of T_a = 300 K (point a). Its volume is 0.60 liters. At constant volume, the gas absorbs heat until its temperature reaches T_b where the pressure is 2.5 atm (point b). The gas is then expanded adiabatically until its temperature is 300 K again (point c). (a) Calculate the number of moles in the system. (b) Calculate the temperature T_b. (c) Calculate the volume after the adiabatic expansion to point c. (d) Calculate the heat taken in for the constant volume process connecting points a and b.

Answer: (a) 2.44 x 10^{-2} , (b) 750 K, (c) 3.75 liters, (d) 183 J.

19
THE SECOND LAW OF THERMODYNAMICS

OBJECTIVES

In this chapter the second law of thermodynamics is stated and the limitations that it places on physical processes are given. Heat engines and refrigerators are used to illustrate the concepts on which the second law is based. Your objectives are to:

Calculate the heat taken in and work done per cycle for several heat engines.

Calculate the thermal efficiency of heat engines and the coefficient of performance for refrigerators or heat pumps from the fundamental definitions.

Understand the relationship between the efficiency of a Carnot engine and the absolute temperatures of the hot and cold reservoirs.

Calculate the entropy changes that accompany both reversible and irreversible processes.

REVIEW

The conversion of heat energy into mechanical work is a topic of great practical importance. The sweeping general nature of thermodynamics allows us to know the fundamental limitations of proposed systems without examining each detail of the operation.

The heat engines of greatest importance are those that take the working substance, freon, gasoline, steam, ideal gas, etc., through a complete cycle. The internal energy, U, is a definite function of the thermodynamic coordinates, no matter what the working substance is, so that in a complete cycle we must have $\Delta U = 0$ as the system is returned to its initial state. Applied to this very same cycle, the first law of thermodynamics gives:

$$Q = W$$

This means that the net heat taken in during the entire cycle is equal to the net work done by the engine in the cycle. The quantity Q, the net heat is actually the sum of the heat absorbed per cycle, Q_H, and the heat rejected per cycle, Q_C or $Q = Q_H + Q_C = W$. Since Q_C is a negative number, Q_H will be larger than W. The heat rejected per cycle is usually wasted so the thermal efficiency, e, is defined to be the ratio of the work done by the engine, W, to the heat absorbed, Q_H.

$$e = \frac{W}{Q_H} = \frac{Q_H + Q_C}{Q_H} = 1 + \frac{Q_C}{Q_H} = 1 - \left|\frac{Q_C}{Q_H}\right|$$

We should note here that since Q_C is negative (and less than Q_H in absolute value or W couldn't be positive) that $e \lesssim 1$. See Example 1.

Various types of heat engines can be analyzed in order to gain facility in calculating the quantities needed to obtain the efficiency. Some notable examples are: the internal combustion engine, the Diesel engine, and the steam engine (the cycle for this engine was given in Example 2 in Chapter 18). It is interesting to note that the efficiencies of the internal combustion engine and the Diesel engine depend only on the expansion and compression ratios (see Example 3) and not directly on the temperatures.

If the heat engine is run backwards (only reversible engines can be run backwards), it becomes a refrigerator (or a heat pump) in that mechanical work is supplied, heat is extracted from some 'cold' reservoir and delivered, along with the work, to the 'hot' reservoir. The coefficient of performance, K, of a refrigerator is the (negative) ratio of the heat extracted from the cold source to the mechanical work needed to do this.

$$K = -\frac{Q_C}{W} = -\frac{Q_C}{Q_H + Q_C}$$

The coefficient of performance for a heat pump is

$$K' = \frac{Q_H}{W}$$

These coefficients are always positive (as Q_H and W are negative for the refrigerator) but can easily be greater than unity. From the K and e definitions, it is clear that they are related. In fact

$$K = \frac{1 - e}{e}$$

also

$$K' = K + 1$$

so that an inefficient Carnot heat engine, run backwards as a heat pump or refrigerator, has a high performance coefficient.

The second law of thermodynamics is stated in the following way: 'It is impossible for any system to undergo a process in which it absorbs heat from a reservoir at a single temperature and converts it completely into mechanical work, while ending in the same state in which it began.' There are several alternative statements but it has been shown that they are all equivalent. The second law places stringent limitations on physical processes and tells us that there are some things that we cannot do even though they are allowed by the conservation of energy, momentum, angular momentum, etc.

Closely connected with any discussion of the second law is the Carnot cycle in which one uses an ideal gas as a working substance and considers two adiabatic processes and two isothermal processes. We used this cycle in Chapter 18, Example 5. Since the irreversible process of heat flow due to a temperature gradient is avoided by letting heat enter or leave the system only during isothermal processes, this cycle has the highest efficiency of any thermodynamic cycle. We can use this fact to set limits of performance on real heat engines (which can never be as efficient as the idealized Carnot engines). See Examples 3 and 4.

It is important to remember that

$$e = 1 - \frac{T_C}{T_H}$$

is the efficiency of a Carnot engine where T_C and T_H are the absolute temperatures of the cold and hot reservoirs respectively. Clearly

$$K = \frac{T_C}{T_H - T_C} .$$

This last expression for the Carnot efficiency forms the basis of the absolute temperature scale. This scale is identical, for all practical purposes, to the ideal gas temperature scale but in fact is independent of the nature of the working substance.

Numerous examples of irreversible processes, processes that can proceed only in one direction in nature, are given in the text. All processes involving friction are irreversible (but there are many others as well). The reverse process to a sliding block coming to rest on a rough surface would be that of a block initially at rest on this surface to gain kinetic energy (and speed) by extracting heat energy from the surface thereby cooling it. The opposite of an irreversible process is a reversible one (the direction being unimportant) and such a process can be thought of as the sum of a series of infinitesimal processes that take the system from one equilibrium state to another nearby equilibrium state. Irreversible processes start and end at equilibrium states, but in between, the system is out of equilibrium.

It has been pointed out that work and heat are not simply functions of the thermodynamic coordinates, as internal energy is, but depend on the actual path taken between the initial and final states. The entropy, S, of a system is like the internal energy in that the entropy change in a given process depends only on the coordinates of the initial and final states and not on the path. For a reversible process the infinitesimal change in entropy can be defined as:

$$TdS = dQ$$

For a reversible process in which the system goes from state 1 to state 2, the entropy change, ΔS, is given by:

$$\Delta S = \int_1^2 dQ/T$$

Any path connecting 1 and 2 may be used to evaluate the integral. For an ideal gas the entropy is obtained using this definition in Example 5 and the entropy change occurring during a reversible isothermal expansion is calculated in Example 6.

As many of the processes we wish to study are irreversible ones, we need a technique for finding the entropy changes occurring there as our previous formula allows us to find ΔS only for a reversible process. If a system goes from state 1 to a state 2 by means of an irreversible process, we can still find the entropy change (since it depends only on the coordinates of the initial and final states) by inventing a reversible process that connects states 1 and 2 and then calculating the previous integral. This is illustrated in Example 7.

The critical statement in the text regarding entropy and its connection with the second law of thermodynamics is: 'When all systems taking part in a process are included, the entropy either remains constant or increases.' No process is possible in which the entropy of the universe decreases. The arrow representing the direction of time (from past to present to future) is oriented by the existence of irreversible processes.

PROBLEM-SOLVING STRATEGIES

For cycles, the change in internal energy is zero. Nevertheless it is easier to calculate the work done in an adiabatic process by calculating the change in internal energy <u>for that process</u> and then using the first law of thermodynamics. This approach will also work for constant volume processes where ΔQ can be obtained from a calculation of ΔU.

For irreversible processes the entropy change ΔS can be calculated by replacing the actual process by a reversible process connecting the initial and final states.

EXAMPLES AND SOLUTIONS

Example 1

Calculate the efficiency of a heat engine using an ideal monatomic gas as the working substance, following the cycle shown in Fig. 19-1. Take $\gamma = 1.4$ and let the ratio (r) of V_b to V_a be 2.5.

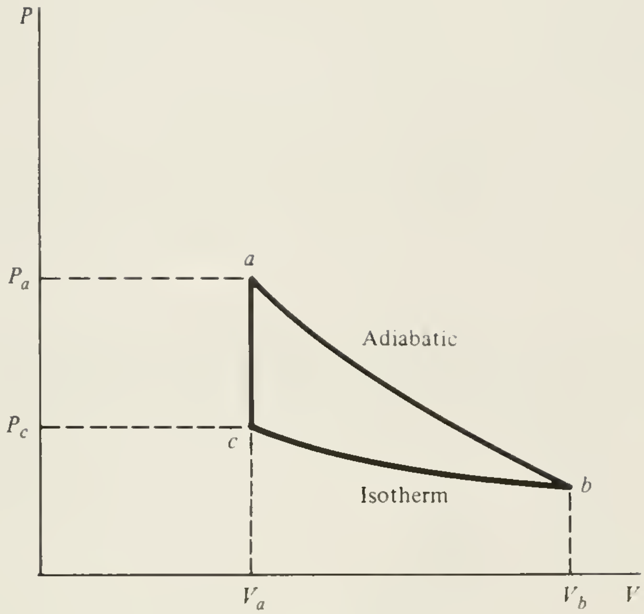


Figure 19-1

Solution:

No heat is taken in or rejected along ab. Since $\Delta U = 0$ along bc, we have

$$Q_C = W_{bc} = nRT_c \ln(V_a/V_b)$$

Since $V_a < V_b$ both Q_C and W_{bc} are negative. Along ca, the heat taken in is:

$$Q_H = nC_v(T_a - T_c).$$

Q_H is positive here since $T_a > T_c$.

There is no net change in U for this cyclic process so the net work done by the gas is equal to the sum of the above two terms. Along ab, we have

$$T_aV_a{}^{\gamma-1} = T_cV_b{}^{\gamma-1}$$

so

$$T_a = T_c(V_b/V_a)^{\gamma-1}.$$

If the ratio of the two volumes is given as $r = V_b/V_a$, the efficiency can be written in terms of r.

$$e = 1+ \frac{Q_C}{Q_H}$$

$$e = 1 - \frac{(\gamma-1)\ln r}{(r^{\gamma-1}-1)} = 0.172$$

Increasing the ratio r will increase the efficiency.

Example 2

Calculate the efficiency of the Diesel cycle shown in Fig. 19-3 of the text, and below in Fig. 19-2, in terms of the temperatures T_a, T_b, T_c, and T_d.

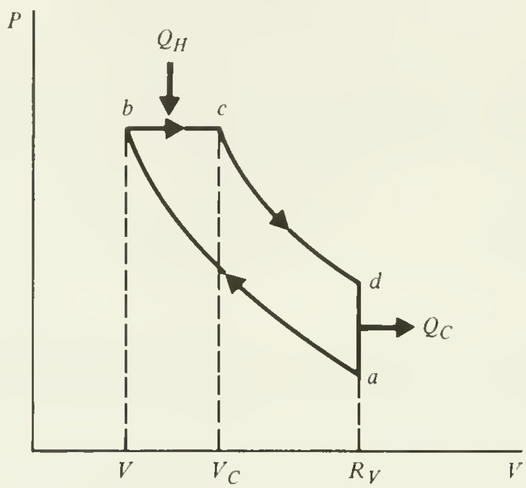


Figure 19-2

Solution:

R is the compression ratio and $E = \dfrac{RV}{V_c}$ is the expansion ratio.

Although these two ratios are the same in the internal combustion cycle, they are not in the Diesel cycle. Processes ab and cd are adiabatics so we have

$$Q_H = nC_p(T_c - T_b) > 0$$

and

$$Q_C = nC_v(T_a - T_d) < 0.$$

As the two processes where heat enters or leaves are constant pressure and constant volume processes respectively, the efficiency, e, is given by

$$e = \frac{Q_H + Q_C}{Q_H} = 1 + \frac{Q_C}{Q_H}$$

$$= 1 - \frac{T_d - T_a}{\gamma(T_c - T_b)}$$

since $C_p/C_v = \gamma$. This is the complete answer but it can also be written in terms of the expansion and compression ratios, as will be seen in the next example.

Example 3

Evaluate the efficiency of the Diesel cycle of the previous example using R = 18, E = 6, and γ = 1.4.

Solution:

Using the relationships for adiabatics, we have

(a) $$\frac{V}{T_b} = \frac{V_c}{T_c} \quad \text{and } E = RV/V_c \quad \text{so}$$

$$T_c = T_b \frac{R}{E}$$

(b) Along the adiabatic,

$$T_b V^{\gamma-1} = T_a (RV)^{\gamma-1}.$$

Thus

$$T_b = T_a R^{\gamma-1} \text{and } T_c = T_a R^{\gamma-1} \frac{R}{E}$$

(c) Finally along the adiabatic cd, we have

$$T_c V_c{}^{\gamma-1} = T_d (RV)^{\gamma-1}.$$

Thus

$$T_c = T_d E^{\gamma-1}$$

and we can write

$$T_d = E^{1-\gamma} R^{\gamma} E^{-1} T_a = \left(\frac{R}{E}\right)^{\gamma} T_a.$$

Now all temperatures are expressed in terms of T_a so the efficiency is:

$$e = 1 - \frac{1}{\gamma}\frac{[(R/E)^{\gamma} - 1]}{[R^{\gamma}/E - R^{\gamma}/R]}$$

This can also be written as:

$$e = 1 - \frac{1}{\gamma}\;\frac{[(1/E)^{\gamma} - (1/R)^{\gamma}]}{(1/E - 1/R)}$$

$$= 1 - \frac{1}{1.4}\;\frac{[(1/6)^{1.4} - (1/18)^{1.4}]}{[(1/6) - (1/18)]} = 0.59.$$

Example 4

The mean temperature of deep ground water is 5°C whereas the mean house temperature is about 20°C.

(a) What is the efficiency of a Carnot engine operating between these two temperatures?

(b) What is the coefficient of performance of a Carnot refrigerator operating between these two temperatures?

(c) What is the coefficient of performance of a Carnot heat pump here?

Solution:

(a) The efficiency is given in terms of Kelvin temperatures so the above Celsius temperatures must be converted.

$$e = 1 - \frac{T_C}{T_H} = 1 - \frac{278}{293}$$

$$e = 0.051 \qquad \text{or} \qquad 5.1\%$$

(b) The coefficient of performance, K, is given by:

$$K = \frac{T_C}{T_H - T_C} = \frac{278}{293 - 278} = 18.5$$

(c) The coefficient of performance for the heat pump is:

$$K' = K + 1 = 18.5 + 1 = 19.5$$

This device is practical for heating a house in winter.

Example 5

Calculate the entropy, S, for a system that obeys the ideal gas law.

Solution:

For an ideal gas, the internal energy depends on the temperature alone, so:

$$dU = nC_V dT$$

$$dW = pdV = \frac{nRT}{V} V$$

Since the entropy is related to dQ through TdS = dQ (for a reversible process only) the first law yields:

$$dQ = dU + dW$$

$$TdS = nC_V\ dT + nRT\ \frac{dV}{V}$$

Dividing by T and treating n, C_V, and R as constants,

$$dS = d(nC_V \ln T + nR \ln V + \text{const.})$$

We can regard the entropy as the quantity in the bracket above. It can be put in a more useful form by noting that

$$\frac{R}{C_V} = \gamma - 1$$

Then we have

$$S = nC_V(\ln T + (\gamma - 1) \ln V) + \text{const.}$$

$$= nC_V \ln (TV^{\gamma-1}) + \text{const.}$$

In an adiabatic process, where the entropy remains constant, it is apparent from this form that $TV^{\gamma-1}$ must remain constant.

Example 6

Calculate the entropy change of an ideal gas during an isothermal expansion from an initial volume of V to a final volume of 2V.

Solution:

For the ideal gas dU = 0 in an isothermal process, so we have

$$T\ dS = dQ = 0 + dW$$

$$dW = pdV = \frac{nRT}{V}\ V$$

which yields

$$dS = nR\ \frac{dV}{V}$$

Integrating from the initial volume to the final volume, we have

$$\Delta S = nR\ \ln 2V - nR\ \ln V = nR\ \ln 2$$

If we use the expression for the ideal gas entropy,

$$S = n\ C_v\ \ln TV^{\gamma-1} + K,$$

then,

$$S_f - S_i = \Delta S = nC_v\ \ln T(2V)^{\gamma-1} - nC_v\ \ln TV^{\gamma-1}$$

$$\Delta S = nC_v\ \ln T + (\gamma - 1)nC_v\ \ln 2V - nC_v\ \ln T - (\gamma - 1)nC_v\ \ln V$$

$$= n\ R\ \ln 2$$

Example 7

Equal values of water at 80°C and 20°C are mixed together. Calculate the increase in entropy for a total volume of 1 m^3.

Solution:

Assuming the heat capacities are all constant and independent of temperature, the final temperature is midway between the above two temperatures: the heat lost by one volume is gained by the other. To calculate the entropy change in this irreversible process, we must replace the actual process by a fictitious reversible one that still connects the initial and final states. In this case, let the volume remain fixed and find the heat taken in (or rejected) in a process that would take the system from the initial temperature to the final temperature. Since $dQ = C_v dT = TdS$ for this process,

$$\Delta S = C_v \int_{T_i}^{T_f} \frac{dT}{T} = C_v\ \ln(T_f/T_i)$$

For the 'hot' water, T_f = 333 K while T_i = 353 K whereas for the 'cold' water, T_f = 333 K and T_i = 293 K. Numerically

$$\frac{\Delta S}{C_v} = 2 \ln 333 - (\ln 353 + \ln 293) = 0.0696$$

For C_v we can take 4.18×10^6 $J \cdot m^{-3} \cdot C^{-1}$. Since the volume is 1 m^3, the final result for ΔS is

$$\Delta S = 2.91 \times 10^5 \ J(C^{o})^{-1}.$$

QUIZ

1. For a Carnot engine with operating temperatures of 800 K and 300 K, (a) calculate the efficiency. (b) To improve the efficiency, it is decided to either raise the higher temperature by 10 K or to lower the lower temperature by 10 K. Which would result in the largest improvement?

Answer: (a) e = 0.625 (b) Reducing the lower temperature by 10 K gives e = 0.6375 whereas raising the higher temperature by 10 K only gives e = 0.6296.

2. In a thermally isolated container, 60 g of water at 80 C is mixed with 40 g of water at 20 C. Assume the specific heat of water is constant. (a) Calculate the final temperature of the mixture and (b) the entropy change of the system.

Answer: (a) T_f = 56 C or 329 K, (b) ΔS = 1.72 $J \cdot K^{-1}$.

3. An ideal gas with $\gamma = 5/3$ is used for a thermodynamic cycle that starts at point a where p,V, and T are p_o,V_o, and T_o. The gas expands to point b along a straight line path until both the pressure and volume are double that at point a. Next the gas goes to point c, where the pressure is p_o along a constant volume path. Finally the gas returns to point a along a constant pressure path. The figure is a triangle in the pV plane. Calculate the thermodynamic efficiency of this cycle.

Answer: e = (1/12).

4. The pressures, volumes, and temperatures of three points in the pV plane are: p_a = 3 atm, V_a = 6 L, T_a = 500 K, p_b = 2 atm, V_b = 9 L, T_b = T_a, p_c = 1.53 atm, V_c = V_b, T_c = 381.6 K. A thermodynamic cycle of an ideal gas (γ = 5/3) starts at point a and proceeds to b by an isothermal process. From b to c, a constant volume process is followed. From c to a, the process is adiabatic. (a) Calculate the efficiency of this cycle and (b) that of a Carnot engine operating between 500 K and 381.6 K.

Answer: (a) e = 0.124, (b) e_c = 0.237.

20
MOLECULAR PROPERTIES OF MATTER

OBJECTIVES

In this chapter a microscopic model, the kinetic theory of gases, is used to derive the some of the observed macroscopic properties of ideal gases. Your objectives are to:

Recognize that ordinary matter is composed of enormous number of molecules and that the observed macroscopic properties are average properties.

Calculate the mean square speed of gas molecules as a function of temperature and the mass of the molecule.

Calculate the observable properties such as molar heat capacity at constant volume and the ratio C_p/C_v for monatomic (and other gases) using kinetic theory.

REVIEW

The atomic nature of matter and the formation of molecules from atoms are well established. The attractive forces between electrically neutral molecules lead to the clustering of molecules in the liquid state and then in the solid state as the mean separation between particles gets smaller. These attractive forces are electrical in their origin and arise from the fact that while the average charge at a point in a molecule is zero, its instantaneous value is non-zero. This leads to distortions of the charge clouds of molecules and non-zero forces. In Example 1, the average spacing between molecules in a gas, liquid, and solid is calculated.

One very important observation concerning the atomic nature of matter is that one mole of a pure substance contains a fixed number of identical molecules. This number is called Avagadro's number, N_A, and is equal to 6.022×10^{23} mol^{-1}. This fact coupled with the knowledge that 1 mole of an ideal gas occupies 22.4 liters at standard temperature and pressure (S.T.P.) enables you to calculate the number of molecules present in an ideal gas under any conditions of temperature and pressure. See Example 2.

The kinetic theory of an ideal gas was an early attempt to calculate the observed macroscopic properties of gases from a simple microscopic model. The usual approach (as in the text) is to find the pressure in a dilute (non-interacting) gas due to the molecules colliding with the walls. This pressure is equal to the time average force per unit area on the selected wall due to one molecule multiplied by the number of molecules in the volume of interest. In Example 3 the time average force is shown to be equal to the average momentum change per collision with the wall divided by the mean time between collisions (with the same wall). If the wall (of area L_yL_z) lies in the yz plane, the change in momentum per collision is $2mV_x$ as the y and z components are unaffected. Furthermore the mean time between collisions (with the same wall) is $2L_x/V_x$. Since the pressure, p_1, equals force per unit area,

$$p_1 = \frac{F}{L_y\ L_z}$$

$$= \frac{2\ mv_x}{(\ L_y\ L_z)(2\ L_x\ /\ v_x)}$$

$$= \frac{mv_x{}^2}{V}$$

where $L_xL_yL_z = V$. This is the partial pressure due to one molecule. Since the effects due to the other molecules (N in number) are additive, the total pressure p is:

$$p = \frac{Nmv_x{}^2}{V}$$

On average $v_x{}^2 = v_y{}^2 = v_z{}^2$, so $v_x{}^2$ is replaced by

$$\frac{1}{3}\ v^2.$$

Since the average kinetic energy per molecule is $mv^2/2$, this result can be written:

$$pV = \frac{2}{3}\ (\ \frac{1}{2}\ mv^2)$$

This is the end of the road for this model. All other results are obtained by appealing to the ideal gas law. In particular these key results are:

(a) The internal energy of this monatomic gas is,

$$U = \frac{3}{2}\ nRT$$

(b) The average kinetic energy per molecule depends only on T,

$$1/2\ mv^2 = \frac{3}{2} T \frac{R}{N_A} = \frac{3}{2} kT$$

This last result is used then to calculate the root-mean-square velocity for an ideal gas with molecules of mass m. See Examples 4, 5, and 6.

The first result can be used to calculate the molar heat capacity of an ideal gas. Note that U depends only on the temperature and $\Delta U = 3/2\ R\ \Delta T$. At constant volume $\Delta Q = \Delta U$ from the first law so for one mole, $Cv = 3/2\ R$. Furthermore, since $Cp = Cv + R$, we have

$$C_p = \frac{5}{2} R \text{ and } \gamma = \frac{C_p}{C_v} = \frac{5}{3} = 1.67.$$

This is the high point for the simple kinetic theory presented here, as all these results agree well with experimental values. The extension of this approach to more complicated molecules is more complex and is done by appealing to the principle known as 'the equipartition of energy'.

In its most general form, this principle assigns a 'degree of freedom', f, to each term in the total energy (sum of kinetic plus potential) which is a pure squared term in a coordinate or momentum. Since both the coordinates and momenta are as often negative as positive, the average over a long time of either one will be zero but the average of the square will be non-zero. For each such degree of freedom there is associated a thermal energy of kT/2. Based on this approach, a gas of N molecules, each with f degrees of freedom, has an internal energy of

$$U = \frac{f}{2} NkT$$

and a molar heat capacity at constant volume of

$$Cv = \frac{dU}{dT} = \frac{f}{2} Nk = \frac{f}{2} R$$

(since $N_A k = R$). While this approach is clear enough, it requires some help from a more advanced theory, quantum mechanics, in order to give satisfactory agreement with experiment. For a monatomic gas, we obtain f = 3 so U = 3nRT/2 as required. To agree with room temperature measurements on <u>diatomic gases</u> (H_2, N_2, O_2, etc.) a value of f = 5 is needed. Diatomic gases actually have 8 degrees of freedom, not 5. This problem was resolved with the discovery of Quantum Mechanics, which modifies the equipartition principle.

EXAMPLES AND SOLUTIONS

Example 1

Find the average spacing between H_2 molecules, assuming the molecules are at the vertices of a fictitious cubic structure, in:
(a) gaseous H_2 at S.T.P.
(b) liquid H_2 at 20K where the density is 41060 mole$\cdot$m^{-3}
(c) solid H_2 at 4.2K where the molar volume is $22.91 x 10^{-6} m^3 \cdot mole^{-1}$.

Solution:

(a) We assign a volume of a^3 to each molecule. This is extremely crude but useful in producing order-of-magnitude estimates. Since one mole, N_A molecules, occupies 22.4 l at S.T.P., we write:

$$N_A\ a^3 = 22.4\ l \cdot mol^{-1} = 2.24\ x\ 10^{-3}\ m^3 \cdot mol^{-1}$$

Substituting N_A, we have

$$a^3 = 3.72\ x\ 10^{-26}\ m^3$$

$$a = 3.34\ x\ 10^{-9}\ m$$

(b) The density is specified in moles per cubic meter so one cubic meter contains 41060 moles of liquid H_2. Letting a^3 be the volume per molecule, we have

$$1\ m^3 = 41060\ N_A a^3$$

$$N_A a^3 = 2.44\ x\ 10^{-5}\ m^3 \cdot mol^{-1}$$

$$a = 3.43\ x\ 10^{-10}\ m$$

(c) Here the molar volume is specified, yielding

$$N_A\ a^3 = 22.91\ x\ 10^{-6}\ m^3 \cdot mol^{-1}$$

$$a^3 = 3.80\ x\ 10^{-29}\ m^3$$

$$a = 3.36\ x\ 10^{-10}\ m$$

Thus the average spacing between molecules in the liquid and solid is about the same but this spacing in the condensed states is about a factor of ten smaller than the spacing in the gas at S.T.P.

Example 2

A common type of laboratory vacuum pump produces an ultimate (lowest) pressure of 10 microns. How many molecules (how many moles) of gas would be present in a volume of 0.15 m^3 reduced to that pressure at 300 K?

Solution:

Here the pressure unit is tricky as it is based on the fact that atmospheric pressure is 76 cm of mercury (Hg). Thus 1 atm = 0.76m (of Hg). The unit, micron (μm), is one millionth of a meter of mercury so:

$$10\ \mu m = 10 \times 10^{-6}\ m\ (1\ atm/0.76\ m) = 1.32 \times 10^{-5}\ atm$$

The volume 0.15 m^3 is equal to 1.5×10^2 l. In these units we can write the molar gas constant, R, as:

$$R = \frac{(1\ atm)(22.4\ l)}{(273K)(1\ mole)}$$

Since the number of moles, n, is given by n = pV/RT we have:

$$n = \frac{(1.32 \times 10^{-5}\ atm)(1.5 \times 10^2\ l)(273K)(1\ mole)}{(1\ atm)(22.4\ l)(300K)}$$

$$= 8.04 \times 10^{-5}\ mole$$

There are 6.02×10^{23} molecules per mole so there are still 4.84×10^{19} molecules left in this 'evacuated' volume.

Example 3

Using the impulse-momentum theorem, show that the time average force is equal to the ratio of the average momentum change per collision to the mean time between collisions.

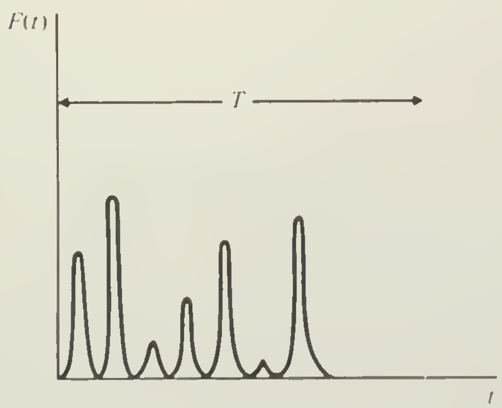


Figure 20-1

Solution:

The time average force, F, is given by:

$$F = \frac{1}{T} \int_0^T F(t)dt$$

where the integral gives us the total area under the F(t) spikes (see Fig. 20-1) without looking at the details of each collision. The impulse, I, is equal to the total change in momentum, and also given by the above integral,

$$I = \int_0^T F(t)dt = \Delta p_T$$

Thus $FT = \Delta p_T$. The total change in momentum is equal to the average change in momentum in one collision (Δp) multiplied by the number of collisions, N. Thus

$$F = \Delta p\left(\frac{N}{T}\right) = \frac{\Delta p}{T/N}$$

where T/N is the mean time between collisions.

Example 4

In a gas at 300 K that is a mixture of the diatomic molecules H_2 ($M = 2 \times 10^{-3}$ $kg \cdot mol^{-1}$) and D_2 ($M = 4 \times 10^{-3}$ $kg \cdot mol^{-1}$) calculate the root-mean-square velocities of the molecules.

Solution:

The root-mean-square velocity is given by:

$$v_{rms} = \left[\frac{3RT}{M}\right]^{1/2}$$

For H_2 (regular hydrogen)

$$v_{rms} = \left[\frac{(3)(8.314\ J \cdot mol^{-1} \cdot K^{-1})(300\ K)}{2 \times 10^{-3}\ kg \cdot mol^{-1}}\right]^{1/2}$$

$$= 1.93 \times 10^3\ m \cdot s^{-1}.$$

For D_2 (heavy hydrogen or deuterium)

$$v_{rms} = [\frac{(3)(8.314\ J\cdot mol^{-1}\cdot K^{-1})(300\ K)}{4 \times 10^{-3}\ kg\cdot mol^{-1}}]^{1/2}$$

$$= 1.37 \times 10^3\ m\cdot s^{-1}.$$

Because of the large relative difference in these speeds, it is relatively easy to separate the isotopes H_2 and D_2 by a process called effusion.

Example 5

A one liter flask at 293 K contains a mixture of 3 g of N_2 gas and 3 g of H_2 gas. Assuming the gases behave as ideal gases,
(a) calculate the partial pressures exerted by N_2 and H_2,
(b) calculate the root-mean-square speeds of N_2 and H_2.

Solution:

The mass of a mole of N_2 is 28 g while that of H_2 is 2 g.
(a) In the flask there are

$$n(N_2) = \frac{3}{28} \text{ moles of } N_2,$$

and

$$n(H_2) = \frac{3}{2} \text{ moles of } H_2.$$

Applying the ideal gas law to each component <u>separately</u>, the partial pressures $p(N_2)$ and $p(H_2)$ are:

$$p(N_2) = \frac{n(N_2)RT}{V}$$

$$= \frac{3}{28}\ \frac{(8.314\ J\cdot mol^{-1}\cdot K^{-1})(293\ K)}{10^{-3}\ m^3}$$

$$= 2.61 \times 10^5\ Pa \quad \text{(about 2 atm)}$$

$$p(H_2) = \frac{n(H_2)RT}{V}$$

$$= \frac{3}{2} \frac{(8.314\ \mathrm{J \cdot mol^{-1} \cdot K^{-1}})(293\ \mathrm{K})}{10^{-3}\ \mathrm{m^3}}$$

$$= 36.5 \times 10^5\ \mathrm{Pa} \quad \text{(about 36.5 atm)}$$

(b) The root-mean-square speeds are calculated from the expression

$$v_{rms} = [\frac{3RT}{M}]^{1/2}$$

For N_2,

$$v_{rms} = [\frac{(3)(8.314\ \mathrm{J \cdot mol^{-1} \cdot K^{-1}})(293\ \mathrm{K})}{28 \times 10^{-3}\ \mathrm{kg \cdot mol^{-1}}}]^{1/2}$$

$$= 5.11 \times 10^2\ \mathrm{m \cdot s^{-1}}.$$

For H_2,

$$v_{rms} = [\frac{(3)(8.314\ \mathrm{J \cdot mol^{-1} \cdot K^{-1}})(293\ \mathrm{K})}{2 \times 10^{-3}\ \mathrm{kg \cdot mol^{-1}}}]^{1/2}$$

$$= 1.91 \times 10^3\ \mathrm{m \cdot s^{-1}}.$$

Example 6

The element carbon has atomic masses of 9, 10, 11, 12, 13, 14, 15 and 16. Only three of these isotopes live long enough to study by thermal means in the laboratory: 12, 13 and 14. Calculate the root-mean-square velocity of carbon 12 at 300 K and compare it with the same quantity for carbon 9 and carbon 16.

Solution:

The expression for the speed, v_{rms}, is:

$$v_{rms} = \left(\frac{3kT}{m} \right)^{1/2}.$$

For carbon 12, the mass of an atom is 12 times that of a hydrogen atom so:

$$m = 12(1.673 \times 10^{-27}\ \mathrm{kg}) = 2.01 \times 10^{-26}\ \mathrm{kg}.$$

Thus we have

$$v_{rms} = \left[\frac{3(1.38 \times 10^{-23} \text{ J K}^{-1})(300 \text{ K})}{2.01 \times 10^{-26} \text{ kg}} \right]^{1/2}$$

$$v_{rms} = 7.87 \times 10^2 \text{ m}\cdot\text{s}^{-1}$$

For carbon 16, this speed is lower by the factor $(12/16)^{1/2}$ or

$$v_{rms} = 6.81 \times 10^2 \text{ m}\cdot\text{s}^{-1}$$

For the isotope with mass 9, the speed is higher than that of carbon 12 by the factor $(12/9)^{1/2}$ giving:

$$v_{rms} = 9.08 \times 10^2 \text{ m}\cdot\text{s}^{-1}$$

Example 7

Calculate the 'escape' velocities at the surfaces of the earth and sun and determine whether any molecules have root-mean-square thermal velocities comparable to these escape velocities at 300 K.

Solution:

To just escape, we want the total energy at infinite separation to be zero. This condition determines the kinetic energy at the surface (of the earth or sun).

$$0 = \frac{1}{2} mv^2 - \frac{GMm}{R}$$

The formula for v is then:

$$v = \left(\frac{2GM}{R} \right)^{1/2}.$$

For the sun, $M = 2 \times 10^{30}$ kg and $R = 7 \times 10^8$ m. $G = 6.67 \times 10^{-11}$ $\text{N}\cdot\text{m}^2\cdot\text{kg}^{-2}$. Numerically, for the sun, we have

$$v_{sun} = 6.18 \times 10^5 \text{ m}\cdot\text{s}^{-1}$$

For the earth, we can use $GM = gR_E^2$ where $g = 9.8$ $\text{m}\cdot\text{s}^{-2}$ and $R_E = 6.4 \times 10^6$ m. Substituting these values into the general formula, we obtain:

$$v_{earth} = 1.12 \times 10^4 \text{ m}\cdot\text{s}^{-1}$$

In example 3 in the text, the thermal velocity, at 300 K, for an H_2 molecule is found to be 1.927×10^3 $\text{m}\cdot\text{s}^{-1}$. The thermal velocity of a hydrogen atom would be $(2)^{1/2}$ larger than this or 2.7×10^3 m·s^{-1}. This is less than the escape velocity on the earth and much less than the escape velocity on the sun. The sun's temperature is more like 6000 K so the thermal velocity of a hydrogen atom at 6000 K would be 1.2×10^4 m·s^{-1} which is still well below the escape velocity.

QUIZ

1. One mole of helium (M = 4 g per mole) at S.T.P. is in a cubic container. Calculate the mean time between collisions of a given molecule with the same wall.

Answer: $t = 7.48 \times 10^{-4}$ s

2. Assuming conduction electrons in a metal behave as ideal gas particles, calculate the mean thermal velocity of such an electron at 300 K and 4 K.

Answer: At 300 K, $v_{rms} = 1.17 \times 10^5$ m·s^{-1}
At 4 K, $v_{rms} = 1.35 \times 10^4$ m·s^{-1}.

3. A particle of mass 6×10^{-27} kg, traveling with a constant velocity of 1600 m·s^{-1} in the x direction (between collisions with the walls) strikes walls of cross-sectional area $A = 0.04$ m^2 at $x = 0$ and $x = 0.2$ m reversing its velocity without loss of energy. Ignoring gravity, calculate the following quantities: (a) the magnitude of the change in linear momentum per collision with the wall, (b) the mean time between collisions of one particle with the same wall, (c) the net average force exerted by this particle on one wall, and (d) the number of such particles required to produce a pressure of 10^5 N·m^{-2} on one wall.

Answer: (a) $\Delta p = 1.92 \times 10^{-23}$ kg·m·s^{-1}, (b) $\Delta t = 2.5 \times 10^{-4}$ s, (c) $F_1 = 7.68 \times 10^{-20}$ N, (d) $N = 5.21 \times 10^{22}$ molecules.

4. The internal energy of one mole of a particular ideal gas is given by

$$U = (5/2)RT + 3RT_o$$

where T_o is a constant. Calculate (a) the specific heat at constant volume for this gas and (b) the value of $\gamma = C_p/C_v$.

Answer: (a) $C_v = (5/2)R$, (b) $\gamma = (7/5) = 1.4$.

21
MECHANICAL WAVES

OBJECTIVES

In this chapter your objectives are to:

Classify the various types of wave motions.

Define frequency, period, and wavelength.

Describe mathematically simple harmonic traveling waves, and find the displacement, given position and time.

Recognize the form of the general wave equation and apply it to a variety of problems.

Calculate the velocity of propagation for waves in a few specific media.

REVIEW

In contrast to the random, chaotic motion studied in the last chapter, this chapter is concerned with the organized, cooperative motion of a large group of particles in a wave that propagates in space. This motion results from a non-equilibrium situation and the departure from equilibrium can be perpendicular to the direction of propagation: transverse waves (e.g. waves on a string and electromagnetic waves such as light, radio, T.V.); or the departure from equilibrium can be in the same direction as the wave propagation: longitudinal waves (e.g. sound waves). Water waves are also considered briefly in this chapter. They have both a transverse and a longitudinal character so they don't fit either classification.

The waves considered in this chapter are periodic waves (to be contrasted with pulses) such that the departure from equilibrium y repeats after a definite time interval τ, the period. Here y can represent a coordinate that gives the displacement of a particular segment of a string or it can represent a pressure variation in a gas. If we choose the x direction as the direction of propagation, then $y(x,t)$ can be called the wave function as it gives the departure from equilibrium, y, at each point x as a function of the time. If, for fixed x, the departure from equilibrium is a simple harmonic oscillation, then the wave function for a traveling wave in the x direction is of the form

$$y(x,t) = A \sin[2\pi(ft - x/\lambda) + \phi]$$

where A is the amplitude (maximum excursion from equilibrium) and must carry the same units as y; f is the frequency with $2\pi f = \omega$ the angular frequency; and λ is the wavelength with $2\pi/\lambda = k$ the wave number. The entire quantity contained inside the [ ] bracket is called the phase of the wave. The angle ϕ is called the phase constant. If $\phi = 0$ the oscillation follows the sine function, if $\phi = \pi/2$ we get a cosine. In general it allows us to adjust the form of the wave to fit the starting conditions of the problem. In terms of the wave vector and angular frequency, the wave function becomes:

$$y(x,t) = A \sin(\omega t - kx + \phi)$$

If we pick some particular value of the phase (e.g. if the phase is $\pi/2$ we have a crest of the wave but $3\pi/2$ produces a trough) then the speed at which that point of constant phase advances in the x direction is called the phase velocity, c. To find it, we simply take the phase in the above expression (ϕ is a constant), equate it to a constant, and calculate the first time derivative.

$$\frac{d}{dt}(\omega t - kx + \phi) = \frac{d}{dt}(\text{constant}) = 0$$

Then we have

$$c = \frac{dx}{dt} = \frac{\omega}{k} = \frac{2\pi f\lambda}{2\pi} = f\lambda$$

Had we written the phase as $\omega t + kx + \phi$, the above procedure would have given $c = -\omega/k$ which corresponds to a wave propagating in the − x direction. See Examples 1 and 2.

Thus at a given point in space the disturbance from equilibrium is a simple harmonic oscillation whereas (if ϕ is set equal to zero) at a given time the disturbance is a sine wave in space. This separation, which fixes x but allows time to vary in the first case and then fixes time while x is allowed to vary, is accomplished mathematically by means of the <u>partial derivative</u>.

For a wave propagating on a string in the x direction, the <u>particle velocity</u>, v, is actually in the y direction and calculated by taking the partial derivative of the wave function y(x,t) with respect to time. For instance:

$$v = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}[A \sin(\omega t - kx)] = \omega A \cos(\omega t - kx).$$

The maximum particle velocity is equal to ωA and is not the same as the propagation velocity of the wave, c.

By calculating the second partial derivatives with respect to x and t of the wave function in the harmonic form given, it can be shown that y(x,t) obeys the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \left(\frac{\partial^2 y}{\partial t^2} \right)$$

This form is very easy to remember if you note that the speed c must be paired with time t to give a quantity with the dimensions of x (i.e. ct is a length). Various solutions to this very important equation are explored in Examples 3 and 4.

The velocity of propagation c depends on specific parameters that characterize a particular problem. For each situation, this velocity must be calculated. Several such examples are given in the text, such as the speed of transverse waves on a string (see Example 5), sound waves in gas, or compressional waves in a metal (see Example 6). Although it is a challenging problem to calculate the velocity of propagation of a wave in a given medium, a thoughtful analysis based on the dimensions (units) of the pertinent physical parameters usually gives at least an approximate value for the propagation velocity.

Dimensionally we have seen (recall the formula for kinetic energy) that an energy divided by a mass is a velocity squared. For a string of mass M and length L, the only other physical quantity that is used to characterize the string is its tension, S (a force). The tension divided by the mass per unit length (μ = M/L) has the same units as energy (force times distance) divided by mass, that is a velocity squared. As shown in the text, the velocity of propagation of a wave on a string is equal to the square root of the ratio of the tension to the mass per unit length.

$$c = \left(\frac{SL}{M} \right)^{1/2} = \left(\frac{S}{\mu} \right)^{1/2}$$

For a gas, the parameters that characterize its state, T, p, V, and the density ρ are not all independent but are connected by an equation of state. Since RT is dimensionally an energy, the ratio RT/M where M is the mass per mole is a velocity squared. Also the ratio of pressure to density has the dimensions of a velocity squared. For an ideal gas, this ratio is identical to RT/M. Because the sound oscillations are fast compared to the times necessary to reach local thermal equilibrium in a gas, the compressions and rarefactions are adiabatic--not isothermal--so sound in an ideal gas has a velocity of propagation

$$c = \left[\gamma\left(\frac{p}{\rho} \right)\right]^{1/2} = \left[\gamma\left(\frac{RT}{M} \right)\right]^{1/2}$$

where $\gamma = C_p/C_v$.

EXAMPLES AND SOLUTIONS

Example 1

Given that a transverse wave disturbance is of the form

$$y = (0.015\ \text{m})\ \sin 2\pi\left(\frac{t}{10^{-3}\ \text{s}} - \frac{x}{1.2\ \text{m}}\right)$$

where t is expressed in seconds and x in meters, identify the following quantities:

(a) amplitude
(b) wavelength
(c) frequency
(d) speed of propagation
(e) direction of propagation

Solution:

We compare the above form with the standard form,

$$y = A\ \sin\left(2\pi ft - \frac{2\pi x}{\lambda}\right)$$

and see that

(a) the amplitude is $A = 0.015$ m

(b) the wavelength is $\lambda = 1.2$ m

(c) the frequency is $f = (10^{-3}\ \text{s})^{-1} = 1000\ \text{s}^{-1}$

(d) the velocity of propagation is

$$v = \lambda f = (1.2\ \text{m})(1000\ \text{s}^{-1}) = 1.2 \times 10^{3}\ \text{m}\cdot\text{s}^{-1}$$

(e) the direction of propagation is the + x direction

Example 2

Write the equation $y(x,t)$ describing a traveling transverse wave that propagates in the + x direction and satisfies the following conditions:

(a) The maximum disturbance from equilibrium at any point is 1 cm.
(b) The wavelength is 2 m.
(c) The period is 0.02 s.
(d) At $t = 0$ and $x = 0.5$ m, the instantaneous particle velocity is $\pi/2$ $\text{m}\cdot\text{s}^{-1}$ down (or negative).

Solution:

The form to be used is

$$y = A \sin [2\pi ft - kx + \phi]$$

where the minus sign was chosen to make the wave go in the + x direction. Rewritten in terms of the period and wavelength, the displacement is

$$y = A \sin [2\pi(\frac{t}{\tau} - \frac{x}{\lambda}) + \phi]$$

Since τ and λ are given in the problem, only ϕ must be found. The amplitude A is equal to the maximum disturbance from equilibrium or 0.01 m. To determine ϕ, we must calculate the instantaneous particle velocity.

$$v_y = \frac{\partial y}{\partial t}(x,t) = \frac{2\pi}{\tau} A \cos [2\pi(\frac{t}{\tau} - \frac{x}{\lambda}) + \phi]$$

This expression must be evaluated at t = 0 and x = 0.5 m.

$$v_y(0.5\text{m},0) = \frac{2\pi A}{\tau} \cos[- \frac{\pi}{2} + \phi] = \frac{2\pi A}{\tau} \sin \phi$$

$$- \pi/2 = \pi \sin \phi$$

$$\sin \phi = - (1/2)$$

$$\phi = - 30^\circ$$

$$= - \pi/6 \text{ radians}$$

The final result is:

$$y(x,t) = (0.01 \text{ m}) \sin 2\pi[(50 \text{ s}^{-1})t - (0.5 \text{ m}^{-1})x - (1/12)]$$

Example 3

Calculate $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ for

(a) $y = A(x - vt)^n$ where A is a constant and $n > 1$
(b) $y = A \exp(-kx)\sin \omega t$ where A, k, and ω are constants

Solution:

(a) To calculate the partial derivative of y with respect to x, treat A, n, v, and t as constants. Thus

$$\frac{\partial y}{\partial x} = nA(x - vt)^{n-1}$$

and

$$\frac{\partial}{\partial x}\frac{\partial y}{\partial x} = \frac{\partial^2 y}{\partial x^2} = n(n - 1)A(x - vt)^{n-2}.$$

To calculate the partial derivative of y with respect to t, treat everything except t as a constant.

$$\frac{\partial y}{\partial t} = nA(x - vt)^{n-1}\ (-v)$$

$$\frac{\partial}{\partial t}\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial t^2} = n(n - 1)A\ (x - vt)^{n-2}(-v)^2.$$

In this case note that

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

so that this function <u>is</u> an acceptable solution of the wave equation.

(b) For the function $y = A \exp(-kx) \sin \omega t$, the partial derivative with respect to x is taken by treating all other variables (such as t) as constants.

$$\frac{\partial y}{\partial x} = (-k)\ A \exp(-kx) \sin \omega t$$

$$\frac{\partial^2 y}{\partial x^2} = k^2\ A \exp(-kx) \sin \omega t$$

For the partial derivative with respect to t, treat x as constant.

$$\frac{\partial y}{\partial t} = A \exp(-kx) \cos \omega t$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A \exp(-kx) \sin \omega t$$

$$-\frac{1}{\omega^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{k^2}\frac{\partial^2 y}{\partial x^2}$$

The minus sign is critical here and we do not have a function that obeys the wave equation.

Example 4

A rope under tension provided by a hanging mass of 20 kg has a velocity of wave propagation of 30 $m \cdot s^{-1}$. If the velocity is to be the same, what mass should be hung from a rope of the same material with

(a) half the diameter of the original rope?

(b) twice the diameter of the original rope?

Solution:

The velocity of propagation v is given by

$$v^2 = \frac{S}{\mu}$$

The tension S is provided by the hanging mass M, where we have

$$S = Mg$$

The mass per unit length, μ, is equal to

$$\mu = \frac{m}{L} = \frac{\rho AL}{L} = \rho A$$

where ρ is the density and is the same for all the ropes in this problem. To make v a constant, we must have M/A equal to a constant.

(a) If the diameter is halved, the area decreases by a factor 4 so M = 5 kg.

(b) If the diameter is doubled, the area increases by a factor 4 so M = 80 kg.

Example 5

Two strings of mass per unit length μ_1 and μ_2 are joined together at point A. The tension in the two strings is the same. If the wavelength in the string with $\mu_1 = 0.005\ kg \cdot m^{-1}$ is 0.03 m, what is the mass per unit length, μ_2, if the wavelength in that string is .05 m? Assume the frequencies in the two strings are the same.

Solution:

The velocity of propagation, c, is given by

$$c = \left(\frac{S}{\mu} \right)^{1/2}$$

where S is the tension and μ is the mass per unit length. Since the tension is the same in the two strings, squaring the above equation gives:

$$c_1^2 \mu_1 = c_2^2 \mu_2.$$

The speeds are not given but the frequency is the same in each string and the wavelengths are known ($c_1 = \lambda_1 f$, $c_2 = \lambda_2 f$) so we have

$$\lambda_1^2 f^2 \mu_1 = \lambda_2^2 f^2 \mu_2.$$

leading to

$$\frac{\mu_2}{\mu_1} = \left(\frac{\lambda_1}{\lambda_2} \right)^2 = \left(\frac{.03 \text{ m}}{.05 \text{ m}} \right)^2 = 0.36.$$

and yielding

$$\mu_2 = 1.8 \times 10^{-3} \text{ kg} \cdot \text{m}^{-1}.$$

Example 6

Suppose a wave motion is set up in copper (a typical metal), air, and deep water (where the depth $h >> \lambda$) with a wavelength of 1 m in each case. What would be the approximate frequency of each of these waves?

Solution:

(a) For copper the Young's modulus Y is 1.1×10^{11} $N \cdot m^{-1}$ and the density ρ is 8.9×10^3 $kg \cdot m^{-3}$. The velocity of propagation, c, is:

$$c = \left(\frac{Y}{\rho} \right)^{1/2} = \left(\frac{1.1 \times 10^{11}}{8.9 \times 10^3} \right)^{1/2} = 3.52 \times 10^3 \text{m} \cdot \text{s}^{-1}$$

This corresponds to a frequency of 3.52 Khz.

(b) For air at S.T.P., treating it as an ideal diatomic gas ($\gamma = 1.4$),

$$c = \left(\frac{\gamma RT}{M} \right)^{1/2}, \text{ where M = mass of a mole.}$$

If we take 29×10^{-3} kg = M as a weighted average of the N_2 and O_2 constituents of air, then:

$$c = \left(\frac{1.4 \times 8.31 \times 273}{29 \times 10^{-3}} \right)^{1/2} = 331 \text{ m}\cdot\text{s}^{-1}.$$

This corresponds to a freqnency of 331 Hz.

(c) For water waves the situation is more complex. The speed of a water wave is:

$$c^2 = \left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda} \right) \tanh\left(\frac{2\pi h}{\lambda} \right).$$

Since $h \gg \lambda$, the tanh reduces to unity, so we only have to evaluate the first two terms as:

$$c^2 \sim \left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda} \right).$$

For λ = 1 m, $g\lambda/2\pi = 1.56 \text{ m}^2\cdot\text{s}^{-2}$. The surface tension γ (see Table 12-2) at 20 C is equal to 7.28×10^{-2} $\text{N}\cdot\text{m}^{-1}$ and $\rho = 10^3$ $\text{kg}\cdot\text{m}^{-3}$.

$$\frac{2\pi\gamma}{\rho\lambda} = 4.57 \times 10^{-4} \text{ m}^2\cdot\text{s}^{-2}$$

This is negligible compared to the first term so:

$$c = 1.25 \text{ m}\cdot\text{s}^{-1}.$$

The corresponding frequency is 1.25 Hz.

Thus for the same wavelength, the velocities of propagation in a metal, a gas, and water are so different that the frequencies span three orders of magnitude.

QUIZ

1. Given that the transverse wave disturbance is of the form

$$y(x,t) = (0.025\ \text{m}) \cos 2\pi\left(\frac{t}{10^{-2}\ \text{s}} + \frac{x}{1.5\ \text{m}}\right)$$

where t is in seconds and x is in meters, identify or calculate the following quantities:
- (a) the amplitude
- (b) the wavelength
- (c) the frequency
- (d) the velocity of propagation
- (e) the direction of propagation.

Answer: (a) 0.025 m,
(b) 1.5 m,
(c) 100 s^{-1},
(d) 150 $m \cdot s^{-1}$,
(e) $-$ x direction.

2. Write an equation that correctly describes a traveling wave disturbance y(x,t) propagating in the + x direction that satisfies the following conditions:
- (a) The maximum disturbance from equilibrium is 0.02 m.
- (b) The wave velocity is 50 $m \cdot s^{-1}$.
- (c) The frequency is 80 Hz.
- (d) At t = 0 and x = 0, y(0,0) = $-$ 0.02 m.

Answer: $y(x,t) = (0.02\ \text{m}) \sin 2\pi[(80\ s^{-1})t - (1.6\ m^{-1})x + 0.75]$
$= -(0.02\ \text{m}) \cos 2\pi[(80\ s^{-1})t - (1.6\ m^{-1})x]$

3. A mass of 10 kg hangs vertically, supported by a string of length 1.5 m and mass of 45 grams.
(a) Calculate the velocity of propagation, c, for this string.
(b) Find the mass of a second string of the same length that would give a wave speed double that of the first string (for the same tension).

Answer: (a) c = 57.2 $m \cdot s^{-1}$, (b) m_2 = 11.25 grams

4. A wave traveling in the $-x$ direction is described by the wave function
$y(x,t) = (0.05\ \text{m}) \sin 2\pi[(100\ s^{-1})t + (2\ m^{-1})x]$.
(a) Calculate the maximum particle speed, v.
(b) Calculate the particle speed at x = 0.3 m and t = 10^{-3} s.

Answer: (a) $v_{max} = 10\pi\ m \cdot s^{-1}$, (b) v(0.3 m, 10^{-3} s) = 0.

22
VIBRATING BODIES

OBJECTIVES

In this chapter the transverse vibrations of a wave or a pulse on a string are considered in detail. The mathematical results for this simple physical system can be used to develop intuition that will be of great use in analysing more complicated and more interesting physical systems. Your objectives here are to:

Apply the rules for reflection of a wave or pulse on a string (from a free end or a fixed end) to a variety of problems.

Apply the boundary conditions responsible for reflection to produce standing waves when the initial wave train is a harmonic wave. Obtain the normal modes of vibration.

Apply the boundary conditions used for transverse waves on a string to the longitudinal standing waves produced by reflections at open and closed ends in organ pipes and obtain the normal modes of vibration.

Superimpose two or more waves with the same frequency and a constant phase relationship to produce interference phenomena.

REVIEW

As seen in the last chapter, there are many possible solutions to the wave equation. Even sums and differences of these solutions are themselves solutions. Restrictions placed on these solutions by the physical problem at hand pick out the one appropriate solution from the numerous possible ones. These restrictions are called boundary conditions and are somewhat like the 'initial conditions' we used in studying the simple harmonic oscillator. The transverse waves on a string are used to obtain the most important results. These results are:

(1) A wave (or pulse) reflected from a 'fixed end' of a string suffers a 180° phase reversal. If the fixed end is at $x = 0$, the boundary condition is that $y(0,t) = 0$.

(2) A wave (or pulse) reflected from an 'open end' or unattached end

suffers no phase change. If the open end is at $x = 0$, the boundary condition is that $\partial y/\partial x = 0$ at $x = 0$.

In both cases the principle of superposition was used to get the most general solution before applying the boundary condition. A function of the form $f(x - ct)$ or $g(x + ct)$ by itself satisfies the wave equation. Furthermore, addition (or superposition) of these solutions also gives a solution. The most general solution is the one that covers all the possibilities while the boundary conditions pick out the specific choice for the problem at hand.

A node in a wave form is a place where the displacement vanishes. For harmonic waves when the string is fixed at one end (say at $x = 0$), there is a node in the wave function at that point. The next node occurs one half wavelength away and in between these two nodes is an antinode. An antinode is a place where the displacement is a maximum. If we specify a boundary condition at the other end of the string as well, this restriction, coupled with the one from the fixed end, is so stringent that only certain wave vectors k are possible, leading to <u>standing waves</u>. We examine three cases.

(1) The string is fixed at $x = 0$ and $x = L$ so that it has at least two displacement nodes. The mathematical condition for this case is that the allowed wavelengths, λ_n, are given by:

$$\lambda_n = \frac{2L}{n} \qquad \text{where } n = 1,2,\ldots\text{etc.}$$

It is instructive to sketch these solutions.

(2) The string has unattached or open ends at $x = 0$ and $x = L$. The string then has at least two displacement antinodes (with one or more nodes in between). The allowed wavelengths in this case are precisely the ones given above although at first glance the sketches appear different. While this case is not an especially interesting one for a string, it is quite useful in discussing pressure-displacement longitudinal waves in a gas.

(3) The string has one fixed end at $x = 0$ and one open end at $x = L$. The boundary condition at the open end is that $\partial y/\partial x = 0$ at $x = L$. The allowed wavelengths, λ_n, for standing waves are:

$$\lambda_n = \frac{4L}{(2n-1)} \qquad \text{where } n = 1,2,3,\ldots\text{etc.}$$

For the longest wavelength possible, only a quarter wavelength fits into L, with a displacement node at the fixed end and a displacement antinode at the open end.

For the longitudinal waves in a gas, the wave function may represent either the physical displacement of particles from their equilibrium positions (y) or the departure of the pressure from its average value, Δp. These two quantities are not independent but (as is shown in the next chapter) are related by the equation

$$\Delta p = - B \left(\frac{\partial y}{\partial x}\right)$$

where B is the bulk modulus. For a string with an open end, $\partial y/\partial x = 0$, so a wave function using Δp as the variable would have a node at the open end. When Δp has an antinode, the displacement wave, $y(x,t)$ has a node. This makes it really quite simple to analyse the vibrations of organ pipes which are totally analagous to the three cases considered for the vibrating string. The three possibilities are:

(1) The organ pipe is closed at both ends (small openings are needed to let the sound escape). This is like the string fixed at both ends. There must be a displacement node (and hence a pressure antinode) at each end. Since $\lambda_n = 2L/n$, the allowed frequencies, f_n, are:

$$f_n = \frac{c}{\lambda_n} = n \left(\frac{c}{2L}\right).$$

The fundamental frequency is $c/2L$ and all multiples (or harmonics) are present.

(2) The organ pipe is open at both ends. This is like the string with both ends unattached. Since Δp is the departure of the pressure from the average value and the bulk air at both ends of the pipe is at the average pressure inside the pipe, then there must be a pressure node (and hence a displacement antinode) at each end. Again, the allowed frequencies are:

$$f_n = n \left(\frac{c}{2L}\right).$$

Just like the pipe closed on both ends, the fundamental frequency is $c/2L$ and all harmonics are allowed.

(3) The organ pipe has one closed end and one open end. This is analagous to the string with one end free and one end fixed. At the closed end there will be a displacement node (and a pressure antinode) whereas at the open end there will be a pressure node (and a displacement antinode). Since the allowed wavelengths are

$$\lambda_n = \frac{4L}{(2n - 1)} \qquad \text{where } n = 1,2,3,..\text{etc.},$$

the allowed frequencies are:

$$f_n = \frac{(2n - 1)\, c}{4L}.$$

The fundamental frequency is $c/4L$ (lower by a factor of two than the previous cases) and only the odd harmonics are allowed.

If two travelling waves with the same frequency and a constant difference of phase constants are superimposed, the waves may reinforce or cancel each other. When this occurs, it is termed <u>interference</u>. Constructive interference occurs (the waves reinforce each other) when the two waves have a path difference of 0, λ, 2λ,... etc., implying a phase difference of 0, 2π, 4π, ...etc. Destructive interference occurs whenever the path difference of the two waves is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, etc., implying a phase difference of π, 3π, 5π, etc. A useful result is the following:

$$\frac{\text{(path difference)}}{\lambda} = \frac{\text{(phase difference)}}{2\pi}$$

The systems treated here, the stretched string and the organ pipe, can exhibit the phenomenon of resonance if the driving 'force' happens to oscillate at or near one of the special frequencies (called normal modes) of the system. If you blow air across the top of a bottle, several possible sound frequencies are present in the stream of air but the bottle will pick out one (usually) that corresponds to one of its normal modes (dependent on the level of liquid in the bottle) and make a relatively loud sound at that frequency. This is one example of a resonance.

EXAMPLES AND SOLUTIONS

Example 1

A uniform string of length L = 0.5 m is fixed on both ends. Calculate the wavelength of the fundamental mode of vibration. If the wave speed is 300 $m \cdot s^{-1}$, calculate the frequency of the fundamental mode and the next possible mode.

Solution:

For the fundamental mode we have $L = \lambda/2$, so since L = 0.5 m, the wavelength of the fundamental mode is:

$$\lambda_1 = 2L = 1 \text{ m}.$$

For a wave speed of 300 $m \cdot s^{-1} = c$, the frequency of the fundamental mode, f_1, will be:

$$f_1 = \frac{c}{\lambda_1} = \frac{300 \text{ m}\cdot\text{s}^{-1}}{1 \text{ m}} = 300 \text{ Hz}.$$

The next possible mode will have $L = \lambda_2$.

$$\lambda_2 = L = 0.5 \text{ m}$$

The associated frequency is

$$f_2 = \frac{c}{\lambda_2} = \frac{300\ \text{m}\cdot\text{s}^{-1}}{0.5\ \text{m}} = 600\ \text{Hz}.$$

Example 2

The string in the previous example, of length L = 0.5 m, is fixed at one end but free at the other end. Calculate the wavelength of the fundamental mode of vibration. For a wave speed of 300 $\text{m}\cdot\text{s}^{-1}$ calculate the frequency of the fundamental mode and the next possible mode.

Solution:

For the string fixed at one end but free at the other, the fundamental mode is characterized by:

$$\frac{\lambda_1}{4} = L$$

$$\lambda_1 = 4(0.5\ \text{m}) = 2\ \text{m}.$$

The corresponding frequency is:

$$f_1 = \frac{c}{\lambda_1} = \frac{300\ \text{m}\cdot\text{s}^{-1}}{2\ \text{m}} = 150\ \text{Hz}.$$

For the next possible mode,

$$\frac{3}{4}\lambda_2 = L$$

$$\lambda_2 = \frac{4}{3}\ L$$

$$= \frac{4}{3}\ (0.5\ \text{m}) = \frac{2}{3}\ \text{m}.$$

The frequency of this first overtone is:

$$f_2 = \frac{c}{\lambda_2} = \frac{300\ \text{m}\cdot\text{s}^{-1}}{(2/3)\ \text{m}} = 450\ \text{Hz}.$$

The second harmonic is missing and the third harmonic is the first overtone.

Example 3

It is desired to make an organ pipe that will produce middle C on the 'even tempered scale', or $f = 261.6$ Hz. Take $c = 345\ \text{m}\cdot\text{s}^{-1}$.

(a) If the tube is open at both ends, how long should it be?

(b) If the tube is open at one end but closed at the other, how long should it be?

Solution:

(a) A tube open at both ends has the same relationship between λ and L as a tube closed at both ends or a string fixed at both ends.

$$\frac{\lambda_1}{2} = L$$

The wavelength corresponding to middle C is:

$$\lambda_1 = \frac{c}{f_1} = \frac{345\ \text{m}\cdot\text{s}^{-1}}{261.6\ \text{s}^{-1}} = 1.319\ \text{m}.$$

Solving for the length, we have

$$L = \frac{\lambda_1}{2} = 0.659\ \text{m}.$$

(b) For a tube open at one end but closed at the other, the wavelength is

$$\frac{\lambda_1}{4} = L$$

The wavelength is unchanged from part (a) so

$$L = \frac{1.319\ \text{m}}{4} = 0.330\ \text{m}.$$

Example 4

Obtain the allowed frequencies (normal modes) of the two organ pipes in the previous problem where $f_1 = 261.6$ Hz and $c = 345\ \text{m}\cdot\text{s}^{-1}$.

Solution:

(a) For the pipe open on both ends, of length L = 0.659 m, the possible wavelengths satisfy:

$$\frac{\lambda_1}{2} = L$$

$$\lambda_2 = L$$

$$\frac{3}{2}\lambda_3 = L$$

or

$$\lambda_n = \frac{2}{n} L \qquad \text{where } n = 1, 2, 3, \text{ etc.}$$

The frequencies of the harmonics are:

$$f_n = \frac{c}{\lambda_n} = \frac{c}{2L}(n) \qquad n = 1, 2, 3, \text{ etc.}$$

$$f_1 = \frac{c}{2L} = 261.6 \text{ Hz}$$

$$f_2 = \frac{c}{L} = 2f_1 = 523.2 \text{ Hz}$$

$$f_3 = \frac{3c}{2L} = 3f_1 = 784.8 \text{ Hz.}$$

All harmonics are present.

(b) For the pipe open at one end,

$$\frac{\lambda_1}{4} = L$$

$$\frac{3\lambda_2}{4} = L$$

$$\frac{5\lambda_3}{4} = L$$

The general term is:

$$\lambda_n = \frac{4L}{2n - 1} \qquad \text{where } n = 1, 2, 3, \text{ etc.}$$

This gives for the normal mode frequencies

$$f_n = \frac{c}{\lambda_n} = \frac{c}{4L}(2n - 1) \qquad n = 1, 2, 3, \ldots.$$

$$f_1 = 261.6 \text{ Hz}$$

$$f_2 = 784.8 \text{ Hz}$$

$$f_3 = 1308 \text{ Hz.}$$

All odd harmonics are present but the even ones are missing.

Example 5

In a Kundt's tube (see Fig. 22-7 of the text), the spacing between mounds of powder is measured to be 4.4 cm when the source frequency is 4000 Hz.

(a) Calculate the sound velocity in the tube.

(b) Assuming the gas in the tube to be air ($M = 29 \times 10^{-3}$ kg·mol^{-1}, $\gamma = 1.4$) calculate the Kelvin temperature.

Solution:

(a) The distance d between pressure antinodes where the powder mounds accumulate is equal to one-half wavelength ($\lambda/2$).

$$d = \frac{\lambda}{2}$$

Thus we have

$$\begin{aligned} c &= \lambda f \\ &= 2df \\ &= 2(0.044 \text{ m})(4000 \text{ s}^{-1}) \\ &= 352 \text{ m}\cdot\text{s}^{-1} \end{aligned}$$

(b) The speed of sound in a gas is related to the temperature by:

$$c^2 = \frac{\gamma RT}{M}$$

Solving for T, we have

$$T = \frac{Mc^2}{\gamma R}$$

$$= \frac{(29 \times 10^{-3}\ \text{kg}\cdot\text{mol}^{-1})(352\ \text{m}\cdot\text{s}^{-1})^2}{(1.4)(8.314\ \text{J}\cdot\text{mol}^{-1}\cdot\text{K}^{-1})}$$

$$= 308.7\ \text{K} \quad (\text{or } 35.7\ \text{C})$$

Example 6

The gas in a Kundt's tube is a mixture of H_2 and O_2 ($\gamma = 1.4$ for each). The temperature of the gas mixture is 20 C. At a source frequency of 5000 Hz, the distance between mounds of powder is 5.57 cm. Calculate the fraction of H_2 present in the gas.

Solution:

First the speed of sound must be calculated from the given data. In the previous example it was shown that:

$$c = 2df$$

Substituting numerically we have

$$c = 2(0.0557\ \text{m})(5000\ \text{s}^{-1}) = 557\ \text{m}\cdot\text{s}^{-1}.$$

The molecular mass, M, can be calculated from the equation

$$c^2 = \frac{\gamma RT}{M}$$

As c, T, and γ are known, we can solve for M:

$$M = \frac{\gamma RT}{c^2}$$

$$= \frac{(1.4)(8.314\ \mathrm{J\cdot mol^{-1}\cdot K^{-1}})(293\ \mathrm{K})}{(557\ \mathrm{m\cdot s^{-1}})^2}$$

$$= 10.99 \times 10^{-3}\ \mathrm{kg\cdot mol^{-1}}.$$

$$= 10.99\ \mathrm{g}.$$

The molecular mass M can be written in terms of the known molecular masses of H_2 and O_2 and the fraction x of H_2 present.

$$M = x(2\ \mathrm{g\cdot mol^{-1}}) + (1 - x)(32\ \mathrm{g\cdot mol^{-1}})$$

For 1 mole we have

$$10.99\ \mathrm{g} = 32\ \mathrm{g} - x(30\ \mathrm{g})$$

Solving for x, we have

$$x = \frac{21.0\ \mathrm{g}}{30\ \mathrm{g}} = 0.700$$

or 70% of the mixture is H_2.

Example 7

For an acoustic interferometer filled with air (see Fig. 22-13 of the text) calculate the frequency of vibration of the source that will produce successive maxima and minima for a horizontal motion of the slide of 2 cm.

Solution:

The path change corresponding to a horizontal motion of the slide through a distance d is

$$\Delta x = 2d$$

To go from a minimum to a maximum, this path change must correspond to one half wavelength:

$$\Delta x = 2d = \frac{\lambda}{2}$$

or

$$\lambda = 4d$$

$$= 4(0.02\ \mathrm{m}) = 0.08\ \mathrm{m}$$

The frequency will be

$$f = \frac{c}{\lambda} = \frac{345\ \text{m}\cdot\text{s}^{-1}}{0.08\ \text{m}} = 4125\ \text{Hz}.$$

Example 8

In an attempt to find where a plug is in a tube containing air, a plumber blows air across the opening to the tube and hears a resonance at a frequency of 80 Hz. If this is the fundamental mode, how far away is the plug? (Take $c = 345\ \text{m}\cdot\text{s}^{-1}$.)

Solution:

Treating this as an 'organ pipe' with one end open and one end closed, let L be the distance from the open end to the plug. This distance is related to the fundamental wavelength, λ_1, by

$$\frac{\lambda_1}{4} = L$$

The fundamental frequency, f_1, is given by:

$$f_1 = \frac{c}{\lambda_1} = \frac{345\text{m}\cdot\text{s}^{-1}}{4\ L}$$

Taking $f_1 = 80$ Hz, we have

$$L = \frac{345\ \text{m}\cdot\text{s}^{-1}}{4(80\ \text{s}^{-1})} = 1.08\ \text{m}.$$

QUIZ

1. Calculate the distance between pressure antinodes (or displacement nodes) in a Kundt's tube filled with a mixture containing 60% Helium (mass 4) and 40% Neon (mass 20) at 20 C for a source frequency of 5000 Hz.

Answer: d = 6.25 cm.

2. A long thin pipe connects the water in a well with the surface. Sound resonates in this pipe at a lowest frequency of 24 Hz. If $c = 345\ \text{m}\cdot\text{s}^{-1}$, calculate how deep the well is by treating this system as an 'organ pipe' open on one end but closed on the other.

Answer: depth = 3.59 m.

3. Write the equation of a wave $y_2(x,t)$ that when superimposed with the wave

$$y_1(x,t) = A \sin(\omega t - kx)$$

will produce a standing wave, $y_t = y_1 + y_2$, with nodes at $x = 0$ and $x = \lambda/2$.

Answer: $y_2 = -A \sin(\omega t + kx)$

4. An organ pipe open at one end but closed at the other produces a frequency f = 230 Hz. This frequency is the second overtone (or fifth harmonic) of the fundamental. Take the speed of sound in air to be 345 $m \cdot s^{-1}$.
(a) Calculate the length of the pipe.
(b) Calculate the fundamental frequency.

Answer: (a) L = 1.875 m, (b) f_1 = 46 Hz.

23
ACOUSTIC PHENOMENA

OBJECTIVES

In this chapter concepts of waves in gases developed in the last chapter are applied to sound waves or acoustics. Your objectives are to:

Relate the pressure amplitude of a sound wave to the displacement amplitude.

Calculate the intensity in a wave from either the pressure or displacement amplitude (and the bulk modulus).

Calculate the beat frequency resulting from the superposition of two harmonic waves with different frequencies.

Calculate the apparent frequency change due to source and/or listener motion arising from the Doppler effect.

REVIEW

Sound waves in gases can be characterized by the displacement y of a given mass from an equilibrium position or by a pressure difference Δp from the equilibrium or average pressure. The amplitude of the pressure wave is proportional to the local change in the displacement wave function, $\partial y/\partial x$, the proportionality constant being the adiabatic bulk modulus B of the gas:

$$\Delta p = -B \frac{\partial y}{\partial x}$$

For an ideal gas where $pV^\gamma = K$ for an adiabatic process we have $B = \gamma p$, where p is the average pressure. If we express the wave function $y(x,t)$ as a harmonic wave, then the maximum value of the partial derivative will be kA where k is the wave vector and A is the amplitude of the displacement wave. This means that the maximum pressure amplitude can be related to the maximum displacement amplitude by $\Delta p_m = \gamma pkA$, so that a calculation of either one completely characterizes the problem. The smallness of A is illustrated in Example 1.

The intensity I of travelling waves of the harmonic type considered here is

$$I = \frac{1}{2} \ kBA^2$$

The intensity in general is equal to the time average power per unit area (units are watts per square meter). The factor 1/2 in the above expression comes from the fact that the time average of a sine (or cosine) squared over a cycle is equal to 1/2. An alternative form for the intensity, written in terms of the pressure amplitude p is:

$$I \ \frac{1}{2} \left(\frac{cp}{B} \right)^2$$

The range of sound intensities that the human ear can tolerate is so large that a logarithmic scale (base 10) is used to describe the intensity level. The intensity level β is defined by:

$$\beta = (10 \text{ db}) \log_{10} \frac{I}{I_o}$$

where $I_o = 10^{-12}$ $W \cdot m^{-2}$ is the hearing threshold. See Example 2. The intensity levels that the human ear responds to, from the hearing threshold to the level where pain begins, span twelve orders of magnitude or 120 db.

A vibrating string, set into motion by plucking or bowing, can support many frequencies of vibration at the same time. Our perception of the sound that originates from a stringed instrument depends on the various frequencies present and on their relative intensities. The 'quality' of a sound depends, in a subjective way, on this intensity distribution. The 'pitch' of a sound is subjectively connected with the frequency but also depends somewhat on the intensity especially at low frequencies.

Superposition of waves of the same frequency traveling in opposite directions can lead to standing waves. Superposition of waves of the same frequency traveling in the same direction leads to constructive and destructive interference. Superposition of waves traveling in the same direction with nearly equal frequencies leads to a phenomenon known as 'beats'.

The mathematical expression that predicts the beat frequency for two frequencies f_1 and f_2 contains a sum term $(f_1 + f_2)/2$, which represents the average frequency, and a difference term that gives the beats $(f_1 - f_2)/2$. The ear is a power detector--it senses the square of the amplitude--so we hear two maxima and two zeroes per cycle. Thus the beat frequency we hear is $\Delta f = f_1 - f_2$. See Example 4. Beats are a superposition in the time domain.

The Doppler effect for sound waves is very complicated as it involves three reference frames and three velocities: the speed of sound c, the velocity of the sound source, v_s, and the velocity of the observer or listener, v_L. The sign convention employed in the text can be visualized in the following way: let the observer be at the origin and draw the positive x axis from the observer to the source. If v_L and v_s point in the + x direction (we only consider motion on this line in the text), they are positive, otherwise negative.

One way of obtaining the formula for the Doppler effect is to imagine that the source emits sharp sound pulses with a period T so the source frequency $f_s = T^{-1}$. If the origins of the coordinate systems on the listener and source coincide at t = 0 when the first pulse is emitted, there is no delay between sending and receiving the first pulse. When the second pulse is emitted by the source (at time T), the distance between source and observer is equal to $(v_s - v_L)T$. The velocity of this sound pulse relative to the listener is $c + v_L$ so if $\Delta t = t - T$ is the time interval necessary for the pulse to reach the listener, we have:

$$\Delta t = \frac{(v_s - v_L)T}{c + v_L}$$

Substituting $\Delta t = t - T$ and rearranging the above equation gives:

$$t(c + v_L) = T(c + v_s)$$

Since the observer received the first pulse from the source at t = 0 and the second at time t, then t is the period of the source according to the listener. Accordingly $f_L = t^{-1}$ and since $f_s = T^{-1}$ the frequencies are related by:

$$\frac{f_L}{(c + v_L)} = \frac{f_s}{(c + v_s)} .$$

This expression is correct for sound waves but incorrect for light waves (or electromagnetic waves). The problem with light occurs because the method used to find the relative velocity here breaks down at high speeds. Light travels with a velocity c in any inertial frame wheras sound velocity is frame dependent. The Doppler effect for light depends only on the relative velocity between source and observer, which is not the case for sound waves.

The sign convention used in the Doppler effect is illustrated in Example 5.

EXAMPLES AND SOLUTIONS

Example 1

The 'sound intensity level' of ordinary conversation at an average frequency of 500 Hz is 65 db above the hearing threshold (where the intensity is 10^{-12} W m^{-2}). Calculate the displacement amplitude A and compare it with the mean spacing between air molecules at S.T.P.

Solution:

(a) To find the intensity I write

$$65 \text{ db} = 10 \text{ db} \log \frac{I}{I_o}$$

where $I_o = 10^{-12}$ W·m^{-2}. Solving this gives $I = 3.16 \times 10^{-6}$ W·m^{-2}. The intensity is related to the displacement amplitude by Eq. (23-11):

$$I = \frac{1}{2} BkA^2$$

If we take $c = 340$ m·s^{-1}, $\lambda = 0.68$m, and $B = 1.42 \times 10^5$ Pa, then A is equal to:

$$A = \frac{1}{2\pi} \left[\frac{2(3.16 \times 10^{-6})(0.68)}{(500)(1.42 \times 10^5)} \right]^{1/2}$$

$$= 3.92 \times 10^{-8} \text{ m}$$

(b) To find the average spacing (a) between air molecules at S.T.P., since 1 mole occupies 22.4 = 22.4×10^{-3} m^3 and $N_A = 6.02 \times 10^{23}$ mol^{-1}, write:

$$N_A a^3 = 22.4 \times 10^{-3}$$

or

$$a = 3.34 \times 10^{-9} \text{ m}$$

The displacement amplitude for ordinary conversation is more than 10 times larger than the average spacing between gas molecules at S.T.P.

Example 2

Two sound waves are characterized by pressure amplitudes of $P_1 = 20$ Pa and $P_2 = 0.20$ Pa. Calculate the ratio of their intensities and the difference in their sound intensity levels.

Solution:

Since the intensities are proportional to the squares of the respective pressure amplitudes,

$$\frac{I_1}{I_2} = \left(\frac{P_1}{P_2} \right)^2 = \left(\frac{20}{.2} \right)^2 = 10^4.$$

To obtain the difference in sound intensity levels, calculate:

$$\beta = \beta_1 - \beta_2 = 10 \text{ db}[\log(\frac{I_1}{I_o}) - \log(\frac{I_2}{I_o})]$$

$$= (10 \text{ db})\log \frac{I_1}{I_2}$$

Since we have

$$\log \frac{I_1}{I_2} = 4,$$

$\Delta\beta$ = 40 dB. The absolute intensities can be calculated if desired from the known pressure amplitude, the speed of sound, and the bulk modulus.

Example 3

A one meter long tube open at one end and closed at the other contains water to a depth d. Assuming the sound waves have a displacement node at the water surface and an antinode at the open end, calculate the liquid depth necessary to make the tube resonate at C′ above middle C, middle C, and a note an octave below middle C. Take the speed of sound to be 340 m s^{-1}. See fig. 23-1.

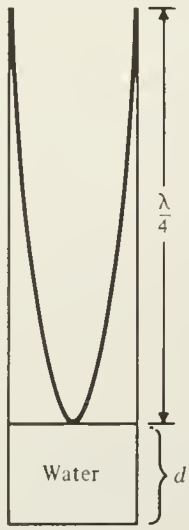


Figure 23-1

Solution:

The frequencies can be obtained from Table 23-2. C′ has a frequency of 528 Hz, middle C has a frequency of 264 Hz, and a note one octave below middle C would have a frequency of 1/2(264)Hz = 132 Hz.

For resonance, we must be able to set up a standing wave in the tube. The lowest mode that can be excited has one quarter wavelength just fitting in the available space.

$$\frac{\lambda}{4} = 0.161 \text{ m} \quad \text{for } C'$$

$$= 0.322 \text{ m} \quad \text{for } C$$

$$= 0.644 \text{ m} \quad \text{for the note at 132 Hz.}$$

When using the note at 132 Hz, only one resonance will occur and that is when

$$d = 1 \text{ m} - \lambda/4 = (1 - .644) \text{ m} = 0.356 \text{ m.}$$

When using C, we can obtain two resonances, one occurring for

$$d_1 = 1 \text{ m} - \lambda/4 = (1 - .322) \text{ m} = 0.678 \text{ m}$$

and a second one occurring for

$$d_2 = 1 \text{ m} - 3\lambda/4 = (1 - 0.966) \text{ m} = 0.034 \text{ m}$$

since the next mode will also fit into one meter.

For the note at 528 Hz, we hear resonances at:

$$d_1 = 1 \text{ m} - \frac{\lambda}{=} = 0.839 \text{ m}$$

$$d_2 = 1\text{m} - \frac{\xi\lambda}{4} = 0.517 \text{ m}$$

$$d_3 = 1\text{m} - \frac{5\lambda}{4} = 0.195 \text{ m}$$

Since $\frac{7\lambda}{4}$ is greater than 1 m, we only obtain three resonances.

Example 4

A detector at $x = 0$ responds to two traveling waves

$$y_1 = A \sin(k_1 x - \omega_1 t)$$

$$y_2 = A \sin(k_2 x - \omega_2 t)$$

where $f_1 = 400$ Hz and $f_2 = 396$ Hz. How many beats per second are obtained if the detector is a power detector?

Solution:

Superimposing these two solutions to give the total displacement y_T, we have

$$y_T = y_1 + y_2 = A[\sin(k_1x - \omega_1t) + \sin(k_2x - \omega_2t]$$

Using the identity

$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2} .$$

the quantity in the [ ] can be rewritten as

$$y_T = 2 A \cos 1/2(\Delta kx - \Delta\omega t) \sin(\bar{k}x - \bar{\omega}t)$$

where

$$\bar{k} = \frac{k_1 + k_2}{2} , \bar{\omega} = \frac{\omega_1 + \omega_2}{2} , \Delta k = k_1 - k_2;$$

and

$$\Delta\omega = \omega_1 - \omega_2.$$

For the detector at x = 0, the displacement is:

$$y_T = 2 A \cos \frac{1}{2} [(\Delta\omega t)] \sin \bar{\omega}t$$

Since the detector is sensitive to power, it measures y_T^2. For instance, zeroes in the intensity occur whenever $\cos (\Delta\omega t/2) = 0$. The term $\sin^2 \bar{\omega}t$ can be considered to supply a constant time averaged background signal that is modulated by the term $\cos (\Delta\omega t/2)$. The times, t_n, where we get nulls, obey the equation:

$$\frac{\Delta\omega}{2} t_n = \frac{(2n-1)}{2}\pi \qquad n = 1,2,3, \text{ etc.}$$

Explicitly we have

$$t_1 = \frac{2\pi}{\Delta\omega} \qquad \text{and} \qquad t_2 = \frac{3\pi}{\Delta\omega}$$

The time difference between these nulls is

$$\Delta t = \frac{2\pi}{\Delta\omega} = \frac{1}{\Delta f}$$

The number (n) of beats in a time T is

$$n = \frac{T}{\Delta t} = T(\Delta f)$$

So if T is one second, we hear Δf beats per second. Numerically, for this example, we have $\Delta f = (400 - 396)\text{Hz} = 4\ \text{Hz}$ so there are four beats per second.

Example 5

Consider a source that produces a sound with frequency of σρρ mz. If the speed of sound in air is $c = 340\ \text{m}\cdot\text{s}^{-1}$ and the source and listener both move along the line joining them with speeds of $25\ \text{m}\cdot\text{s}^{-1}$ (about 56 mph), what is the frequency heard by the listener?

Solution:

The formula for the Doppler shift given in the text is:

$$\frac{f_L}{c + v_L} = \frac{f_s}{c + v_s} .$$

This example is an exercise in using the sign conventions on v_L and v_s.

(a) Let both v_L and v_s point in the positive x direction. They are both then positive as the positive direction is from the listener to the source. Since both speeds are numerically equal, $f_L = f_s = 500$ Hz. Here there is no relative motion of the source and listener and there is no Doppler shift.

(b) Let v_L point in the negataive x direction and v_s point in the positive x direction. Then we have

$$\frac{f_L}{(340 - 25)\text{m}\cdot\text{s}^{-1}} = \frac{f_s}{(340 + 25)\text{m}\cdot\text{s}^{-1}}$$

$$f_L = 500\ \text{Hz}\ \frac{365}{315} = 432\ \text{Hz}$$

(c) Let v_L point in the positive x direction and v_s point in the negative x direction. Then we have

$$\frac{f_L}{(340 + 25)\text{m}\cdot\text{s}^{-1}} = \frac{f_s}{(340 - 25)\text{m}\cdot\text{s}^{-1}}$$

$$f_L = 500 \text{ Hz } \frac{365}{315} = 579 \text{ Hz}$$

(d) If both v_L and v_s point in the negative x direction, then again there is no relative motion of source and listener so $f_L = f_s = 500$ Hz.

QUIZ

1. The pressure amplitude (P) is related to the velocity amplitude (V) by P = ZV where $Z = (B\rho)^{1/2}$ is the mechanical impedance. If Z, for air, is 429 $\text{kg}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$, and the pressure amplitude for a 100 Hz sound is 3×10^{-5} Pa, (a) find the velocity amplitude (V) and (b) the displacement amplitude (A).

Answer: (a) $V = 7.0 \times 10^{-8}\ \text{m}\cdot\text{s}^{-1}$, (b) $A = 1.11 \times 10^{-10}$ m.

2. A sound source radiates isotropically in all directions. The acoustic power of the source is 200 Watts. (a) Find the intensity at a distance of 1 meter from the source. (b) At what distance from the source will the intensity level be 20 dB lower than at R = 1 m?

Answer: (a) $I = 15.9\ \text{W}\cdot\text{m}^{-2}$, (b) R = 10 m.

3. Two identical loudspeakers that radiate isotropically in all directions are oscillating in phase at 300 Hz. Take $c = 330\ \text{m}\cdot\text{s}^{-1}$. Their acoustic power output is 10^{-3} W. (a) What is their minimum separation if a point 1.1 m from one of the speakers, on the line joining the speakers, is a point of <u>minimum sound intensity</u>? (b) Determine the sound intensity at that point if the power to the nearest speaker is shut off. (c) Determine the sound intensity at that point if the power to the most distant speaker is shut off.

Answer: (a) $d_{min} = 1.65$ m, (b) $I = 6.58 \times 10^{-5}\ \text{W}\cdot\text{m}^{-2}$, (c) $I = 2.63 \times 10^{-4}\ \text{W}\cdot\text{m}^{-2}$.

4. Assume each automobile below carries a sound source with frequency of 20,000 Hz. Take $c = 330\ \text{m}\cdot\text{s}^{-1}$ and assume the listener is at rest. (a) What frequency would be 'heard' by a listener if the car moved toward the listener at a speed of 112 $\text{km}\cdot\text{hr}^{-1}$ (about 70 mph)? (b) What frequency would be 'heard' by a listener if the car moved away from the listener at the same speed? (c) If the listener also had a 20,000 Hz source, what would be the beat frequency in the two cases?

Answer: (a) 22,081 Hz, (b) 18,277 Hz, (c) 2081 Hz and 1722 Hz.

24
COULOMB'S LAW

OBJECTIVES

In this short chapter you will be introduced to electric charge and the Coulomb force between positive and negative charges. Atoms will be described as made up of negatively charged electrons and a nucleus made up of positively charged protons and neutral neutrons. Insulating and conducting qualities are discussed. Your objectives are to:

Learn how bodies become charged, positively and negatively, by physical processes involving charge separation.

Observe that like charges repel; unlike attract.

Identify charged bodies as having an electron surplus or deficit.

Charge an electroscope by physical charge transfer or induction.

Calculate the force between point charges, using Coulomb's law.

REVIEW

When two dissimilar materials are rubbed together, one material often becomes negatively charged and the other positively charged. The materials are originally neutral (uncharged), containing a charge balance between negatively charged electrons and the positively charged nucleus. When rubbed, the negative electrons accumulate on one material and are removed from the other. The charge is said to have been separated. The material with a surplus of electrons is negatively charged. The material with a deficit of electrons is positively charged.

Bodies with like charges repel each other. Bodies with unlike charges attract each other.

The magnitude of the force between two charged objects is given by Coulomb's law,

$$F = \frac{k[qq']}{r^2} = \frac{1}{4\pi\varepsilon_o}\frac{[qq']}{r^2}$$

$$k = \frac{1}{4\pi\varepsilon_o} = 8.99 \times 10^9 \ N \cdot m^2 \cdot C^{-2}$$

$$\varepsilon_o = 8.85 \times 10^{-12} \ C^2 \cdot N^{-1} \cdot m^{-2}$$

yielding F in newtons (N) when q and q' are in coulombs (C) and r is in meters (m). The coulomb is a new, independent unit, not previously encountered. It has an independent status, as do m, kg, s, in the SI system.

Matter is generally made up of neutral atoms. A neutral atom consists of Z negatively charged electrons surrounding a positively charged nucleus containing Z positively charged protons and A-Z neutrons. Z is the atomic number of the nucleus.

The charge of an electron is the negative number -e, where

$$e = 1.60 \times 10^{-19} \ C.$$

The mass of an electron is

$$\text{Mass of electron} = 9.10 \times 10^{-31} \ kg.$$

The charge on a proton is e, and the mass of a proton is

$$\text{Mass of proton} = 1.67 \times 10^{-27} \ kg = 1.67 \times 10^{-24} \ g$$

$$= \text{about 1800 electron masses.}$$

The mass of the neutron is approximately that of the proton,

$$\text{Mass of neutron} = 1.67 \times 10^{-27} \ kg.$$

The charge of the neutron is zero.

The atomic number Z of an atom is the number of protons in its nucleus. A gram-atom or mole of a pure substance is an amount of the substance equal in grams to its atomic weight. The mass of a gram-atom of hydrogen is primarily due to the protons within it, with small corrections for the electrons and binding energies. Thus the number of protons in a gram-atom of hydrogen is

$$\frac{(1.008 \ g)}{(1.673 \times 10^{-24} \ g)} = 6.02 \times 10^{23} = \text{Avogadro's number.}$$

EXAMPLES AND SOLUTIONS

Example 1

The force between two identical point charges 1 cm apart has a magnitude of 2 N. What is the magnitude of the point charges?

Solution:

Using Coulomb's law, one can solve for the charge in terms of the force and separation:

$$F = \frac{kq^2}{r^2} \qquad q = \left(\frac{Fr^2}{k} \right)^{1/2}$$

$$q = \left[\frac{(2\ \text{N})(0.01\ \text{m})^2}{8.99 \times 10^9\ \text{N}\cdot\text{m}^2\cdot\text{C}^{-2}} \right]^{1/2}$$

$$= \pm\ 1.49 \times 10^{-7}\ \text{C}.$$

Since the charges are identical they are both positive or both negative. The force is repulsive in either case. It is evident that tiny fractions of a coulomb exert sizable forces at macroscopic distances.

Example 2

Electrons are removed from an originally neutral sphere and placed on another originally neutral sphere. When 1 cm apart the <u>small</u> spheres attract each other with a force of 10^{-6} N. How many electrons were transferred?

Solution:

The magnitude of the repulsive force is

$$F = \frac{kqq'}{r^2} \qquad q = -q' = ne \qquad e = 1.60 \times 10^{-19}\ \text{C},$$

where n is the number of electrons removed and e is the electronic charge. (The charges are equal and opposite because the surplus of electrons on one is equal to the deficit on the other.)

$$F = \frac{k(ne)^2}{r^2} \qquad n = \left(\frac{r^2F}{ke^2} \right)^{1/2}$$

$$n = \left[\frac{(0.01\ \text{m})^2(10^{-6}\ \text{N})}{(8.99 \times 10^9\ \text{N}\cdot\text{m}^2\ \text{C}^{-2})(1.60 \times 10^{-19}\ \text{C})^2} \right]^{1/2}$$

Thus the number of electrons is

$$n = 6.59 \times 10^8.$$

Example 3

What is the total charge of all electrons in a gram-atom of hydrogen? If two such charges were separated by a kilometer, what would be the force between them? Convert this force to lb.

Solution:

A gram-atom or mole of hydrogen has a mass of 1.008 g and contains Avogadro's number of atoms, with a total charge of 6.02×10^{23} (x e) = $6.02 \times 10^{23} \times 1.6 \times 10^{-19}$ C = 9.63×10^4 C.

Two such charges separated by one km repel each other with a force = 8.34×10^{13} N, as you may find using Coulomb's law.

Since a lb is $(2.2)^{-1}$ of the weight of a kg or

$$1\ \text{lb} = (2.2)^{-1}(1\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2}) = 4.45\ \text{N}.$$

we find the total force in lb:

$$F = (8.34 \times 10^{13}\ \text{N})\left(\frac{1\ \text{lb}}{4.45\ \text{N}} \right) = 1.87 \times 10^{13}\ \text{lb}$$

$$= 9.35 \times 10^9\ \text{tons}.$$

Example 4

Positive point charges of equal magnitude $Q = 10^{-8}$ C are distributed at the points with coordinates (0,a) and (0,−a) in the x-y plane, with a = 1 cm. Find the total force these two charges exert on a negative charge −Q located on the x axis at the point with coordinates (x,0). Evaluate this force for x = 0 and x = a.

Solution:

Referring to Fig. 24-1,

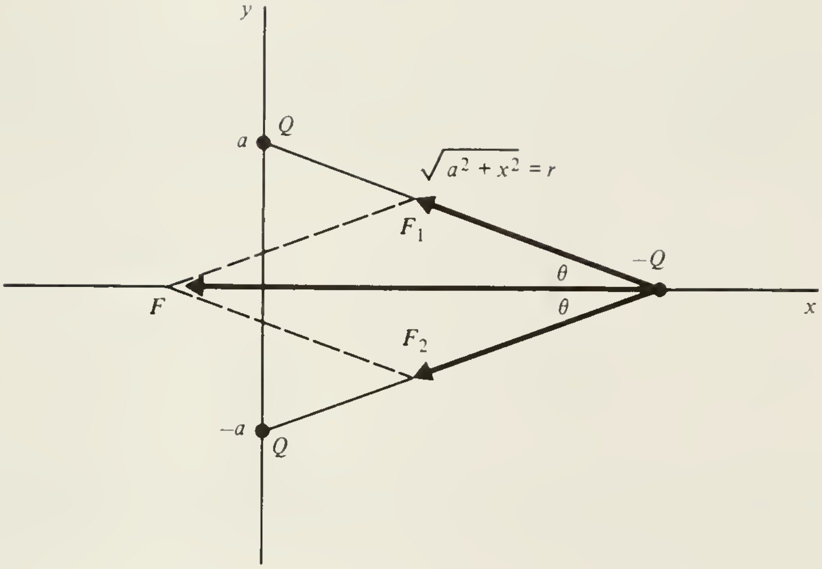


Figure 24-1

we see that each of the positive charges on the y axis attracts the negative charge on the x axis with a force

$$|\mathbf{F}_1| = |\mathbf{F}_2| = \frac{kQ^2}{r^2} = \frac{kQ^2}{a^2 + x^2}$$

The total force on the x axis charge is the <u>vector</u> sum of the two forces shown,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2. \qquad F_y = 0 \quad \text{because } F_{1y} = -F_{2y}.$$

$$F_x = -\,2|F_1|\cos\theta = \frac{-2kQ^2}{a^2 + x^2}\,\frac{x}{(a^2 + x^2)^{1/2}}$$

$$= -\,\frac{2kQ^2 x}{(a^2 + x^2)^{3/2}}$$

At the origin ($x = 0$) $F_x = 0$. At the point $(a,0) = (1\text{ cm},0)$,

$$F_x = -\,\frac{2kQ^2 a}{(2a^2)^{3/2}} = -\,\frac{2kQ^2}{2^{3/2}a^2}$$

$$= -\,\frac{(2)(8.99 \times 10^9\ \text{N}\cdot\text{m}^2\cdot\text{C}^{-2})(10^{-8}\ \text{C})^2}{2^{3/2}(0.01\ \text{m})^2}$$

$$= -6.36 \times 10^{-3}\ \text{N}.$$

Example 5

In the last problem find the total force of the charges at (0,±a) on a negative charge of equal magnitude located on the y-axis at the point (0,2a) and at the point (0,a/2). As before a = 1 cm.

Solution:

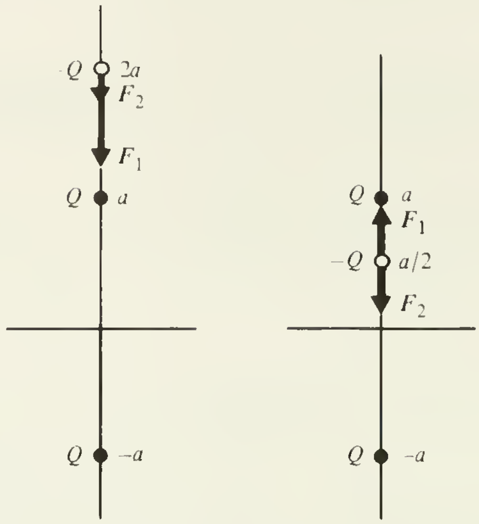


Figure 24-2

The forces on −Q at (0,2a), as shown in Fig. 24-2a, both point in the negative y direction,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2. \qquad F_x = 0.$$

$$F_y = F_{1y} + F_{2y} = \frac{-kQ^2}{a^2} - \frac{kQ^2}{(3a)^2}$$

$$= -\frac{10}{9}\frac{kQ^2}{a^2}$$

$$= \frac{-10}{9}\frac{(8.99 \times 10^9 \text{ N}\cdot\text{m}^2\cdot\text{C}^{-2})(10^{-8}\text{ C})^2}{(0.01\text{ m})^2}$$

$$= -9.99 \times 10^{-3}\text{ N}.$$

The force on −Q at (0, a/2), as shown in Fig. 24-2b, has a contribution in the positive y direction from the charge Q at (0,a) and in the negative y direction from the charge Q at (0,−a). The total force is

$$F_y = \frac{kQ^2}{(a/2)^2} - \frac{kQ^2}{(3a/2)^2} = \frac{kQ^2}{a^2}\left(4 - \frac{4}{9}\right)$$

$$= \frac{32}{9} \frac{kQ^2}{a^2}$$

$$= \frac{(8.99 \times 10^9 \text{ N}\cdot\text{m}^2\cdot\text{C}^{-2})(10^{-8} \text{ C})^2}{(0.01 \text{ m})^2}$$

$$= 3.20 \times 10^{-2} \text{ N}.$$

Example 6

Two small pith balls of mass 5 g each are suspended by light thread of length 30 cm from a common point. They are both negatively charged and repel each other, remaining 4 cm apart at equilibrium. Find the excess number of electrons on each ball.

Solution:

Referring to Fig. 24-3,

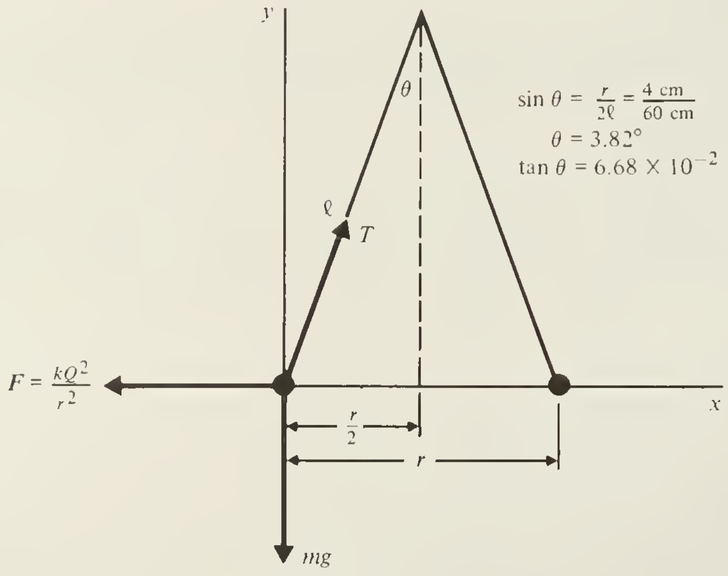


Figure 24-3

we see that the ball at the origin experiences forces of tension, weight, and the Coulomb repulsion of the other pith ball. The equilibrium conditions are:

$$\sum F_x = 0 = T \sin \theta - \frac{kQ^2}{r^2} ,$$

$$\sum F_y = 0 = T \cos \theta - mg.$$

Eliminating T we have

$$\tan\theta = \frac{kQ^2}{r^2mg} = \frac{k(ne)^2}{r^2mg}$$

where n is the number of excess electrons. Thus we find

$$n^2 = \frac{Q^2}{e^2} = \frac{r^2\ mg\ \tan\theta}{ke^2}$$

$$= \frac{(0.04\ \text{m})^2(5 \times 10^{-3}\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(6.68 \times 10^{-2})}{8.99 \times 10^9\ \text{N}\cdot\text{m}^2\cdot\text{C}^{-2})(1.6 \times 10^{-19}\ \text{C})^2}$$

$$n = 1.5 \times 10^{11}\ \text{electrons.}$$

Example 7

An electron rotates in a circular orbit about a heavy fixed proton. The radius of the orbit is 10^{-8} cm. Find the velocity of the electron.

Solution:

The centripetal force is supplied by the Coulomb attraction. Setting this force equal to the mass times the centripetal acceleration, we find

$$\frac{mv^2}{r} = \frac{ke^2}{r^2}$$

$$v^2 = \frac{ke^2}{mr}$$

$$v = \left(\frac{ke^2}{mr}\right)^{1/2}$$

$$= \left[\frac{(8.99 \times 10^9\ \text{N}\cdot\text{m}^2\cdot\text{C}^{-2})(1.6 \times 10^{-19}\ \text{C})^2}{(9.11 \times 10^{-31}\ \text{kg})(10^{-10}\ \text{m})}\right]^{1/2}$$

$$= 1.59 \times 10^6\ \text{m}\cdot\text{s}^{-1}.$$

This velocity is less than a percent of the velocity of light.

QUIZ

1. At a point 1.3 of the way along a line from charge Q to charge Q′, a third charge experiences no electrical force.

   (a) What is the relative sign of Q and Q′?

   (b) What is the relative size of Q and Q′?

Answer: (a) Q and Q′ are both positive or both negative.
(b) Q/Q′ = 1/4

2. A positive charge of magnitude 2×10^{-9} C is located 4×10^{-2} m from a negative charge of magnitude 3×10^{-9} C. What is the force between the charges?

Answer: 0.34×10^{-9} N, attractive

3. Calculate the ratio of the Coulomb force to the gravitational force of two electrons for each other.

Answer: 4.2×10^{42}

25
THE ELECTRIC FIELD

OBJECTIVES

In this key chapter, based on the Coulomb force law, the electric field, a vector, is introduced. The field strength at a point is calculated by direct summation of the fields due to all other charges, or by use of Gauss' law, in charge distributions of high symmetry. Your objectives are to:

Calculate the electric field caused by a distribution of point charges by vector summation of the electric fields of the individual point charges.

Calculate the electric field of continuous charge distributions by summing the field of its parts. Examples are a line charge, an infinite plane, or the axial field of a circular loop of charge.

By use of Gauss' Law, calculate the electric field of symmetrical charge distributions, such as charged surfaces of spherical and cylindrical conductors, and of continuous volume charge distributions of insulating bodies with high symmetry.

Establish that stationary charge may exist only on the surfaces of conductors. Establish that the static electric field always is zero within a conductor, and perpendicular to a conducting surface.

REVIEW

An electric field E exists at a point in space if there is a force F of electric origin on a test charge q' at rest at that point.

By 'electric origin' is meant that the force, and electric field, is due to all other charged bodies present. Since the test charge itself may disturb the surrounding charges, it is desirable to minimize this effect by making q' as small as possible. The electric field is defined as

$$\mathbf{E} = \mathrm{Lim}_{q' \to 0} \frac{\mathbf{F}}{q'}$$

where F is the Coulomb force on charge q'.

The electric field points in the direction that a positive test charge would move if placed in the field at rest. The static field lines of force are continuous in charge free regions; for static distributions of charges they begin on positive charges and end on negative charges.

If the force F is in newtons (N) and q′ is in coulombs (C) the field E has units of $N \cdot C^{-1}$.

The field due to a point charge of magnitude q is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o} \frac{q\hat{\mathbf{r}}}{r^2} = \frac{kq\hat{\mathbf{r}}}{r^2}$$

where, as shown in Fig. 25-1, $\hat{\mathbf{r}}$ is a unit vector pointing radially outward from the charge at the field point P. If the charge q is positive, E has the direction of r. If the charge is negative, E has the opposite direction.

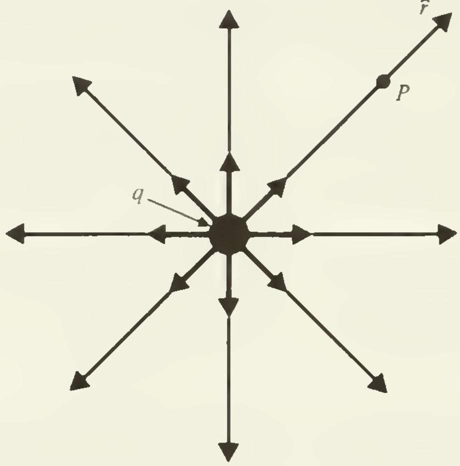


Figure 25-1

The field strength is inversely proportional to the square of the distance from P to q; the field strength is proportional to the number of field lines per unit area perpendicular to the field direction. It is greatest near the charge q where the field lines are closely spaced.

The field of a set of point charges q_i is the <u>vector sum</u> (or 'superposition') of the fields of the individual point charges,

$$\mathbf{E} = \sum_i \frac{1}{4\pi\varepsilon_o} \frac{q_i\hat{\mathbf{r}}_i}{r_i^2}$$

$$= \sum_i k\left(\frac{q_i\hat{\mathbf{r}}_i}{r_i^2} \right).$$

If the charge distribution is continuous, the sum becomes an integral

$$E = \sum_i \frac{1}{4\pi\varepsilon_o} \frac{q_i \hat{r}_i}{r_i^2} - \frac{1}{4\pi\varepsilon_o} \int \left(\frac{\hat{r}\, dq}{r^2} \right),$$

where $\hat{r}$ points from the charge increment dq to the field point as in Fig. 25-1.

The axial field of a charge Q uniformly distributed on a ring of radius a is

$$E_x = \frac{1}{4\pi\varepsilon_o} \frac{Qx}{(x^2 + a^2)^{3/2}},$$

as shown in Fig. 25-2. (Components not in the axial (x) direction cancel, as shown.)

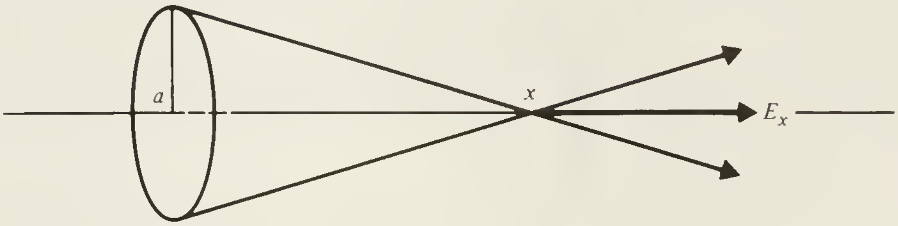


Figure 25-2

As a $\longrightarrow$ 0 the loop behaves like a point charge, and E_x approaches the field due to point charge, $\frac{1}{4\pi\varepsilon_o} \frac{Q}{x^2}$.

The field of a long line charge of magnitude

$$\lambda = \frac{dQ}{dL} = \text{charge per unit length}$$

is radially outward if the charge is positive, of magnitude

$$E = \frac{1}{4\pi\varepsilon_o} \frac{2\lambda}{r}$$

as shown in Fig. 25-3.

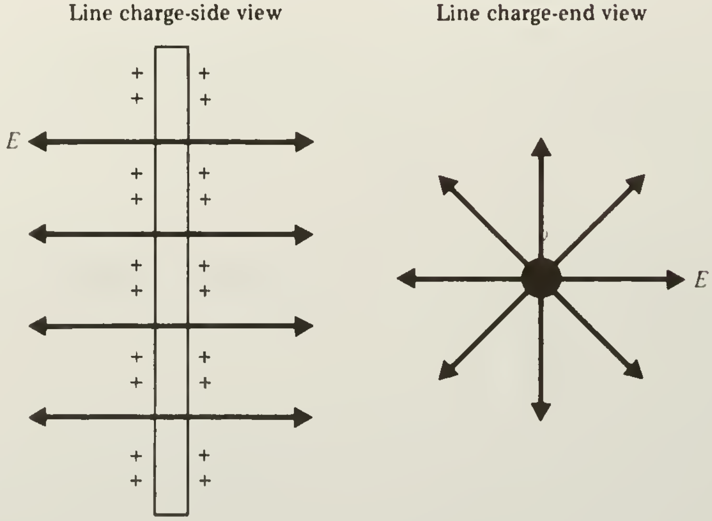


Figure 25-3

The charge density of an infinite sheet of charge, uniformly distributed, with charge density

$$\sigma = \frac{dQ}{dA} = \text{charge per unit area}$$

is perpendicular to the plane, has the magnitude

$$E = \frac{\sigma}{2\varepsilon_o}$$

and is directed outward from a positive sheet, as shown in Fig. 25-4.

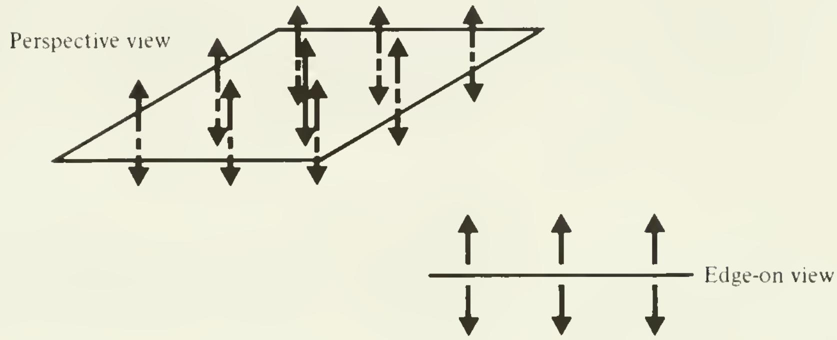


Figure 25-4

This field has the same strength, no matter how far from the plane.

The field near a conducting surface is twice this field, or

$$E = \frac{\sigma}{\varepsilon_o}$$

The previous field distributions may be calculated using the methods of the integral calculus, from the basic expression

$$E = \int \frac{dq\ \hat{r}}{r^2}$$

which comes from Coulomb's law. Another consequence of Coulomb's law is Gauss' law,

$$\int E_n dA = \frac{Q}{\varepsilon_o} = \oint \mathbf{E} \cdot d\mathbf{A}.$$

Once mastered, Gauss' law provides a simple and powerful method of calculating the electric fields of symmetrical charge distributions without the labor of direct integration.

$E_n dA = \mathbf{E} \cdot d\mathbf{A}$ is the product of an area increment and the perpendicular component of field pointing through it. $E_n dA$ is counted positive if the field points out of the enclosed surface in the $\int$ integral and negative otherwise.

The symbol $\oint$ means that the integral is to be done over a closed surface. Q is the net charged enclosed by the surface. $E_n dA = 0$ when the field is parallel to the surface and $E_n dA = EdA$ when the field is perpendicular to the surface.

Care must be taken in distinguishing between charges on a conductor (which always reside on a surface of the conductor in the form of σ = charge per unit area) and idealized charge distributions in insulating bodies which may take the forms:

$$\rho = \frac{dQ}{dV} = \text{charge per unit volume (volume charge)}$$

$$\sigma = \frac{dQ}{dA} = \text{charge per unit area (surface or layer charge)}$$

$$\lambda = \frac{dQ}{dL} = \text{charge per unit length (line charge)}$$

Q = point charge.

EXAMPLES AND SOLUTIONS

Example 1

Find the electric field at a distance 1 cm from a positive point charge of magnitude $q = 10^{-10}$ C.

Solution:

The force on a test charge q' a distance r from q is

$$F = \frac{1}{4\pi\varepsilon_o} \frac{qq'}{r^2} = k \frac{q'q}{r^2} .$$

The electric field is the force per unit test charge, or

$$E = \frac{F}{q'} = \frac{kq}{r^2} = \frac{(9 \times 10^9\ N\cdot m^2\cdot C^{-2})(10^{-10}\ C)}{(0.01\ m)^2}$$

$$= 9000\ N\cdot C^{-1}.$$

The direction of the field is the direction of the force a positive test charge at rest would experience; it is radially <u>outward</u> from q in this case.

Example 2

A positive charge of magnitude $q = 10^{-9}$ C is placed at a point where the electric field is

$$E_x = 10^7\ N\cdot C^{-1} \qquad E_y = E_z = 0.$$

(a) Find the force on the charge.
(b) What is the force on a negative charge of the same magnitude?

Solution:

(a) $F = qE;\qquad F_x = qE_x;\qquad F_y = F_z = 0$

$$F_x = (10^{-9}\ C)(10^7\ N\cdot C^{-1}) = 10^{-2}\ N$$

(b) $F_x = -10^{-2}$ N; $\qquad F_y = F_z = 0$

Example 3

Two positive point charges of magnitude $Q = 10^{-10}$ C are placed on the y axis at positions (0,a) and (0,−a). a = 3 cm.

(a) Find the electric field on the x axis at the point with coordinates (x,0) for x = 1 cm and x = 10 cm.

(b) Find the electric field on the y axis at the point with coordinates (0,y) for y = 2 cm and y = 6 cm.

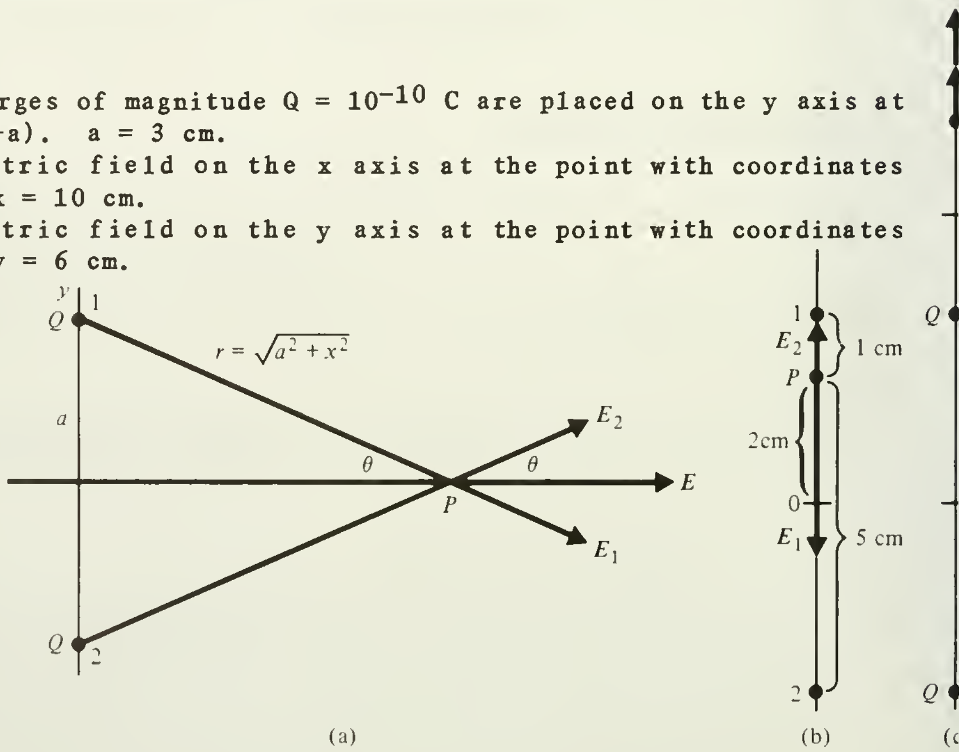


Figure 25-5

Solution:

(a) Referring to Fig. 25-5a, we calculate the field on the x axis as the vector sum of the field $\mathbf{E}_1$ due to the upper charge and the field $\mathbf{E}_2$ due to the lower charge.

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

$$|E_1| = |E_2| = \frac{kQ}{r^2} = \frac{kQ}{a^2 + x^2}$$

$$E_x = E_{1x} + E_{2x} = |E_1|\cos\theta + |E_2|\cos\theta$$

$$= 2|E_1|\cos\theta = \left(\frac{2kQ}{a^2 + x^2}\right)\left(\frac{x}{(a^2 + x^2)^{1/2}}\right)$$

$$= \frac{2kQx}{(a^2 + x^2)^{3/2}} \text{ where } a = 3 \text{ cm.}$$

$$E_y = E_{1y} + E_{2y} = |E_1|\sin\theta - |E_2|\sin\theta = 0.$$

At (1 cm,0)

$$E_x = \frac{(2)(9 \times 10^9\ \mathrm{N\cdot m^2\cdot C^{-2}})(10^{-10}\ \mathrm{C})(0.01\ \mathrm{m})}{[(0.03\ \mathrm{m})^2 + (0.01\ \mathrm{m})^2]^{3/2}}$$

$$= 5.69 \times 10^2\ \mathrm{N\cdot C^{-1}}$$

At (10 cm,0)

$$E_x = \frac{(2)(9 \times 10^9\ \mathrm{N\cdot m^2\cdot C^{-2}})(10^{-10}\ \mathrm{C})(0.1\ \mathrm{m})}{[(0.03\ \mathrm{m})^2 + (0.1\ \mathrm{m})^2]^{3/2}}$$

$$= 1.58 \times 10^2\ \mathrm{N\cdot C^{-1}}.$$

(b) Referring to Fig. 25-5b, we see that the field $\mathbf{E}_1$ at (0,2 cm) of Q_1 points down and the field $\mathbf{E}_2$ of Q_2 points up. (To determine how the field points, imagine a positive test charge at P, the point in question; the direction of E_1 is down because Q_1 would exert a downward force on such a charge; correspondingly the force of Q_2 on such a charge would be upward and hence E_2 points up.) Thus

$$E_y = E_{1y} + E_{2y} = \frac{dQ}{r_1^2} - \frac{kQ}{r_2^2} = kQ\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

$$= 9 \times 10^9 \text{ N}\cdot\text{m}^2\cdot\text{C}^{-2} \cdot 10^{-10} \text{ C} \left(\frac{1}{(0.01 \text{ m})^2} - \frac{1}{(0.05 \text{ m})^2}\right)$$

$$= 8.64 \times 10^3 \text{ N}\cdot\text{C}^{-1}.$$

The fields at (0,6cm), on the other hand point in the same direction. Referring to Fig. 25-5c, we find for this case that

$$E_y = E_{1y} + E_{2y} = \frac{kQ}{r_1^2} + \frac{kQ}{r_2^2} = kQ\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)$$

$$= (9 \times 10^9 \text{ N}\cdot\text{m}^2\cdot\text{C}^{-2})(10^{-10} \text{ C}) \left[\frac{1}{(.03 \text{ m})^2} + \frac{1}{(.09 \text{ m})^2}\right]$$

$$= 1.11 \times 10^3 \text{ N}\cdot\text{C}^{-1}.$$

Example 4

If in Fig. 25-5a, the charge at position 2 (on the negative y axis) is −2Q instead of Q, find the field on the x axis as a function of x.

Solution:

Referring to Fig. 25-6

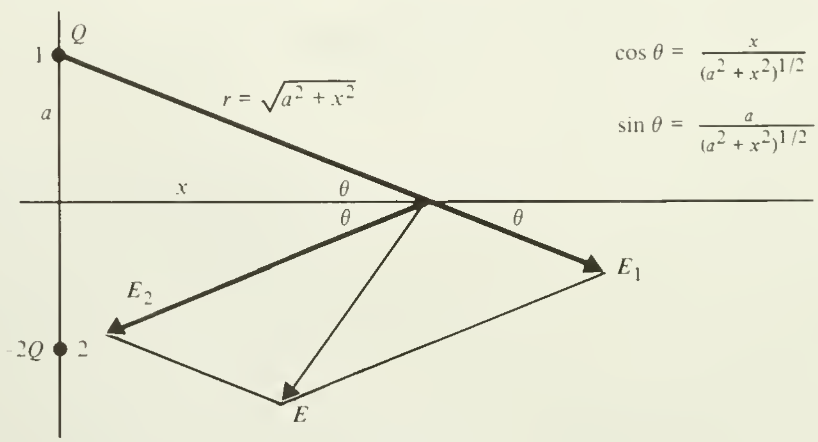


Figure 25-6

we have

$$\cos\theta = \frac{x}{(a^2 + x^2)^{1/2}}$$

$$\sin\theta = \frac{a}{(a^2 + x^2)^{1/2}}$$

Thus

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

$$E_x = E_{1x} + E_{2x} = |\mathbf{E}_1|\cos\theta - |\mathbf{E}_2|\cos\theta$$

$$= \frac{kQ}{a^2 + x^2}\,\frac{x}{(a^2 + x^2)^{1/2}} - \frac{k(2Q)}{(a^2 + x^2)}\,\frac{x}{(a^2 + x^2)^{1/2}}$$

$$= \frac{-kQx}{(a^2 + x^2)^{3/2}}$$

$$E_y = E_{1y} + E_{2y} = -|\mathbf{E}_1|\sin\theta - |\mathbf{E}_2|\sin\theta$$

$$= \frac{-kQ}{a^2 + x^2}\,\frac{a}{(a^2 + x^2)^{1/2}} - \frac{kQ}{(a^2 + x^2)}\,\frac{a}{(a^2 + x^2)^{1/2}}$$

$$= \frac{-3kQa}{(a^2 + x^2)^{3/2}}$$

$$E = (E_x^2 + E_y^2)^{1/2} = \frac{kQ}{(a^2 + x^2)^{3/2}}(x^2 + 9a^2)^{1/2}$$

Example 5

An electron enters a region of space between two conducting plates, 2 cm long and 1 cm wide, which has a uniform electric field of magnitude 5000 $N\cdot C^{-1}$ as shown in Fig. 25-7. What is the minimum velocity it must have to escape hitting one of the plates?

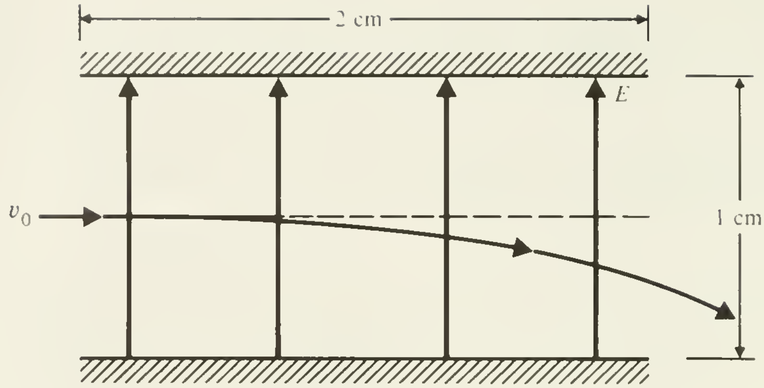


Figure 25-7

Solution:

The electric force on the electron is

$$\mathbf{F} = q\mathbf{E} = -e\mathbf{E}$$

and thus is directed down in Fig. 25-7, opposite to the field direction. The equations of motion are

$$F_x = 0 = ma_x \qquad a_x = 0 \qquad x = x_o + v_{ox}t$$

$$F_y = -eE = ma_y \qquad a_y = -eE/m$$

$$y = y_o + v_{oy}t - \frac{eE}{2m}t^2$$

Since $x_o = 0$, $y_o = 0$, $v_{ox} = v_o$, and $v_{oy} = 0$,

$$x = v_o t$$

we have

$$y = \frac{-eE}{2m}t^2 = \frac{-eE}{2m}\left(\frac{x}{v_o}\right)^2.$$

To just hit the plate as shown, $x = 2$ cm when $y = -0.5$ cm. Solving the last expression for v_o,

$$v_o = \left(\frac{-eE}{2\ m}\ \frac{x^2}{y}\right)^{1/2}$$

with

$$e = 1.6 \times 10^{-19}\ \text{C}$$

$$m = 9.1 \times 10^{-31}\ \text{kg}$$

$E = 5000\ N \cdot C^{-1}$.

We find

$$v_o = \left[\frac{(-1.6 \times 10^{-19}\ C)(\ 5000\ N \cdot C^{-1})(0.02\ m)^2}{(2)(9.1 \times 10^{-31}\ kg)(-0.005\ m)} \right]^{1/2}$$

$$= 5.93 \times 10^6\ m \cdot s^{-1}.$$

Example 6

A uniform electric field of magnitude E exists between two conducting plates separated by a distance d. An electron is released from rest at the negative plate and is attracted to the positive plate.

(a) Find its kinetic energy when it collides with the positive plate, in terms of E, d, and the electronic charge.

(b) If $E = 10\ N \cdot C^{-1}$ and $d = 1$ cm, find the velocity of impact.

Solution:

Referring to Fig. 25-8, we have

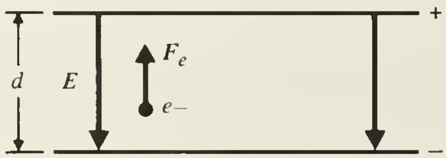


Figure 25-8

$$F_e = eE = ma \qquad\qquad a = \frac{eE}{m}$$

For constant acceleration

$$v^2 = v_o^2 + 2ax.$$

If the electron starts from rest ($v_o = 0$) and travels a distance d,

$$v^2 = 2ad = 2\ \frac{eE}{m}\ d.$$

Its final kinetic energy is

$$K = 1/2(mv^2) = eEd.$$

(b) Its velocity is

$$v = \left(\frac{2eE}{m} d \right)^{1/2}$$

$$= \left[\frac{2(1.6 \times 10^{-19} \text{ C})(10 \text{ N}\cdot\text{C}^{-1})(0.01 \text{ m})}{9.1 \times 10^{-31} \text{ kg}} \right]^{1/2}$$

$$= 1.88 \times 10^5 \text{ m}\cdot\text{s}^{-1}.$$

Example 7

Find the electric field on the axis of and a distance z away from a circular disk of radius a and of uniform charge density σ per unit area.

Solution:

Referring to Fig. 25-9, we divide the disk into thin concentric rings of thickness dr, as shown.

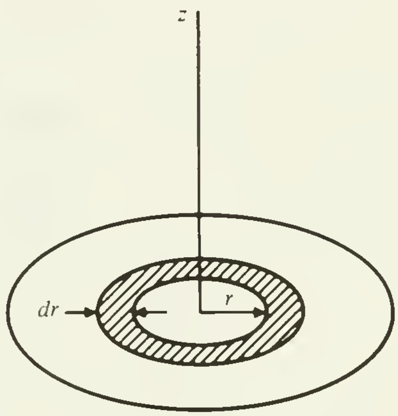


Figure 25-9

Each ring has an area $2\pi r$ dr and hence a total charge

$$dQ = \sigma \, dA = \sigma 2\pi r \, dr.$$

Such a ring of charge contributes an axial field at z of magnitude

$$dE_z = \frac{1}{4\pi\varepsilon_o} \frac{\sigma 2\pi r \, dr \, z}{(z^2 + r^2)^{3/2}}$$

To find the total field, sum all ring contributions,

$$E_z = \frac{1}{4\pi\varepsilon_o} \; 2\pi\sigma z \int_o^a \frac{rdr}{(z^2 + r^2)^{3/2}}$$

With the substitution

$$x = r^2 \qquad dx = 2rdr$$

the integral is

$$E_z = \frac{\sigma z}{2\varepsilon_o} \int_o^{a^2} \frac{(1/2)dx}{(z^2 + x)^{3/2}} = \frac{\sigma z}{4\varepsilon_o} \left| \frac{(z^2 + x)^{-1/2}}{-1/2} \right|_0^{a^2}$$

$$= \frac{\sigma z}{2\varepsilon_o} \left[\frac{1}{z} - \frac{1}{(z^2 + a^2)^{1/2}} \right] = \frac{\sigma}{2\varepsilon_o} \left[1 - \frac{z}{(z^2 + a^2)^{1/2}} \right].$$

Note for $z \ll a$ the field approaches that of an infinite sheet,

$$\lim_{z \to o} E_z = \frac{\sigma}{2\varepsilon_o}$$

Example 8

A thin wire of length L carries λ charge per unit length. Find the electric field on the axis of the wire a distance a from an end. What is the magnitude of this field when

$$\frac{L}{a} \ll 1.$$

Solution:

Referring to Fig. 25-10,

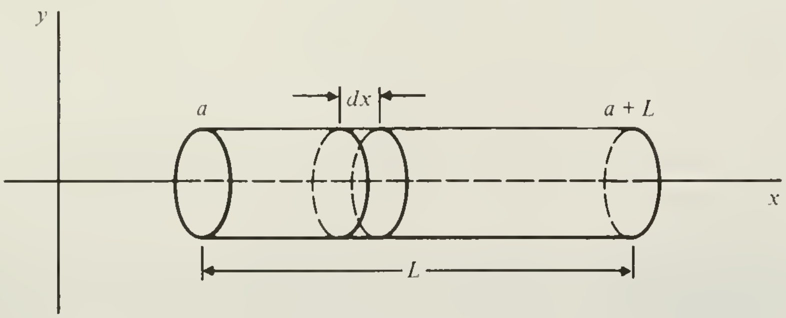


Figure 25-10

we first find the field at the origin due to the bit of thin wire at dx,

$$dE_x = \frac{1}{4\pi\varepsilon_o}\frac{\lambda dx}{x^2}$$

where λdx is the charge in the small bit of length dx. The total field is

$$E_x = \frac{-\lambda}{4\pi\varepsilon_o}\int_a^{L+a}\frac{dx}{x^2}$$

$$= \frac{-\lambda}{4\pi\varepsilon_o}\left|\frac{-1}{x}\right|_a^{L+a}$$

$$= \frac{-\lambda}{4\pi\varepsilon_o}\left[\frac{1}{a} - \frac{1}{L+a}\right]$$

For $\frac{L}{a} << 1$, the field has the magnitude

$$E_x = \frac{-\lambda}{4\pi\varepsilon_o}\left[\frac{1}{a} - \frac{1}{a(1+L/a)}\right]$$

$$= \frac{-\lambda}{4\pi\varepsilon_o}\frac{1}{a}\left[1 - \left(1+\frac{L}{a}\right)^{-1}\right]$$

$$= \frac{-\lambda}{4\pi\varepsilon_o}\left[1 - 1 + \frac{L}{a} + \ldots\right]$$

$$= \frac{-\lambda L}{4\pi\varepsilon_o a^2} = \frac{-Q}{4\pi\varepsilon_o a^2}$$

where $Q = \lambda L$ is the net charge on the wire. The minus sign indicates that the field points in the negative x direction. Far away, the piece of wire looks like a point charge.

Example 9

A long cylindrical solid conductor has radius R and λ charge per unit length. Find the field inside and outside of the conductor.

Solution:

Inside the conductor the field is zero. Charges on conductors reside on the surface.

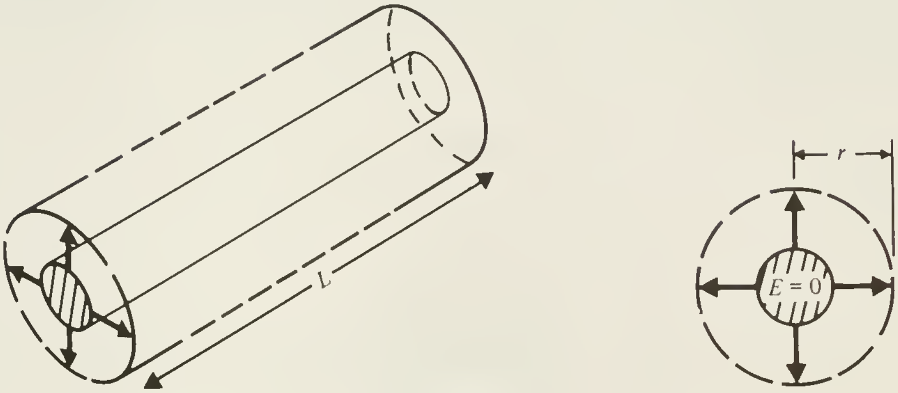


Figure 25-11

Referring to Fig. 25-11 we apply Gauss' law to a cylindrical surface coaxial with the charge conductor of radius $r > R$. By the cylindrical symmetry the field must point radially outward. Thus

$$\oint E_n dA = EA = E\,2\pi rL = \frac{Q}{\varepsilon_o} = \frac{\lambda L}{\varepsilon_o}$$

$$E = \frac{2\lambda}{4\pi r\varepsilon_o} \qquad r > R$$

$$E = 0 \qquad r < R$$

The exterior field is same as that of a thin wire on the cylinder axis, with the same charge per unit length.

Example 10

A long insulating cylinder of radius R contains a uniform distribution of charge density per unit volume ρ. Find the electric field inside and outside of the cylinder.

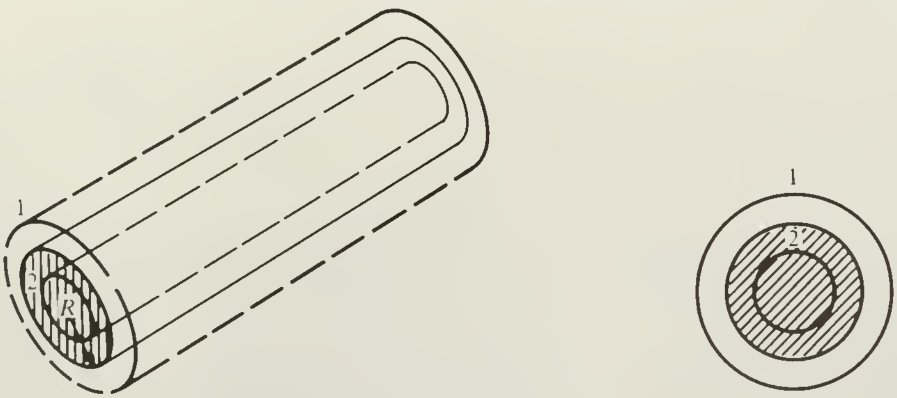


Figure 25-12

Solution:

Referring to Fig. 25-12 we apply Gauss' law in the two regions inside and outside of the (cross hatched) uniformly charged cylinder. As before the field is radial and constant on a surface of fixed radius r, and end caps have no contribution because $E_n dA = 0$ on them.

For $r > R$ we have, applying Gauss' Law to surface 1, outside of the charged cylinder

$$\oint E_n dA = EA = E\, 2\pi r L = \frac{Q}{\varepsilon_0}$$

The charge enclosed Q is the charge density ρ times the volume containing the charge

$$Q = \rho\ \pi R^2 L$$

and thus

$$E\, 2\pi r L = \frac{\rho \pi R^2 L}{\varepsilon_0} \qquad E = \frac{\rho R^2}{2 r \varepsilon_0}\ .$$

For $r < R$ we have, applying Gauss' Law to surface 2,

$$\oint E_n dA = EA = E\, 2\pi r L.$$

However, the charge enclosed Q' is now less than before,

$$Q' = \rho V' = \rho \pi r^2 L$$

since a part of the charge distribution is outside of the surface. Thus we find

$$E\, 2\pi r L = \frac{Q'}{\varepsilon_0} = \frac{\rho \pi r^2 L}{\varepsilon_0}$$

$$E = \frac{\rho r}{\varepsilon_0}\ .$$

Note at $r = R$ the inner field matches the outer field. The field is plotted in Fig. 25-13

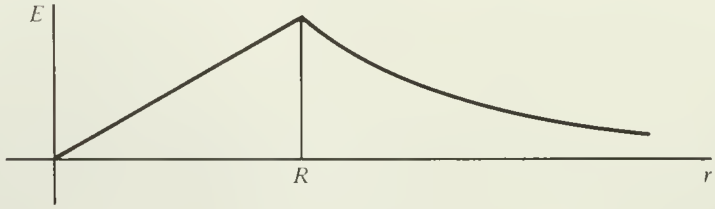


Figure 25-13

Example 11

A <u>conducting sphere</u> of radius R contains a net charge Q. Find the field inside and outside of the sphere.

Solution:

Referring to Fig. 25-14

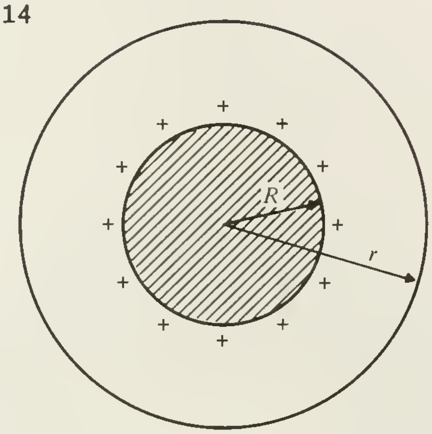


Figure 25-14

we apply Gauss' law to a spherical surface concentric with the sphere. Because we have a conductor, all of the charge resides <u>on the surface</u> of the sphere and the inner field is zero; if there were an inner field it would be radial, but then

$$\oint E_n dA = EA = \frac{Q}{\varepsilon_0} = 0$$

$$E = 0 \qquad r < R$$

verifying that the field must be zero inside the conductor. For $r > R$ the field is radial and

$$\int E_n dA = EA = E\,4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad r > R$$

Thus a <u>sphere</u> carrying a charge Q has the same field as a <u>point charge</u> Q when $r > R$ (outside of the sphere).

Example 12

A conducting spherical shell of outer radius 2R is electrically neutral. The conductor contains a concentric spherical cavity of radius R. At the center of the cavity is suspended a point charge Q. Find the electric field for regions $0 < r < R$, $R < r < 2R$, and $2R < r$.

Solution:

Referring to Fig. 25-15, where the conductor is cross-hatched,

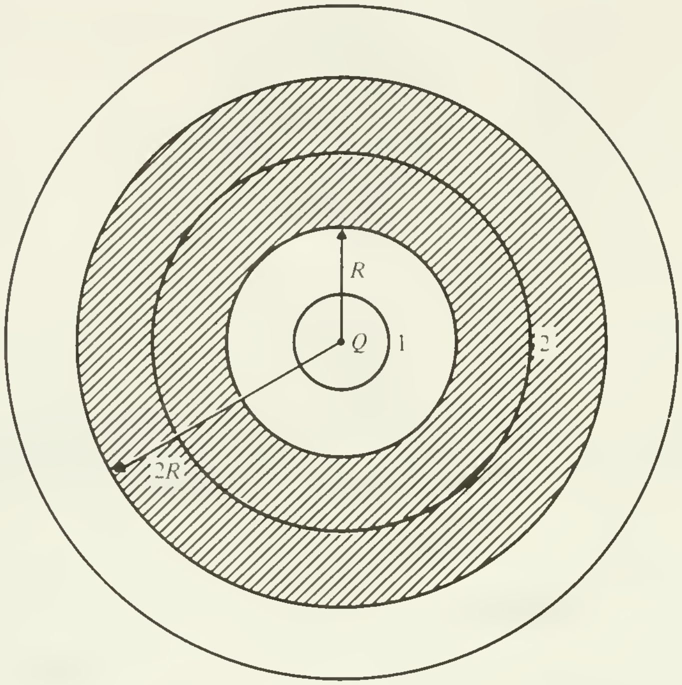


Figure 25-15

we apply Gauss' law to three concentric spherical surfaces, first to the one inside the cavity (1),

$$\oint_1 E_n dA = \frac{Q}{\varepsilon_o}$$

$$E\,4\pi r^2 = \frac{Q}{\varepsilon_o} \qquad r < R$$

$$E = \frac{Q}{4\pi r^2 \varepsilon_o} \qquad r < R$$

Inside the conductor E = 0 and Gauss' law applied to surface 2 implies

$$\oint_2 E_n dA = \frac{Q_{enc}}{\varepsilon_o} = 0 \qquad R < r < 2R \qquad E = 0$$

Surface 2 encloses the point charge Q. Evidently an equal but opposite charge −Q is spread over the <u>inner</u> surface of the conductor at r = R.

Since the conductor is electrically neutral, a charge Q must reside on its outer surface at r = 2R. Thus for the region outside of the conductor

$$\oint_3 E_n dA = \frac{Q_{total}}{\varepsilon_0} = \frac{Q - Q + Q}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

$$E\, 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi r^2 \varepsilon_0} \qquad r > 2R$$

Example 13

An insulating sphere has a uniform charge density ρ per unit volume. Find the electric field inside and outside of the sphere.

Solution:

Referring to Fig. 25-16,

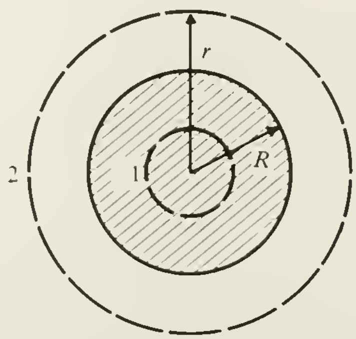


Figure 25-16

we apply Gauss' law to concentric spherical surfaces with radii r < R and r > R. The field is radial and everywhere perpendicular to these surfaces. For r < R

$$\oint_1 E_n dA = \frac{Q}{\varepsilon_0} \qquad Q = \rho V = \rho\, \frac{4}{3}\, \pi r^3$$

$$E\, 4\pi r^2 = \frac{4\rho\pi r^3}{3\varepsilon_0}$$

$$E = \frac{\rho r}{3\varepsilon_o} . \qquad r < R$$

Note in calculating Q, only the charge <u>within</u> the surface is included.

For $r > R$

$$\oint_2 E_n dA = \frac{Q}{\varepsilon_o} \qquad Q = \rho V = \rho \frac{4}{3} \pi R^3$$

$$E\, 4\pi r^2 = \frac{4\rho\pi R^3}{\varepsilon_0}$$

$$E = \frac{R^3}{3r^2\varepsilon_0} \rho \qquad r > R$$

Note at $r = R$ the expressions for the inner and outer fields match.

QUIZ

1. Two opposite charges of magnitude 2×10^{-14} C are separated by 1 cm. Find the electric field on the line passing through the charges, 1 m from the midpoint of the charges.

Answer: 3.6×10^{-6} $N \cdot C^{-1}$. The field is along the line passing through the points, in the negative to positive direction.

2. A long solid cylinder of charge has a non-uniform charge density which varies with the radial coordinate,

$$\rho = \rho_o \frac{r}{R}$$

where R is the radius of the cylinder. Using Gauss's Law find the field inside the cylinder.

Answer: $E = \frac{\rho_o r^2}{3R\varepsilon_o}$, radially outward if $\rho_o > 0$.

3. A modest sized electric field near a conducting surface is 1250 $N \cdot C^{-1}$. What is the surface charge density in terms of electron per square meter?

Answer: 7×10^{10} electrons per m^2

4. Find the electric field on the perpendicular bisector of a uniformly distributed line of charge of magnitude Q and length a as a function of the distance x from the charge.

Answer: $\dfrac{Q}{4\pi\varepsilon_o x\,[(a/2)^2 + x^2]^{1/2}}$

26
ELECTRICAL POTENTIAL

OBJECTIVES

This chapter is based on the conservative nature of the the coulomb force. According to the analysis of Chapter 7, a potential energy function U may then be introduced when the forces are conservative, and the total energy K + U is conserved if no other forces act. The electrical potential V is the potential energy per unit charge, $V = U/q$. Your objectives are to:

Find the potential of a distribution of point charges.

Calculate the potential at a point by calculating the work per unit charge, the line integral of the electric field.

Evaluate the potential difference between charged parallel plates.

Evaluate the potential difference between charged spherical concentric conductors.

Evaluate the potential difference between charged coaxial cylindrical conductors.

Calculate the work done when a charge is moved in an electric field from one potential to another.

Apply the conservation of kinetic and electric potential energy to problems involving charged bodies moving in regions where there is an electric field.

REVIEW

The work done by a conservative force F on a body that moves along a path from a $\longrightarrow$ b is given by the line integral

$$W_{a \longrightarrow b} = \int_a^b \mathbf{F} \cdot d\mathbf{l} = \int_a^b F \cos \theta \, dl = U_a - U_b.$$

The work of the conservative force is path independent and equal to the decrease of the potential energy:

$$U_a - U_b.$$

For a point charge q in the field of a point charge Q, moved from r_a to r_b, the corresponding change in potential energy is

$$W_{a-b} = \int_{r_a}^{r_b} F\ dr = \int_{r_a}^{r_b} \frac{Qq}{4\pi\varepsilon_o} \frac{dr}{r^2} = \frac{Qq}{4\pi\varepsilon_o} \left[- \frac{1}{r}\right]_{r_a}^{r_b}$$

$$= \frac{Qq}{4\pi\varepsilon_o} \left[\left(\frac{1}{r_a}\right) - \left(\frac{1}{r_b}\right)\right] = U_a - U_b.$$

The potential energy of q in the field of Q is thus

$$U = \frac{Qq}{4\pi\varepsilon_o r}$$

where the potential energy is referred to a base level $U = 0$ at $r = \infty$.

The <u>electric potential</u> V is the potential energy per unit charge, defined as the <u>difference</u>

$$V_a - V_b = \frac{U_a - U_b}{q} = \int_a^b \frac{\mathbf{F}\cdot d\mathbf{l}}{q} = \int_a^b \mathbf{E}\cdot d\mathbf{l}.$$

The electric potential is a <u>scalar</u>, unlike the field **E**, which is a vector.

<u>For a point charge</u> the potential difference between points distance r_a and r_b from the charge is:

$$V_a - V_b = \frac{Q}{4\pi\varepsilon_o} \left[\left(\frac{1}{r_a}\right) - \left(\frac{1}{r_b}\right)\right]$$

Taking $r_b = \infty$ to be a reference point where $V_b = 0$, we find the potential at any point $r = r_a$ is

$$V(r) = \frac{Q}{4\pi\varepsilon_o} \frac{1}{r}, \qquad V = 0 \text{ at } r = \infty.$$

For a <u>distribution of point charges</u> q_i at distances r_i, the potential is the sum of the point charge potentials

$$V = \sum_i \frac{q_i}{4\pi\varepsilon_o} \frac{1}{r_i} .$$

Electric potential V and potential energy U are strictly defined only up to an additive constant. The potential V_b may be assigned any value. Often it is convenient to let b be the point at infinity and set $V_b = 0$; then

$$V_a = \int_a^{\infty} \mathbf{E}\cdot d\mathbf{l}.$$

If the potential depends on a single radial coordinate, and the field only has a component in this radial direction, as is the case for a point charge or distributions with spherical or cylindrical symmetry, then

$$V(r) = \int_r^{\infty} \mathbf{E}\cdot d\mathbf{r} = \int_r^{\infty} E\, dr$$

For a point charge of magnitude q

$$V(r) = \int_r^{\infty} \frac{q\, dr}{4\pi\varepsilon_o r^2} = \frac{q}{4\pi\varepsilon_o} \left[-\frac{1}{r}\right]_r^{\infty} = \frac{q}{4\pi\varepsilon_o r}$$

The differential of the potential in general can be written in terms of the <u>gradient</u> operator:

$$dV = \mathbf{grad\ V}\cdot d\mathbf{l} = -\,\mathbf{E}\cdot d\mathbf{l},$$

We retrieve the original definition of potential by integrating dV along a path,

$$\int_a^b dV = V_b - V_a = -\int_a^b \mathbf{E}\cdot d\mathbf{l}$$

$$V_a - V_b = \int_a^b \mathbf{E}\cdot d\mathbf{l}.$$

Thus we have

$$\mathbf{E} = -\ \text{grad}\ V$$

$$E_x = -\left(\frac{\partial V}{\partial x}\right), \quad E_y = -\left(\frac{\partial V}{\partial y}\right), \quad E_Z = -\left(\frac{\partial V}{\partial Z}\right)$$

The SI unit of electric potential is the volt (V). Useful conversions are

$$1\ V = 1\ N\cdot C^{-1}\cdot m = 1\ J\cdot C^{-1}$$

Potential and potential differences are sometimes colloquially referred to as voltage or voltage differences.

EXAMPLES AND SOLUTIONS

Example 1

Two large parallel plates are separated by 0.02 m and have an electric field of magnitude 1000 $N \cdot C^{-1}$ between them.

(a) Find the potential difference between the plates.

(b) Assigning the positively charged plate a potential zero, what is the potential of the negatively charged plate?

(c) Can the potential at the negatively charged plate be positive?

(d) Find the potential at a distance r from the positive plate.

Solution:

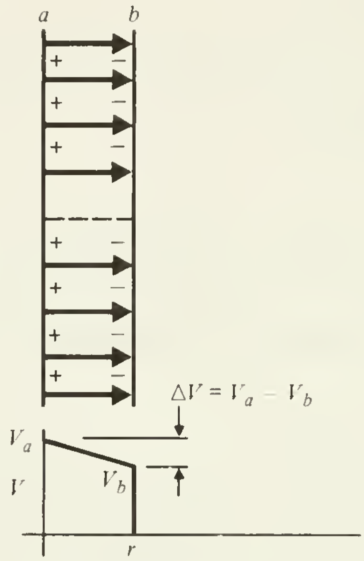


Figure 26-1

(a) The potential (see Fig. 26-1) is given by

$$V_a - V_b = \int_a^b \mathbf{E} \cdot d\mathbf{r}$$

where for the path we choose the one for which the evaluation of the integral is simplest, namely the dashed line perpendicular to the plates and parallel to the field:

$$V_a - V_b = \int_a^b E\, dr = E \int_a^b dr = ER$$

where R is the plate separation. The potential difference is

$$V_a - V_b = (1000\ N \cdot C^{-1})(0.02\ m) = 20\ V.$$

(b) If $V_a = 0$, then plate b is at potential

$$V_b = -20\ V$$

(c) If $V_b = 0$, $V_a = 20$ V.

If $V_a = 10$ V, $V_b = V_a - 20 \text{ V} = 10 \text{ V} - 20 \text{ V} = -10$ V.

If $V_b = +10$ V, $V_a = 20 \text{ V} + V_b = 30$ V.

In the last case V_b is positive. All are possible assignments. In all cases $V_a - V_b = 20$ V.

(d)

$$V_a - V_r = \int_a^r \mathbf{E}\cdot d\mathbf{r} = rE$$

$$V_r = V_a - rE = V_a - r\left(\frac{V_a - V_b}{R}\right)$$

The potential falls linearly from V_a with a slope $-E$, that is

$$\frac{dV_r}{dr} = -E = -\left(\frac{V_a - V_b}{R}\right)$$

V(r) is plotted in Fig. 26, below the parallel plates.

Example 2

Find the electrical field between two large parallel plates, 1 cm apart, with a potential difference of 220 V.

Solution:

The electric field is uniform, as in Fig. 26-1. The potential difference is

$$220 \text{ V} = V_a - V_b = \int_a^b \mathbf{E}\cdot d\mathbf{l} = El$$

$$E = \frac{220 \text{ V}}{1 \text{ cm}} = 220 \text{ V}\cdot\text{cm}^{-1} = 22{,}000 \text{ V}\cdot\text{m}^{-1}$$

$$= 22{,}000 \text{ N}\cdot\text{C}^{-1}.$$

Example 3

Find the potential inside and outside of a solid spherical conductor of radius R = 10 cm carrying a charge $Q = 10^{-9}$ C. Let V = 0 at infinity. Evaluate your result for r = 5 cm and r = 20 cm. Sketch the potential as a function of r.

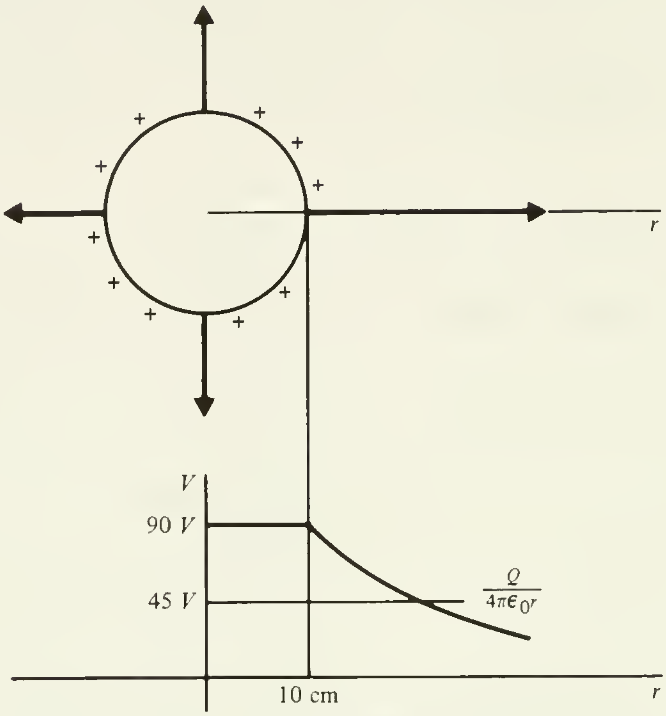


Figure 26-2

Solution:

Referring to Fig. 26-2, we note that the field is radially outward for r > R (the symmetry requires this) and zero within the conductor, r < R (the field is always zero within a conducting region.) Choosing the dashed line as the path for the line integral,

$$V_a - V_b = \int_a^b \mathbf{E}\cdot d\mathbf{l}$$

we set $V_b = 0$ at $b = \infty$ and allow the point a to be at r, $V_a = V(r)$. Thus we have

$$V(r) = \int_r^\infty \mathbf{E}\cdot d\mathbf{r} = \int_r^\infty E_r dr,$$

with

$$E = 0 \qquad r < R$$

$$E_r = \frac{Q}{4\pi\varepsilon_o r^2} \qquad r > R$$

Thus for $r > R$ the integral can be evaluated as

$$V(r) = \int_r^\infty \frac{Q\,dr}{4\pi\varepsilon_o r^2} = \frac{Q}{4\pi\varepsilon_o}\left[-\frac{1}{r}\right]_r^\infty = \frac{Q}{4\pi\varepsilon_o r}$$

At $r = 20$ cm, we find

$$V(20\text{ cm}) = V(0.2\text{ m}) = \frac{10^{-9}\text{ C}}{4\pi(8.85 \times 10^{-12}.C^2.N^{-1}.m^{-2})(0.2\text{ m})}$$

$$V = 45\text{ V}$$

For $r < R$ we break the integral into two parts

$$V(r) = \int_r^R \mathbf{E}\cdot d\mathbf{l} + \int_R^\infty \mathbf{E}\cdot d\mathbf{l}$$

$$= 0 + \int_R^\infty \frac{Q}{4\pi\varepsilon_o r^2}\,dr$$

The first term is zero because $E = 0$ when $r < R$. Thus anywhere inside the solid conductor we have the same potential as at the surface,

$$V = \frac{Q}{4\pi\varepsilon_o}\left[-\frac{1}{r}\right]_R^B = \frac{Q}{4\pi\varepsilon_o R}$$

$$= \frac{10^{-9}\text{ C}}{4\pi(8.85 \times 10^{-12}\ C^2.N^{-1}.m^{-2})(0.1\text{ m})}$$

$$= 90\text{ V}.$$

A sketch of V(r) is given in Fig. 26-2b.

Example 4

A positive charge q of mass m is released from rest at the positive capacitor plate of Fig. 26-1.

(a) Find the acceleration of the charge, and its velocity and kinetic energy when it strikes the other plate. Express your results in terms of V_a, V_b, q, the plate separation L, and the mass m.

(b) Find the work done by the electric field on the charge.

(c) Find the change in potential V and the change in potential energy U of the charge.

(d) What are the relations among the results (a) - (c)?

Solution:

(a) The constant acceleration is

$$a = \frac{F}{m} = \frac{qE}{m} = \frac{q}{m} \frac{V_a - V_b}{L} .$$

If v is the final velocity, and the initial velocity is equal to zero,

$$v^2 = 2aL = 2\left(\frac{q}{m}\right)\left(\frac{V_a - V_b}{L}\right)L = \frac{2q}{m}(V_a - V_b)$$

$$K = \frac{1}{2} mv^2 = q(V_a - V_b).$$

(b) The electric field does the work

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{L} = FL = qEL = q\left(\frac{V_a - V_b}{L}\right) L$$

$$= q(V_a - V_b).$$

(c) The change in potential is

$$\Delta V = V_b - V_a.$$

The change in potential energy is

$$\Delta U = q\,\Delta V = q(V_b - V_a).$$

(d) The work done by the field is the negative of the change in potential energy,

$$W = -\Delta U = -q\,\Delta V$$

The work energy relation is

$$W = \Delta K = -\Delta U$$

or

$$\Delta K + \Delta U = \Delta(K + U) = 0$$

implying

$$\Delta E = \frac{1}{2} mv^2 + q(V_b - V_a) = 0$$

or

$$\frac{1}{2} mv^2 = q(V_a - V_b)$$

as verified above.

Example 5

Find the potential on the x axis at a point P with coordinates (x,0) when two positive point charges of magnitude $Q = 10^{-9}$ C are on the y axis at (0,a) and (0,-a), where a = 1 cm.

(a) Calculate this by superposition of the potentials due to each charge.

(b) Calculate this by direct integration of $\int \mathbf{E}\cdot d\mathbf{l}$. Evaluate the results for x = a.

(c) Find the work done by the electric field of the two charges when a third charge of equal magnitude is moved from the point (2a,0) to (a,0).

Solution:

(a) Refer to Fig. 26-3:

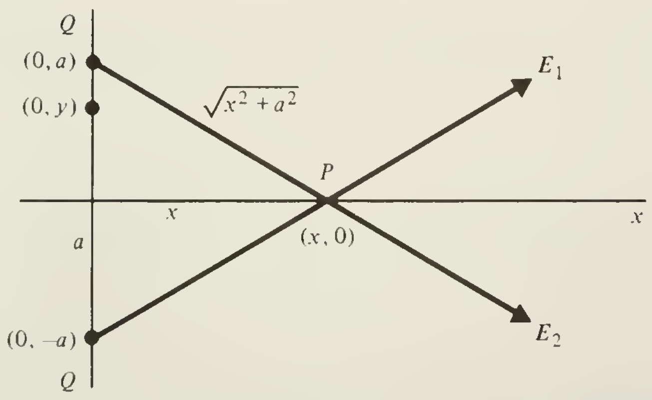


Figure 26-3

The potential at point (x,0) arising from the charges at $(0,\pm a)$ is

$$V = \frac{Q}{4\pi\varepsilon_o(x^2 + a^2)^{1/2}} + \frac{Q}{4\pi\varepsilon_o(x^2 + a^2)^{1/2}}$$

$$= \frac{2\,Q}{4\pi\varepsilon_o(x^2 + a^2)^{1/2}}$$

(b) The field **E** on the x-axis at x has only an x component,

$$E_x = E_{1x} + E_{2x} = 2|E_1|\cos\theta$$

$$= 2\,\frac{Q}{4\pi\varepsilon_o(x^2 + a^2)}\;\frac{x}{(x^2 + a^2)^{1/2}}$$

$$= \frac{2Qx}{4\pi\varepsilon_o(x^2 + a^2)^{3/2}}$$

Thus we have

$$V = \int_x^\infty E_x dx = \frac{2Q}{4\pi\varepsilon_o}\int_x^\infty \frac{xdx}{(x^2 + a^2)^{3/2}}$$

$$= \frac{2Q}{4\pi\varepsilon_o}\left[\frac{-1}{(x^2 + a^2)^{1/2}}\right]_x^\infty$$

$$= \frac{2Q}{4\pi\varepsilon_o}\;\frac{1}{(x^2 + a^2)^{1/2}} \quad \text{as in (a).}$$

At $x = a = 1\ \text{cm} = 10^{-2}$ m,

$$V = \frac{2 \times 10^{-9}\ \text{C}}{4\pi(8.85 \times 10^{-12}\ \text{C}^2\cdot\text{N}^{-1}\cdot\text{m}^{-2})[2(0.01\ \text{m})^2]^{1/2}}$$

$$= 1270\ \text{V}.$$

(c) $$W = \int_a^b \mathbf{F}\cdot d\mathbf{l} = \int_a^b Q\,\mathbf{E}\cdot d\mathbf{l} = Q(V_a - V_b)$$

where

$$V_a = \frac{2Q}{4\pi\varepsilon_o[(2a)^2 + a^2]^{1/2}} = \frac{2Q}{4\pi\varepsilon_o a(5)^{1/2}}$$

$$V_b = \frac{2Q}{4\pi\varepsilon_o(a^2 + a^2)^{1/2}} = \frac{2Q}{4\pi\varepsilon_o(2)^{1/2}}$$

$$W = \frac{2Q^2}{4\pi\varepsilon_o a} \left[\frac{1}{(5)^{1/2}} - \frac{1}{(2)^{1/2}} \right]$$

$$= \frac{(10^{-9}\ C)^2\ (-0.26)}{2\pi(8.85 \times 10^{-12}\ C^2 \cdot N^{-1} \cdot m^{-2})(0.01\ m)}$$

$$= -4.68 \times 10^{-7}\ J.$$

The work done is negative because the displacement is opposite to the direction of the electric force.

Example 6

For the charge distribution of Example 5, Fig. 26-3,

(a) Find the potential at a point (0,y) on the y-axis between the two charges.

(b) If an electron is released from rest at the point (0,0) and given a slight nudge, what is its velocity at (0,a/2)?

Solution:

(a) The potential is

$$V = \frac{Q}{4\pi\varepsilon_o(a - y)} + \frac{Q}{4\pi\varepsilon_o(a + y)}$$

$$= \frac{2aQ}{4\pi\varepsilon_o(a^2 - y^2)} .$$

(b) The potential energy of the electron (with charge −e) is

$$U = -eV = \frac{-2aeQ}{4\pi\varepsilon_o(a^2 - y^2)} .$$

The total energy is conserved,

$$E = K + U$$

$$\Delta E = 0$$

$$\Delta K = \frac{1}{2} mv^2 = -\ \Delta U = [U(0) - U(a/2)]$$

Thus the gain in kinetic energy is

$$= \frac{-2aeQ}{4\pi\varepsilon_o a^2} + \frac{2aeQ}{4\pi\varepsilon_o[a^2 - (a/2)^2]}$$

$$= \frac{2aeQ}{4\pi\varepsilon_o a^2}\left[-1 + \frac{4}{3}\right]$$

$$= \frac{eQ}{6\pi\varepsilon_o a} = \frac{1}{2}mv^2$$

and the velocity is

$$v = \left[\left(\frac{1}{m}\right)\frac{eQ}{3\pi\varepsilon_o a}\right]^{1/2}$$

$$= \left[\frac{(1.6 \times 10^{-19}\ C)(10^{-9}\ C)}{3\pi(9.10 \times 10^{-31}\ kg)(8.85 \times 10^{-12}\ C^2 \cdot N^{-1} \cdot m^{-2})(0.01\ m)}\right]^{1/2}$$

$$= 1.45 \times 10^7\ m \cdot s^{-1}$$

Example 7

In the previous example, find the field at a point on the y axis and verify that

$$E_y = -\frac{dV}{dy}\ .$$

Solution:

The total field is the sum of the fields arising from each charge,

$$E_y = \frac{-Q}{4\pi\varepsilon_o(a - y)^2} + \frac{Q}{4\pi\varepsilon_o(a + y)^2}$$

This can be verified by calculating the negative y derivative of the potential,

$$-\frac{dV}{dy} = -\frac{d}{dy}\left[\frac{Q}{4\pi\varepsilon_o(a - y)} + \frac{Q}{4\pi\varepsilon_o(a + y)}\right]$$

$$= \frac{-Q}{4\pi\varepsilon_o}\left[\frac{-1}{(a - y)^2}(-1) + \frac{-1}{(a + y)^2}\right]$$

$$= \frac{-Q}{4\pi\varepsilon_o(a - y)^2} + \frac{Q}{4\pi\varepsilon_o(a + y)^2} = E_y.$$

Example 8

A coaxial cable consists of an inner solid cylindrical wire conductor of radius $r_1 = 0.2$ mm surrounded by a thin cylindrical coaxial outer sleeve of radius $r_2 = 0.5$ cm. The potential difference between the inner and outer conductors is 220 V. Find the charge per unit area on the inner and outer conductors.

Solution:

The field is radial and depends on the radial coordinate r only. Using Gauss' law on a coaxial cylindrical surface of length L with radius r, $r_1 < r < r_2$,

$$\int \mathbf{E}\cdot d\mathbf{A} = E2\pi rL = \frac{Q}{\varepsilon_o} = \frac{\lambda L}{\varepsilon_o}$$

where λ is the charge per unit length on the inner conductor. Thus the field inside the cable is

$$E = \frac{\lambda}{2\pi\varepsilon_o r} = E_r.$$

The potential difference is

$$V_1 - V_2 = \int_1^2 \mathbf{E}\cdot d\mathbf{l} = \int_{r_1}^{r_2} E_r dr = \int_{r_1}^{r_2} \frac{\lambda dr}{2\pi\varepsilon_o r}$$

$$= \frac{\lambda}{2\pi\varepsilon_o} \ln\left(\frac{r_2}{r_1}\right) = \Delta V.$$

The charge per unit length is

$$\lambda = \frac{2\pi\varepsilon_o\Delta V}{\ln(r_2/r_1)} = \frac{2\pi(8.85 \times 10^{-12}\ C^2\cdot N^{-1}\cdot m^{-2})(220\ V)}{\ln(0.5\ cm/0.02\ cm)}$$

$$= 3.80 \times 10^{-9}\ C\cdot m^{-1}.$$

The charge per unit area is related to the charge per unit length by

$$dq = \sigma\ dA = \lambda\ dl$$

$$\sigma = \lambda \frac{dl}{dA} = \frac{\lambda \, dl}{d(2\pi r l)} = \frac{\lambda}{2\pi r}$$

$$\sigma_1 = \frac{\lambda}{2\pi r_1} = \frac{3.80 \times 10^{-9}\ \mathrm{C \cdot m^{-1}}}{2\pi(2 \times 10^{-4}\ \mathrm{m})}$$

$$= 3.02 \times 10^{-6}\ \mathrm{C \cdot m^{-2}}$$

$$\sigma_2 = \frac{\lambda}{2\pi r_2} = \frac{3.80 \times 10^{-9}\ \mathrm{C \cdot m^{-2}}}{2\pi\ 5 \times 10^{-3}\ \mathrm{m}}$$

$$= 1.21 \times 10^{-7}\ \mathrm{C \cdot m^{-2}}$$

Example 9

As shown in Fig. 26-4, a solid conducting sphere of radius r_a is concentric with a hollow spherical shell of inner radius r_b and outer radius r_c. The potential difference between the spheres is V_{ab}.

(a) Find the field between the spheres and the charge on the inner sphere.

(b) Find the charge on the surface at r_b.

(c) Find the charge on the surface at r_c if $E = 0$ for $r > r_c$.

(d) Find the field outside if the outer conductor is neutral, that is has zero net charge.

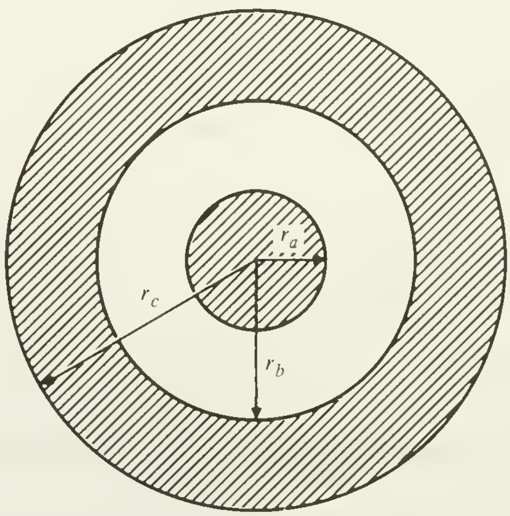


Figure 26-4

Solution:

(a) The field is radial and depends on r only. Let Q be the total charge on the inner sphere, residing on the surface at r_a. We don't know Q, but can find it knowing V_{ab}. By Gauss' law

$$\int \mathbf{E}\cdot d\mathbf{A} = E4\pi r^2 = \frac{Q}{\varepsilon_o} \qquad r_a < r < r_b$$

so that the field in the cavity is

$$E = \frac{Q}{4\pi\varepsilon_o r^2}.$$

The potential difference between r_a and r_b is

$$V_{ab} = V_a - V_b = \int_a^b \mathbf{E}\cdot d\mathbf{l} = \int_{ra}^{rb} \frac{Q}{4\pi\varepsilon_o} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_o} \left[\frac{-1}{r} \right]_{r_a}^{r_b}$$

$$= \frac{Q}{4\pi\varepsilon_o} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]$$

Thus the charge on the sphere is

$$Q = 4\pi\varepsilon_o V_{ab} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]^{-1}$$

and the field is also given by

$$E = \frac{Q}{4\pi\varepsilon_o r^2} = \frac{V_{ab}}{r^2} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]^{-1}$$

(b) The field inside the inner sphere and between r_b and r_c is zero because the material is a conductor. By Gauss' law, the net charge enclosed by a sphere of radius r, $r_b < r < r_c$, must be zero. Thus the charge on the surface at r_b is equal and opposite to the charge on the inner sphere,

$$Q = -4\pi\varepsilon_o V_{ab} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]^{-1}$$

(c) Since the field outside of both conductors is zero, the net charge enclosed in a sphere of radius $r > r_c$ is zero, by Gauss' law. Thus there is no charge on the surface at r_c.

(d) If the outer conductor is neutral, a charge Q must be on the surface at r_c to balance the charge -Q on the surface at r_b. Outside of the system at $r > r_c$

an application of Gauss' law yields

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{Q - Q + Q}{\varepsilon_o} = \frac{Q}{\varepsilon_o} = E4\pi r^2$$

$$E = \frac{Q}{4\pi\varepsilon_o r^2}$$

Example 10

Sketch the potential V as a function of r for the system of Example 9 when
(a) the outer conductor has net charge -Q and
(b) the outer conductor is electrically neutral.

Solution:

(a) We choose the reference potential to be $V = 0$ at $r = \infty$. Thus

$$V(r) = \int_r^{\infty} \mathbf{E}\cdot d\mathbf{l}$$

$$V(r) = \int_r^{r_a} Edr + \int_{r_a}^{r_b} Edr + \int_{r_b}^{r_c} Edr + \int_{r_c}^{\infty} Edr, \qquad 0 < r < r_a$$

$$V(r) = \int_r^{r_b} Edr + \int_{r_b}^{r_c} Edr + \int_{r_c}^{\infty} Edr, \qquad r_a < r < r_b$$

$$V(r) = \int_r^{r_c} Edr + \int_{r_c}^{\infty} Edr, \qquad r_b < r < r_c$$

$$V(r) = \int_r^{\infty} Edr, \qquad r_c < r$$

The line integral has been split into parts as a convenience because the expressions for the fields may differ in each region. Since $E = 0$ for $r < r_a$ and $r_b < r < r_c$, and $E=0$ for $r > r_c$,

$$V(r) = \int_{r_a}^{r_b} Edr, \qquad 0 < r < r_a$$

$$V(r) = \int_r^{r_b} Edr, \qquad r_a < r < r_b$$

$$V(r) = 0, \qquad r_b < r < r_c$$

$$V(r) = 0, \qquad r_c < r$$

Thus

$$V(r) = \int_{r_a}^{r_b} E dr = \frac{Q}{4\pi\varepsilon_o} \left[\frac{-1}{r} \right]_{r_a}^{r_b} = \frac{Q}{4\pi\varepsilon_o} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]$$

$$= V_{ab}, \qquad r \leq r_a$$

$$V(r) = \int_{r_a}^{r_b} E dr = \frac{Q}{4\pi\varepsilon_o} \left[\frac{-1}{r} \right]_{r}^{r_b} = \frac{Q}{4\pi\varepsilon_o} \left[\frac{1}{r} - \frac{1}{r_b} \right]$$

$$= V_{ab} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]^{-1} \left[\frac{1}{r} - \frac{1}{r_b} \right] \quad r_a < r < r_b$$

$$V = 0 \qquad r > r_b$$

A sketch is given in Fig. 26-5a

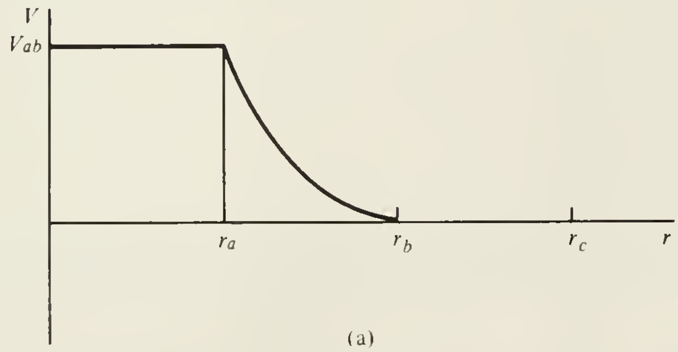


Figure 26-5a

(b) When the outer conductor is neutral, there is a field outside the system, and

$$V(r) = \int_{r_a}^{r_b} E dr + \int_{r_c}^{\infty} E dr, \qquad 0 < r < r_a$$

$$V(r) = \int_{r}^{r_b} E dr + \int_{r_c}^{\infty} E dr, \qquad r_a < r < r_b$$

$$V(r) = \int_{r_c}^{\infty} E dr, \qquad r_b < r < r_c$$

$$V(r) = \int_r^\infty E\,dr, \qquad r_c < r$$

Thus

$$V(r) = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_a} - \frac{1}{r_b}\right] + \int_{r_c}^{\infty} \frac{Q}{4\pi\varepsilon_o}\frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_a} - \frac{1}{r_b} + \frac{1}{r_c}\right] \qquad 0 < r < r_a$$

$$V(r) = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r} - \frac{1}{r_b} + \frac{1}{r_c}\right] \qquad r_a < r < r_b$$

$$V(r) = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_c}\right] \qquad r_b < r < r_c$$

$$V(r) = \frac{Q}{4\pi\varepsilon_o}\frac{1}{r} \qquad r_c < r$$

For a sketch see Fig. 26-5b

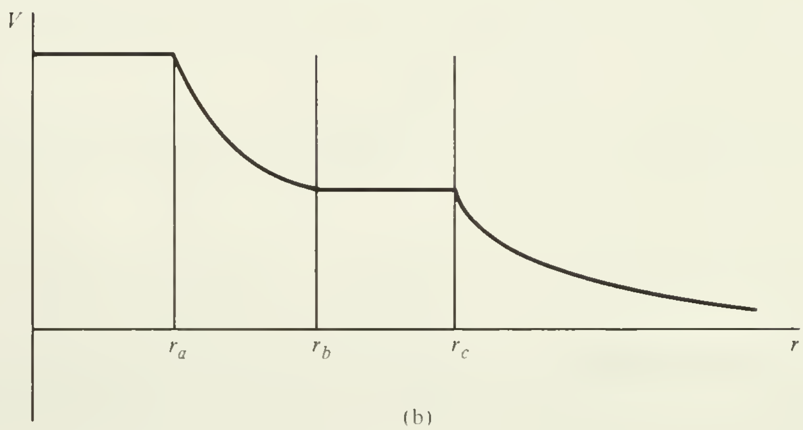


Figure 26-5b

Example 11

(a) Estimate the energy necessary to separate completely an electron and proton bound by the Coulomb force in a hydrogen atom, normally a distance 10^{-8} cm apart?
(b) Through what voltage must an electron be accelerated to gain this energy?

Solution:

(a) The work done is equal to the increase in potential energy

$$W = U = -eV = -e[V(\infty) - V(r)]$$

$$= -e[0 - \frac{e}{4\pi\varepsilon_o r}]$$

$$= \frac{e^2}{4\pi\varepsilon_o r}$$

$$= [\frac{(1.6 \times 10^{-19}\ C)^2}{4\pi(8.85 \times 10^{-12}\ C^2 \cdot N^{-1} \cdot m^{-2})(10^{-10}\ m)}]$$

$$W = 2.3 \times 10^{-18}\ J$$

(b) $$e\Delta V = 2.3 \times 10^{-18}\ J$$

$$\Delta V = \frac{2.3 \times 10^{-18}\ J}{1.6 \times 10^{-19}\ C} = 14.4\ V.$$

QUIZ

1. A positive charge of magnitude 10^{-9} C is released from rest in a constant electric field of magnitude $4 \times 10^4\ N \cdot C^{-1}$. What is its kinetic energy when it is 6×10^{-2} m from its initial position?

Answer: 2.4×10^{-6} J

2. Find the potential on the axis of a uniformly charged disk of radius a, a distance x from the disk.

Answer: $V = \frac{\sigma}{2\varepsilon_o}[(a^2 + x^2)^{1/2} - x]$

3. The conductors of a coaxial conducting cable consist of an inner cylinder of radius one mm and an outer cylinder of radius 5 mm. What is the charge per unit length on the cable when the voltage across the conductors is 8 V?

Answer: $2.8 \times 10^{-10}\ C \cdot m^{-1}$

4. A pair of charges $\pm$ Q are on the x-axis at $x = \pm$ s. Find the potential at a point

(a) on the x-axis
(b) on the y-axis

Answer: (a) $\dfrac{Qx}{2\pi\varepsilon_o(s^2 - x^2)}$

(b) zero

27
CAPACITANCE AND DIELECTRICS

OBJECTIVES

In this chapter the idea of capacitance between conductors and the effects of dielectric materials on electric fields and potentials are developed. Your objectives are to:

Calculate the capacitance of simple systems such as parallel plates, concentric spheres and coaxial cylinders.

Calculate the equivalent capacitance of networks of capacitors in series and parallel.

Calculate the energy stored in a capacitor by finding the work done to charge it.

Describe the effect of dielectric materials on capacitance by the dielectric constant K.

Interpret this effect as due to induced charge Q_i and a polarization P.

Define the displacement **D** and obtain a Gauss' law for **D**.

REVIEW

A capacitor usually consists of two conductors, often with equal and opposite charges, Q, -Q, separated by an insulator. If the potential difference between the conductors is V, the capacitance C is the ratio of charge to potential

$$C = \frac{Q}{V}$$

The unit of capacitance is the coulomb $(\text{volt})^{-1}$ = farad (F).

The capacitance of a parallel plate capacitor is

$$C = \frac{\varepsilon_o A}{d}$$

where A is the plate area and d the plate separation.

The equivalent capacitances of two capacitors in series or parallel are given in Fig. 27-1.

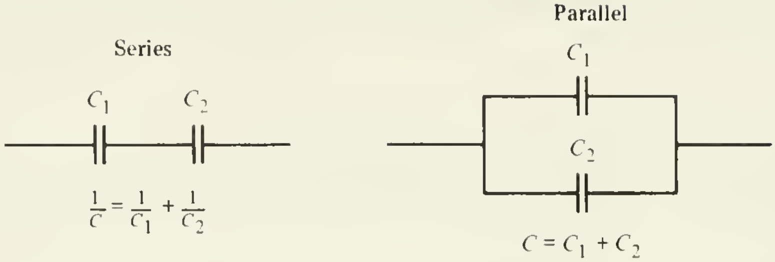


Figure 27-1

The energy needed to place a charge Q on a capacitor is

$$W = \frac{Q^2}{2C} = 1/2(CV^2) = 1/2(QV).$$

This energy can be thought of as stored in the region of space where there is an electric field. The energy per unit volume is

$$u = 1/2(\varepsilon_o E^2).$$

The capacitance increases when dielectric material is inserted in the region between the plates of a capacitor; that is, for a given potential, the charge on the plates increases when a dielectric is inserted. The dielectric constant of the material may be measured as the ratio of the increased capacitance to the original capacitance:

$$K = \frac{C}{C_o} = \frac{\varepsilon}{\varepsilon_o}$$

where C is the capacitance with dielectric material and C_o the capacitance in a vacuum. When the charge on the plates is fixed, the effect of the material is to reduce the field (and hence the potential) between the plates for a given charge Q. The capacitance of a parallel plate capacitor with dielectric material between the plates of an area A and separation L is

$$C = K\, C_o = K \frac{A\varepsilon_o}{L} = \frac{A\varepsilon}{L}, \qquad \varepsilon = K\varepsilon_o.$$

The electric field is reduced because the molecules of the material line up with the electric field in such a way that part of the field due to the charge Q is cancelled by the molecular fields arising from induced or bound charge Q_i. This lining up is called polarization, and is measured by the dipole moment per unit volume P. The Gauss' law for P is

$$\oint \mathbf{P}\cdot d\mathbf{A} = -Q_i$$

If Q is the charge on conductors ('free charge'), the total charge is $Q + Q_i$ and the Gauss' law for **E** is

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{1}{\varepsilon_o}(Q + Q').$$

The vector electric displacement **D** is defined as

$$\mathbf{D} = \varepsilon_o\mathbf{E} + \mathbf{P}$$

and **D** satisfies the Gauss' law

$$\oint \mathbf{D}\cdot d\mathbf{A} = Q.$$

In dielectrics where P is proportional to E, and has the same direction

$$\mathbf{D} = \varepsilon\mathbf{E} = K\varepsilon_o\mathbf{E}.$$

EXAMPLES AND SOLUTIONS

Example 1

(a) Find the capacitance of a parallel plate capacitor with plate separation 3 mm and plate area 100 cm^2.
(b) If the potential across the plates is 110 V, find the charge on each plate.
(c) Find the field between the plates.
(d) Find the energy stored in the capacitor. (Unless otherwise specified, assume that the material between the plates has K=1, $\varepsilon = \varepsilon_o$, true for a vacuum and a good approximation for air.)

Solution:

(a) $$C = \frac{\varepsilon_o A}{L} = \frac{(8.85 \times 10^{-12}\ C^2\cdot N^{-1}\cdot m^{-2})(100 \times 10^{-4} m^2)}{0.003\ m}$$

$$= 2.95 \times 10^{-11}\ F = 29.5\ pF$$

(b) $$Q = VC = (110\ V)(2.95 \times 10^{-11}\ F) = 3.25 \times 10^{-9}\ C$$

(c) $$E = \frac{\sigma}{\varepsilon_o} = \frac{Q}{A\varepsilon_o}$$

$$= \frac{3.25 \times 10^{-9}\ C}{(100 \times 10^{-4}\ m^2)(8.85 \times 10^{-12}\ C^2 \cdot N^{-1} \cdot m^{-2})}$$

$$= 3.67 \times 10^4\ V \cdot m^{-1}$$

Alternatively

$$E = \frac{V}{L} = \frac{110\ V}{0.003\ m} = 3.667 \times 10^4\ V \cdot m^{-1}$$

(d) $$W = \frac{Q^2}{2C} = \frac{(3.25 \times 10^{-9}\ C)^2}{(2)(2.95 \times 10^{-11})} = 1.79 \times 10^{-7}\ J$$

Example 2

A parallel plate capacitor has a charge 10^{-9} C when the plates are 1 cm apart and it is connected to a voltage source of 12 V.

(a) If the plates are pulled to 2 cm separation while keeping the potential constant, what is the new charge?

(b) If the plates are disconnected from the voltage source and pulled to 2 cm separation, while keeping the charge constant, what is the new potential?

Solution:

The original capacitance is

$$C_1 = \frac{\varepsilon_o A}{L} = \frac{Q_1}{V_1}$$

The final capacitance is

$$C_2 = \frac{\varepsilon_o A}{2L} = \frac{Q_2}{V_2} = \frac{C_1}{2}$$

(a) If $V_1 = V_2$ (this is the case if the capacitor is connected to a 12 V battery while the plates are pulled apart),

$$Q_2 = C_2 V_2 = C_2 V_1 = \frac{C_1 V_1}{2} = \frac{Q_1}{2} = 0.5 \times 10^{-9}\ C.$$

(b) If $Q_1 = Q_2$ (this would be the case if the insulated plates were disconnected from the battery before being pulled apart.)

$$V_2 = \frac{Q_2}{C_2} = \frac{2Q_1}{C_1} = 2V_1$$

$$= 24 \text{ V}.$$

Example 3

A capacitor with capacitance $C_1 = 30\ \mu F$ is charged to a potential of 500 V, disconnected from the source, and then connected in parallel to an uncharged capacitor with capacitance $C_2 = 10\ \mu F$.

(a) Find the charge on the 30 μF capacitor before the connection is made.
(b) Find the final charge on each of the capacitors.
(c) Find the energy lost when the connection is made.
(d) What happens to the energy?

Solution:

(a) Originally we have the charge

$$Q = VC = (500 \text{ V})(30 \times 10^{-6} \text{ F}) = 1.50 \times 10^{-2} \text{ C}$$

(b) When the capacitors are connected in parallel the original charge redistributes itself across the two capacitors as shown in Fig. 27-2.

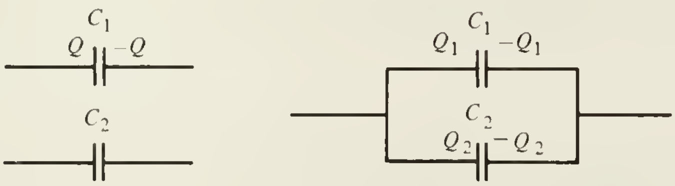


Figure 27-2

$$Q = Q_1 + Q_2$$

In parallel connection, the potential across each capacitor is the same,

$$V_1 = V_2$$

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q - Q_1}{C_2}$$

$$Q_1 \left[\frac{1}{C_1} + \frac{1}{C_2} \right] = \frac{Q}{C_2}$$

$$Q_1 = \frac{Q}{C_2}\left[\frac{1}{C_1} + \frac{1}{C_2}\right]^{-1} = \left(\frac{C_1}{C_1 + C_2}\right)Q$$

Another way to get this result is to note that the equivalent capacitance of the combination is $C = C_1 + C_2$. The voltage across the system is equal to the voltage across each capacitor,

$$V = \frac{Q}{C} = \frac{Q}{C_1 + C_2} = \frac{Q_1}{C_1}$$

$$Q_1 = \left(\frac{C_1}{C_1 + C_2}\right)Q$$

$$= \left(\frac{30\ \mu F}{30\ \mu F + 10\ \mu F}\right)1.50 \times 10^{-2}\ C$$

$$= 1.13 \times 10^{-2}\ C$$

$$Q_2 = Q - Q_1 = 1.50 \times 10^{-2}\ C - 1.13 \times 10^{-2}\ C$$

$$= 0.37 \times 10^{-2}\ C$$

(c) The original energy is

$$W = \frac{Q^2}{2C_1} = \frac{(1.5 \times 10^{-2}\ C)^2}{(2)(30 \times 10^{-6}\ F)} = 3.75\ J.$$

The final energy is

$$W = \frac{Q_1{}^2}{2C_1} + \frac{Q_2{}^2}{2C_2}$$

$$= \frac{(1.13 \times 10^{-2}\ C)^2}{(2)(30 \times 10^{-6}\ F)} + \frac{(0.37 \times 10^{-2}\ C)^2}{(2)(10 \times 10^{-6}\ F)}$$

$$= 2.13\ J + 0.68\ J$$

$$= 2.81\ J$$

Another way to calculate the energy is to use the equivalent capacitance, $C = C_1 + C_2$, yielding the same result

$$W = \frac{Q^2}{2C} = \frac{(1.5 \times 10^{-2}\ C)^2}{(2)(40 \times 10^{-6}\ F)} = 2.81\ J.$$

The energy lost in connecting the capacitors is

$$\Delta W = (3.75 - 2.81)J = 0.94 \text{ J}.$$

(d) The energy is lost to dissipative processes as the connection is made.

Example 4

Find the capacitance of a capacitor consisting of two thin concentric conducting shells of inner radius r_a = 1 cm and outer radius r_b = 2 cm.

Solution:

If Q is the charge on the inner conductor and -Q is the charge on the outer conductor, the field between the shells is

$$E = \frac{Q}{4\pi\varepsilon_o r^2} .$$

The potential difference between the shells is

$$V = V_a - V_b = \int \frac{Q dr}{4\pi\varepsilon_o r^2} = \frac{Q}{4\pi\varepsilon_o} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]$$

The capacitance is then

$$C = \frac{Q}{V} = 4\pi\varepsilon_o \left[\frac{1}{r_a} - \frac{1}{r_b} \right]^{-1}$$

$$= 4\pi(8.85 \times 10^{-12}\ C^2 \cdot N^{-1} \cdot m^{-2}) \left[\frac{1}{0.01\ m} - \frac{1}{0.02\ m} \right]^{-1}$$

$$= 2.22 \times 10^{-12}\ F = 2.22\ pF$$

$$(1\ pF = 10^{-12}\ F \qquad 1\ \mu F = 10^{-6}\ F)$$

Example 5

Find the capacitance of two concentric conducting cylinders of inner radius r_a = 1 mm and outer radius r_b = 2 mm and length L = 1 m.

Solution:

The field between the cylinders is

$$E = \frac{\lambda}{2\pi\varepsilon_o r}$$

where λ is the charge per unit length on the inner conductor. The potential between the conductors is

$$V = \int_{r_a}^{r_b} \frac{\lambda dr}{2\pi\varepsilon_o r} = \frac{\lambda}{2\pi\varepsilon_o} \ln\left(\frac{r_b}{r_a}\right)$$

The capacitance is

$$C = \frac{Q}{V} = \frac{\lambda L}{(\lambda/2\pi\varepsilon_o)\ln(r_b/r_a)} = \frac{2\pi\varepsilon_o L}{\ln(r_b/r_a)}$$

$$= 8.02 \times 10^{-11} \text{ F} = 80.2 \text{ pF}$$

Example 6

Find the equivalent capacitance between points a and b of the network of Fig. 27-3a.

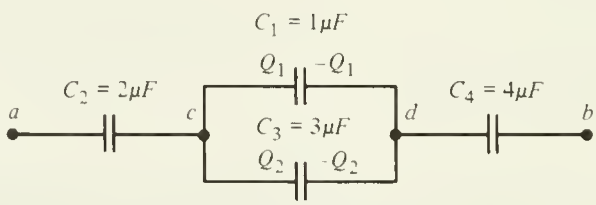


Figure 27-3a

Solution:

The 1 μF and 3 μF are in parallel and their equivalent capacitance is 1 μF + 3 μF = 4 μF. This reduces the network to the three series capacitances shown in Fig. 27-3b. The equivalent capacitance of this network is C, with

$$\frac{1}{C} = \frac{1}{2\ \mu F} + \frac{1}{4\ \mu F} + \frac{1}{4\ \mu F} = \frac{2 + 1 + 1}{4\ \mu F}$$

$$= \frac{1}{\mu F}; \qquad C = 1\ \mu F$$

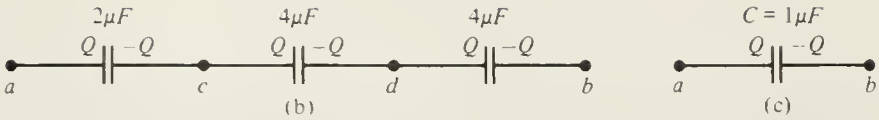


Figure 27-3bc

Example 7

In the network of Fig. 27-3, V_{ab} = 1000 V. Find the charges on each capacitor and the potential drops V_{ac}, V_{cd}, and V_{db}.

Solution:

The charge on the equivalent capacitor (see Fig. 27-3c) is

$$Q = VC = (1000\ V)(1\ \mu F) = 10^{-3}\ C.$$

The charge on each of the series reduced capacitors is the same, as shown in Fig. 27-3b. The potential drops are thus

$$V_{ac} = \frac{Q}{C_{ac}} = \frac{10^{-3}\ C}{2\ \mu F} = 500\ V$$

$$V_{cd} = \frac{Q}{C_{cd}} = \frac{10^{-3}\ C}{4\ \mu F} = 250\ V$$

$$V_{db} = \frac{Q}{C_{db}} = \frac{10^{-3}\ C}{4\ \mu F} = 250\ V$$

Note $\quad V_{ac} + V_{cd} + V_{db} = V_{ab}.$

Now we can find the charge on the 1 μF and 3 μF capacitors. Since they are in parallel they have a common potential drop V_{cd}. Thus

$$Q_1 = C_1V_{cd} = (1\ \mu F)(250\ V) = 0.25 \times 10^{-3}\ C$$

$$Q_3 = C_3V_{cd} = (3\ \mu F)(250\ V) = 0.75 \times 10^{-3}\ C$$

Note

$$Q_1 + Q_3 = 10^{-3}\ C = Q.$$

Example 8

Suppose now that the charged capacitors C_1 and C_3 of Fig. 27-3a, Examples 6 and 7, are disconnected and reconnected with terminals of unlike sign together. Find the new charge on each capacitor and the potential across each.

Solution:

We originally have the configuration of Fig. 27-4a and then for a moment the configuration of Fig. 27-4b after the crossed reconnection; the charges then redistribute so that the potential across each capacitor is equal, as shown in Fig. 27-4c. Since total charge is conserved,

$$Q_1' + Q_3' = Q_1 - Q_3 = (0.25 - 0.75)10^{-3}\ C$$

$$= -\ 0.50 \times 10^{-3}\ C$$

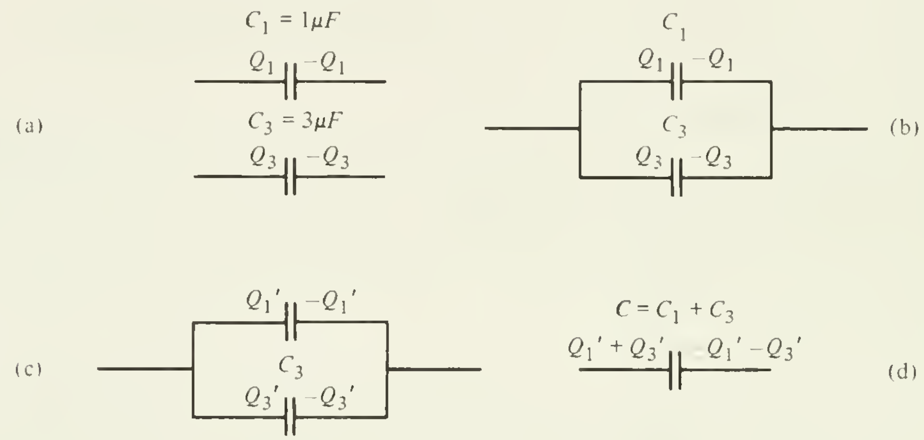


Figure 27-4

The potential across the combination may be calculated by noting that the equivalent capacitor shown in Fig. 27-4d has capacitance,

$$C = C_1 + C_3 = 4\ \mu F$$

so that the potential across it is

$$V = \frac{Q}{C} = \frac{0.50 \times 10^{-3}\ C}{4\ \mu F} = 125\ V.$$

Since $Q_1' + Q_3'$ is negative the potential across the combination is $V' = -125$ V. The charge on each capacitor is

$$Q_1' = V'C_1 = -\ 125\ V\ (1\ \mu F) = -125 \times 10^{-6}\ C$$

$$Q_3' = V'C_3 = -\ 125\ V\ (3\ \mu F) = -375 \times 10^{-6}\ C$$

Note

$$Q_1' + Q_3' = -\ 500 \times 10^{-6}\ C \text{ as calculated above.}$$

Example 9

A parallel plate capacitor of area A has charge Q and separation x.

(a) Find the potential energy stored in the field in terms of Q, A and x.

(b) Find the force of attraction between the plates by calculating the field at one plate arising from the charge on the other plate.

(c) Find the work necessary to separate the plates an additional distance Δx in terms of the force F and Δx. By equating this to the increase in potential energy, find F, checking the results of (b).

Solution:

(a) The potential energy is

$$W = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2}{C} ; \qquad C = \frac{A\varepsilon_o}{x}$$

$$= \frac{1}{2} \frac{Q^2 x}{A\varepsilon_o} .$$

(b) The field due only to one plate is half the total field,

$$E' = \frac{\sigma}{2\varepsilon_o} = \frac{Q}{2A\varepsilon_o}$$

The force of one plate on the other is the product of the field due to one plate and the charge on the other plate,

$$F = QE' = \frac{Q^2}{2A\varepsilon_o}$$

(c) The work done by an external force when the plates are separated a distance Δx is

$$F\Delta x = \Delta W = \frac{1}{2} \frac{Q^2 \Delta x}{A\varepsilon_o}$$

$$F = \frac{Q^2}{2A\varepsilon_o}$$

Example 10

A parallel plate capacitor has a capacitance of 1 pF = 10^{-12} F and is filled with dielectric material with a dielectric coefficient K = 3. The dielectric breaks down and becomes conducting at an electric field of 2×10^5 V·cm^{-1}.

(a) If the plate separation is 1 mm, what is the area of the capacitor?

(b) What is the maximum potential that may be put across the capacitor before it breaks down?

Solution:

(a) The capacitance of the parallel plate capacitor when filled with dielectric material is

$$C = KC_o = \frac{KA\varepsilon_o}{L}$$

Thus

$$A = \frac{LC}{K\varepsilon_o} = \frac{(10^{-3}\ \text{m})(10^{-12}\ \text{F})}{(3)(8.85 \times 10^{-12}\ \text{C}^2 . \text{N}^{-1} . \text{m}^{-2})}$$

$$= 3.77 \times 10^{-5}\ \text{m}^2 = 0.37\ \text{cm}^2$$

(b) The field between the plates is

$$E = V/L$$

$$E_{max} = \frac{V_{max}}{L}$$

$$V_{max} = LE_{max} = (10^{-3}\ \text{m})(2 \times 10^5\ \text{V}\cdot\text{cm}^{-1})$$

$$= 2 \times 10^4\ \text{V}$$

Example 11

A parallel plate capacitor of area A and plate separation L is filled with a removable dielectric slab of dielectric constant K. The capacitor is given a charge Q with the slab removed, disconnected from the battery, and then the slab is inserted.

(a) Find the potential difference without the slab.

(b) Find the potential difference with the slab.

(c) Find the field between the plates without the slab.

(d) Find the field with the slab.

(e) Find the induced or bound charge Q_i on the dielectric.

(f) Find the displacement D with and without the slab.

(g) Find the polarization P of the dielectric slab.

Solution:

(a) $$Q = VC \qquad V = \frac{Q}{C} = \frac{QL}{A\varepsilon_o}$$

(b) $$V' = \frac{Q}{C'} = \frac{QL}{A\varepsilon}$$

$$= \frac{QL}{KA\varepsilon_o} = \frac{V}{K} < V$$

(c) $$E = \frac{V}{L} = \frac{Q}{A\varepsilon_o}$$

(d) $$E' = \frac{V'}{L} = \frac{Q}{A\varepsilon} = \frac{Q}{KA\varepsilon_o} = \frac{E}{K} < E$$

(e) $$E' = \frac{Q + Q_i}{A\varepsilon_o} = \frac{Q}{A\varepsilon}$$

$$Q + Q_i = \frac{\varepsilon_o}{\varepsilon} Q = \frac{Q}{K}$$

$$Q_i = Q[\frac{1}{K} - 1]$$

Note the bound charge on the slab surface is opposite in sign to the contiguous free charge on the plate.

(f) $$D = \varepsilon E = \varepsilon_o E = \frac{Q}{A} \qquad \text{(no slab)}$$

$$D' = \varepsilon E = \frac{\varepsilon Q}{A\varepsilon_o} = \frac{KQ}{A} \qquad \text{(slab)}$$

(g) $$P = D - \varepsilon_o E = 0 \qquad \text{(no slab)}$$

$$P = \varepsilon E - \varepsilon_o E = (\varepsilon - \varepsilon_o)\frac{Q}{A\varepsilon_o} \qquad \text{(slab)}$$

$$= (K - 1)\frac{Q}{A}$$

Example 12

In the last example suppose the capacitor is always connected to a battery keeping it at a voltage V.

(a) Find the charge on the plates <u>without</u> the slab.
(b) Find the charge on the plates <u>with</u> the slab inserted.
(c) Find the field without the slab.
(d) Find the field with the slab.
(e) Find the bound or induced charge Q_i when the slab is inserted.
(f) Find the displacement vector with and without the slab.
(g) Find the polarization of the dielectric slab.

Solution:

(a) $$Q = VC = \frac{VA\varepsilon_o}{L}$$

(b) $$Q' = VC' = \frac{VA\varepsilon}{L} > Q \quad \text{(increases)}$$

(c) $$E = \frac{V}{L}$$

(d) $$E' = \frac{V}{L} \qquad \text{(same)}$$

(e) $$E = \frac{Q_{total}}{A\varepsilon_o} = \frac{Q}{A\varepsilon_o} = \frac{V}{L} \quad \text{(without slab)}$$

$$E' = \frac{Q_{total}}{A\varepsilon_o} = \frac{Q' + Q_i'}{A\varepsilon_o} \quad \text{(with slab)}$$

Equating the two fields by (c) and (d), and using the results of (a) and (b), we have

$$\frac{Q' + Q_i'}{\varepsilon_o} = \frac{VA}{L}$$

$$\frac{VA\varepsilon/1 + Q_i'}{\varepsilon_o} = \frac{VA}{L}$$

$$Q_i' = (\varepsilon_o - \varepsilon)\frac{VA}{L}$$

(f) $$D = \varepsilon_o E = \frac{\varepsilon_o V}{L} = \frac{Q}{A} \quad \text{(no slab)}$$

$$D = \varepsilon E = \frac{\varepsilon V}{L} = \frac{Q'}{A} \quad \text{(slab)}$$

(g) $$P = (\varepsilon - \varepsilon_o)E = (\varepsilon - \varepsilon_o)\frac{V}{L}$$

QUIZ

1. A parallel plate capacitor has a plate area of 4×10^{-2} m^2 and a plate separation of 10^{-2} m. The potential difference across the plates is 25 V.
   (a) Find the capacitance of the capacitor.
   (b) Find the charge on the plates.
   (c) Find the electric field between the plates.
   (d) Find the energy stored in the capacitor.

Answer: 35.4×10^{-12} F, 0.88×10^{-9} C, 2500 $V \cdot m^{-1}$, 1.1×10^{-8} J

2. A 1 μF and a 2 μF capacitor are connected in parallel across a 600 V line.
   (a) Find the charge on each capacitor and the voltage across each.
   (b) The charged capacitors are then disconnected from the line and each other, and reconnected with the terminals of unlike sign together. Find the final charge on each and the voltage across each.

Answer: (a) 1.2 mC, 0.6 mC, V = 600 V. (b) 200 V, 0.2 mC, 0.4 mC

3. An air-gap capacitor remains connected to a battery. How do the following change when a dielectric slab of constant K = 2 is inserted between the plates:
   (a) electric field.
   (b) charge on the plates.
   (c) potential across the plates.

Answer: (a) remains the same (b) doubles (c) remains the same

4. An air-gap capacitor is given a charge Q and then disconnected from the battery. How do the following change when a dielectric slab of constant $K = 2$ is inserted between the plates:

(a) electric field.

(b) charge on the plates.

(c) potential across the plates.

Answer: (a) halves (b) remains the same (c) halves

28
CURRENT, RESISTANCE, AND ELECTROMOTIVE FORCE

OBJECTIVES

In this chapter you will be introduced to the basic elements of circuits: current, resistance, potential differences, emf's and power dissipated. Your objectives are to:

Define electric current I and electric current density J. Relate them to the density and velocities of moving charge carriers.

Apply Ohm's law for I and J, in terms of resistivity ρ and resistance R. Note change of resistance with temperature.

Define an electromotive force (emf). Find the open and closed circuit potential difference across a battery of given emf and internal resistance.

Apply Kirchhoff's loop rule to a simple circuit with emf's and resistances.

Develop and apply the idea of work and power in elements of a simple circuit.

Gain a picture of how and why metals conduct electric current.

REVIEW

When a charge flows through an area there is a current

$$I = \frac{dQ}{dt} = \sum_i n_i q_i v_i A,$$

where

n_i = particles of type i per unit volume

q_i = charge on particles of type i

v_i = velocity of particles of type i

A = cross section area through which charge flows.

The unit of current I is coulomb per second = ampere (A).

The current per unit area is the <u>current density</u>.

$$J = \frac{I}{A} = \frac{1}{A} \cdot \frac{dQ}{dt} = \sum_i n_i q_i v_i$$

The unit of current density is ampere $(\text{meter})^{-2} = A \cdot m^{-2}$.

In many conductors the current density J is proportional to the electric field E. The constant of proportionality is the <u>resistivity</u> ρ,

$$\rho = \frac{E}{J} = \frac{EA}{I} = \frac{VA}{LI}$$

where V is the potential drop along a field line of length L. The units of ρ are $(\text{volt} \cdot \text{meter})(\text{ampere})^{-1}$ = ohm meter. The potential drop is

$$V = I\left(\frac{L\rho}{A}\right) = IR$$

where the resistance R of a wire of length L and cross section A is

$$R = \frac{L\rho}{A} .$$

$E = \rho J$ and $V = IR$ are two forms of <u>Ohm's</u> law. The unit of resistance is volt $(\text{ampere})^{-1}$ = ohm (Ω).

The resistivity and resistance of metals increase with temperature; for a temperature around some reference temperature T_o, the change is described by

$$\rho_T = \rho_o[1 + \alpha(T - T_o)]$$
$$R_T = R_o[1 + \alpha(T - T_o)]$$

where α is the temperature coefficient of resistivity.

To maintain a steady current in a conductor with resistance, one requires a closed circuit containing an emf (electromotive force) which maintains the potential difference across the resistance. A battery may produce the emf. (See Fig. 28-1).

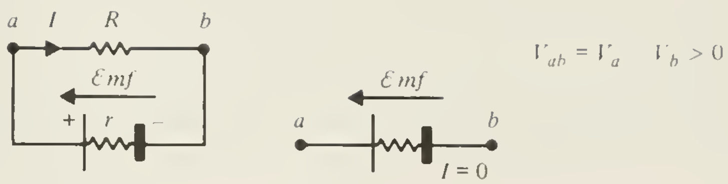


Figure 28-1

The battery has a positive and negative terminal, and may have an internal resistance r. The internal and external resistances are indicated by the sawtooth symbol in the diagram in Fig. 28-1a. The electromotive force is in the direction indicated, raising the potential from V_b to V_a when crossing from the negative to the positive terminal. The potential then drops as the current flows along the resistor, in the direction of the electric field. The potential rise across the emf is equal to the potential drop across the resistor R. The emf of such a battery is the 'open circuit' potential difference

$$V_{ab} = \text{emf}(\mathcal{E}).$$

If the emf has no internal resistance then, when connected as in Fig. 28-1a,

$$\mathcal{E} = Ir \qquad\qquad I = \frac{\mathcal{E}}{R}$$

Otherwise, the internal resistance also contributes to the potential drop:

$$\mathcal{E} = IR + Ir$$

$$I = \frac{\mathcal{E}}{R + r}$$

$$V_{ab} = \mathcal{E} - Ir = IR.$$

If the battery is 'short circuited' R = 0 and

$$I = \frac{\mathcal{E}}{r}$$

$$V_{ab} = \mathcal{E} - Ir = 0$$

All these relations are summarized in the Kirchhoff's loop rule,

$$\sum \mathcal{E} - \sum IR = 0 \qquad \text{(closed circuit)}$$

which states that the sum of all the emf's about a closed circuit minus the sum of all the potential drops, IR, is zero.

The power P necessary to maintain a current I across a potential difference V_{ab} is

$$P = V_{ab}\, I = I^2R = \frac{V^2}{R}$$

If the potential difference is produced by a battery, the power delivered to the external resistance R is

$$P = (\mathcal{E} - Ir)I = \mathcal{E}I - I^2r.$$

Since

$$\mathcal{E}I = I^2R + I^2r$$

the power developed by a battery of emf ($\mathcal{E}$) goes partly into the external circuit (I^2R) and is partly lost to the internal resistance (I^2r).

HINTS AND PROBLEM-SOLVING STRATEGIES

When applying Kirchhoff's loop rule:

(1) Decide on a positive direction of the circuit. (Clockwise or counterclockwise).

(2) Note $\mathcal{E}$ is positive if its open circuit voltage increases along this direction.

(3) Note I is positive if I points in this direction.

EXAMPLES AND SOLUTIONS

Example 1

Find the resistance of a 10 cm long aluminum bar of cross section area 1 cm^2. (see table 28-1, SZY.)

Solution:

$$R = \frac{\rho L}{A} = \frac{2.63 \times 10^{-8}\ \Omega\ m(0.1\ m)}{(10^{-2}\ m)^2}$$

$$= 2.63 \times 10^{-5}\ \Omega$$

Example 2

An antique vacuum tube carries an electron current of 10 mA across a flow cross section of area 1 cm^2. Find the density of electrons at a point in the gap where their velocity is 10^8 $m \cdot s^{-1}$.

Solution:

$$J = \frac{I}{A} = nve$$

$$n = \frac{I}{Ave} = \frac{10 \times 10^{-3}\ A}{(10^{-2}\ m)^2(10^8\ m \cdot s^{-1})(1.6 \times 10^{-1}\ C)}$$

$$= 6.25 \times 10^{12}\ m^{-3}$$

Example 3

A copper wire of cross section area 5 $(mm)^2$ carries a current of 5 A. The density of free electrons is 10^{29} m^{-3}.

(a) How many electrons pass through a cross section of the wire per unit time?

(b) What is the current density in the wire?

(c) What is the drift velocity of the electrons?

(d) If the wire is 1 m long, what is its resistance?

(e) What is the electric field in the wire?

(f) What is the potential drop along the wire. Verify Ohm's law.

Solution:

(a) If the number of conducting electrons is N, the charge is Q = Ne and the current is

$$I = \frac{dQ}{dt} = e\frac{dN}{dt}$$

The rate at which electrons pass through the wire is

$$\frac{dN}{dt} = \frac{I}{e} = \frac{5\ A}{1.6 \times 10^{-19}\ C}$$

$$= 3.13 \times 10^{19} \text{ electrons per second.}$$

(b) $$J = \frac{I}{A} = \frac{5\ A}{5(10^{-3}\ m)^2} = 10^6\ A\cdot m^{-2}$$

(c) $$J = nve$$

$$v = \frac{J}{ne} = \frac{10^6\ A\cdot m^{-2}}{(10^{29}\ m^{-3})(1.6 \times 10^{-19}\ C)}$$

$$= 6.25 \times 10^{-5}\ m\cdot s^{-1}.$$

(d) The resistivity of copper (see table 28-1, SZY) is

$$\rho = 1.72 \times 10^{-8}\ \Omega\cdot m = 1.72 \times 10^{-8}\ V\cdot m\cdot A^{-1}$$

$$R = \frac{\rho L}{A} = \frac{(1.72 \times 10^{-8}\ \Omega\cdot m)(1m)}{5(10^{-3})^2}$$

$$= 3.44 \times 10^{-3}\ \Omega.$$

(e) $$E = \rho J = (1.72 \times 10^{-8}\ V\cdot m\cdot A^{-1})(10^6\ A\cdot m^{-2})$$

$$= 1.72 \times 10^{-2}\ V\cdot m^{-1}.$$

(f) $$V = EL = (1.72 \times 10^{-2}\ V\cdot m^{-1})(1\ m) = 1.72 \times 10^{-2}\ V$$

$$IR = (5\ A)(3.44 \times 10^{-3}\ \Omega) = 1.72 \times 10^{-2}\ V$$

$V = IR$ verified

Example 4

The current in a wire varies with time according to

$$i = i_o e^{-t/\tau} \qquad \tau = 10^{-6}\ s, \quad i_o = 2\ A$$

Find the total charge which passes through the wire
(a) between $t = 0$ and $t = \tau$
(b) between $t = 0$ and $t \gg \tau$.

Solution:

(a) $$i = i_o e^{-t/\tau} = \frac{dQ}{dt}$$

$$\int_0^Q dQ = \int_0^t i_o e^{-t/\tau}\, dt$$

$$Q = i_o\left[\frac{e^{-t/\tau}}{-\tau^{-1}}\right]_0^t = i_o\tau(1 - e^{-1})$$

$$= (2\ \text{A})(10^{-6}\ \text{s})(0.63)$$

$$= 1.26 \times 10^{-6}\ \text{C}$$

(b) $$Q = i_o\left[\frac{e^{-t/\tau}}{-\tau^{-1}}\right]_0^\infty = i_o\tau = 2 \times 10^{-6}\ \text{C}$$

Example 5

A copper wire has a resistance of $10^{-2}\ \Omega$ at 20° C. What is its resistance at 100° C?

Solution:

The resistivity varies with temperature T according to

$$\rho = \rho_o[1 + \alpha(T - T_o)]$$

where

ρ_o = resistivity at 20° C

$T_o = 20^\circ$ C

$\alpha = 0.00393\ ^\circ C^{-1}$ (Table 28-2, SZY)

The resistance of the wire is

$$R = \frac{\rho L}{A} = \frac{\rho_o L}{A}[1 + \alpha(T - T_o)]$$

$$= R_o[1 + \alpha(T - T_o)]$$

where R_o is the resistance at $T_o = 20^\circ$ C. Thus at $T = 100^\circ$ C the resistance is

$$R = 10^{-2}\ \Omega[1 + 0.00393^\circ\ C^{-1}(100^\circ\ C - 20^\circ\ C)]$$

$$= 1.31 \times 10^{-2}\ \Omega$$

Example 6

A parallel plate capacitor whose capacitance in air is C_0 is filled with a dielectric of constant K and resistivity ρ. Find the leakage current in terms of the charge Q on the capacitor, ρ, ε_0, and K.

Solution:

The field between the plates is

$$E = \frac{V}{L} = \rho J = \frac{\rho I}{A}$$

where L is the plate separation and A the plate area. The current I is thus

$$I = \frac{AV}{\rho L} = \frac{KA\varepsilon_0}{L} \quad \frac{V}{\rho\varepsilon_0 K} = \frac{KC_0V}{\rho\varepsilon_0 K} = \frac{CV}{\rho\varepsilon_0 K}$$

$$= \frac{Q}{\rho\varepsilon_0 K}$$

Example 7

A battery has an open circuit potential difference of 3.0 V and a short-circuit current of 10 A. Find its internal resistance.

Solution:

The open circuit potential difference is the emf $\mathcal{E} = 3.0$ V. In the short circuit condition the only resistance is the internal resistance:

$$\sum \mathcal{E} = \sum IR$$

$$\mathcal{E} = Ir$$

$$r = \frac{\mathcal{E}}{I} = \frac{3.0\ \text{V}}{10\ \text{A}} = 0.3\ \Omega$$

Example 8

If the battery of Example 7 is connected to an external resistance R and the current is 1 A, what is the external resistance?

Solution:

$$\sum \mathcal{E} = \sum IR$$

$$\mathcal{E} = Ir + IR$$

$$\frac{\mathcal{E} - Ir}{I} = R = \frac{3.0 \text{ V} - (1\text{A})(0.3\ \Omega)}{1 \text{ A}}$$

$$= 2.7\ \Omega$$

Example 9

(a) Find the current in the circuit of Fig. 28-2.

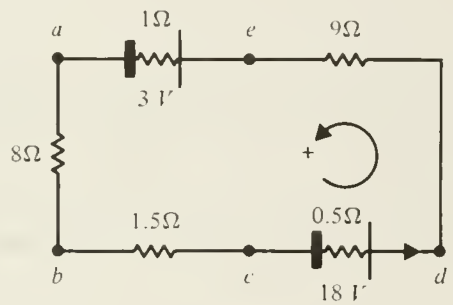


Figure 28-2

(b) Find the potential differences V_{ab}, V_{bc}, V_{cd}, V_{de}, and V_{ea}. Verify that they sum to zero around the closed loop.

Solution:

(a) $\sum \mathcal{E} = \sum IR$ is the loop rule. To use this expression a positive <u>circuit direction</u> must first be chosen. This has been taken to be counterclockwise in Fig. 28-2. Then a direction of the current must be assumed. This also has been taken counterclockwise in Fig. 28-2. Emf's are taken positive in the loop rule if they point in the positive circuit direction, negative otherwise. The potential drop IR is positive if I points in the positive direction, negative otherwise. In Fig. 28-2, the 3 V emf is a negative one and the 18 V emf is a positive one. All IR drops are positive.

$$\sum \mathcal{E} = 18 \text{ V} - 3 \text{ V} = 15 \text{ V}$$

$$\sum IR = I(1\ \Omega + 9\ \Omega + 0.5\ \Omega + 1.5\ \Omega + 8\ \Omega)$$

$$= (I)(20\ \Omega)$$

Thus

$$15\ \text{V} = (I)(20\ \Omega)$$

$$I = \frac{15\ \text{V}}{20\ \Omega} = 0.75\ \text{A}$$

(b) $$V_{ab} = (0.75\ \text{A})(8\ \Omega) = 6\ \text{V}$$

$$V_{bc} = (0.75\ \text{A})(1.5\ \Omega)\ 1.13\ \text{V}$$

$$V_{cd} = Ir - \mathcal{E} = (0.5\ \Omega)(0.75\ \text{A}) - 18\ \text{V} = -\ 17.63\ \text{V}$$

(The battery tends to make the potential higher at d than c, hence its emf contributes negatively to $V_{cd} = V_c - V_d$)

$$V_{de} = (0.75\ \text{A})(9\ \Omega) = 6.75\ \text{V}$$

$$V_{ea} = Ir + \mathcal{E}$$

$$= (0.75\ \text{A})(1\ \Omega) + 3\ \text{V} = 3.75\ \text{V}$$

(The 3 V battery is said to be charging because the current is passing through it opposite in direction to the emf.) It may be readily verified that

$$V_{ab} + V_{bc} + V_{cd} + V_{de} + V_{ea} = 0 = V_{aa}.$$

The sum of the potential differences around any closed loop is zero.

Example 10

A 6 V battery has an internal resistance of 0.5 Ω and is connected to a 5.5 Ω resistance.

(a) What is the current in the circuit when the circuit is closed ?

(b) What is the potential difference across the battery terminals when the circuit is closed ?

(c) What is the potential difference across the battery when the circuit is open ?

Solution:

(a) $$\sum \mathcal{E} = \sum IR$$

$$= Ir + IR = I(6\ \Omega)$$

$$I = \frac{6\ \text{V}}{6\ \Omega} = 1\ \text{A}$$

(b) $V = \mathcal{E} - Ir = 6\ V - (1\ A)(0.5\ \Omega) = 5.5\ V$

(c) $V = \mathcal{E} = 6\ V$

Example 11

(a) What is the resistance of a 60 W light bulb designed for use on 120 V lines?
(b) What power would it draw if it were operated on a 240 V line?

Solution:

(a) $$P = IV = \frac{V^2}{R} = \frac{(120\ V)^2}{R} = 60\ W$$

$$R = \frac{(120\ V)^2}{60\ W} = 240\ \Omega$$

(b) $$P = \frac{V^2}{R} = \frac{(240\ V)^2}{240\ \Omega} = 240\ W$$

(The bulb would quickly overheat and burn out.)

Example 12

A 12 V battery with internal resistance 0.3 Ω is discharging through a 11.7 Ω resistor? What power is dissipated in the 11.7 Ω resistor?

Solution:

$$\mathcal{E} = I(r + R)$$

$$I = \frac{\mathcal{E}}{r + R} = \frac{12\ V}{0.3\ \Omega + 11.7\ \Omega} = \frac{12\ V}{12\ \Omega} = 1\ A$$

$$P = VI = I^2R = (1\ A)^2(11.7\ \Omega) = 11.7\ W$$

Example 13

If the battery in Example 12 is charged by connecting it to a 24 V source, what power does the source develop?

Solution:

Referring to Fig. 28-3,

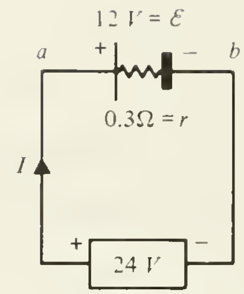


Figure 28-3

we see that the potential difference between a and b is:

$$V_{ab} = Ir + \mathcal{E}$$

$$I = \frac{V_{ab} - \mathcal{E}}{r} = \frac{24\ V - 12\ V}{0.3\ \Omega} = 40\ A$$

$$P = V_{ab}I = (24\ V)(40\ A) = 960\ W$$

(In the discharging configuration, the current reverses and $V_{ab} = \mathcal{E} - Ir$.)

QUIZ

1. An aluminum wire is one quarter as long as a copper wire of diameter 2 mm. If the aluminum wire has the same resistance as the copper wire, what is its diameter?

Answer: 1.24 mm

2. A battery has an emf of 12 V and an internal resistance of 2 Ω.

(a) What is the open circuit potential difference across the terminals of the battery

(b) What is the current in the battery when it is short circuited?

(c) What is the current when the battery is connected to an external resistance of 4 Ω?

(d) What is the potential difference across the terminals of the battery under the conditions of (c)?

Answer: 12 V, 6 A, 2 A, 8 V

3. A flashlight consists of two 1.5 V batteries connected in series to a bulb with resistance 15 Ω.

(a) What is the internal resistance of <u>each</u> battery when the power delivered to the bulb is 0.6 W?

(b) What is the internal resistance of <u>each</u> battery when the power delivered to the bulb is 0.3 W?

Answer: (a) zero (b) 3.1 Ω

4. 25% of the power developed by a battery in a circuit is lost to internal resistance heating. What is the ratio of the internal to the external resistance?

Answer: 1:3

29
DIRECT-CURRENT CIRCUITS

OBJECTIVES

The circuit rules of the previous chapter are generalized to include networks with branches and more than one current loop. Your objectives are to:

Find the equivalent resistance of resistors in series and parallel.

Apply Kirchhoff's rules to circuits with more than one loop, including electrical instruments for measuring potential, current, and resistance.

Design an ammeter and voltmeter for a given full scale deflection, given a galvanometer and its full scale deflection current.

Find how charges, currents and potentials change with time in an R-C circuit.

Define a displacement current and find its value in a charging or discharging capacitor and in a resistance carrying a varying current.

Gain an understanding of grounded parallel wiring for household and automobile electric power distribution.

REVIEW

If resistances R_1 and R_2 are connected in series, their equivalent resistance R is given by

$$R = R_1 + R_2. \qquad \text{(series)}$$

If the same resistances are connected in parallel, their equivalent resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{(parallel)}$$

Kirchhoff's rules are:

(1) The algebraic sum of the currents <u>toward</u> any branch point is zero.

$$\sum I = 0.$$

A current is counted positive if it flows into the branch point and negative if it flows out of the branch point. For examples, see Fig. 29-1.

Figure 29-1

$$I_1 - I_2 - I_3 = 0 \qquad I_1 + I_2 - I_3 - I_4 = 0 \qquad I_1 - I_2 - I_3 = 0$$

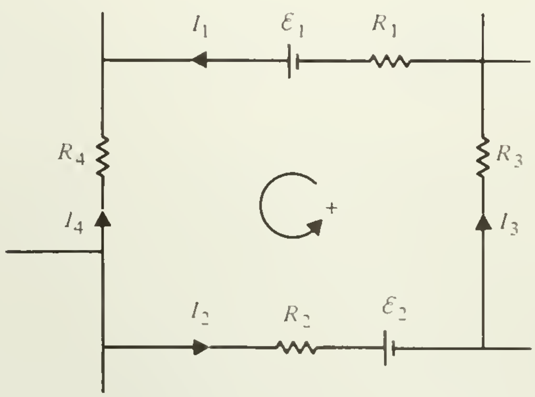
$$I_1 = I_2 + I_3 \qquad I_1 + I_2 = I_3 + I_4 \qquad I_1 = I_2 + I_3$$

(2) The algebraic sum of all the potential differences in any loop, including those associated with emf's and resistances, is zero. When IR is a potential drop, it counts as a negative potential difference.

$$\sum \mathcal{E} - \sum IR = 0$$

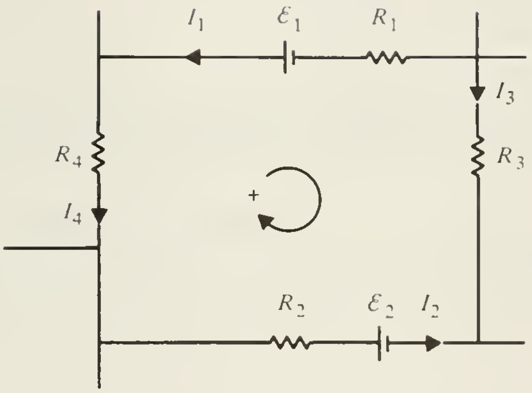
$$\sum \mathcal{E} = \sum IR$$

As in the last chapter, an emf is counted positive if it points in the loop direction and negative otherwise. I is taken positive in the rule if it points in the circuit direction, negative otherwise. For examples see Fig. 29-2, where the arbitrarily chosen positive direction is indicated in each loop by the circulating arrow.



$$\mathcal{E}_1 - \mathcal{E}_2 = I_1 R_1 - I_4 R_4 + I_2 R_2 + I_3 R_3$$

Figure 29-2a



$$\mathcal{E}_2 - \mathcal{E}_1 = -I_1R_1 - I_4R_4 - I_2R_2 + I_3R_3$$

Figure 29-2b

$$\mathcal{E}_1 - \mathcal{E}_2 = I_1R_1 - I_4R_4 + I_2R_2 + I_3R_3$$

$$\mathcal{E}_2 - \mathcal{E}_1 = -I_1R_1 - I_4R_4 - I_2R_2 + J_3R_3$$

To convert a galvanometer whose full scale current is I_c and whose resistance is R_c into an ammeter whose full scale current is I, put it in parallel with a shunt resistance

$$R_{sh} = R_c \frac{I_c}{I - I_c}$$

To convert the same galvanometer into a voltmeter whose full scale voltage is V_{ab}, put it in series with a resistance

$$R = \frac{V_{ab}}{I_c} - R_c$$

An R-C series circuit has a time varying charge q and voltage i. If the capacitor is charging,

$$q = VC(1 - e^{-t/RC}) = Q_f(1 - e^{-t/RC})$$

$$i = \frac{dq}{dt} = \frac{V}{R} e^{-t/RC} = I_o e^{-t/RC}$$

where V is the charging voltage, Q_f the final charge, and I_o the initial current.

If the capacitor is discharged through a resistance of magnitude R,

$$i = I_o e^{-t/RC}$$

$$q = Q_o e^{-t/RC}$$

where Q_o is the initial charge and I_o the initial current.

A <u>displacement current density</u> exists everywhere there is a changing electric field; its direction is the direction of the change of E and its magnitude is

$$J_D = \varepsilon_o\left(\frac{dE}{dt} \right).$$

Between the plates of a parallel plate capacitor the <u>displacement current</u>

$$I_D = J_D A = \varepsilon_o \frac{dE}{dt} A = \varepsilon_o \frac{d}{dt}\left(\frac{\sigma}{\varepsilon_o} \right) A$$

is

$$= \frac{d(\sigma A)}{dt} = \frac{dQ}{dt} = I_C$$

where I_C is the conduction current leading into the capacitor. Thus the total current, which is the sum of the displacement current and the conduction current, is continuous, the same across the capacitor gap as in the connecting wires.

EXAMPLES AND SOLUTIONS

Example 1

Find the equivalent (R_{equ}) of the network of resistors in Fig. 29-3a.

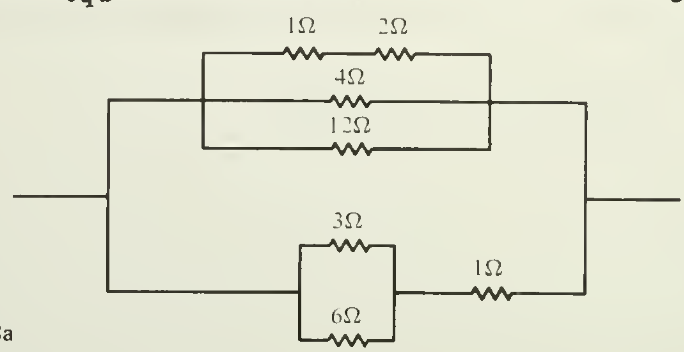


Figure 29-3a

Solution:

The successive reduction of the circuit is shown in Fig. 29-3 a,b,c. In Fig. 29-b the 1 Ω and 2 Ω in the upper branch of Fig. 29-3a have been combined according to the series rule,

$$1\ \Omega + 2\ \Omega = 3\ \Omega,$$

whereas the 3 Ω and 6 Ω of the lower branch of Fig. 29-3a have been combined according to the rule for adding reciprocals,

$$\frac{1}{6\ \Omega} + \frac{1}{3\ \Omega} = \frac{1}{2\ \Omega} \qquad R_{equ} = 2\ \Omega$$

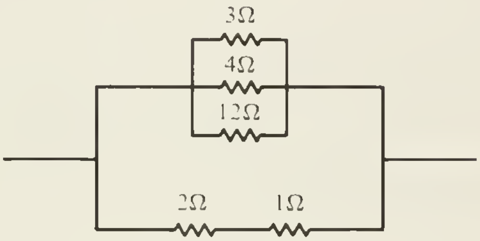


Figure 29-3b

In Fig. 29-3c the parallel 3 Ω, 4 Ω, and 12 Ω resistances in the upper branch of Fig. 29-3b have been combined according to the parallel rule

$$\frac{1}{3\ \Omega} + \frac{1}{4\ \Omega} + \frac{1}{12\ \Omega} = \frac{8}{12\ \Omega} = \frac{2}{3\ \Omega}$$

$$R_{equ} = \frac{3}{2}\ \Omega$$

while the series resistances of 2 Ω and 1 Ω have been combined into an equivalent 3 Ω resistance.

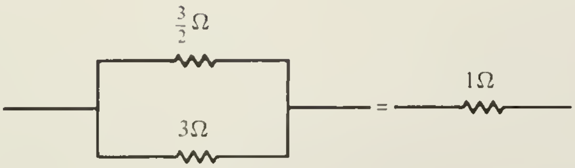


Figure 29-3c

Finally on the right side of Fig. 29-3c, the two parallel resistances have been combined,

$$\frac{1}{(3/2)\,\Omega} + \frac{1}{3\ \Omega} = \frac{1}{1\ \Omega} \qquad R_{equ} = 1\ \Omega$$

into the equivalent resistance 1 Ω.

Example 2

In the circuit of Fig. 29-4, V_{ab} = 12 V.
(a) Find the currents I_1, I_2, and I_3.
(b) Find the power dissipated in each resistance and in the entire network.

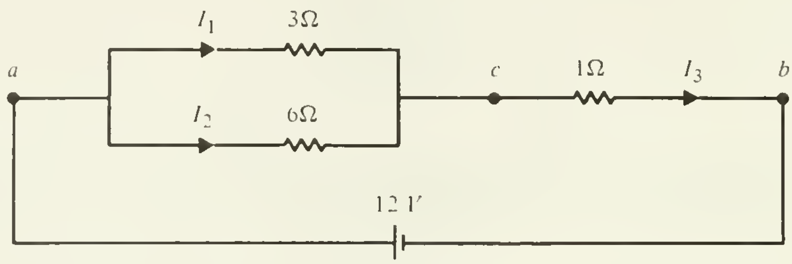


Figure 29-4

Solution:

According to Kirchhoff's rules,

$$I_1 + I_2 - I_3 = 0.$$

The parallel resistances can be reduced to 2 Ω (because 1/3 Ω + 1/6 Ω = 1/2 Ω), and this resistance can be combined with the 1 Ω resistance, resulting in an equivalent resistance (R_{equ}) of 2 Ω + 1 Ω = 3 Ω through which the current I_3 flows. Then

$$V_{ab} = 12\ \text{V} = R_{equ}\, I_3 = 3\ \Omega I_3$$

$$I_3 = \frac{12\ \text{V}}{3\ \Omega} = 4\ \text{A}.$$

The potential drop across cb is thus

$$V_{cb} = I_3(1\ \Omega) = 4\ \text{V}.$$

Thus the potential drop V_{ac} is given by

$$V_{ac} + V_{cb} = 12\ \text{V}$$

$V_{ac} = 12\ V - 4\ V = 8\ V.$

Thus

$V_{ac} = 8\ V = I_1(3\ \Omega);$ $\qquad$ $I_1 = 2.67\ A$

$V_{ac} = 8\ V = I_2(6\ \Omega);$ $\qquad$ $I_2 = 1.33\ A$

Note the junction rule $I_1 + I_2 - I_3 = 0$ is obeyed.

The power dissipated in the entire network is

$P = V_{ab}I_3 = I_3{}^2R_{equ}$ $\qquad$ $P = V_{ab}I_3 = I_3{}^2\ R_{equ}$

$= 12\ V(4\ A) = 48\ W$

The power dissipated in the 3 Ω, 6 Ω, and 1 Ω resistances is, respectively,

$P_3 = I_1{}^2(3\ \Omega) = (2.67\ A)^2(3\ \Omega) = 21.4\ W$

$P_6 = I_2{}^2(6\ \Omega) = (1.33\ A)^2(6\ \Omega) = 10.6\ W$

$P_1 = I_3{}^2(1\ \Omega) = (4\ A)^2(1\ \Omega) = 16.0\ W$

Note $P_3 + P_6 + P_1 = 48$ W; the power dissipated in the entire network is equal to the sum of the power dissipated in each resistance.

Example 3

In the network of Fig. 29-5a,

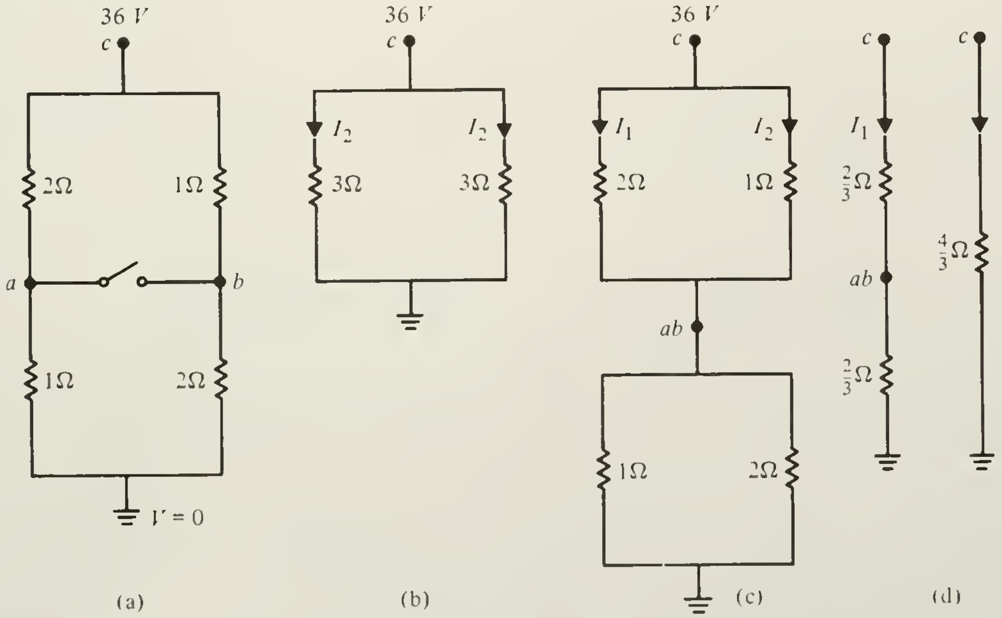


Figure 29-5

(a) Find the current in each branch when the switch is open.
(b) Find the potential drops V_{ca} and V_{cb} when the switch is open.
(c) Find the current in each branch when the switch is closed.
(d) Find the potential drops V_{ca} and V_{cb} when the switch is closed.

Solution:

(a) When the switch is open the circuit may be reduced to that of Fig. 29-5b. Thus

$$V_c = 36\ C = I_1(3\ \Omega) = I_2(3\ \Omega)$$

$$I_1 = I_2 = \frac{36\ V}{3\ \Omega} = 12\ A$$

(b) $$V_{ca} = I_1(2\ \Omega) = 12\ A(2\ \Omega) = 24\ V$$

$$V_{cb} = I_1(1\ \Omega) = 12\ A(1\ \Omega) = 12\ V$$

Thus

$$V_{ba} = V_{ca} - V_{cb} = 12\ V$$

(c) When the switch is closed (now $V_{ba} = 0$) the circuit may be reduced to the equivalent circuits in Fig. 29-5 c and d. Thus

$$V_c = 36\ V = I(4/3\ \Omega)$$

$$I = \frac{3}{4\ \Omega}(36\ V) = 27\ A$$

$$V_{ca} = I(\frac{2}{3}\ \Omega) = 27\ A(\frac{2}{3}\ \Omega) = 18\ V = V_{cb}$$

$$I_1 = \frac{V_{ca}}{2\ \Omega} = \frac{18\ V}{2\ \Omega} = 9\ A$$

$$I_2 = \frac{V_{ca}}{1\ \Omega} = \frac{18\ V}{1\ \Omega} = 18\ A \qquad \text{note: } I_1 + I_2 = I$$

Example 4

In the circuit shown in Fig. 29-6, find the currents I_1, I_2, and I_3.

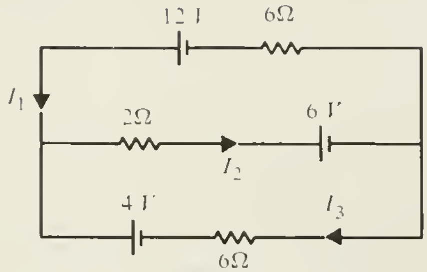


Figure 29-6

Solution:

Adopting the positive circuit conventions of Fig. 29-6, we write Kirchhoff's rules for each circuit loop, $\sum \mathcal{E} = \sum IR$:

$$\text{(i)} \quad 12 \text{ V} - 6 \text{ V} = I_1(6\ \Omega) + I_2(2\ \Omega)$$

$$\text{(ii)} \quad 6 \text{ V} - 4 \text{ V} = -I_2(2\ \Omega) - I_3(6\ \Omega)$$

$$\text{(iii)} \quad 12 \text{ V} - 4 \text{ V} = I_1(6\ \Omega) - I_3(6\ \Omega)$$

These are three equations in three unknowns:

$$6 = 6I_1 + 2I_2$$

$$2 = -2I_2 - 6I_3$$

$$8 = 6I_1 - 6I_3$$

Are they independent? No, because the sum of the first two yields the third; an independent relation among I_1, I_2, and I_3 is given by Kirchhoff's junction rule

$$I_1 - I_2 + I_3 = 0$$

From the first two loop relations, we find

$$I_1 = \frac{6 - 2I_2}{6} \qquad\qquad I_3 = \frac{-2 - 2I_2}{6}$$

Substituting these in the junction rule yields

$$\frac{6 - 2I_2}{6} - I_2 + \frac{-2 - 2I_2}{6} = 0$$

$$I_2 = \frac{2}{5} \text{ A.}$$

Substituting this result in the first 2 loop relations yields

$$I_1 = \frac{13}{15} \text{ A}$$

$$I_3 = -\frac{7}{15} \text{ A.}$$

The fact that I_3 turns out negative means that the actual current direction is opposite to the arrow in Fig. 29-6.

To check these results, substitute them back into the original Kirchhoff's rules and see if they are verified.

Example 5

For the circuit of Fig. 29-7

(a) Write down Kirchhoff's rules for the three subcircuits (i), (ii) and (iii) in terms of I_1, I_2, and I_3, as shown. Adopt the sign conventions as indicated by the circulating arrows.

(b) Solve for I_1, I_2, and I_3.

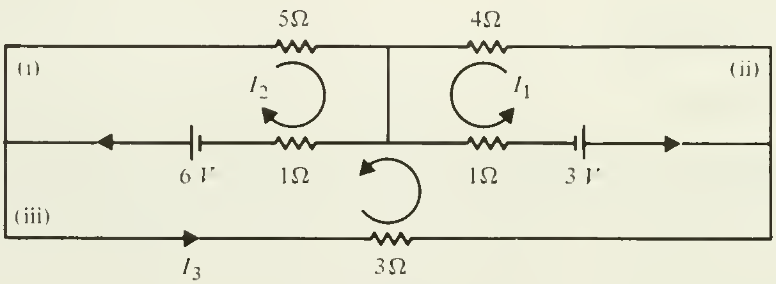


Figure 29-7

Solution:

(a) By Kirchhoff's junction rule, the current in the 5 Ω resistance is $I_2 - I_3$ and the current in the 4 Ω resistance is $I_1 + I_3$. Thus we have

(i) $6 \text{ V} = I_2(1\ \Omega) + (I_2 - I_3)5\ \Omega$

(ii) $3 \text{ V} = I_1(1\ \Omega) + (I_1 + I_3)4\ \Omega$

(iii) $(6 - 3)\text{V} = I_2(1\ \Omega) - I_1(1\ \Omega) + I_3(3\ \Omega)$

(b) $I_1 = 0.15$ A $\qquad I_2 = 1.47$ A $\qquad I_3 = 0.56$ A

Note that this can not be solved by parallel-series reduction methods.

Example 6

A voltmeter of resistance 500 Ω, placed in the circuit of Fig. 29-8 as shown, reads 50 V.

(a) Find the resistance R and the currents I, I_1, and I_2.

(b) What is the potential at point A when the voltmeter is not in the circuit? (Equivalently, what does the voltmeter read if its resistance is infinite?)

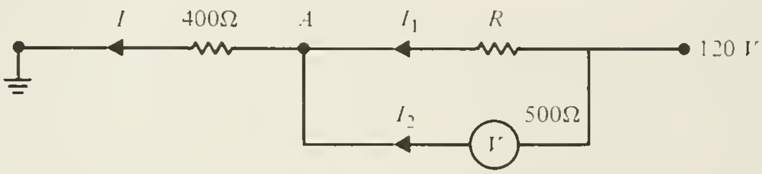


Figure 29-8

Solution:

(a) Point A is at a potential 120 V - 50 V = 70 V with respect to the ground where V = 0. Thus the current I through the 400 Ω resistor is given by

$$I(400\ \Omega) = 70\ \text{V}$$

$$I = \frac{7}{40}\ \text{A}$$

From Kirchhoff's junction rule we have

$$I_1 + I_2 = I = \frac{7}{40}\ \text{A}.$$

Across the parallel branches the potential drops are

$$I_1 R = 50\ \text{V}$$

$$I_2(500\ \Omega) = 50\ \text{V}$$

The last three equations involve the unknowns R, I_1, and I_2. Their solution is

$R = 667\ \Omega \qquad I_1 = 0.075$ A $\qquad I_2 = 0.10$ A

(b) The equivalent resistance of the two resistors (without the voltmeter) is

$$R_{equ} = 400\ \Omega + 667\ \Omega = 1067\ \Omega.$$

The current through the circuit is

$$I = \frac{V}{R_{equ}} = \frac{120\ V}{1067\ \Omega} = 0.113\ A$$

The voltage difference between the 120 V point and A is

$$V = RI = 667\ \Omega(0.113\ A) = 75\ V$$

(The voltmeter, having relatively low resistance, disturbs the circuit considerably. A better voltmeter will have a much higher resistance and read much closer to 75 V).

Example 7

The resistance of a galvanometer is 25 Ω. A current of 100 μA produces full scale deflection. Find the shunt resistance required to convert it to an ammeter reading 10 A at full scale.

Solution:

Referring to Fig. 29-9,

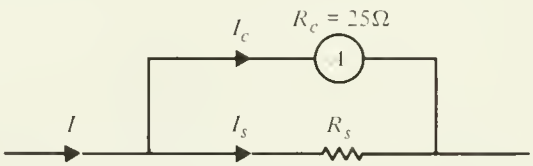


Figure 29-9

we wish to adjust the shunt resistance so that

$I_c = 100\ \mu A$ when $I = 10$ A. From

$I_cR_c = I_sR_s$ (parallel voltage drops equal)

$I_c + I_s = I$ (Kirchhoff's junction rule)

we see

$$R_s = R_c\left(\frac{I_c}{I_s}\right) = R_c\left(\frac{I_c}{I - I_c}\right)$$

$$= 25\ \Omega\ \frac{100\ \mu A}{10\ A - 100\ \mu A} \simeq 2.5 \times 10^{-4}\ \Omega.$$

Example 8

Show how to convert the galvanometer of **Example** 7 to a voltmeter reading 300 V at full scale.

Solution:

Referring to Fig. 29-10,

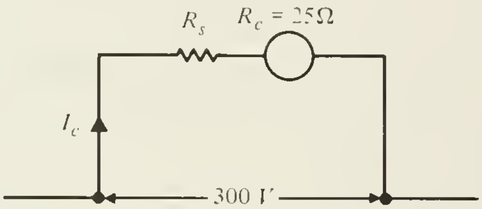


Figure 29-10

we seek a shunt resistance R_s such that a current I_c = 100 μA passes through the coil when 300 V is put across the shunt plus voltmeter combinations:

$$(R_s + R_c)I_c = 300\ V$$

$$R_s = \frac{300\ V}{I_c} - R_c = \frac{300\ V}{100\ \mu A} - 25\ \Omega$$

$$= 3 \times 10^6\ \Omega$$

Example 9

The circuit of Fig. 29-11 is used to charge the two capacitors.

(a) Find the final charge on each capacitor and the potential across each after the switch has been closed for a long time.

(b) Find the time it takes for the charges and potential differences to reach half their final values.

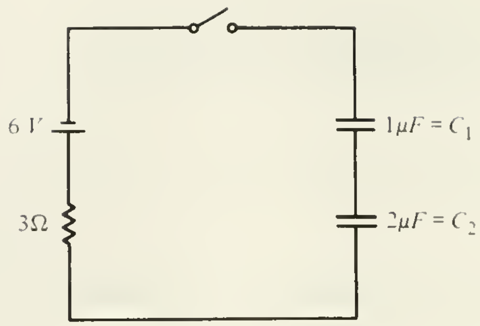


Figure 29-11

Solution:

(a) The equivalent capacitance of the two capacitors is given by

$$\frac{1}{C} = \frac{1}{1\ \mu F} + \frac{1}{2\ \mu F} = \frac{3}{2\ \mu F}$$

$$C = \frac{2}{3}\ \mu F.$$

Thus the final charge on each is

$$Q_f = V_f C = 6\ V\left(\frac{2}{3} \times 10^{-6}\ F\right)$$

$$= 4 \times 10^{-6}\ C = 4\ \mu C$$

The final potentials across the capacitors are

$$V_a = \frac{Q_f}{C_1} = \frac{4\ \mu C}{1\ \mu F} = 4\ V$$

$$V_2 = \frac{Q_f}{C_2} = \frac{4\ \mu C}{2\ \mu F} = 2\ V$$

(b) The time constant of the circuit is

$$RC = 3\ \Omega\ (4\ \mu C) = 12 \times 10^{-6}\ s.$$

For a charging circuit, the charges and potentials across the capacitors vary as

$$Q = Q_f(1-e^{-t/RC})$$

$$V = \frac{Q}{C} = \frac{Q_f}{C}(1 - e^{-t/RC}) = V_f\,(1-e^{-t/RC})$$

For the charges and voltages to reach half their final values,

$$\frac{Q}{Q_f} = \frac{V}{V_f} = 1-e^{-t/RC} = \frac{1}{2}$$

$$\frac{1}{2} = e^{-t/RC}$$

$$\ln(1/2) = -\ln 2 = -t/RC$$

$$t = RC \ln 2$$

$$= 12 \times 10^{-6}\ \text{s}(0.69)$$

$$= 8.32 \times 10^{-6}\ \text{s}$$

Example 10

For the capacitors in Example 9 calculate the displacement current in the capacitor gaps and show it is equal to the conduction current in the wire.

Solution:

The electric field in the capacitors is

$$E = \frac{\sigma}{\varepsilon_o} = \frac{Q}{A\varepsilon_o}$$

The displacement current is

$$I_D = J_D A = \varepsilon_o \frac{dE}{dt} A.$$

Since

$$E = \frac{\sigma}{\varepsilon_o} = \frac{Q}{A\varepsilon_o}; \qquad \frac{dE}{dt} = \frac{1}{A\varepsilon_o}\frac{dQ}{dt}$$

We have for I_D,

$$I_D = \varepsilon_o \left(\frac{dQ}{dt} \right) \left(\frac{A}{A\varepsilon_o} \right) = \frac{dQ}{dt} = I_c$$

$$= \frac{d}{dt} Q_f(1-e^{-t/RC})$$

$$= \frac{Q_f}{RC} e^{-t/RC} = I_c$$

Where I_c is the conduction current in the wire (assumed perfectly conducting; see the next problem for I_D in a real conductor.)

Example 11

Find the displacement current in the copper resistor of Example 9 at time t = 0.

Solution:

Across the resistor we have

$$V = IR = EL$$

where E is the field along the current flow and L the length of the current flow. Thus within the resistor,

$$E = \frac{IR}{L}$$

The displacement current is

$$I_D = A\, J_D = A\varepsilon_o \left(\frac{dE}{dt} \right) = A\varepsilon_o \left(\frac{R}{L} \right) \left(\frac{dI}{dt} \right)$$

$$= \frac{AR}{L} \varepsilon_o \left(\frac{dI}{dt} \right) = \rho\varepsilon_o \left(\frac{dI}{dt} \right)$$

where $\rho = \frac{AR}{L}$ is the resistivity of the copper resistor. We have

$$\rho = 1.72 \times 10^{-8}\ \Omega \cdot \text{m}$$

$$\frac{dI}{dt} = \frac{d}{dt}\frac{Q_f}{RC} e^{-t/RC} = -\frac{Q_f}{(RC)^2} \quad \text{at } t = 0,$$

$$I_D = -\rho\varepsilon_o\left(\frac{Q_f}{(RC)^2}\right)$$

Thus the displacement current in the wire is quite small:

$$I_D = \frac{(-1.72 \times 10^{-8}\ \Omega\cdot m)(8.85 \times 10^{-12}\ C^2\ N^{-1}\ m^2)(4 \times 10^{-6}\ C)}{(12 \times 10^{-6}\ s)^2}$$

$$= -\ 4.2 \times 10^{-15}\ A = -\ 4.2 \times 10^{-3}\ pA.$$

Example 12

You wish to install a radio in a car with a 12 V, negatively grounded electrical system. The power is to be drawn from the fuse box. Trace the complete circuit and the connections that need to be made.

Solution:

The negative terminal of the battery is connected to the ground, often the matallic chassis of the auto. Starting here we trace the circuit through the battery from ground (−) to hot (+) side to the fuse box where power may be distributed to any number of accessories, which are in parallel. A wire from the fuse box brings power to the radio's power supply on-off switch and through the radio to a ground wire, which completes the circuit to the negative side of the battery.

QUIZ

1. Three pieces of wire of the same length have resistances of 1 Ω, 2 Ω, and 3 Ω. They are braided together to produce a single resistor, which is connected across a 12 V battery. What is the current in each strand of the braided wire?

Answer: 12 A, 6 A, 4 A

2. Suppose that in Fig. 29-6a all emf's are kept the same, but each resistance is replaced by one with half its original resistance. What are the currents I_1, I_2, and I_3?

Answer: 13/30 A, 2/10 A, −7/30 A

3. The resistance of a galvanometer is 50 Ω. A current of 200 μA produces full scale deflection. Find the shunt resistance required to convert it to an ammeter reading 10 A at full scale.

Answer: 10^{-3} Ω

4. A 30 mF capacitor is charged by connecting it to a 12 V battery. The capacitor is then allowed to discharge by short circuiting its plates with a wire of resistance 20 Ω. How long does it take the charge on the capacitor to decay to 1/10 its original value?

Answer: 1.4 s

30
MAGNETIC FIELD AND MAGNETIC FORCES

OBJECTIVES

In this chapter your objectives are to:

Calculate the vector force on a moving charged particle in a specified magnetic field using the right hand rule. The magnetic force does no work on charged particles.

Calculate the flux of a known magnetic field through a given surface.

Calculate the radius of the circular orbit that results when the particle velocity is perpendicular to the magnetic field.

Apply the expression for the magnetic force on a single charged particle to the calculation of the force on a conductor carrying current in a constant magnetic field.

Calculate the torque on a rectangular coil of wire carrying current in a constant magnetic field in terms of the magnetic moment of a current distribution.

Calculate the Hall emf (voltage) for a conductor in a constant, uniform magnetic field.

REVIEW

In this chapter the concept of a magnetic field is introduced in analogy to the electric field studied previously. A <u>static</u> charge produces an <u>electric field</u> which in turn produces a force on a second static charge. A <u>moving charge</u> or current produces a <u>magnetic field</u> which in turn causes a force to act on a second moving charge or current.

The electrostatic force depends on the particle's position and charge: Coulomb's law does <u>not</u> contain the particle's speed. The magnetic force on the other hand depends not only on the speed, v, but also on the relative orientation of the velocity vector, **v**, and the magnetic field, **B**. This dependence is given by:

$$\mathbf{F}_M = q\mathbf{v} \times \mathbf{B}$$

where q is the charge on the particle. If an electric field represented by vector E is also present then we must add these two forces vectorially:

$$F_{total} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}.$$

The direction of the magnetic force is found by using the right hand rule to evaluate the cross product **v** x **B**. Thus $\mathbf{F_M}$ is always perpendicular to both **v** and **B**. One important consequence of the fact that $\mathbf{F_M}$ is perpendicular to **v** is that magnetic forces do no work on moving charged particles. Since $\mathbf{F_M}$ is perpendicular to **B**, the lines of **B** are not lines of force as they were with the electric field but will be called <u>magnetic field lines</u>.

The infinitesimal flux, $d\Phi$, through a surface element dA is calculated by taking the product of the normal component of **B** (the tangential component does not contribute to the flux) with dA. For a closed surface, the normal is chosen to be the outward normal. For an open surface, it will be defined more carefully in a later chapter. The total flux Φ, is obtained by integrating $d\Phi = \mathbf{B}\cdot d\mathbf{A}$ over the surface in question. This is illustrated in Example 4.

The S.I. unit for magnetic field is the tesla. From the equation for the force, since qv has dimensions of coulomb·meters·$(sec)^{-1}$ and a coulomb per second is an ampere, one tesla is equal to one newton per ampere·meter. As flux is dimensionally a magnetic field multiplied by an area, it will have units of newton·meters per ampere. This unit is given the name of a weber. Hence one weber is equal to one tesla multiplied by one meter squared ($1\ T\cdot m^2$).

The motion of a charged particle in a constant magnetic field is very important to understand as it forms the basis for many later applications. The simplest way to understand this motion for an arbitrary orientation of the vectors **v** and **B** is to resolve v into components parallel to **B** (v_t) and perpendicular to **B** (v_n). No magnetic force is exerted because of $\mathbf{v_t}$ so that motion parallel to **B** proceeds at constant velocity. For the component $\mathbf{v_n}$, the magnetic force (of magnitude qv_nB) results in a uniform circular motion about the magnetic field direction. Since the motion associated with $\mathbf{v_t}$ is unaccelerated and that associated with $\mathbf{v_n}$ is uniform circular motion, the general trajectory is a helix (see Example 2). In the discussion to follow, we will omit the motion associated with $\mathbf{v_t}$ and concentrate just on the uniform circular motion. This will correspond to the special case where **v** is perpendicular to **B**. We treat that case now.

In Example 3 it is shown that the magnetic force can do no work on a moving charged particle so the kinetic energy cannot increase or decrease (i.e. **v** is a vector of fixed length). The time rate of change of **v** or the acceleration **a** due to the magnetic force is thus perpendicular to **v** at each instant and points to the center of the circle. For uniform motion on a circle, the acceleration must have a magnitude of v^2/R where R is the radius. Since **v** is perpendicular to **B**, the acceleration resulting from the magnetic force is:

$$a = \left(\frac{1}{m} \right) qvB.$$

For uniform circular motion, we must have:

$$\left(\frac{1}{m}\right)qvB = \frac{v^2}{R}$$

Solving this for the radius we have: $R = mv/qB$. In our earlier study of uniform circular motion, we saw that there was an angular velocity, ω, associated with this motion such that $R\omega = v$. For the charged particle in a uniform magnetic field, since $\omega = v/R$, we deduce that $\omega = qB/m$. This angular velocity or angular frequency is called the cyclotron frequency.

The period of revolution, τ, is equal to the distance around the circle divided by the speed, so $\tau = 2\pi R/v$. Using the above value for R we find that $\tau = 2\pi m/qB$ or $\tau = 2\pi/\omega$. It is important to note that this period of revolution is independent of the particle's speed. Particles with higher speeds travel in larger circles but the time for a revolution is the same for all.

Applications of these ideas to the determination of the charge to mass ratio for electrons and the measurements of the relative masses of isotopes are excellent examples and discussed fully in the text.

The magnetic force on a single moving charge is used to calculate the <u>net</u> force on a conductor carrying current I in a uniform magnetic field B. Charge carriers of both signs (with different drift velocities) are considered and it is shown that the force always points in the same direction independent of the sign of the charge since the velocities are in opposite directions. The macroscopic current depends on the number of charge carriers per unit volume, the charge carried, the average drift velocity, and the cross-sectional area of the conductor. By adding up the individual forces on the charges, the total force is found to depend on B, I, and L, the length of conductor. If we define a vector **L** that points in the direction of the current flow and has magnitude equal to the length (L) considered, then:

$$\mathbf{F} = I(\mathbf{L} \times \mathbf{B}).$$

The use of this expression is illustrated in Examples 7b, 8, and 10.

If the direction (or magnitude) of the magnetic field varies over the length of the conductor, as it would if the conductor were not simply a long straight wire, then it is preferable to write the element of force, **dF**, as,

$$d\mathbf{F} = I(d\mathbf{L} \times \mathbf{B})$$

where **dL** is a vector pointing in the direction of current flow and small enough so that B is essentially constant over its infinitesimal length, dL. This expression can then be integrated over the configuration of the conductor to obtain the total force. This approach is illustrated for a wire bent in the shape of a semi-circle in Example 9.

The magnetic force on a segment of conductor can be used to calculate the total force on a complete circuit in a constant, uniform magnetic field. This total force is zero and independent of the geometric shape of the circuit. See Example 7b. The net torque (Γ) however is not in general zero. This torque is first calculated just as it was in the study of mechanics by writing $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$

or more simply by finding the product of the force with the lever arm. This is illustrated in Example 7c. In the text this calculation is also done for a rectangular loop of wire. Once the torque has been calculated by finding **r** x **F**, it can be recognized that by defining a new vector **M**, the magnetic moment, the torque can be alternatively expressed by the cross product **M** x **B**. Although the definition **M** = I**A** was made with reference to the rectangular loop, it is perfectly general and applies to any shape of closed circuit. The method for proving this statement is indicated in the text and the proof itself is given in most advanced texts on electromagnetic theory. The potential energy of a magnetic moment in a magnetic field is a very useful expression for many calculations and it is given in the text.

The Hall effect is a consequence of the magnetic force on individual charge carriers. The internal non-electrostatic electric field set up by this force and the electrostatic force due to the charge separation necessary to maintain electrical neutrality is perpendicular to the magnetic field and the direction of current flow. The magnitude of this field is the same as that needed in a velocity filter (E = vB) except here the velocity is the average drift velocity. The direction of this field depends on the sign of the charge carrier. This is emphasized in the text and in Example 5. Measurement of the sign of the Hall emf can determine whether the majority charge carriers in a material are electrons or holes. Measurement of the magnitude of the Hall emf can determine the number of charge carriers per unit volume.

In addition to its usefulness in obtaining fundamental information about conductors and semiconductors, the Hall effect has some very practical applications. Since voltages are easy to measure with accuracy and the Hall emf is proportional to the magnetic field, 'Hall probes' are used to measure laboratory magnetic fields. For the same reason, the Hall effect can be used to regulate magnetic fields by using the difference between the actual Hall emf and the desired Hall emf as an 'error signal' to feed back to the current generator used to produce the magnetic field.

The number of extremely important practical devices that draw on the concepts presented in this chapter is very large. Three simple examples (the pivoted coil galvanometer, the d.c. motor, and the electromagnetic pump) were selected from this long list to underscore the practical nature of this material.

EXAMPLES AND SOLUTIONS

Since nearly all the problems encountered here involve the magnetic force, which is proportional to the cross product of two vectors, it is essential to review the cross product discussion in Chapter 1.

Example 1

Given vectors $\mathbf{A} = 3\ \mathbf{i} + 4\ \mathbf{j}$ and $\mathbf{B} = 2\ \mathbf{j} + 3\ \mathbf{k}$,
(a) calculate the vector product $\mathbf{A} \times \mathbf{B}$,
(b) calculate the angle θ between $\mathbf{A}$ and $\mathbf{B}$,
Refer to Fig. 30-1.

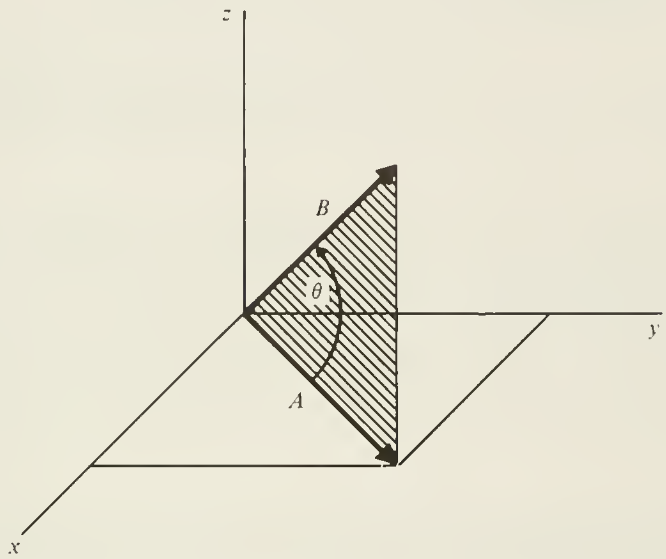


Figure 30-1

Solution:

(a)
$$\mathbf{A} \times \mathbf{B} = (3\ \mathbf{i} + 4\ \mathbf{j}) \times (2\ \mathbf{j} + 3\ \mathbf{k})$$
$$= 3\ \mathbf{i} \times (2\ \mathbf{j} + 3\ \mathbf{k}) + 4\ \mathbf{j} \times (2\ \mathbf{j} + 3\ \mathbf{k})$$
$$= 6\ \mathbf{k} - 9\ \mathbf{j} + 12\ \mathbf{i}$$
$$= 12\ \mathbf{i} - 9\ \mathbf{j} + 6\ \mathbf{k}$$

(b) To find θ we can use the fact that $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$ or that

$$|\mathbf{A} \times \mathbf{B}| = AB \sin \theta.$$

Using the first approach we note

$$\mathbf{A} \cdot \mathbf{B} = (3 \cdot 0 + 4 \cdot 2 + 0 \cdot 3) = 8$$

$$A = (3^2 + 4^2)^{1/2} = 5$$

$$B = (2^2 + 3^2)^{1/2} = (13)^{1/2}$$

yielding

$$\cos\theta = \frac{8}{5(13)^{1/2}} \qquad \text{or } \theta = 63.7^\circ.$$

The second approach gives

$$\sin\theta = \frac{(261)^{1/2}}{5(13)^{1/2}} \quad \text{or } \theta = 63.7^\circ, \text{ as before.}$$

Example 2

A particle with charge $q = 1.6 \times 10^{-19}$ C, mass 2×10^{-27} kg, and velocity

$$\mathbf{v} = v_o(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

where $v_o = 2 \times 10^5$ m·s^{-1}, enters a region of constant and uniform magnetic field

$$\mathbf{B} = B_o\mathbf{j}$$

with $B_o = 2.5$ T.

(a) Calculate the force on this moving charge due to the magnetic field.
(b) Calculate the path of the particle in this field.

Solution:

(a) The force is given by

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

$$= qv_oB_o[(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \times (\mathbf{j})]$$

$$= qv_oB_o[\mathbf{k} + 2(-\mathbf{i})]$$

The magnitude of this force is equal to

$$|\mathbf{F}| = qv_oB_o(1 + 4)^{1/2} = qv_oB_o(5)^{1/2}$$

$$= qv_nB_o$$

where $v_n = (5)^{1/2}v_o$ is the magnitude of the velocity component perpendicular (or normal) to **B**. Substituting the given values, the force is

$$|\mathbf{F}| = (1.6 \times 10^{-19}\text{ C})(2.5\text{ T})(2 \times 10^5\text{ m·s}^{-1})(5)^{1/2}$$

$$|\mathbf{F}| = 1.79 \times 10^{-13} \text{ N.}$$

(b) The velocity component parallel to the field $v_o(2j)$ is <u>unaccelerated</u> so that the motion in the y direction proceeds at constant velocity.

In the xz plane the orbit is a circle with radius R obtained from

$$\frac{mv_n^2}{R} = qv_nB_o$$

$$R = \frac{mv_n}{qB_o}$$

$$= \frac{(2 \times 10^{-27} \text{ kg})(2 \times 10^5 \text{ m}\cdot\text{s}^{-1})(5)^{1/2}}{(1.6 \times 10^{-19} \text{ C})(2.5 \text{ T})}$$

$$= 2.24 \times 10^{-3} \text{ m.}$$

The two motions when combined, the unaccelerated motion in the y direction and the circular motion in the xz plane, lead to a path called a helix.

Example 3

Calculate the work done on a charge q moving with velocity $\mathbf{v}$ in a constant magnetic field $\mathbf{B}$.

Solution:

The time rate of doing work (or the power) of a force, F, is given by the expression

$$P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}$$

If we use for $\mathbf{F}$ the magnetic force, $\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$, then

$$\frac{dW}{dt} = q(\mathbf{v} \times \mathbf{B})\cdot\mathbf{v}$$

The right hand side is zero since $\mathbf{v} \times \mathbf{B}$ is a vector perpendicular to both $\mathbf{B}$ and $\mathbf{v}$ so its scalar product with $\mathbf{v}$ vanishes. Since $dW/dt = 0$, <u>no work is done by the magnetic force on charge q</u>. The work done on a charge q for a particular motion in an electric field was used to compute the change in potential energy. The above result for the magnetic field implies that there is no scalar quantity analogous to potential energy in electrostatics that characterizes the magnetic field.

Example 4

Calculate the flux through the surfaces ABCD and AEFD shown in Fig. 30-2 for a constant and uniform magnetic field $\mathbf{B} = 0.8$ T(j).

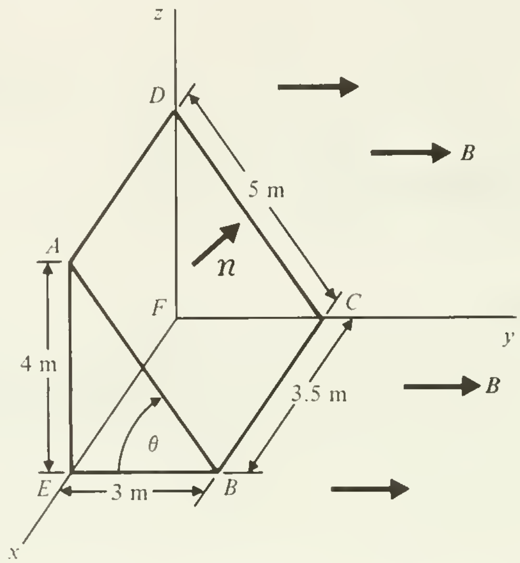


Figure 30-2

Solution:

The outward normal to the surface ABCD can be written:

$$\mathbf{n}_1 = \cos(90 - \theta)\mathbf{j} + \sin(90 - \theta)\mathbf{k}$$

$$= 0.8\mathbf{j} + 0.6\mathbf{k}$$

$$\mathbf{B}\cdot\mathbf{n}_1 = (0.8 \text{ T } \mathbf{j})\cdot(0.8 \ \mathbf{j} + 0.6 \ \mathbf{k}) = 0.64 \text{ T}.$$

The flux through ABCD is

$$\phi = (\mathbf{B}\cdot\mathbf{n}_1)A_{ABCD} = (0.64 \text{ T})(5 \text{ m})(3.5 \text{ m})$$

$$= 11.2 \text{ T}\cdot\text{m}^2 = 11.2 \text{ webers}$$

The outward normal to the surface AEFD is equal to

$$\mathbf{n}_2 = -\mathbf{j}$$

The flux through this surface is

$$\phi = (\mathbf{B}\cdot\mathbf{n}_2)A_{AEFD} = (0.8 \text{ T } \mathbf{j})(-\mathbf{j})(4 \text{ m})(3.5 \text{ m})$$

$\phi = - 11.2 \ T \cdot m^2$

The flux through <u>any</u> of the other surfaces in Fig. 30-2 is <u>zero</u> because the normals to all the other surfaces are perpendicular to **B**. Adding the two contributions calculated above, it is seen that the <u>total</u> flux through this closed surface is zero.

Example 5

A charge q moving with velocity v in a uniform magnetic field B is deflected from its original trajectory (where it would have hit the screen at S_o) and strikes the screen at S_1. What magnitude and direction of electric field E must exist between the plates P_1 and P_2 in order to give the charge a deflection of zero (i.e. to return it to S_o)? Refer to Fig. 30-3.

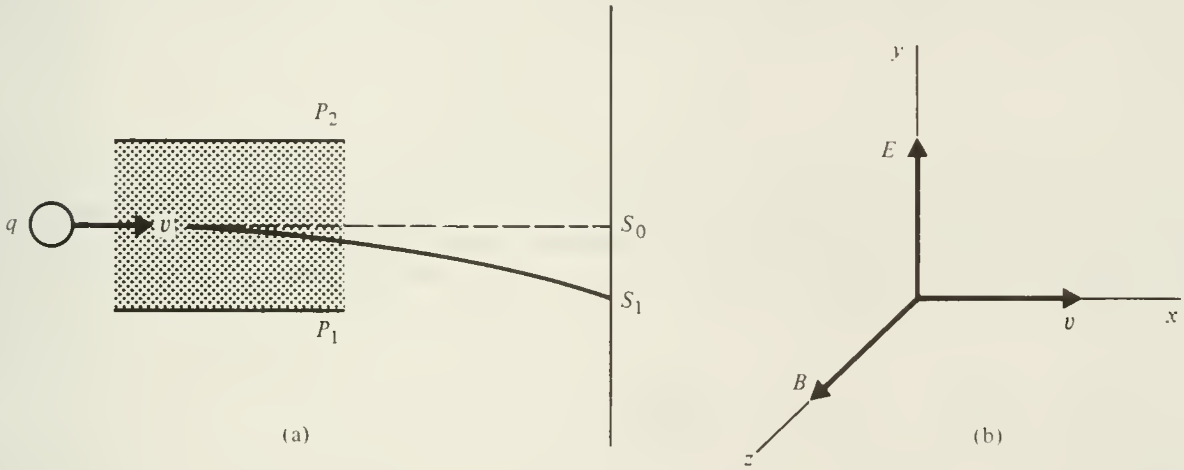


Figure 30-3

Solution:

Choose a coordinate system in which the initial velocity **v** points in the **i** direction, **v** = v**i**; the magnetic field points in the **k** direction, **B** = B**k**, and (if P_1 is positive with respect to P_2) **E** = E**j**. The total force on the charge q from the combined electric and magnetic field (we ignore gravitation here) is:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = q[E\mathbf{j} + vB(\mathbf{i} \times \mathbf{k})]$$

$$= q[(E - vB)\mathbf{j}]$$

For zero deflection, this force must vanish, so E = vB is the condition for F = 0.

Note that:

(i) We have here a method for measuring the charged particle's velocity--by just measuring E and B since $v = E/B$.

(ii) We can use this arrangement of crossed (perpendicular) electric and magnetic fields as a <u>velocity-filter</u>.

If instead of a screen we had a small slit at S_0, only those particles whose velocity satisfied $v = E/B$ would pass through the slit. All others would suffer some deflection and be stopped.

Example 6

A given mass spectrometer can detect only ions that have a radius of curvature equal to that of helium of atomic mass 4, $_2He^4$, in a uniform magnetic field in which charged ions move in circles of radius R. How can the spectrometer be modified to make it detect helium of mass 3, $_2He^3$, assuming the atoms are singly ionized?

Solution:

Assume the spectrometer can detect only ions with a charge to mass ratio the same as a singly ionized helium atom of mass four because <u>the radius of curvature is fixed</u> by the instrument geometry and the fixed strength B_0 of the magnetic field. For circular motion in the constant magnetic field B_0 we have (with v perpendicular to B):

$$\frac{mv^2}{R} = qvB_0 \qquad \text{or} \qquad R = \frac{mv}{qB_0}$$

The only variable parameter in the expression for R is the velocity v in this example. (The velocity v is determined by the 'filter' condition $v = E/B$ where B is a second constant magnetic field (not equal to B_0). Thus if m is reduced from four mass units to three mass units, v must be increased by a factor of 4/3 to keep R constant (i.e. $m_3v_3 = m_4v_4$). Thus $v_3 = (4/3)v_4$. This is best done by increasing the electric field E in the filter. For parallel plates $Ed = V$ where V is the potential difference and d is the plate separation. Thus the potential difference in the filter section should be increased by the same factor 4/3 to convert the spectrometer from mass 4 to mass 3.

Example 7

For the square coil geometry shown in Fig. 30-4,

(a) calculate the flux through the open surface with the normal defined as shown;

(b) calculate the force on segments ab, bc, cd, and da and show that F = 0 where F is the vector sum of these four forces;

(c) calculate the torque about an axis parallel to the x axis through the center of the coil.

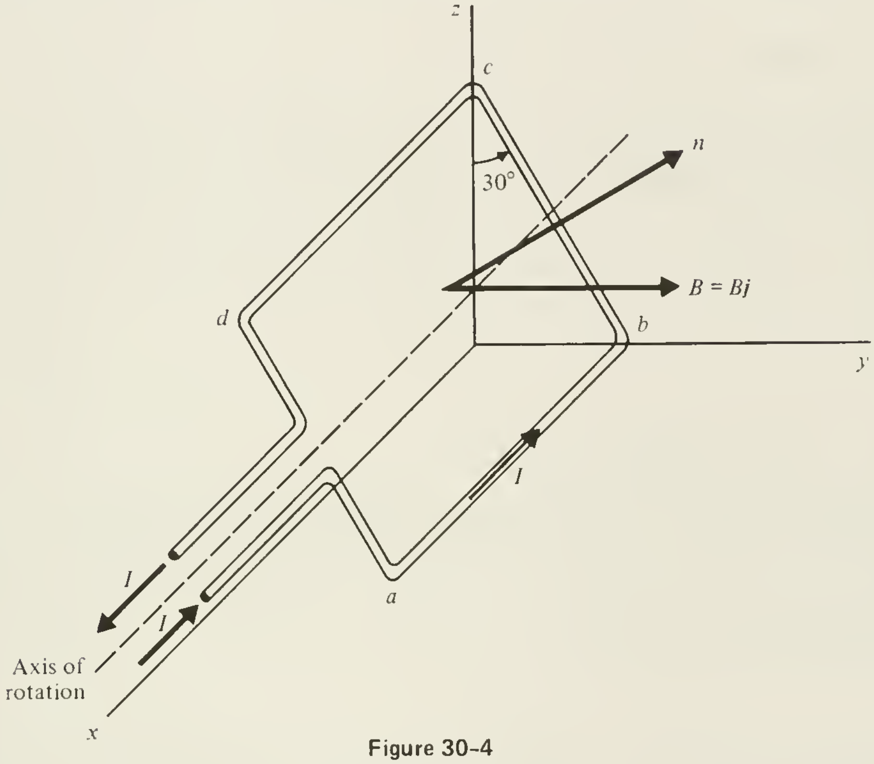


Figure 30-4

Solution:

(a) $$\Phi = (\mathbf{B}\cdot\mathbf{n})A, \qquad \text{since B is constant.}$$

$$= |\mathbf{B}|\cdot|\mathbf{n}|\cdot \cos 30^\circ = B\cos 30^\circ$$

where

$$\mathbf{n} = \cos 30^\circ\ \mathbf{j} + \sin 30^\circ\ \mathbf{k}$$

Thus since $A = L^2$ we have

$$\Phi = BL^2(\cos 30^\circ)$$

(b) For segment ab, we have $\mathbf{L}_{ab} = L(-\mathbf{i})$ so

$$\mathbf{F}_{ab} = I\mathbf{L}_{ab} \times \mathbf{B} = IBL(-\mathbf{i}\times\mathbf{j}) = BIL(-\mathbf{k})$$

For segment bc,

$$\mathbf{L}_{bc} = L \sin 30^\circ\ (-\mathbf{j}) + L \cos 30^\circ\ (\mathbf{k})$$

and

$$\mathbf{L}_{bc} \times \mathbf{B} = LB \cos 30^\circ\ (\mathbf{i})$$

so

$$\mathbf{F}_{bc} = BIL \cos 30^\circ\ (\mathbf{i})$$

For segment cd, we have $\mathbf{L}_{cd} = L\mathbf{i}$ (just the opposite of $\mathbf{L}_{ab}$) so

$$\mathbf{F}_{cd} = -\ \mathbf{F}_{ab} = BIL(\mathbf{k})$$

For segment da, ignore the slight break due to the bend in the wires and note that

$$\mathbf{L}_{da} = -\ \mathbf{L}_{bd}$$

so

$$\mathbf{F}_{da} = -\ \mathbf{F}_{bc} = -BIL \cos 30^\circ\ (\mathbf{i}).$$

The forces cancel in pairs and the net magnetic force vanishes.

(c) Taking an edge view of the coil, since torque equals force times lever arm, we have, referring to Fig. 30-5,

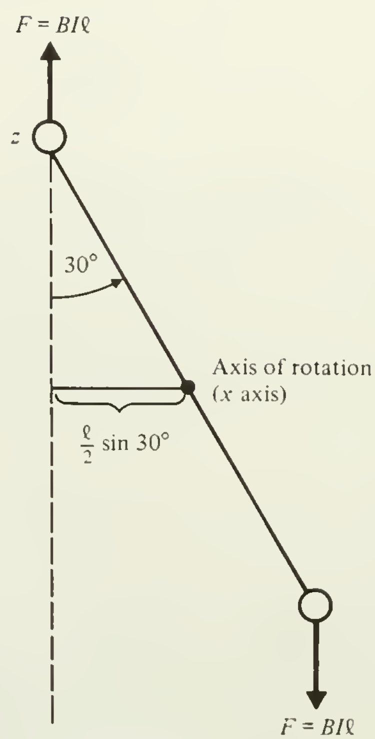


Figure 30-5

$$\Gamma = (BIL)\left(\frac{L}{2}\sin 30^\circ\right) \times 2$$

$$= BIL^2 \sin 30^\circ.$$

However

$$IL^2 = M, \text{ the magnetic moment,}$$

$$\Gamma = BM \sin 30^\circ = \frac{1}{2} BIL^2$$

where the torque has been taken about the x axis and has a contribution only from the forces on the dc and ab segments.

Another way to do this is to write

$$\mathbf{M} = IA\mathbf{n} = IL^2 [\cos 30^\circ (\mathbf{j}) + \sin 30^\circ (\mathbf{k})]$$

and calculate the torque Γ,

$$\Gamma = \mathbf{M} \times \mathbf{B} = IL^2 B \sin 30^\circ (\mathbf{k} \times \mathbf{j}) = BIL^2 \sin 30^\circ(-\mathbf{i}).$$

The vector $(-\mathbf{i})$ indicates that a clockwise rotation would take place about the x axis as counter-clockwise torques are positive.

Example 8

Find the force on a segment of straight wire, L, carrying a current, I, in a magnetic field B with arbitrary orientation with respect to the wire. Refer to Fig. 30-6.

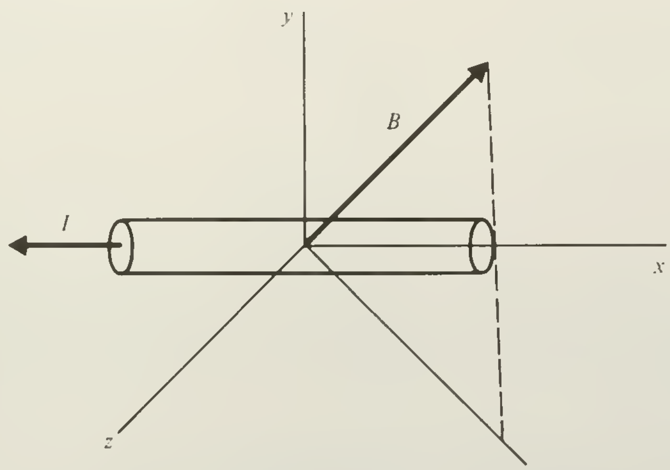


Figure 30-6

Solution:

Since B has arbitrary orientation, we write

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k},$$

and then by choosing different values for B_x, B_y, and B_z we can obtain any desired orientation of B. We have $\mathbf{L} = L(-\mathbf{i})$ since the current is in the negative x direction.

$$\mathbf{F} = I(\mathbf{L} \times \mathbf{B}) = IL\ (-\mathbf{i}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

$$= IL\ [B_y(-\mathbf{k}) + B_z(\mathbf{j})]$$

This is the final answer. We see that B_x is unimportant: the component of B parallel to the wire causes no force. In general then, the force vector lies in the YZ plane (the plane is perpendicular to the wire), the specific orientation depending on the relative magnitudes of B_y and B_z.

Example 9

Calculate the force on a wire of semi-circular shape carrying current I in a constant magnetic field B pointing in the + z direction. Refer to Fig. 30-7. Note that the force obtained is the same as that found for a straight wire connecting the end points a,b. See Example 8.

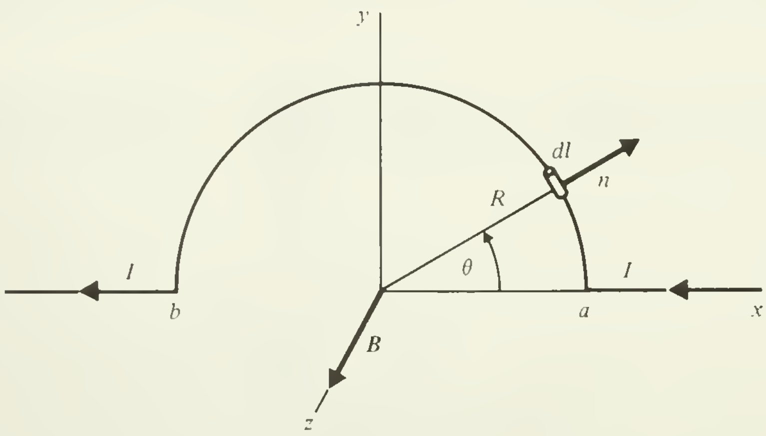


Figure 30-7

Solution:

The infinitesimal force on the segment dL carrying current I is

$$d\mathbf{F} = I\ d\mathbf{L} \times \mathbf{B}$$

Referring to Fig. 30-7, where dL is perpendicular to **B**

$$\mathbf{dL} \times \mathbf{B} = (dL)(B)\mathbf{n}$$

where

$$\mathbf{n} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}$$

Replacing dL by Rdθ, the element of force becomes

$$\mathbf{dF} = I(Rd\theta)(B)(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j})$$

The total force on this semi-circular piece of wire is obtained by integrating **dF** from $\theta = 0$ to $\theta = \pi$.

$$\mathbf{F} = \int_0^\pi (\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$$

Note that

$$\int_0^\pi \cos\theta\ d\theta = 0$$

since sin θ vanishes at both limits, but that

$$\int_0^\pi (-)\sin\theta\ d\theta = -2.$$

Therefore after integration, we have

$$\mathbf{F} = 2BIR(\mathbf{j})$$

Since we have 2R = diameter = length of wire between points a and b, this result is identical to Example 8.

If we consider a complete circle, then F = 0 because the sines and cosines will have both gone through a complete cycle.

Example 10

Suppose the magnetic field due to the earth can be represented by the vector

$$\mathbf{B} = B\cos 70^\circ\ (-\mathbf{i}) + B\sin 70^\circ\ (-\mathbf{k})$$

where $B = 2.5 \times 10^{-5}$ Tesla. A straight wire 1 m long carrying 500 A runs parallel to the ground (lies in the xy plane). Refer to Fig. 30-8.

(a) Calculate the force when the current is in the (− x) direction.

(b) Calculate the force when the current is in the (+ y) direction.

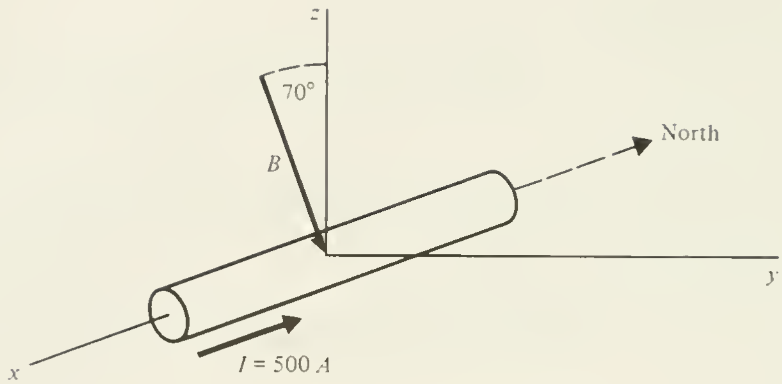


Figure 30-8

Solution:

(a) When the direction of the current is the − x direction,

$$\mathbf{L} = L(-\mathbf{i})$$

so that

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}$$

becomes

$$\mathbf{F} = IL(-\mathbf{i}) \times [B \cos 70^\circ(-\mathbf{i}) + B \sin 70^\circ(-\mathbf{k})]$$

$$= ILB \sin 70^\circ(-\mathbf{j})$$

This force is in the negative y direction and of magnitude

$$|\mathbf{F}| = (500\ \text{A})(1\ \text{m})(2.5 \times 10^{-5}\ \text{T})(0.940)$$

$$= 1.18 \times 10^{-2}\ \text{N}.$$

(b) When the current is in the + y direction,

$$\mathbf{L} = L(\mathbf{j})$$

The force becomes

$$\mathbf{F} = IL(\mathbf{j}) \times [B \cos 70^\circ(-\mathbf{i}) + B \sin 70^\circ(-\mathbf{k})]$$

$$= BIL[\cos 70^\circ(\mathbf{k}) + \sin 70^\circ(-\mathbf{i})]$$

$$= BIL[(0.940)(-\mathbf{i}) + (0.342)(\mathbf{k})]$$

making an angle of 20° with the (−x) direction in the xz plane. The magnitude of F is just BIL which equals 2.5×10^{-2} N.

Since a transmission line carries an alternating current, the force will oscillate, causing the line to vibrate.

Example 11

A loop of wire carries current, I, to produce a magnetic moment, M, as in Fig. 30-9. The loop has mass m and its center of gravity is located L units from the z axis (the coil is pivoted in a frictionless manner about the z axis). A constant magnetic field **B** = B(−**i**) points in the (−x) direction. If θ is 37° what value of B is needed for equilibrium? Would this value of B give equilibrium for any value of θ?

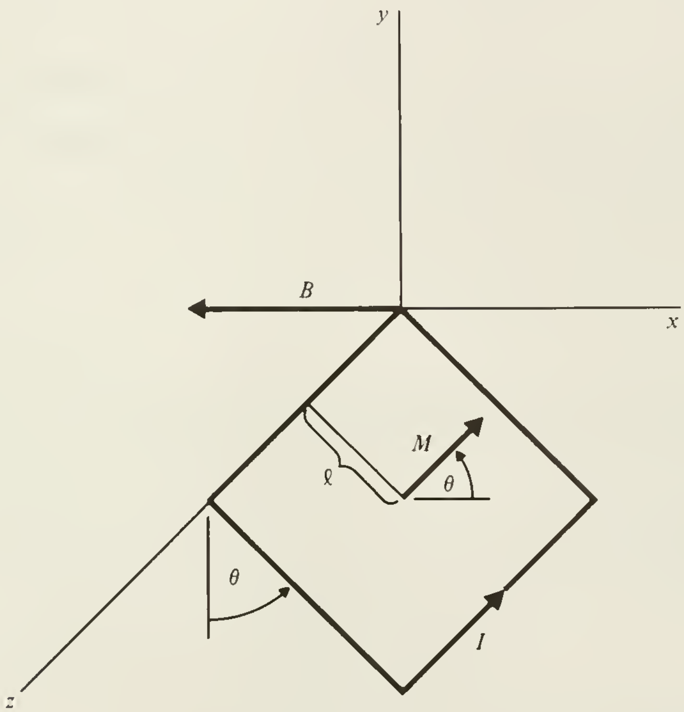


Figure 30-9

Solution:

$$\mathbf{B} = B(-\mathbf{i}) \text{ and } \mathbf{M} = M(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j})$$

Therefore the torque on **M** due to **B** is:

$$\Gamma = \mathbf{M} \times \mathbf{B} = BM\ (\sin\theta\ \mathbf{k}).$$

This tends to produce a c.c.w. rotation about the z axis. The torque due to the weight at the center of gravity, mg, is equal to **r** x **F** where

$$\mathbf{r} = L[\sin\theta\,(\mathbf{i}) + \cos\theta\,(-\mathbf{j})]$$

is the vector to the center of gravity from the axis of rotation and **F** = mg(-**j**). This torque is equal to:

$$\Gamma = L[\sin\theta\,(\mathbf{i}) + \cos\theta\,(-\mathbf{j})] = mgL\sin\theta\,(-\mathbf{k})$$

This would produce a c.w. rotation about the z axis. Requiring that

$$\Gamma_{total} = 0$$

we have:

$$BM\sin\theta - mgL\sin\theta = 0$$

Thus sin θ cancels out, and if $B = \frac{mgL}{M}$ we have zero torque (and hence neutral equilibrium) at any angle! This result could have been anticipated by looking at the expressions for the potential energy.

Example 12

A fictitious material with 4×10^{28} charge carriers per m^3 (shown in Fig. 30-10) is in a constant magnetic field in the (-z) direction of 3 T. Let x_1 = 3 cm, y_1 = 2.5 cm, and z_1 = 1 mm.

(a) If I = 10 A and the charge of each carrier is $q = -1.6 \times 10^{-19}$ C, calculate the magnitude and <u>sign</u> of the potential difference between the top face and the bottom face.

(b) Do the same calculation but for positive charge carriers $q = +1.6 \times 10^{-19}$ C.

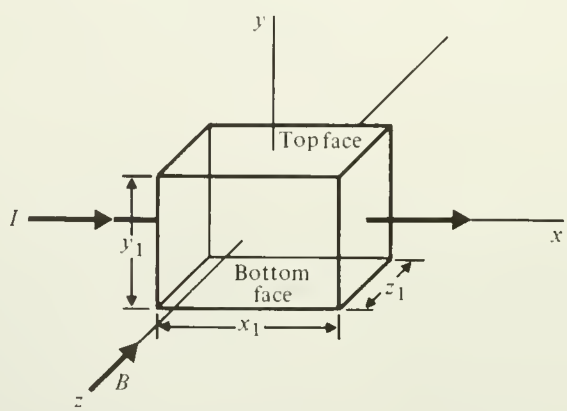


Figure 30-10

Solution:

To carry a current in the positive x direction, a negative charge carrier must have a drift velocity in the (− x) direction. $\mathbf{v} \times \mathbf{B}$ points in the (− y) direction but since the charge is negative, the magnetic force points in the (+ y) direction. Thus negative charge accumulates on the top face (and positive charge on the bottom face) until the internal electric field satisfies the equation $E = vB$. In this case, E points in (+ y) direction. The potential difference $V_H = Ey_1 = vBy_1$. The positive terminal ot the voltmeter should be attached to the bottom face. To obtain the magnitude of V_H, eliminate v, the drift velocity by noting that the current per unit area (J) is equal to the product of nqv. Thus

$$v = \frac{J}{nq} = \frac{(I/y_1z_1)}{nq}$$

and

$$V_H = vBy_1 = \frac{IB}{nqz_1}$$

Note V_H is independent of x_1 and y_1, but making the material thinner increases V_H!

Numerically, the product BI is 30 in S.I. units. Since force $F = BIL$, the units are $N\cdot m^{-1}$. We have

$$nqz_1 = (4 \times 10^{28}\ m^{-3})(1.6 \times 10^{-19}\ C)(10^{-3}\ m)$$

$$= 6.4 \times 10^{6}\ C\cdot m^{-2}.$$

Thus

$$V_H = \frac{30\ N\cdot m^{-1}}{6.4 \times 10^{6}\ C\cdot m^{-2}} = 4.69 \times 10^{-6}\ \text{volts}.$$

(b) If the charge carriers were positive, the magnitude of V_H would remain the same but the sign would be reversed (top face is positive) since the magnetic force still points in (+ y) direction. Thus positive charge builds up on the top face. By measuring the sign of the Hall voltage V_H, one can determine whether the dominant charge carriers in a material are positive or negative.

QUIZ

1. A particle with charge $q = +3.2 \times 10^{-19}$ C, mass $m = 3 \times 10^{-27}$ kg, and velocity $\mathbf{v} = v_0\mathbf{j}$ where $v_0 = 5 \times 10^5\ \text{m}\cdot\text{s}^{-1}$ enters a region of constant and uniform magnetic field $\mathbf{B} = B_0\mathbf{i}$ where $B_0 = 1.6$ T.

   (a) Compute the magnitude of the magnetic force.

   (b) Calculate the radius of the resulting circular orbit.

Answer: (a) $F = 2.56 \times 10^{-13}$ N
(b) $R = 2.93 \times 10^{-3}$ m.

2. A negatively charged particle with velocity $\mathbf{v} = 3 \times 10^6\ \text{m}\cdot\text{s}^{-1}(\mathbf{i})$ enters a region of constant and uniform electric field $\mathbf{E} = E_0\mathbf{j}$ with $E_0 = 300\ \text{N}\cdot\text{C}^{-1}$ and constant and uniform magnetic field perpendicular to both $\mathbf{v}$ and $\mathbf{E}$. Calculate the magnitude and direction of $\mathbf{B}$ necessary to give the particle zero deflection.

Answer: $\mathbf{B} = (10^{-4}\ \text{T})(-\mathbf{k})$.

3. A square coil (side = 4 cm) with 12 turns lies in the xy plane. Its sides are parallel to the axes and its center is at the origin. A constant and uniform magnetic field exists in the + y direction with

$$\mathbf{B} = (2.0\ \text{T})\mathbf{j}.$$

Calculate the current needed to produce a net torque about the x axis of 1.92×10^{-3} N·m.

Answer: I = 0.05 A (remember there are 12 turns).

4. When there is a current of 1.5 A in a thin film (thickness = 10^{-6} m) a Hall voltage of 0.144 mv is developed across the face of width 1.2 cm. A 3.0 T magnetic field is applied perpendicular to the face.

   (a) Calculate the drift velocity of the charge carriers.

   (b) Calculate the current density, J.

   (c) Calculate the charge density of the carriers, nq.

Answer: (a) $v = 4 \times 10^{-5}\ \text{m}\cdot\text{s}^{-1}$
(b) $J = 1.25 \times 10^8\ \text{A}\cdot\text{m}^{-2}$
(c) $nq = 3.12 \times 10^{12}\ \text{C}\cdot\text{m}^{-3}$.

31
SOURCES OF MAGNETIC FIELD

OBJECTIVES

In this chapter your objectives are to:

Calculate the magnetic field at an arbitrary position in space produced by a single moving charge.

Calculate the contribution to the magnetic field at an arbitrary position in space due to a small element of current carrying conductor.

Apply Ampere's law to highly symmetric current distributions in order to calculate the magnetic field produced by that current distribution.

Distinguish between paramagnetism, diamagnetism, and ferromagnetism and learn how to modify the expressions obtained for the magnetic field due to a given current distribution if magnetic materials are present.

Specific problems that will be encountered include calculation of the magnetic field due to a long straight wire, the field due to a circular loop of current, the field due to a toroid, and the field at the center of a solenoid.

The cross products used in the last two chapters are used frequently here. The ideas of symmetry used in connection with Gauss' law are also very useful in the applications that involve Ampere's law.

REVIEW

In this chapter, three expressions, all equivalent, are presented for calculating the magnetic field due to a particular motion of the charges. The first expression, based on experimental observations is:

$$\mathbf{B} = \left(\frac{\mu_o}{4\pi} \right) q \frac{(\mathbf{v} \times \mathbf{r})}{r^2}$$

where $(\mu_o/4\pi) = 10^{-7}$ in S.I. units, **v** is the velocity of charge q and **r** is a unit vector from the position of the charge (source point) to the point where B is evaluated (field point).

Using the model previously employed in Chapter 31 to relate the force on a moving charge to the force on a conductor carrying current in a uniform magnetic field, the above expression is generalized to:

$$\mathbf{dB} = \left(\frac{\mu_o}{4\pi}\right) I \frac{\mathbf{dl \times r}}{r^2}$$

where as before dl is a vector pointing in the direction of current flow and of magnitude dl. This form is extremely useful since, when integrated, it gives the magnetic field at any desired point for any given current distribution. Example 2 illustrates this equation, particularly its vector nature. In the text this formula is applied to the current distributions presented by a long straight wire and a circular loop of current. These are two very important results as judicious application of these results to other, more complicated geometries, frequently leads to a good qualitative estimate of the magnetic field (even if an exact calculation isn't feasible).

For the long straight wire (assumed infinitely long) the lines of B form closed circles concentric with the wire. On any given circle of radius,

$$B = \left(\frac{\mu_o}{4\pi}\right) \frac{2I}{r}$$

An application of this formula with superposition of the fields of two currents is given in Example 1. In Example 4 the field due to a straight wire of finite length is calculated and this result is shown to agree with the above formula as the length becomes infinite. For the circular loop, the magnetic field is calculated in the text only on a line perpendicular to the plane of the coil and passing through its center. On this line, B is parallel to the line (but not so, off the line). The result for B on this line is found to be

$$B = \left(\frac{\mu_o}{4\pi}\right) \frac{2I(\pi a^2)}{(a^2+x^2)^{3/2}}$$

where a is the radius of the loop and $(a^2+x^2)^{1/2}$ is the distance from the field point to any point on the loop. Since $I(\pi a^2)$ is the magnetic moment M, of the current loop,

$$B = \left(\frac{\mu_o}{4\pi}\right) \frac{2M}{(a^2+x^2)^{3/2}}$$

This form gives a very good indication of how quickly B falls off with increasing distance from the loop. Again it is seen that the lines of B (although not circles in this case) are closed curves with no beginning and no ending. This is always true for magnetic field lines and must be contrasted with the electric field lines which begin on positive charges and end on negative ones. For the magnetic field, there are no point sources, so the lines of B are closed. As there are no 'sources' or 'sinks' of the magnetic field, the flux of B through any closed surface vanishes.

Since the magnetic field lines for a long straight wire are concentric circles about the wire, the force on a second wire carrying current and parallel to the first is easy to calculate. This is illustrated in Example 3. In this case, B produced by the first wire is perpendicular to the current in the second wire so the force on the second wire is either attractive or repulsive depending on the relative current directions. If the two wires carry current in the same direction, they attract whereas if the currents are in opposite directions, they repel. The magnitude of the force on wire 2 carrying current I_2 is:

$$F_2 = B_1 I_2 L_2$$

where B_1 is the magnetic field due to wire 1 evaluated at the position of wire 2. If r is the separation between the two parallel wires, then

$$B_1 = \left(\frac{\mu_o}{4\pi} \right) \frac{2I_1}{r}$$

and

$$F_2 = \left(\frac{\mu_o}{4\pi} \right) \frac{2I_1 I_2 L_2}{r}$$

The force per unit length of wire 2 has the symmetric form,

$$\frac{F_2}{L_2} = \left(\frac{\mu_o}{4\pi} \right) \frac{2I_1 I_2}{r}$$

This expression provides the practical definition of the ampere in the S.I. system. If both I_1 and I_2 are equal to 1 ampere and r is 1 meter, then the force per unit length on either conductor is 2 x 10^{-7} N. The Coulomb is then defined as an ampere-second (i.e. the quantity of charge that flows past a point in one second if the current is 1 ampere).

The third method of calculating the magnetic field is based on Ampere's Law, a relationship analogous to Gauss' law for electrostatics. In the text, this law is justified by appealing to the resultant B for a long straight wire.

$$B_{wire} = \left(\frac{\mu_o}{4\pi} \right) \frac{2I}{r}$$

The lines of B point in a direction tangent to the circle of radius r. Thus

$$\int \mathbf{B} \cdot d\mathbf{r} = B \cdot 2\pi r = \frac{\mu_o}{4\pi} \; I.$$

or

$$\oint \mathbf{B}\cdot d\mathbf{r} = \frac{\mu_o}{4\pi} I.$$

As stated above, this result is totally general and applies to any geometry. In the above equation, I is the current that passes through the surface contained in the contour around which the line integral was performed. Just as we could have started electrostatics by writing down Gauss' law and deriving everything else (including Coulomb's law) from it, we could have started this chapter by writing Ampere's law and obtaining all the other results from it.

As elegant and general as Ampere's law is, its usefulness in calculating B for a given current distribution is confined to cases where the symmetry is so high that the direction and/or magnitude of B is constant on the various parts of the contour so that no actual integration has to be performed. If you cannot exploit the symmetry of the current distribution to write down the value of the line integral $\int \mathbf{B}\cdot d\mathbf{r}$ from physical grounds, then Ampere's law is not useful in obtaining B. Three examples are considered in the text (in addition to the long straight wire): the solenoid, the toroid, and the field between a charging parallel plate capacitor. Examples 5, 6 and 7 deal with aspects of Ampere's law.

Real materials can be classified qualitatively into three magnetic types: ferromagnetic materials, which are strongly attracted to the magnetic field produced by an electromagnet (or permanent magnet); paramagnetic materials, which are weakly attracted to the same magnetic field; and diamagnetic materials, which are weakly repelled by the above magnetic field.

This classification can be made more quantitative by forming the material in question into a 'Rowland ring' and using a toroidal wrapping of wire on this ring. All the flux produced is confined by this geometry and the ratio of the magnetic field produced inside the material (ring), B, to the field that would be produced in vacuum, B_o, by the same current and number of turns per unit length is defined to be the relative permeability, K_m. Thus $K_m = B/B_o$. K_m is exactly equal to one for a non-magnetic material, slightly greater than one for a paramagnetic material, slightly less than one for a diamagnetic material, and much larger than one for a ferromagnetic material. The field inside the material is related to the current distribution which produced it by the equation:

$$B = K_m\, \mu_o\left(\frac{N}{L}\right)I = \mu\left(\frac{N}{L}\right)I,$$

where $\mu = K_m\mu_o$.

EXAMPLES AND SOLUTIONS

Example 1

Two long straight wires as shown in the two sketches of Fig. 31-1a and 31-1b carry currents of 100 A and 200 A in the indicated directions. What is the magnetic field at point P midway between the wires if they are 0.1 m apart?

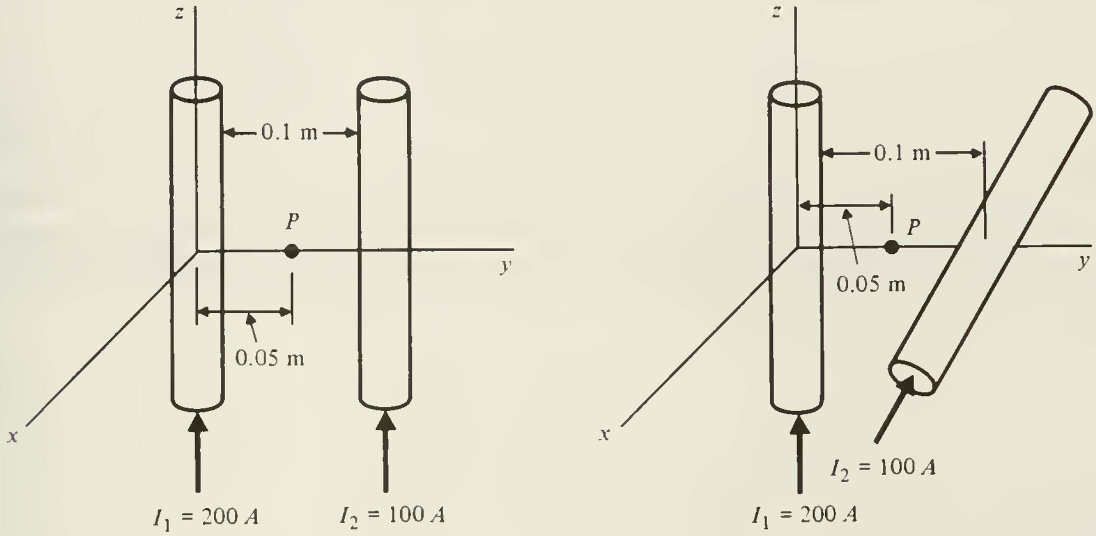


Figure 31-1

Solution:

(a) The magnetic field due to wires 1 and 2 should be calculated separately and then added (superimposed). The field at P due to wire 1 points in the (− x) direction (as can be seen from the right hand rule) and is of magnitude:

$$B_1(P) = \left(\frac{\mu_o}{4\pi}\right)\left(\frac{2I_1}{r}\right)$$

where $(\mu_o/4\pi) = 10^{-7}$ T·m·A^{-1}, I_1 = 200 A and r = 0.05 m; numerically we have

$$B_1(P) = -8 \times 10^{-4} \text{ T } (\mathbf{i})$$

The field at P due to wire 2 points in the (+ x) direction and is half the magnitude of $B_1(P)$ because the current is half as great. Thus

$$B\ (P) = -\ 4 \times 10^{-4}\ (\mathbf{i})$$

(b) The field at P due to wire 1 is unchanged in magnitude and direction:

$$B_1(P) = -\ 8 \times 10^{-4} \text{ T } (\mathbf{i})$$

The field due to wire 2 has the same magnitude at P as in part (a) but the direction is different,

$$B_2(P) = 4 \times 10^{-4}\ T(\mathbf{k}).$$

Thus we have

$$B(P) = 4 \times 10^{-4}\ T(\mathbf{k}) - 8 \times 10^{-4}\ T(\mathbf{i})$$

This is a vector of magnitude 8.94×10^{-4} T making an angle of 26.6° with the (-x) axis.

Example 2

Use the Biot law to calculate the magnetic field at point P due the segment of current carrying conductors shown in Fig. 31-2a and 31-2b. Take the current as I and the radius of the circle as R.

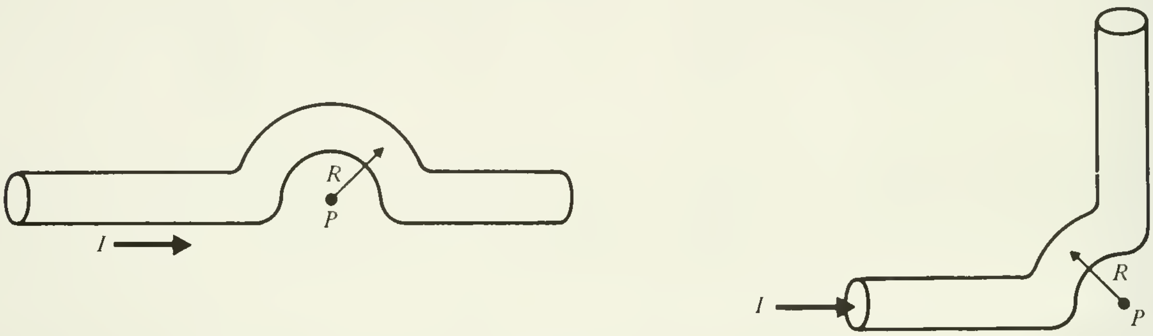


Figure 31-2

Solution:

The vector relationships in the Biot law simplify these calculations. Since P is on the line of the straight segments in both (a) and (b), these segments make no contribution to the magnetic field at P. Note that the **dL** and **r** vectors are parallel on these straight segments so **dL** x **r** = 0. Thus only the portions of the circle contribute to the field at P. For the current direction chosen, |**dL** x **r**| = dL and the direction is into the paper. Thus

$$dB = \left(\frac{\mu_o}{4\pi} I \right) \left(\frac{dL}{R^2} \right)$$

but dL = R dθ and in part (a) the integration over θ is from 0 to π whereas in (b) the integration covers 0 to π/2. Thus we have

$$B = \left(\frac{\mu_o}{4\pi} \right) \int \frac{IR\ d\theta}{R^2} = \left(\frac{\mu_o}{4\pi} \right) \frac{I}{R} \int d\theta$$

$$B = \left(\frac{\mu_o}{4\pi} \right) \frac{I}{R} (\theta_f - \theta_i)$$

The answers for parts (a) and (b) are:

$$\text{(a)} \quad B(P) = \left(\frac{\mu_o}{4\pi} \right) \frac{I\pi}{R} \;; \qquad \text{(b)} \quad B(P) = \left(\frac{\mu_o}{4\pi} \right) \frac{I\pi}{2R}$$

Example 3

Calculate the force on the various parts of the rectangular circuit carrying current I_2 shown in Fig. 31-3 due to the magnetic field of the long straight wire carrying current I_1.

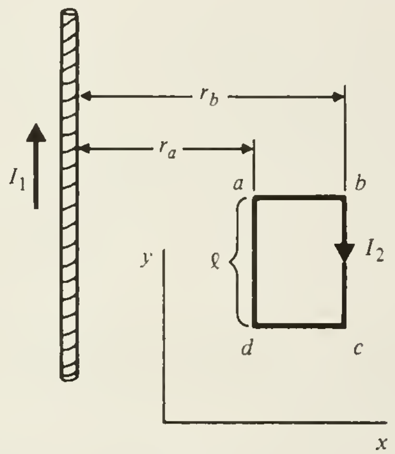


Figure 31-3

Solution:

(a) On segment ab, the field due to I_1 is into the paper and of magnitude

$$B_{ab} = \left(\frac{\mu_o}{4\pi} \right) \frac{2I_1}{r}$$

where r ranges from r_a to r_b. dL x B points in the (+ y) direction. Since B varies with position, we should write $dF = I_2 \, dr \, B(r)$ and integrate over r

$$F_{ab} = \left(\frac{\mu_o}{4\pi} \right)(2I_1 I_2)(\mathbf{j}) \int_a^b \frac{dr}{r}$$

$$\mathbf{F}_{ab} = \left(\frac{\mu_o}{4\pi}\right)(2I_1I_2)\left[\ln \frac{r_b}{r_a}\right](\mathbf{j})$$

(b) On segment bc, dL x B points in the (+ x) direction and B due to I_1 is constant in value and equal to

$$B_1 = \left(\frac{\mu_o}{4\pi} \right)\left(\frac{2I_1}{r_b} \right)$$

so the force is

$$\mathbf{F}_{bc} = B_1 I_2 \; L(\mathbf{i}) = \left(\frac{\mu_o}{4\pi} \right)\left(\frac{2I_1I_2L}{r_b} \right)(\mathbf{i})$$

(c) The considerations in part (a) apply here except now the order of integration is reversed so

$$\mathbf{F}_{cd} = - \mathbf{F}_{ab}$$

Thus there is no net force in the y direction.

(d) This is just like part (b) except

$$B_1 = \left(\frac{\mu_o}{4\pi} \right) \frac{2I_1}{r_a}$$

along path da. Thus we have

$$\mathbf{F}_{da} = \left(\frac{\mu_o}{4\pi} \right) \frac{2I_1I_2L}{r_a} \;(- \;\mathbf{i})$$

Since $r_a < r_b$, there is a net force in the (- x) direction:

$$\mathbf{F} = \mathbf{F}_{bc} + \mathbf{F}_{da}$$

$$= \left(\frac{\mu_o}{4\pi} \right)2I_1I_2L(\mathbf{i})\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

Example 4

Use the result for the magnetic field produced by a straight segment of wire (length L) at the midpoint to calculate the field at the center of a square coil. Refer to Fig. 31-4.

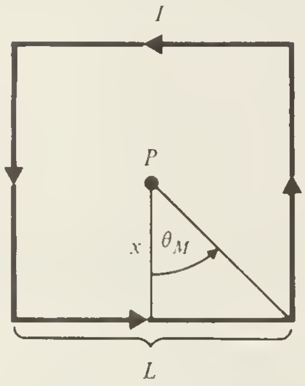


Figure 31-4

Solution:

The field at P due to one segment is equal to

$$B = \left(\frac{\mu_o}{4\pi} \right) \frac{2I}{x} \sin\theta_M$$

For a square coil as shown in Fig. 31-4, B points out of the page. The contributions from all four segments are equal and $\theta_M = 45°$. Thus:

$$B(P) = 4\left(\frac{\mu_o}{4\pi} \right) \frac{I}{(L/2)} \frac{(2)^{1/2}}{2}$$

$$= 4(2)^{1/2}\left(\frac{\mu_o}{4\pi} \right) \frac{I}{L} .$$

This field is larger than the field at the center of a circular loop of diameter L by the factor $(2)^{1/2}$.

Example 5

Use the result for the magnetic field due to a long straight wire to evaluate the integral of $\mathbf{B} \cdot d\mathbf{r}$ round the closed contour shown in Fig. 31-5 in the plane perpendicular to the wire.

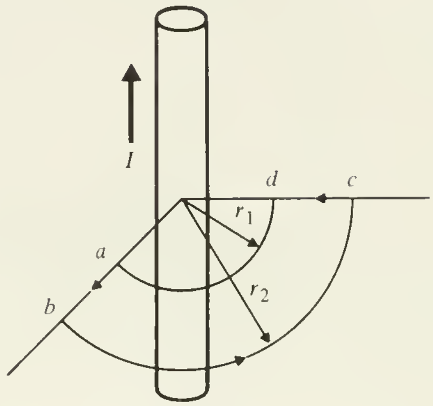


Figure 31-5

Solution:

(a) On segment ab, $\mathbf{B}$ is perpendicular to $d\mathbf{r}$ so $\mathbf{B} \cdot d\mathbf{r} = 0$.

(b) On segment bc, $\mathbf{B}$ is parallel to $d\mathbf{r}$ at every point so $\mathbf{B} \cdot d\mathbf{r} = B\,dr$. Furthermore at the distance r_2 from the wire we have

$$B = \left(\frac{\mu_o}{4\pi} \right) \frac{2I}{r_2}$$

which is the same magnitude at each point on segment bc. The value of B can be taken outside the integral leaving only the integral of dr over the quarter circle of radius r_2. This gives $(1/4)(2\pi r_2)$ for the path length, implying

$$\int \mathbf{B} \cdot d\mathbf{r} = \left(\frac{\mu_o}{4\pi} \quad \frac{2I}{r_2} \right) \frac{\pi r_2}{2} = \left(\frac{\mu_o}{4\pi} \right) \pi I$$

(c) For the segment cd, once again $\mathbf{B}$ is perpendicular to $d\mathbf{r}$ so $\mathbf{B} \cdot d\mathbf{r} = 0$.

(d) On the segment da, $\mathbf{B}$ is antiparallel to $d\mathbf{r}$ so $\mathbf{B} \cdot d\mathbf{r} = -B\,dr$. B is constant in magnitude and equal to

$$B = \left(\frac{\mu_o}{4\pi} \right) \frac{2I}{r_1} .$$

The integral over dr gives $\pi r_1/2$ so:

$$\int_{da} \mathbf{B}\cdot d\mathbf{r} = -\int_{da} B\, dr = -\left(\frac{\mu_o}{4\pi}\right)\frac{2I}{r_1}\cdot\frac{\pi r_1}{2} = -\left(\frac{\mu_o}{4\pi}\right)\pi I$$

Thus if we add results (a) through (d) we have $\int \mathbf{B}\cdot d\mathbf{r} = 0$. This is the result expected from Ampere's law as the current enclosed by the contour abcd is zero.

Example 6

Equal and opposite currents, I, are carried by long concentric thin cylinders of radii R_1 and R_2, as shown in Fig. 31-6a. Calculate the magnetic field between the cylinders and outside the two cylinders.

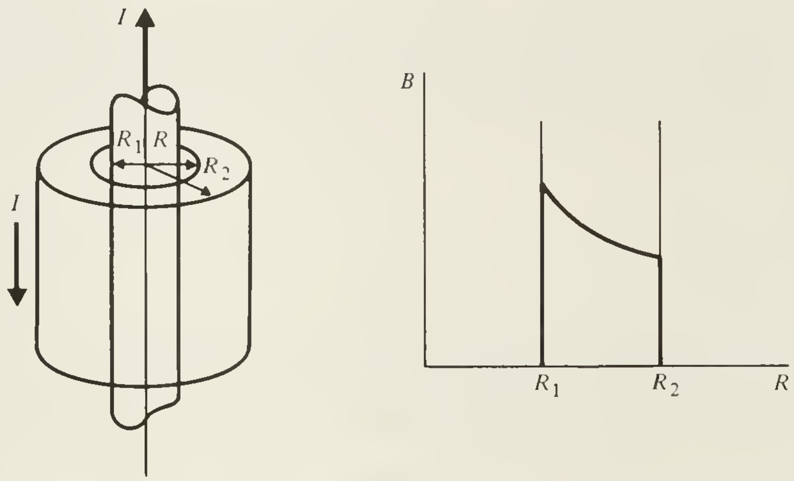


Figure 31-6a

Solution:

Choose a circular contour of radius R where $R_2 > R > R_1$. By symmetry, **B** is tangent to the circle of radius R and parallel to a c.c.w. path element **dL**. Also $\mathbf{B}\cdot d\mathbf{r} = B\, dr$ and B is constant on this circle. Thus

$$\int \mathbf{B}\cdot d\mathbf{L} = B \int dr = B\cdot 2\pi R = \mu_o I,$$

where I is the current carried by the inner conductor. Solving for B we obtain

$$B = \left(\frac{\mu_o}{4\pi}\right)\frac{2I}{R}$$

which is the same result that one would obtain for a long straight wire.

For a circular contour with $R > R_2$, while the direction of B is less obvious since there are two currents penetrating the surface, it is clear from rotational symmetry that B has the same magnitude at each point on the circle.

The direction of B must be along the tangent to the circle since the individual contributions from the two cylinders are each separately tangent to the circle. Designating this constant tangential component of B as B_t, leads to

$$B_t \cdot 2\pi R = \mu_o \text{ (current enclosed)} = \mu_o(I - I) = 0.$$

Thus we have $B_t = 0$ outside the second conductor. This result is similar to electric field calculations for this geometry from Gauss' law.

There is one additional result we could learn from this problem. Suppose we remove the inner conductor, leaving only a hollow cylinder of radius R_2 carrying current, I. For the imaginary surface of radius $R < R_2$ (inside the shell), the same symmetry conditions apply as were discussed previously but now no current cuts this surface. So $B \cdot 2\pi R = 0$ now and $B = 0$.

This result would not hold if the current flowed around the cylindrical surface in circular loops as shown in Fig. 31-6b. Then there would be a magnetic field inside the cylinder pointing along the axis of the cylinder since this geometry is essentially the same as that of a solenoid.

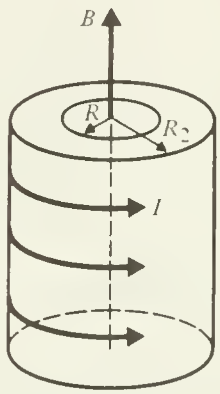
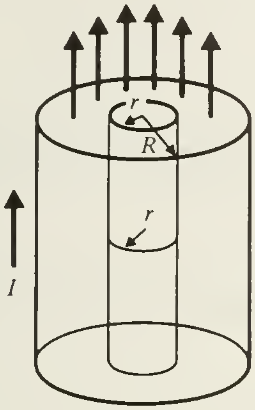


Figure 31-6b

Example 7

Use Ampere's law to calculate the magnetic field both inside and outside a solid cylindrical conductor of radius R that carries current I where the current per unit area (J) is constant. Refer to Fig. 31-7.



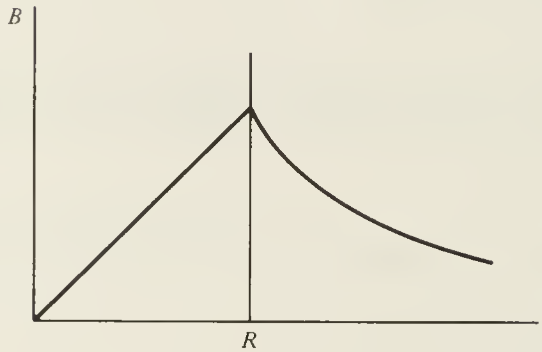


Figure 31-7

Solution:

Choose a circle of radius $r < R$. The region exterior to this circle gives no contribution to B on the circle. This was established in the previous example. For the region interior to the circle, B is tangent to the circle (the long straight wire result) and constant in magnitude. So

$$\int \mathbf{B}\cdot d\mathbf{r} = \int B\, dr = B \int dr = B\cdot 2\pi r$$

The current enclosed by this path is equal to $J\cdot\pi r^2$ since the current per unit area is assumed constant. Thus we have

$$2\pi r B = \mu_o J\, \pi r^2$$

Solving for B we find

$$B = \left(\frac{\mu_o}{4\pi}\right) J(2\pi r) \qquad r \leq R$$

This solution is valid up to the point $r = R$ where we write: (note $I = \pi R^2 J$)

$$B(R) = \left(\frac{\mu_o}{4\pi}\right)\frac{2J}{R}(\pi R^2)$$

$$= \left(\frac{\mu_o}{4\pi}\right)\frac{2I}{R}, \text{ a familiar result.}$$

For a circle of radius $r > R$, the same symmetries apply and

$$2\pi r\, B = \mu_o I$$

since this path encloses all of the current. In this region

$$B = \left(\frac{\mu_o}{4\pi} \right) \frac{2I}{r}$$

Thus B is zero at the center of this current distribution, increases linearly with r, the distance from the center, until $r = R$ (where B is maximum) and then decreases as r^{-1} for $r > R$.

Example 8

A Rowland ring with circumference of 0.25 m is constructed from a material with relative permeability of 500. The winding consists of 1000 turns. For a current of 3 A, calculate

(a) the magnetic intensity H;
(b) the magnetic field B;
(c) the magnetization.

Solution:

(a) The magnetic intensity $H = nI$ where n is the number of turns per unit length and I is the current.

$$H = \left(\frac{1000}{0.25\ \text{m}} \right)(3\ \text{A}) = 1.2 \times 10^4\ \text{A}\cdot\text{m}^{-1}$$

(b) The magnetic field $B = \mu H = K_m \mu_o H = 4\pi K_m \dfrac{\mu_o}{4\pi} H$

$$K_m = 500$$

$$B = 4\pi(500)(10^{-7}\ \text{N}\cdot\text{A}^{-2})(1.2 \times 10^4\ \text{A}\cdot\text{m}^{-1}) = 7.54\ \text{T}.$$

(c) Since $B = \mu_o(H + M)$, $M = \dfrac{B}{\mu_o} - H$, and

$$M = \left(\frac{B}{4\pi} \right) / \left(\frac{\mu_o}{4\pi} \right) - H = \frac{7.54\ \text{T}}{4\pi(10^{-7}\ \text{N}\cdot\text{A}^{-2})} - 1.2 \times 10^4\ \text{A}\cdot\text{m}^{-1}$$

$$= 5.99 \times 10^6\ \text{A}\cdot\text{m}^{-1}$$

QUIZ

1. Two long straight wires carry currents $I_1 = 150$ A and $I_2 = 200$ A <u>in the same direction</u>. The wires are parallel and separated by d = 0.2 m.

(a) Calculate the field at the position of wire 1 due to I_2 and the force per meter on wire 1.

(b) Calculate the field at the position of wire 2 due to I_1 and the force per meter on wire 2.

(c) Show that these two forces are equal in magnitude but opposite in direction. Do they have the same line of action?

Answer: (a) $B = 2 \times 10^{-4}$ T, $F/L = 3 \times 10^{-2}$ $N \cdot m^{-1}$

(b) $B = 1.5 \times 10^{-4}$ T, $F/L = 3 \times 10^{-2}$ $N \cdot m^{-1}$

(c) They are equal and opposite and have the same line of action.

2. For which of the configurations listed below can Ampere's Law be used to calculate the magnetic field?

(a) A wire of infinite length
(b) A wire of finite length
(c) A solenoid at the end
(d) A solenoid at the center
(e) A toroid
(f) A circular loop of wire

Answer: (a), (d), and (e).

3. A toroid consists of N turns wound uniformly on a 'doughnut' shaped form. A constant current of I amps is maintained in the winding. If the inner radins is 23 cm and the outer radius is 25 cm, use Ampere's law to estimate the percentage deviation of the magnetic field from the value at the mean radius of 24 cm.

Answer: Around the circle of radius 23 cm, the field is 4.35% higher than at the center. Around the circle of radius 25 cm, the field is 4 % lower than at the center.

4. The relative permeability of the core material in a Rowland Ring is 5000. The <u>mean</u> circumference of the ring is 0.4 m. The winding consists of 2000 turns and carries a current of 2 A. Calculate

(a) the magnetic intensity H
(b) the magnetic induction B
(c) the magnetization M

Answer: (a) $H = 10^4$ $A \cdot m^{-1}$

(b) $B = 62.8$ T

(c) $M = 5.0 \times 10^7$ $A \cdot m^{-1}$

32 ELECTROMAGNETIC INDUCTION

OBJECTIVES

In this chapter, you are introduced to Faraday's law of electromagnetic induction. Your objectives are to:

Calculate the motional emf that arises from the magnetic force.

Calculate the magnetic flux through an open surface.

Calculate the time rate of change of flux due to a change in effective area or due to a change in the magnetic field itself (or both).

Determine the direction of current in a closed circuit when an induced emf is present.

Summarize the laws of electromagnetism in the form of Maxwell's equations.

REVIEW

In previous chapters, the magnetic force on a moving charge particle has been used to predict orbits in magnetic fields, forces and torques on current carrying conductors, the Hall effect, and a variety of other interesting physical phenomena. Here it is used to establish the existence of a motional electromotive force (emf) for a conductor moving in a magnetic field. The effect of the magnetic force is shown to be equivalent to a non-conservative electric field, E_n, of magnitude $E_n = vB \sin \theta$ where θ is the angle between the velocity v and the field B. If the conductor is not part of a complete circuit, there will be no current but an emf will exist in the conductor. This emf is defined in terms of a line integral of the scalar product of E_n with dl the element of path length.

$$\mathcal{E} = \int \mathbf{E_n} \cdot d\mathbf{l}$$

If the moving conductor forms part of a complete circuit (in which there are no other sources of emf) then there is a current, I. The magnitude of I is given by: $I = \mathcal{E}/R$. This current leads to a force F on the conductor that can be calculated using the methods in Ch. 31 and 32. For a straight wire of length

L moving in a constant field, B, this force is F = BIL and is in such a direction as to oppose the original motion. This last conclusion is reasonable as the opposite situation would have this force assisting the motion so that the velocity of the moving conductor would increase--which would further increase the force and we would have a 'run-away' solution that would violate energy conservation. Experience tells us this is not the case so the force must oppose the motion. The above situation is analysed point-by-point in Example 1.

Another case of great practical importance is that of a coil of wire rotating about an axis perpendicular to a constant external magnetic field. An emf is induced in this case because the angle between the magnetic field and the normal to the plane of the coil changes with time. This time dependent change of flux ($\phi = \mathbf{B}\cdot\mathbf{n}\, A$ in this case) results in an emf that can be taken off the rotating coil (depending on the type of ring used) as either an alternating voltage or a pulsating voltage all of one sign. This device then is a generator and converts mechanical energy (needed to rotate the coil) into electrical energy.

Faraday's law, namely

$$\mathcal{E} = -\frac{d\phi}{dt}$$

is a generalization of the previous idea which states that an emf is developed if there is relative motion between a conductor and a magnetic field. No motion is necessary in Faraday's law, only a time dependent magnetic flux, in order to generate an emf. Thus if the flux through a stationary coil is changed by increasing or decreasing the magnetic field, an emf is generated. This type of emf (no moving parts) is illustrated in Examples 3, 4, and 5.

Several comments need to be made regarding Faraday's law. If the coil has N turns rather than one turn, then the flux through the entire coil

$$\phi = N \int \mathbf{B}\cdot\mathbf{n}\, dA$$

and is N times larger than the flux through 1 turn (assuming all turns are alike). The emf is thus increased by a factor N. If ϕ_1 is the flux through 1 turn only (and all turns are alike), then:

$$\mathcal{E} = -N\,\frac{d\phi_1}{dt}\ .$$

Since $\phi = N\phi_1$, the equation just written is identical in content to the previous one.

The non-conservative electric field, E_n, which is responsible for the emf in Faraday's law,

$$\mathcal{E} = \oint \mathbf{E}_n\cdot d\mathbf{L} = -\frac{d\phi}{dt}$$

is very different from the conservative electric fields that arise from static charge distributions. For such conservative electric fields, the line integral (like the one above) of **E**·d**L** around a closed curve always gives zero. Furthermore the integral of **E**·**n** dS over a closed surface is equal to Q/ε_o where Q is the charge inside the surface (Gauss's law). For our non-conservative electric field, $\mathbf{E}_n$, the line integral around a closed curve is not necessarily zero but depends on the time rate of change of flux. The integral of $\mathbf{E}_n \cdot \mathbf{n}$ dS over any closed surface, however, is always zero.

In our previous discussion of induced emf's, we have been mainly concerned with the magnitude of this emf. Lenz's law is concerned with the sign of the emf. Lenz's law states that the direction (sign) of the induced emf is such as to oppose the motion (or change of flux) that created the emf. The sign convention associated with the calculation of the flux and the positive sense of the line integral (essentially a right hand rule) is carefully discussed in the text and should be adhered to in all problems.

The laws of electromagnetism are neatly summarized in a set of equations known as Maxwell's equations. We have met each of these equations before individually when we studied Gauss's law, the magnetic equivalent of Gauss's law, Ampere's law (with the displacement current included), and Faraday's law. These four equations, together with the Lorentz force law describe all of electromagnetic theory.

EXAMPLES AND SOLUTIONS

Example 1

If the conducting rod AB of length 0.25 m in Fig. 32-1 is moving to the right with a velocity of 3 $m \cdot s^{-1}$ in a constant magnetic field of magnitude 0.8 T (pointing into the paper), what external force must be applied to maintain constant velocity? Assume the complete circuit has a total resistance R = 0.2 Ω.

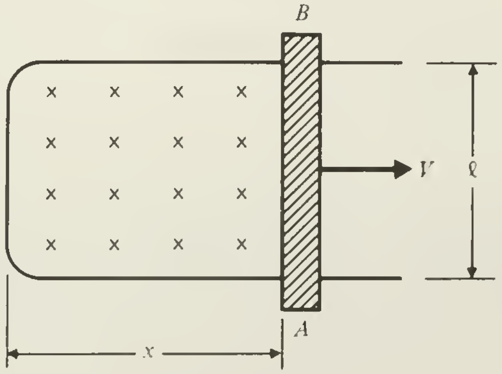


Figure 32-1

Solntion:

(a) The emf generated is vBL and can be calculated by finding the work done per unit positive charge in going from A to B or by calculating the flux change in the circuit due to the motion of rod AB.

$$\phi = xLB, \qquad \frac{d\phi}{dt} = \frac{dx}{dt} LB = vLB = -\mathcal{E}$$

Since the resistance is R, the current $I = \dfrac{vBL}{R} = 3$ A.

(b) This current flows from A to B producing a field for the motion shown so as to the decrease the change in flux. A magnetic force $\mathbf{F_m} = \mathbf{IL} \times \mathbf{B}$ is exerted on this conductor in a direction opposite to **v**. Since **L** and **B** are perpendicular,

$$F_m = BIL = BL\left(\frac{vBL}{R}\right) = \frac{(BL)^2 v}{R} = 0.6 \text{ N}$$

using the value of I obtained in (a). This force opposes the motion and points to the left.

(c) To move the rod AB to the right with constant velocity, v, we must supply an external force, to the right, with magnitude equal to F_m given above.

Example 2

Show that 1 volt is equal to 1 weber per second.

Solution:

The volt was defined as the work done per unit positive charge so one volt is equal to one joule per coulomb.

$$1 \text{ V} = \frac{1 \text{ J}}{1 \text{ C}}$$

The joule is a newton·meter whereas the coulomb is one ampere·sec. Thus

$$1 \text{ V} = \frac{1 \text{ N}\cdot\text{m}}{1 \text{ A}\cdot\text{s}}$$

The tesla is defined so that 1 T equals one newton per ampere·meter.

$$1\ \text{V} = \frac{1\ \text{N}\cdot\text{m}^2}{1\ \text{A}\cdot\text{m}\cdot\text{s}} = \frac{1\ \text{T}\cdot\text{m}^2}{\text{s}}$$

The weber is defined to be 1 $\text{T}\cdot\text{m}^2$ so:

$$1\ \text{V} = \frac{1\ \text{weber}}{1\ \text{s}}$$

Example 3

A high power line carrying a current of $I(t) = I_0 \cos 2\pi ft$ is located near a rectangular coil with dimensions of 5 cm x 10 cm as shown in Fig. 32-2. $I_0 = 250$ A and $f = 60$ Hz. Calculate the emf induced in this coil.

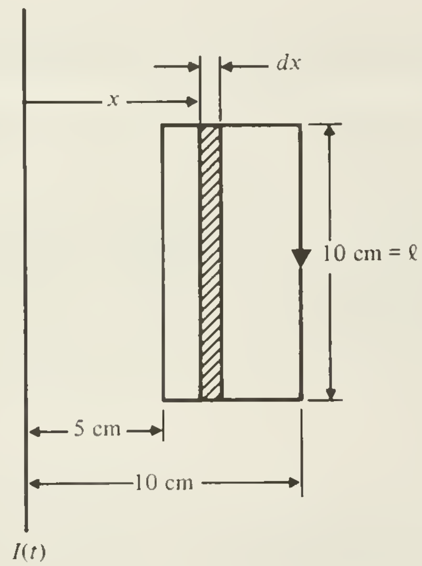


Figure 32-2

Solution:

To solve this problem, first we must use the result for the magnetic field near a long straight wire to calculate the flux through the coil. Then we use Faraday's law to obtain the emf from the time rate of change of the flux.

On the shaded element of area, $dA = L\ dx$, the field has the value

$$B = \frac{\mu_0}{4\pi} \frac{2I(t)}{x} .$$

For the direction chosen for I, B points inward so if we choose $\mathbf{n}$ inward, the flux at $t = 0$ is positive. This means that the positive sense of dL for the line integral is clockwise.

$$\phi = \int B dA = \frac{\mu_o}{4\pi} 2I(t)L \int_{x_i}^{x_f} \frac{dx}{x}$$

$$= \frac{\mu_o}{4\pi} 2I(t)L \ln\left(\frac{x_f}{x_i}\right)$$

Since $x_f/x_i = 2$, we have

$$\phi = \frac{\mu_o}{4\pi} 2I(t)L \ln(2)$$

Using Faraday's Law, we have

$$\mathcal{E} = -\frac{d\phi}{dt} = \frac{\mu_o}{4\pi} 2(\ln 2)L\ 2\pi f I_o \sin 2\pi f t$$

At t = 0, this is positive, so the non-conservative electric field dotted into dL is positive on our contour and the initial induced current flows c.w.

It is instructive to calculate the peak value of this emf to get some idea of the problem posed by 'pick-up' near power lines.

$$\mathcal{E}_{peak} = (10^{-7}\ T\cdot m\cdot A^{-1})(1.386)(.1\ m)(377\ s^{-1})(250\ A)$$

$$= 1.31 \times 10^{-3}\ T\cdot m^2\cdot s^{-1} = 1.31\ mV$$

If the coil had N turns (instead of one), this result would be increased by this factor N, i.e.

$$\mathcal{E}_{peak} = 1.31\ N\ mV\ .$$

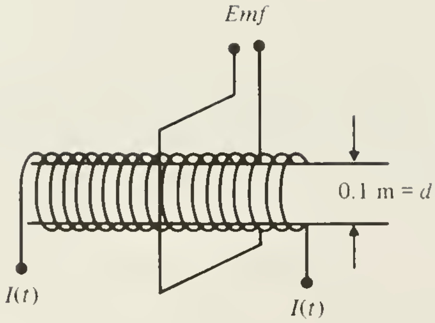
Suppose the nearest edge of the coil is 1 m away from the power line rather than 5 cm. Then $x_f/x_i = 1.05/1.00$ and the factor ln 2 is replaced by ln 1.05. This results in a reduction in peak emf by a factor of 7×10^{-2}.

Example 4

A square coil (l = 0.25 m) is placed around a solenoid of average diameter 0.1 m and length (L) of 0.2 m wrapped with 1000 turns (N) as shown in Fig. 32-3. The coil has its normal parallel to the axis of the solenoid. The current through the solenoid when it is energized obeys the equation

$$I(t) = I_o(1 - e^{-t/\tau})$$

where I_o is equal to 100 A and τ is 5 seconds. Calculate the emf induced in the square coil as a function of time.



$I(t) = I_0\,(1 - e^{-t/\tau})$

Figure 32-3

Solution:

(a) First we need the relationship between the current in the solenoid and the magnetic field B. Using the value appropriate to the center of a long solenoid,

$$B = \frac{\mu_o}{4\pi}\, 4\pi(N/L)I.$$

Thus the magnetic flux through the square coil is equal to the above value of B multiplied by the <u>area of the solenoid</u>, $\pi d^2/4$, not the area of the square coil. (We assume that the solenoid is long enough so that B is essentially zero outside the solenoid volume so that the only region of the square coil that contains magnetic field lines is the part inside the solenoid.)

(b) The emf is found by calculating $d\Phi/dt$.

$$\mathcal{E} = \frac{\mu_o}{4\pi}\, 4\pi\left(\frac{N}{L}\right)\frac{\pi d^2}{4}\left(\frac{dI}{dt}\right)$$

$$= \frac{\mu_o}{4\pi}\, 4\pi\left(\frac{N}{L}\right)\frac{\pi d^2}{4}\left(\frac{I_o}{\tau}\right)e^{-t/\tau}$$

The peak value of the emf is obtained at t = 0. This peak value is equal to 9.87×10^{-4} volts. From t = 0, the emf decreases exponentially with increasing time. If the square coil contains N turns then the emf will be increased by this factor.

Example 5

For the stationary single turn coil (r = 10 cm) shown in the Fig. 32-4 calculate the flux through the coil, the emf induced, and the current in the coil if its resistance is R = 0.1 Ω. Let the time dependence of the magnetic field be given as $\mathbf{B} = B_o \exp(-t/\tau)(\ \mathbf{j}\ )$ where $B_o = 1.5$ T and $\tau = 3$ s.

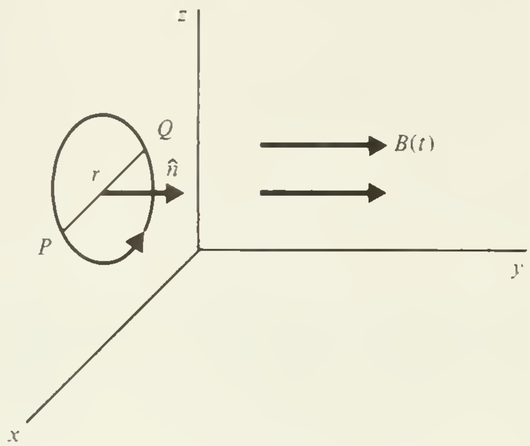


Figure 32-4

Solution:

If we choose the normal to the coil **n** = **j** (as shown), then the positive sense of **dL** is c.c.w. as you face the xz plane.

(a) The flux through the coil is:

$$\phi = B(t)A = B(t)\ \pi r^2$$

(b) The emf is found from Faraday's law

$$\mathcal{E} = -\frac{d\phi}{dt} = -(\pi r^2)\frac{dB}{dt} = -(\pi r^2)(-\frac{1}{\tau} B_o e^{-t/\tau})$$

Thus

$$\mathcal{E} = (\pi r^2)(\frac{B_o}{\tau})e^{-t/\tau}.$$

This emf has a peak value of $\pi r^2 B_o/\tau$ and decays to zero exponentially. Since ξ is positive (or zero), the line integral of $\mathbf{E}_n \cdot dl$ is positive so $\mathbf{E}_n$ is parallel to dl on this circle. Thus the current direction is in the positive sense of dl or c.c.w. here. Again, the flux produced by the current tends to oppose the changing flux of the field. (Lenz's law)

(c) The current $I = \mathcal{E}/R$ so

$$I_{peak} = (\pi r^2) \frac{B_o}{R\tau}$$

For the numerical values given here,

$$\mathcal{E}_{peak} = 1.57 \times 10^{-2} \text{ V and } I_{peak} = 0.157 \text{ A.}$$

Example 6

Suppose an ideal voltmeter is connected by a straight line path between two points P and Q on opposite sides of the circular coil in the previous example. (Refer to Fig. 32-4). Calculate the potential difference read by this meter.

Solution:

For this coil, emf is generated uniformly throughout the circle so no specific part of the coil acts as the source of emf. Similarly the resistance is uniformly distributed around the ring. Thus writing

$$V = \mathcal{E} - IR$$

where V is the potential difference, $\mathcal{E}$ is the emf, and I the current, one sees that $V_{PQ} = 0$. This results from I being equal to $\mathcal{E}/R$. Note that on the path from P to Q, we develop half the total emf and encounter half of the total resistance so $V_{PQ} = (1/2)\,\mathcal{E} - I(1/2\ R) = 1/2(\mathcal{E} - IR) = 0$ as $I = \mathcal{E}/R$.

Example 7

Suppose the ring in the previous example is made of two different metals, the resistance of one part being 0.1 Ω while the other resistance is 0.2 Ω so the total resistance is now 0.3 Ω. Refer to Fig. 32-5. The voltmeter connection is the same as in Example 6. If the induced emf is 0.06 V with the current flowing c.c.w., would the potential difference between P and Q read by the voltmeter still be zero?

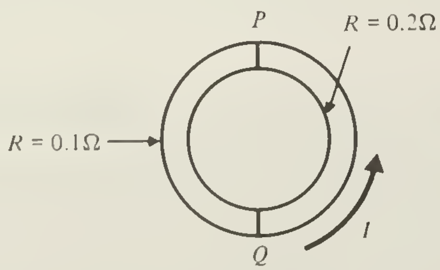


Figure 32-5

Solution:

The current I is continuous and equal to 0.2 A and the emf is uniformly generated but the losses are different on the two halves of the ring. Thus $V_{PQ} = 0.03\ V - (0.2)(0.2)V$ or $V_{PQ} = 0.01\ V$ with P negative with respect to Q.

Example 8

A coil with L = 12 cm and w = 6 cm and mass per unit length of $10^{-2}\ kg\ m^{-1}$ is dropped from a height h as shown in Fig. 32-6. It enters the region of constant magnetic field of B = 0.5 T with the velocity acquired in the fall. The coil resistance is 0.1 Ω. What height h should be chosen so that the velocity of the coil in the magnetic field is constant until the entire coil is in the constant magnetic field?

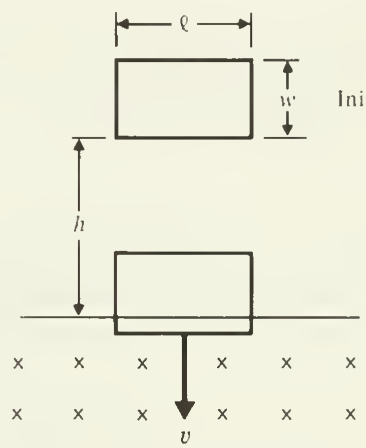


Figure 32-6

Solution:

(a) The flux through the coil, when the edge of the coil is y units into the field is BLy.

(b) The time rate of change of flux is BLv so this is the emf. The current is equal to the emf divided by the resistance.

$$I = BLv/R$$

(c) The force on this current carrying wire is upward and equal to

$$BIL = B\left(\frac{(BLv)}{R}\right)L$$

(d) If the velocity is to remain constant, then this upward force must be just equal to the weight, mg. (i.e. the coil is in free fall until the bottom side enters the field and after that the magnetic force just balances mg.)

$$mg = \frac{(BL)^2}{R} v$$

(e) The velocity v is obtained by equating the change in potential energy, mgh, to the change in kinetic energy, $1/2\ mv^2$. Thus $v^2 = 2gh$ or

$$v = (2gh)^{1/2}.$$

This leaves

$$mg = \frac{(BL)^2}{R} (2gh)^{1/2}$$

an equation we can solve for h since all other parameters are given.

(f) Numerically we have

$m = 3.6 \times 10^{-3}$ kg, $L = 0.12$ m,

$B = 0.5$ T, $R = 0.1\ \Omega$,

$g = 9.8\ m \cdot s^{-2}$

so that

$h = 4.9 \times 10^{-2}$ m or $h = 4.9$ cm.

Example 9

Suppose the coil in the previous example is dropped with the shorter side, w, pointing down. If it is dropped from the same height as the other coil, how do the retarding forces compare?

Solution:

The retarding force on the first coil (F_1) was equal to

$$F_1 = \frac{(BL)^2 v}{R}$$

The retarding force on the second configuration of this coil (F_2) is found by replacing L by w so:

$$F_2 = \frac{(Bw)^2 v}{R}$$

Thus in this problem since $w = 1/2\ L$ we have $F_2 = F_1/4$ and the coil, dropped with the short side down, would still experience a net downward force in the gravitational field. Its acceleration would be $3g/4$ rather than zero and the braking effect of the magnetic field is greatly reduced. This is the general idea behind laminating metal parts that move in magnetic fields to reduce the eddy current effects. In this example, we reduced the length of the conductor carrying the current in the field and this in turn reduced the force considerably.

A practical consequence of Faraday's law is the existence of eddy currents in metallic pieces that move in magnetic fields. These currents can be beneficial and used as a 'brake' on the motion or they can be troublesome and need to be reduced. In Example 9 a simple situation has been presented to illustrate these ideas.

QUIZ

1. A conductor of length $L = 0.2$ m, in a constant and uniform magnetic field, $B = 3$ T, slides over contacts at a constant velocity $v = 0.3\ m\cdot s^{-1}$. The moving conductor is part of a complete circuit. The magnetic field is perpendicular to the plane of the circuit. Refer to Fig. 32-1. If the electrical resistance of the complete circuit is constant and equal to $0.4\ \Omega$,
    (a) calculate the emf generated
    (b) calculate the current in the circuit
    (c) calculate the force on the moving conductor due to this motion in the magnetic field.

Answer: (a) emf = 0.18 volts
(b) $I = 0.45$ A
(c) $F = 0.27$ N

2. A circular loop of wire ($R = 0.15$ m) rotates with constant angular velocity ($\omega = 100\ \pi\ rad\cdot s^{-1}$) in a constant and uniform magnetic field $B = 1.8$ T. The loop rotates about a diameter which is perpendicular to the magnetic field direction. Calculate the maximum emf generated in the coil.

Answer: emf = 40.0 volts

3. The velocity of a car traveling due east is constant and equal to 150 $km\cdot hr^{-1}$. If the earth's magnetic field is vertical at the location of the car and equal to 1.6×10^{-5} T (pointing downward), (a) calculate the magnitude of the emf induced in the axle of the car if the axle is 2 m long. (b) Use Lenz's law to find the direction of the emf.

Answer: (a) The emf = 1.33×10^{-3} volts. (b) A positive charge would be forced north.

4. A coil form of area 3 cm^2 has a total of 15 turns of wire wrapped on it. The leads of the coil are connected together through an external resistance of 1 $M\Omega$. If the coil form, with its normal initially parallel to a constant magnetic field $B = 3$ T, is rotated through 180^0 so that its normal is now antiparallel to the field, calculate the magnitude of the charge that flows through the resistor.

Answer: $Q = 2.7 \times 10^{-8}$ C.

33
INDUCTANCE

OBJECTIVES

In this chapter, inductance is introduced. Like capacitance, it is a property that depends on the geometric dimensions of the circuit element. Your objectives are to:

Calculate, for a particular geometry, both the <u>self-inductance</u> (L) and <u>mutual inductance</u> (M) of a circuit element or elements, from the defining equations for L and M.

Calculate the energy stored in the magnetic field.

Analyse voltage-current relations in simple practical circuits containing self-inductances.

REVIEW

Inductance, like capacitance, is a physical property of components commonly used in electrical circuits. We associate inductance most frequently with wire, coils, wrappings, etc., but virtually all components used in circuits have some inductance. As capacitance could be used to store energy in the electric field so inductance can be used to store energy in the magnetic field established in the particular geometry.

Mutual inductance is a property of components which arises because the magnetic field produced by one element links other elements and can produce emf's in them by virtue of Faraday's law. This emf is thus directly related to the current change in the first component since magnetic fields result from current distributions. If $I_1(t)$ is the current in element (coil) 1, then the magnetic field at element 2 due to 1 can be calculated using the Biot law. Integration of this field over the area of element 2 gives the flux through one turn of element 2 due to the current distribution of element 1. If element 2 has N_2 turns, then according to Faraday's law, the emf induced in it by a current change in element 1 is:

$$(\text{emf})_2 = -N_2 \frac{d\Phi_1}{dt} = -M_{12} \frac{dI_1}{dt}$$

where M_{12}, the mutual inductance of elements 1 and 2, simply relates this induced emf in element 2 to the current change in element 1. Since N_2 and M_{12} are independent of time, we have also:

$$N_2 \phi_1 = M_{12} I_1$$

These relationships are illustrated in Example 1. The mutual inductance of two elements depends only on their geometries and relative position and not upon which element is considered the source of the magnetic field. This statement is easy to prove for two interpenetrating solenoids (see Example 2) but much more difficult to prove for other current distributions.

An element or a coil, by itself, has inductance called self-inductance and denoted by the symbol L. This property is most easily discussed for a toroid but it is a general property of all components. In the case of a toroid, the current in the windings sets up a magnetic field which is constant over the volume of the toroid and zero elsewhere. For a toroid with N turns wound on a core of circumference l this field is approximately

$$B = \mu_o \frac{N}{l} I.$$

This field produces a flux through each turn of magnitude $\phi_1 = BA$ where A is the cross-sectional area of the toroid. The emf induced in the toroid due to a current change would be:

$$(\mathrm{emf})_1 = -N \frac{d\phi_1}{dt} .$$

The self inductance L is defined by the equation

$$(\mathrm{emf})_1 = -L \frac{dI_1}{dt} .$$

Thus we can write (since N and L are independent of time)

$$N\phi_1 = LI_1.$$

This is correct for all current distributions. For our specific example, the toroid, we have

$$L = \frac{N\phi_1}{I} = \frac{N}{I} BA = \mu_o \frac{N^2}{l} A.$$

Apart from the factor μ_o, both L and M are dimensionally lengths since the number of turns carries no dimensions. The self-inductance of virtually every configuration of circuit elements has been calculated and can be regarded as a tabulated quality which could be found if necessary from the Biot law.

Since the power input to a circuit element is the product of the emf and the current, for an inductance, either mutual or self, we have for the energy stored in an inductor:

$$\frac{dW}{dt} = Li \frac{di}{dt} = \frac{d}{dt} \left(\frac{1}{2} Li^2 \right)$$

Thus we can associate an energy of $1/2\ Li^2$ with an inductor. This energy is stored in the magnetic field surrounding the particular element of inductance L, carrying current i.

The unit for mutual and self-inductance is the henry. If a current change of one ampere per second produces an induced voltage of 1 volt, the inductance (self or mutual) is equal to one henry. Apart from the factor μ_o, the dimension of inductance is simply length. We should also note that since a volt per ampere is an ohm that a henry is also equal to an ohm-second. For this reason an inductance divided by a resistance has dimensions of time.

Since a current change in an inductance (or inductor) results in an emf, an inductance is an important practical circuit element. Its role in circuits containing a resistance, R, an R-L circuit and in circuits containing a capacitor and inductance (an L-C circuit) is discussed in the text. A series circuit containing all three elements (an R-L-C circuit) is discussed in a physical way but without detailed mathematical calculation.

For the R-L circuit, the transient behavior is obtained by recognizing that the sum of the emf's is equal to the sum of the potential drops (Kirchhoff's law). If V_o represents a d.c. voltage which is suddenly impressed on a resistor connected in series with an inductor (for instance by closing a switch in the circuit), then the sum of the emf's is equal to $V_o - Ldi/dt$ whereas the sum of the potential drops is just equal to iR. Thus we have

$$V_o - L\left(\frac{di}{dt} \right) = iR,$$

a differential equation that is solved in the text to give i as a function of time. The solution is shown to be

$$i(t) = \frac{V_o}{R} \left(1 - e^{-t/\tau}\right)$$

where τ, the time constant, is equal to L/R and has the same dimensions as time. For very long times, the exponential becomes negligibly small and $I(t) = V_o/R$ as it would in a simple d.c. circuit containing a resistance R only. For short times $t \ll \tau$, the exponential can be approximated by

$$e^{-t/\tau} \simeq 1 - t/\tau$$

so

$$i(t) \simeq \frac{V_o}{R} \quad \frac{t}{\tau} .$$

Thus the current initially rises linearly with time (more slowly for larger L) and then for long times approaches the value V_o/R.

If we charge a capacitor (C) to an initial value Q_o and then connect it to an inductor (L) by closing a switch, oscillation occurs just as with a mass vibrating on a perfect spring. We can use Kirchhoff's law once again to write:

$$-L \frac{di}{dt} = V_{cap} = \frac{1}{C} q(t)$$

Since $i(t) = dq/dt$ we can rewrite this in the form of a second order differential equation

$$\frac{d^2q}{dt^2} + \frac{1}{LC} q(t) = 0.$$

We wrote the spring equation characteristic of single harmonic motion with angular frequency ω_o in the form

$$\frac{d^2x}{dt^2} + \omega_o{}^2 x = 0,$$

so that a comparison of the above two equations shows that the previous equation describes such a simple harmonic oscillation (of the charge on the capacitor and hence the current through the inductor) at an angular frequency

$$\omega_o = \frac{1}{(LC)^{1/2}}$$

Energy stored in the electric field of the capacitor (like the potential energy of the spring) is converted into energy stored in the magnetic field of the inductor (analogous to the kinetic energy of the mass m on the spring). No energy is lost or dissipated in this circuit.

If we add a resistive element, R, to this circuit, then for each cycle of oscillation of the circuit a certain amount of energy is lost by heat generated by the resistor. If the energy loss per cycle is small compared to the total energy in the circuit, the circuit continues to oscillate for many cycles at an altered frequency but the amplitude of the oscillation damps out and eventually the oscillation stops. If the energy loss per cycle is higher and roughly equal to the total energy in the circuit, the amplitude of the oscillation (here it is the stored charge initially on the capacitor) quickly goes to zero and no real oscillation may be discerned. A physical situation analogous to the series R-L-C circuit is found in many systems.

PROBLEM-SOLVING HINTS

To calculate the mutual or self-inductance of a circuit or element, a very systematic procedure can be used:

(1) Use the Biot law or Ampere's law to find the magnetic field (at an arbitrary point) for the given current distribution. This gives B as a function of the position and the current I.

(2) Use the dependence of B on position to calculate the flux enclosed by the circuit or element.

(3) If there are multiple turns involved, equate the product of the number of turns and the flux through a single turn to the inductance multiplied by the current and then solve for the inductance.

In working circuit problems with inductances, include an inductance as a source of emf equal to $-L(di/dt)$ (or M if a mutual inductance is being considered) and then use Ohm's law. A pure (resistanceless) inductance does not dissipate any power.

EXAMPLES AND SOLUTIONS

Example 1

Calculate the mutual inductance between a long straight wire and a rectangular circuit of dimensions 0.1 m by 0.2 m positioned with respect to the wire as shown in Fig. 33-1a, 33-1b, and 33-1c.

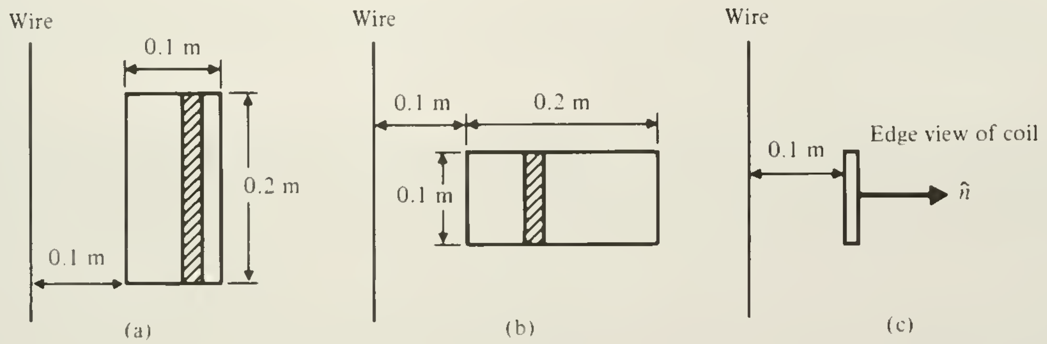


Figure 33-1

Solution:

(a) We assume there is a current I_1 in the long straight wire since this is the easiest magnetic field to calculate in this problem and calculate the flux in the loop due to this current . If we measure the coordinate r radially from the wire, we have

$$B_1 = \frac{\mu_o}{4\pi} \frac{2I_1}{r} .$$

The element of flux, $d\phi_2$, through the cross-hatched element of area, $dA = L\,dr$ (where L is the length of coil parallel to the wire) is:

$$d\phi_2 = \frac{\mu_o}{4\pi} 2I_1 L\left(\frac{dr}{r}\right).$$

If we integrate dr/r from r_1 to r_2 we obtain the flux through element 2 (the rectangular coil) due to the current in element 1 (the wire);

$$\phi_2 = \frac{\mu_o}{4\pi} 2I_1 L \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\mu_o}{4\pi} 2I_1 L \ln(r_2/r_1)$$

Since the rectangular coil has only one turn, $\phi_2 = M_{12}I_1$ and we have:

$$M_{12} = \frac{\mu_o}{4\pi} 2L \ln\left(\frac{r_2}{r_1}\right)$$

Numerically, for part (a), we have

$$M_{12}(a) = (10^{-7})(0.4) \ln 2 \text{ henry.}$$

$$= 2.77 \times 10^{-8} \text{ henry or } 0.0277\ \mu H$$

(b) The general formula developed in (a) can still be used here but the numerical value is changed:

$$M_{12}(b) = (10^{-7})(0.2) \ln 3 \text{ henry}$$

$$= 2.20 \times 10^{-8} \text{ henry.}$$

(c) For this part, since $\mathbf{B}_1 \cdot \mathbf{n} = 0$ due to the orientation of the coil, the flux ϕ_2 is zero and $M_{12}^{(c)} = 0$. From parts a, b, and c we can conclude that the mutual inductance depends critically on the relative orientation of the elements.

Example 2

Consider a short coil (length L_1) of radius R_1 with N_1 turns inside a second long (length $L_2 >> L_1$) solenoid of radius R_2 with N_2 turns. Calculate the mutual inductance of the pair

(a) if the axes of the two coils are parallel

(b) if the axes of the two coils are perpendicular.

Solution:

(a) Assume a current I_2 flows through the outer solenoid. This gives a uniform field

$$B_2 = \mu_o \frac{N_2}{L_2} I_2$$

throughout the region interior to this solenoid and near its center. The flux through the inner solenoid due to I_2 is then B_2 multiplied by the cross-sectional area of the inner solenoid (πR_1^2).

$$\phi_1 = \mu_o \left(\frac{N_2}{L_2} \right) I_2 \ (\pi R_1^2)$$

Since the inner solenoid has N_1 turns, we write

$$N_1 \phi_1 = M_{21} I_2$$

and solving for M_{21} we find that

$$M_{21} = \frac{N_1 \phi_1}{I_2} = \mu_o \frac{N_1 N_2}{L_2} \pi R_1^2$$

(b) If the axes are perpendicular, the flux through coil 1 is zero as the normal to its surface is perpendicular to B_2. The mutual inductance is now zero. Although the coils are the same, their geometric arrangement is different so the mutual inductance is different.

Example 3

The capacitor shown in Fig. 33-2 initially holds a charge Q_o and no current flows until the switch S is closed. Obtain the charge and the current as functions of time.

(a) Evaluate q and i for $t = (\pi/2) \times 10^{-4}$ s and $t = \pi \times 10^{-4}$ s if $Q_o = 10^{-9}$ C, C = 2 μF, and L = 5 mH.

(b) Verify that the energy is conserved.

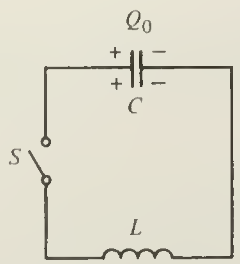


Figure 33-2

Solution:

(a) After S is closed, the inductance L acts as a source of emf equal to $-Ldi/dt$. The voltage drop across the capacitor, V_c, is equal to q/C so Ohm's law gives:

$$-L\left(\frac{di}{dt}\right) = \left(\frac{1}{C}\right)q$$

Since $i = dq/dt$, $di/dt = d^2q/dt^2$ we can rearrange the above equation as

$$\frac{d^2q}{dt^2} + \left(\frac{1}{LC}\right)q = 0.$$

This is to be compared with the equation we obtained for the displacement of a mass m on a spring (with force constant k):

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{d^2x}{dt^2} + \omega_o{}^2x = 0$$

Thus $\omega_o{}^2 = 1/LC$ for the present case. The general solution for the charge is

$$q = A\cos\omega_o t + B\sin\omega_o t$$

and the current is

$$i = \frac{dq}{dt} = -\omega_o A\sin\omega_o t + \omega_o B\cos\omega_o t$$

To find A and B we note that at $t = 0$, we have $i = \omega_o B$. Since $i = 0$ at $t = 0$ then B must be zero. At $t = 0$, we have $q = A$ but since the initial charge is specified as Q_o, then $A = Q_o$ leaving

$$q = Q_o\cos\omega_o t,$$

and

$$i = -\omega_o Q_o\sin\omega_o t.$$

For this problem

$$\omega^2 = \frac{1}{LC} = \frac{1}{(5 \times 10^{-3}\ \mathrm{H})(2 \times 10^{-6}\ \mathrm{F})} = 10^8\ \mathrm{s}^{-2}$$

Thus $\omega = 10^4\ s^{-1}$.

At $t = 0$, $q = Q_o$ and $i = 0$. At $t = (\pi/2) \times 10^{-4}$ s, we have

$$\omega t = (10^4\ s^{-1})(\pi/2)(10^{-4}\ s) = \pi/2$$

and

$$q = Q_o \cos(\pi/2) = 0$$

$$i = -\ \omega Q_o \sin(\pi/2) = -\ \omega Q_o$$

$$= -\ 10^{-5}\ A$$

At $t = \pi \times 10^{-4}$ s, we have

$$q = Q_o \cos(\pi) = -\ Q_o = -\ 10^{-9}\ C$$

$$i = -\ \omega Q_o \sin(\pi) = 0.$$

(b) The energy stored in the capacitor is given by

$$E_C = \frac{1}{2}\left(\frac{q^2}{C}\right) = \frac{1}{2C} Q_o^2 \cos^2 \omega_o t.$$

The energy stored in the inductor is:

$$E_L = \frac{1}{2}(Li^2) = \frac{1}{2} L\, \omega_o^2\, Q_o^2 \sin^2 \omega_o t$$

$E_C + E_L$ is constant since $1/C = L\omega_o^2$ and $\sin^2 \omega_o t + \cos^2 \omega_o t = 1$.

Example 4

Obtain the equivalent single inductance, L_E, to replace two inductances L_1 and L_2 connected (a) in series, and (b) in parallel. See Fig. 33-3a and b.

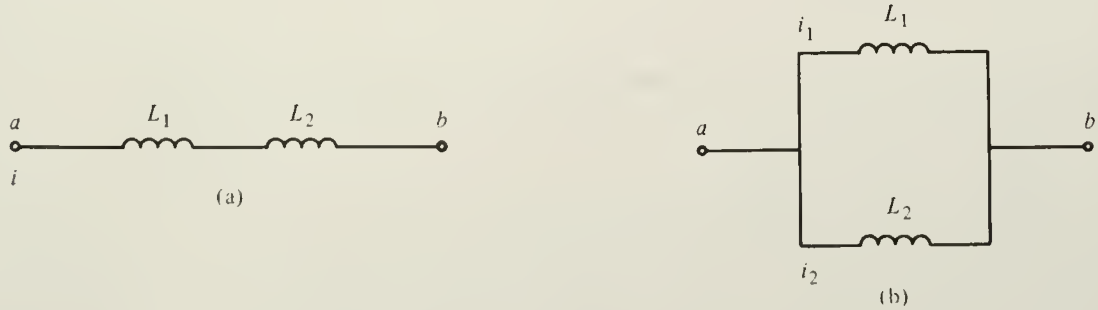


Figure 33-3

Solution:

(a) For two inductances in series, the current i through them is the same and the emf developed between points a and b is equal to:

$$V_{ab} = -L_1 \frac{di}{dt} - L_2 \frac{di}{dt} = -(L_1 + L_2) \frac{di}{dt} .$$

The same emf would be obtained if L_1 and L_2 were replaced by one inductance, L_E provided $L_E = L_1 + L_2$.

(b) For the parallel combination, we want to find L_E such that

$$V_{ab} = - L_E\, di/dt.$$

We have that $V_{ab} = - L_1\, di_1/dt$ and also that $V_{ab} = - L_2\, di_2/dt$. Since $i_1 + i_2 = i$, replace i_2 by $i - i_1$, and write:

$$V_{ab} = - L_1 \frac{di_1}{dt} = -L_2 \frac{d}{dt}(i - i_1);$$

Collecting both terms involving i_1 gives

$$-(L_1 + L_2) \frac{di_1}{dt} = - L_2 \frac{di}{dt}$$

or

$$- \frac{di_1}{dt} = -\left(\frac{L_2}{L_1 + L_2} \right) \frac{di}{dt} .$$

Multiply by L_1 to obtain

$$V_{ab} = -L_1 \frac{di_1}{dt} = - \frac{L_1 L_2}{L_1 + L_2} \frac{di}{dt} = - L_E \frac{di}{dt} .$$

Thus

$$L_E = \frac{L_1 L_2}{L_1 + L_2} .$$

This is exactly like the result obtained for resistors in parallel:

$$\frac{1}{L_E} = \frac{1}{L_1} + \frac{1}{L_2} .$$

Example 5

Calculate the self-inductance per unit length of a pair of concentric thin cylinders of radii R_1 and R_2 where $R_2 > R_1$. See Fig. 33-4a, b.

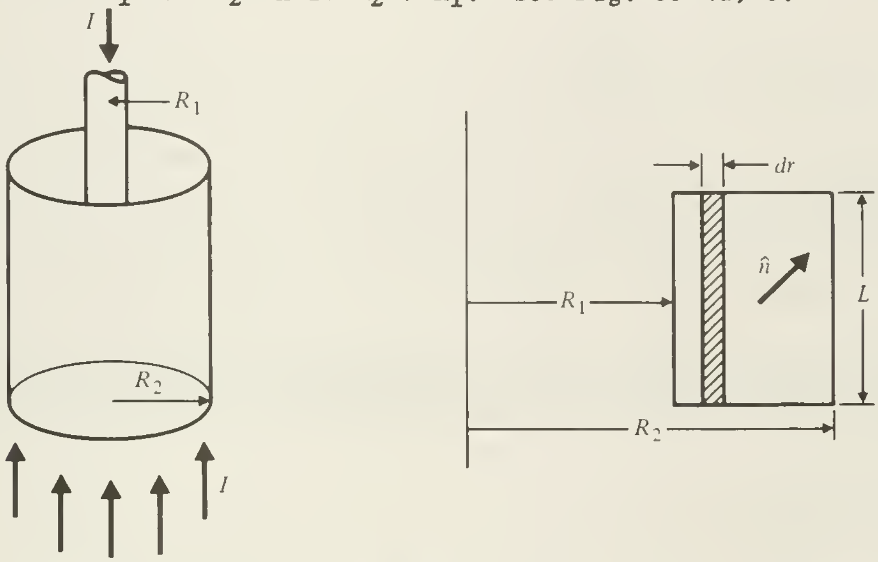


Figure 33-4

Solution:

Before working this problem, review Example 6 in Chapter 31 where the magnetic field B was calculated for this geometry. There it was found that for equal but opposite currents, I, on the two surfaces, that

(a) $$B = \frac{\mu_o}{4\pi}\frac{2I}{r} \qquad \text{for } R_1 < r < R_2$$

This value comes just from the current on the inner cylinder because B due to current on the outer cylinder is zero.

(b) B = 0 outside the pair of cylinders (since the <u>net</u> current through the Amperian surface is zero).

Thus choosing a surface as shown in Fig. 33-b with **n** pointing out of the paper, the flux $d\phi$ through the cross-hatched region (B is constant over this area) is

$$d\phi = \mathbf{B}\cdot\mathbf{n}dA$$

$$= \frac{\mu_o}{4\pi}\frac{2I}{r}\, 1\, dr,$$

so

$$\phi = \frac{\mu_o}{4\pi} 2Il \int_{R_1}^{R_2} \frac{dr}{r}$$

$$= \frac{\mu_o}{4\pi} 2Il \ln \left(\frac{R_2}{R_1} \right).$$

Since we have only 'one turn' involved here, $\phi = LI$ giving for the inductance per unit length, L/l the value;

$$\frac{L}{l} = \frac{\phi}{lI} = \frac{\mu_o}{4\pi} 2 \ln\left(\frac{R_2}{R_1} \right).$$

A more physical approach to this problem involves using the formula for the energy per unit volume in the magnetic field,

$$u = \frac{1}{2\mu_o} B^2 = \frac{U}{V}$$

and the energy stored in an inductance, $U = 1/2\ LI^2$ to obtain L. To do this write

$$dU = \frac{1}{2\mu_o} B^2\, dV.$$

Let $dV = 2\pi r\ l\ dr$ and

$$B = \frac{\mu_o}{4\pi} \frac{2I}{r}$$

as before. Then we have,

$$dU = \frac{1}{2\mu_o} \left(\frac{\mu_o}{4\pi} \frac{2I}{r} \right)^2 2\pi r l\ dr$$

$$= \frac{\mu_o}{4\pi} lI^2 \frac{dr}{r}.$$

Integrating from R_1 to R_2, we have

$$U = \frac{\mu_o}{4\pi} lI^2 \ln \left(\frac{R_2}{R_1} \right) = \frac{1}{2} LI^2$$

so

$$L = \frac{\mu_o}{4\pi} 2l \ln (R_2/R_1) \text{ as before.}$$

Example 6

Suppose we connect a 'lossy' or real inductor (inductance L and resistance r) in series with a light bulb which we represent by a pure resistance R as shown in Fig. 33-5.

(a) Calculate the voltage drop, V, across the light bulb after the switch S is closed.

(b) Assume the switch S has been closed a long time so that there is a steady current. If switch S is now opened, calculate V as a function of time.

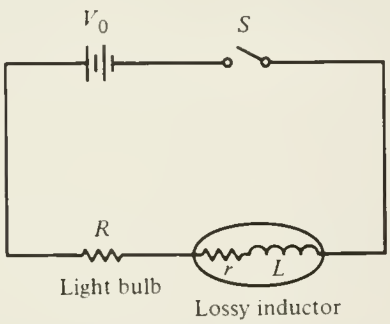


Figure 33-5

Solution:

(a) After S is closed, we regard the inductance as a source of emf and use Ohm's law:

$$V_o - L\left(\frac{di}{dt}\right) = (R + r)i$$

The variables, i and t are separable here so this equation can be integrated as done in the text. For variety, we can solve this by a change of variables. Write

$$\frac{di}{dt} = -\frac{R + r}{L}\left(i - \frac{V_o}{R + r}\right).$$

Letting

$$x = i - \frac{V_o}{R + r}$$

we have

$$\frac{dx}{dt} = \frac{di}{dt} . \qquad \text{Set } \tau = \frac{L}{R + r} .$$

Then we have

$$\frac{dx}{dt} = -\frac{1}{\tau}x \qquad \text{or} \qquad x = x_o\, e^{-t/\tau}.$$

At $t = 0$, there is no current, so

$$x_o = -\frac{V_o}{R + r}$$

giving

$$i - \frac{V_o}{R + r} = -\frac{V_o}{R + r} e^{-t/\tau}$$

or

$$i = \frac{V_o}{R + r}(1 - e^{-t/\tau}).$$

The voltage drop across the light bulb V is:

$$V = i(t)R = \frac{V_oR}{R + r}(1 - e^{-t/\tau}).$$

Thus the bulb is turned on smoothly; the larger the ratio $L/(R + r) = \tau$, the slower the voltage increase.

(b) For very long times, the current reaches the value

$$i = \frac{V_o}{R + r} .$$

If the switch S is now opened, Kirchhoff's law reads (since V_o is no longer in the circuit as an emf),

$$- L\left(\frac{di}{dt} \right) = (R + r)i$$

which we readily integrate as

$$\frac{di}{i} = - \frac{dt}{\tau}$$

to give:

$$i(t) = i(0)e^{-t/\tau}.$$

For i(o) we use the value

$$i(o) = \frac{V_o}{R + r}$$

so the voltage across the light, V, decays exponentially to zero:

$$V(t) = \frac{V_o R}{R + r} e^{-t/\tau}.$$

QUIZ

1. A toroid carrying a current of 3 A has a self inductance of 1 mH. The mean radius of the toroid is 0.075 m and the cross-sectional area is equal to 5×10^{-4} m^2.
   (a) Calculate the number of turns on the toroid.
   (b) Calculate the flux through the toroid.
   (c) Calculate the energy stored in the toroid.

Answer: (a) $N = 7.5 \times 10^5$
(b) flux $= 2.25 \times 10^6$ webers
(c) $E = 4.5 \times 10^{-3}$ J

2. A very long solenoid with radius R = 0.08 m has 4000 turns per meter. A second coil consisting of 150 turns is wound around the center of the solenoid. Calculate the mutual inductance of this pair.

Answer: $M = 1.52 \times 10^{-2}$ H.

3. A toroid of cross-section $A = 5 \times 10^{-4}$ m^2 and mean radius R = 0.2 m carries a current of 5 A. The toroid has 7000 turns. (a) Assuming the magnetic field

inside the toroid is uniform and equal to the value along the circumference, calculate the energy per unit volume stored in the toroid.
(b) Using the expression for the energy stored in an inductance and the result from part (a), calculate the self inductance of the toroid.

Answer: (a) $u = 487.4\ J \cdot m^{-3}$
(b) $L = 2.45 \times 10^{-2}$ H.

4. A 5 H inductor carrying an initial current of 100 A is dischargea through a resistor of 1 Ω. (a) Calculate the initial voltage drop across the resistor and (b) the time needed for the current to reach a value of 10 A.

Answer: (a) V = 100 volts
(b) t = 11.5 s.

34
ALTERNATING CURRENTS

OBJECTIVES

In this chapter you will define inductive and capacitive reactance, quantities that like resistance limit the current flow for given difference in potential. A method is given for analysing simple circuits subject to sinusoidal voltages at a single frequency and consisting of passive elements R, L and C. Your objectives are to:

Obtain the source voltage as the <u>vector sum</u> of voltages across the individual components in a series R-L-C circuit.

Obtain the source current as a <u>vector sum</u> of currents through the individual elements in a parallel R-L-C circuit.

Calculate the phase difference between source voltage and current.

Calculate the time average power consumed by the circuit.

REVIEW

We have previously studied electrical circuits where both steady currents and time dependent currents were present. The passive circuit elements, resistance, capacitance, and inductance have each been studied separately so that we know how to relate the potential difference across each element to the current, the integral of the current with respect to time (the charge) or the time derivative of the current. Thus whenever the current is not constant, we generally must solve a differential equation. We now will study the special (but very important) case where the voltage source produces an emf that is a sinusoidal (sine or cosine) function of time at a single frequency, f. We have

$$v = V \cos \omega t$$

where v is the instantaneous value of the emf, V is constant and equal to the maximum value of the emf, and $\omega = 2\pi f$ is the angular frequency. The general solution to the problem of connecting such a source to a circuit will consist of a transient current, as we found previously for inductors and capacitors, plus a sinusoidal current at the same frequency as the source voltage. We will ignore the transient behavior in this chapter and concentrate on the sinusoidal solution.

If such a source is connected to a resistance element, R, the current is

$$i(t) = \frac{v(t)}{R}$$

so the maximum value of current is V/R and the current and voltage have precisely the same time dependence so they are said to be 'in phase'.

If a capacitor, C, is connected to this alternating current (a.c.) source, the charge on the capacitor oscillates sinusoidally since $q(t) = C\ v(t)$. The current through the capacitor is equal to dq/dt so:

$$i(t) = C \frac{dv(t)}{dt} = - \omega C\ V \sin \omega t.$$

The maximum value of this current, ωCV, can be thought of as V/X_c where X_c, the capacitive reactance, is equal to $1/\omega C$ and plays the role of a resistance in that it relates a maximum current to the maximum voltage. The unit for X_c is the ohm. The above current does not have the same time dependence as the voltage; they are 'out of phase'. We can be quantitative about this by using the identities:

$$\cos(\omega t + \phi) = \cos \omega t \cos \phi - \sin \omega t \sin \phi$$

$$\sin(\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi$$

Thus $- \sin \omega t$ is the same as $\cos(\omega t + 90^\circ)$. We conclude that for a capacitor the voltage and current are 90° 'out of phase', with the current <u>leading</u> the voltage.

For an inductor, L, connected to the a.c. source, since

$$v(t) = L \frac{di}{dt} ,$$

by integrating di/dt, we have

$$i(t) = (\frac{V}{\omega L}) \sin \omega t.$$

Technically since we integrated v(t) to get i, we should include a constant term in i(t) but since the time average of i must be zero, the constant is zero. Here the maximum value of i is given by V/X_L where X_L, the inductive reactance is equal to ωL. Again the unit for X_L is the ohm. Note that $\sin \omega t = \cos(\omega t - 90^\circ)$ so the current <u>lags</u> the voltage in an inductor. This is physically reasonable because of Lenz's law.

If we connect two or more of these elements in series to the a.c. voltage source, the source voltage, v(t), will be equal to the algebraic sum of the voltage drops across the various components. For practical problems however it is preferable to write that the common current, through the elements is

$$i(t) = I \sin \omega t$$

and calculate the voltage by <u>vector addition.</u> This can be illustrated by treating an RLC series combination. The algebraic solution is given in Example 5.

For the resistor,

$$v_R = Ri = RI \sin \omega t,$$

in phase with i(t). For the inductor

$$v_L = \frac{di}{dt} = \omega LI \cos \omega t = IX_L \sin(\omega t + 90^o).$$

For the capacitor

$$v_C = (\frac{1}{C})\ q(t) = \frac{1}{C} \int i\ dt = -\frac{I}{\omega C} \cos \omega t$$

$$= IX_C \sin(\omega t - 90^o).$$

Referring to Fig. 34-1, let ωt be an angle measured c.c.w. from the x axis. Thus if the x axis is used as the reference axis for the current, the voltage across the resistor is a vector (of magnitude IR) along the + x axis. The voltage across the inductor is a vector (of magnitude IX_L) along the + y axis. The voltage across the capacitor is a vector (of magnitude IX_c) along the − y axis.

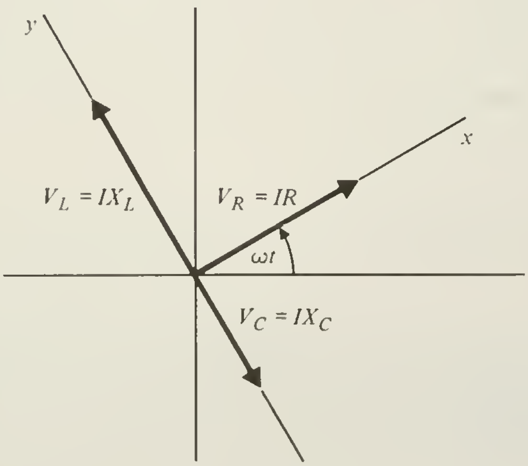


Figure 34-1

Using the rules for vector addition, the resultant voltage magnitude which is the magnitude of the voltage is found to be:

$$V_s = ((IR)^2 + (IX_L - IX_C)^2)^{1/2} = IZ$$

The quantity Z introduced here is called the impedance. It is the single quantity characteristic of the whole circuit that is the ratio of the maximum voltage to the maximum current. Numerical solutions are given in Examples 2 and 3.

Regarding the source voltage as a vector quantity, it is seen that V lies above the + x axis by an angle ϕ where $\tan \phi = (X_L - X_C)/R$. If $X_C > X_L$, V lies below the + x axis. The angle ϕ thus gives the phase difference between the voltage and current in this series circuit such that if

$$i = I \sin \omega t$$

then

$$v = IZ \sin (\omega t + \phi) \text{ where } Z = (R^2 + (X_L - X_C)^2)^{1/2}.$$

The unit for Z is the ohm.

The instantaneous power is the product of i(t) with v(t). For the circuit considered, this is equal to

$$P(t) = I^2Z \sin \omega t \sin(\omega t + \phi).$$

Using the general expression for time averages in the text (see Example 1) and expanding $\sin(\omega t + \phi)$ we find that the time average power is:

$$P = \frac{1}{2} I^2Z \cos \phi.$$

Since $Z \cos \phi = R$, then $P = (1/2)I^2R$, which is the power consumed by the resistor alone. This is not surprising since neither the capacitor nor the inductor consume any power (on time average). The similarity of this expression with that obtained with d.c. circuits leads to the definition of r.m.s. quantities (a.c. instruments are calibrated to read these 'root-mean-square' quantities <u>not</u> the maximum values).

$$P = (I_{r.m.s.})^2R = \frac{1}{2} I^2R$$

so

$$I_{r.m.s.} = \frac{1}{(2)^{1/2}} I.$$

The phenomenon called 'resonance' in such a series R-L-C circuit occurs when $X_L = X_C$ so the impedance Z is equal to R. For a given source voltage, the maximum current is obtained under these conditions as Z is minimum here. Note also that the phase angle ϕ is zero and the source voltage and current are in phase. Maximum power is drawn by the circuit on resonance and the 'power factor', cos ϕ, is unity. See Example 4. Resonance in parallel circuits is treated briefly in Example 6.

An extremely practical device employed in a.c. circuits and invaluable for power transmission is the transformer. If an iron (or equivalent) core is used, the flux linking the primary and secondary portions is the same. If the secondary circuit is open, we have

$$E_p = - N_p \frac{d\Phi}{dt} \text{ and } E_s = - N_s \frac{d\Phi}{dt}$$

so

$$\frac{E_p}{N_p} = \frac{E_s}{N_s} .$$

This enables the transformer to either provide a secondary voltage greater than the primary voltage (step-up transformer $N_s > N_p$) or to provide a secondary voltage less than the primary voltage (step-down transformer $N_s < N_p$). When the secondary circuit is closed and draws current, the primary current must change to keep the flux change fixed. Since the primary impedance is the ratio of the primary voltage to the primary current, this impedance is determined by the secondary impedance and the turns ratio of the transformer. For this reason, changes in the secondary are reflected in the primary. See Example 7.

EXAMPLES AND SOLUTIONS

Example 1

Calculate the time average of $\sin^2 \omega t$ over one period of the motion without doing any integrations.

Solution:

We have

$$1 = \sin^2 \theta + \cos^2 \theta$$

$$1 = \sin^2 \omega t + \cos^2 \omega t$$

Over one cycle of sin ωt, the area under the curve $\sin^2 \omega t$ is equal to the area under the curve of $\cos^2 \omega t$. Sketch this to be sure. Thus if we use $\langle f \rangle$ to denote the time average of a quantity f over one complete cycle,

$$\langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle$$

$$\langle 1 \rangle = 1 = \langle \sin^2 \omega t \rangle + \langle \cos^2 \omega t \rangle = 2\langle \sin^2 \omega t \rangle$$

We conclude that:

$$\langle \sin^2 \omega t \rangle = \frac{1}{2} = \langle \cos^2 \omega t \rangle$$

Example 2

A resistor and inductor as shown in Fig. 34-2a are connected in series to a 60 cycle voltage source which produces a maximum voltage V = 155.6 volts. If R = 200 Ω and L = 0.3 H, calculate

(a) the impedance of the circuit;
(b) the maximum current, I;
(c) the phase angle between the current and the voltage; and
(d) the time average power drawn from the source.

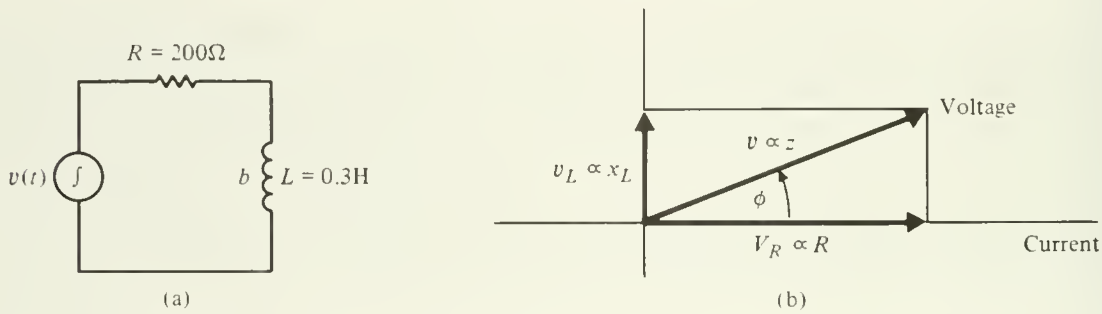


Figure 34-2

Solution:

(a) If the source frequency is 60 HZ. then $\omega = 2\pi(60\ s^{-1}) = 377\ rad\ s^{-1}$. For this angular frequency, the inductive reactance, X_L, is:

$$X_L = \omega L = (377)(0.3)\ \Omega = 113.1\ \Omega$$

The impedance $Z = (R^2 + X_L{}^2)^{1/2}$. For R = 200 Ω we have Z = 230 Ω by direct substitution of these values. Note that Z is the magnitude of the vector that has R as its x component and X_L as its y component.

(b) The maximum current is related to the maximum voltage by:

$$V = IZ$$

Thus

$$I = \frac{V}{Z} = \frac{155.6\ V}{230\ \Omega} = 0.676\ \text{amps}$$

(c) From Fig. 34-2b, the resultant voltage (the source voltage) leads the current in the circuit by the angle ϕ where $\cos\phi = R/Z$. For this case $\phi = 29.6^\circ$.

(d) The voltage across the resistor, v_R, is just Ri. So the instantaneous power through the resistor is $Ri^2 = RI^2 \sin^2 \omega t$. The time average value of $\sin^2\omega t$ is 1/2 so the time average power consumed by the resistor is:

$$P_R = \frac{1}{2} RI^2$$

For the inductor,

$$v_L = L(di/dt) = \omega LI \cos \omega t.$$

The instantaneous power, $P_L(t)$, is just

$$P_L(t) = v_L i = \omega LI^2 \sin \omega t \cos \omega t.$$

Since

$$\sin \omega t \cos \omega t = \frac{1}{2} \sin 2\omega t$$

and all sines and cosines give zero time average over a cycle, the time average power drawn by the inductor is <u>zero</u>.

$$\langle P_L(t)\rangle = 0.$$

Finally, for the entire circuit, since

$$i = I \sin \omega t \text{ and } v = V \sin(\omega t + \phi),$$

the instantaneous power is:

$$iv = IV \sin \omega t \sin(\omega t + \phi).$$

If we write

$$\sin(\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi,$$

then since the time average of $\sin^2 \omega t$ is 1/2 and the time average of $\sin \omega t \cos \omega t$ is zero, we have:

$$P = \frac{1}{2}\ IV \cos \phi.$$

However $\cos \phi = R/Z$ and $V = IZ$ so this gives $P = (1/2)\ RI^2$, just the results obtained for the resistor alone.

Example 3

For the series R-L-C circuit shown in Fig. 34-3 calculate the maximum current and the phase difference between the voltage and the current if the maximum source voltage is 50 V and the frequency is
(a) 5000 Hz and
(b) 2000 Hz.

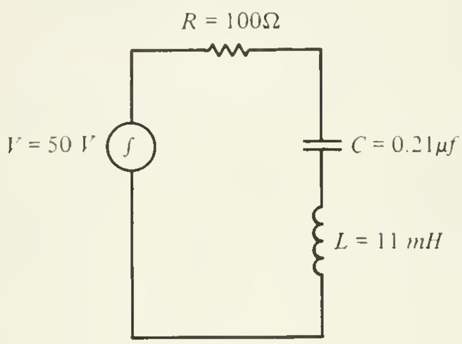


Figure 34-3

Solution:

(a) If $f = 5000$ Hz, $\omega = 3.14 \times 10^4$ rad·s^{-1}. The necessary reactances are

$$X_C = 1/\omega C = [(3.14 \times 10^4\ s^{-1})(2.1 \times 10^{-7}\ F)]^{-1} = 151.6\ \Omega$$

$$X_L = \omega L = (3.14 \times 10^4\ s^{-1})(11 \times 10^{-3}\ H) = 345.6\ \Omega$$

so the impedance Z has the value:

$$Z = ((100)^2 + (345.6 - 151.6)^2)^{1/2}\ \Omega = 218.3\ \Omega.$$

Since $X_L > X_c$, the voltage will lead the current by angle ϕ where

$$\cos \phi = \frac{R}{Z} = 0.458, \qquad \phi = 62.7°.$$

The maximum current is

$$I = \frac{V}{Z} = \frac{50\ \text{V}}{218\ \Omega} = 0.229\ \text{A}.$$

Thus if,

$$i(t) = (0.229\ \text{A}) \sin \omega t$$

then

$$v(t) = (50\ \text{V}) \sin(\omega t + 62.7^\circ)$$

(b) Now f = 2000 Hz so the inductive and capacitive reactances are changed.

$$\omega = 1.257 \times 10^4\ \text{rad}\cdot\text{s}^{-1}$$

$$X_L = \omega L = 138.2\ \Omega$$

$$X_C = (\omega C)^{-1} = 378.9\ \Omega$$

Now the capacitive reactance is larger than the inductive reactance so the voltage will lag behind the current by an angle ϕ.

$$Z = (R^2 + (X_L - X_C)^2)^{1/2}$$

$$= [(100)^2 + (138.2 - 378.9)^2]^{1/2}\ \Omega$$

$$= 261\ \Omega$$

Since $\tan \phi = (X_L - X_C)/R$, then $\phi = -67.44^\circ$. The maximum current is

$$I = \frac{V}{Z} = \frac{50\ \text{V}}{261\ \Omega} = 0.192\ \text{A}.$$

Thus we have

$$i(t) = (0.192\ \text{A}) \sin \omega t$$

$$v(t) = (50\ \text{V}) \sin(\omega t - 67.4^\circ).$$

Example 4

For a series R-L-C circuit carrying current $i = I \sin \omega t$, driven by a voltage source $v = V \sin(\omega t + \phi)$ where V is constant, find the angular frequency ω for which the time average power is a maximum.

Solution:

If we start with the expression for the time average power,

$$\langle P \rangle = \frac{1}{2} VI \cos \phi,$$

we can replace cos ϕ, the power factor, by $\cos \phi = R/Z$ and I by V/Z to get:

$$\langle P \rangle = \frac{1}{2} \left(\frac{V^2}{Z^2} \right) R.$$

For the impedance Z, use $Z^2 = R^2 + (\omega L - 1/\omega C)^2$. Since V and R are constants, the maximum power will be obtained when Z is a minimum. This occurs at a frequency ω_o where

$$\omega_o L = 1/\omega_o C \qquad \text{or} \qquad \omega_o{}^2 = \frac{1}{LC}$$

At this frequency Z = R and the maximum power, P_{max}, is:

$$P_{max} = \frac{1}{2} \frac{V^2}{R} .$$

Example 5

For a series R-L-C circuit carrying current $i = I \sin \omega t$, obtain the voltage across the circuit using the sum of the algebraic expressions for the individual voltages.

Solution:

Writing the source voltage, v(t), as the algebraic sum of the voltage drops,

$$v(t) = v_R(t) + v_L(t) + v_C(t)$$

where

$$v_R(t) = R\, i(t) = RI \sin \omega t$$

$$v_L(t) = L\left(\frac{di}{dt} \right) = \omega LI \cos \omega t$$

and

$$v_C(t) = \frac{1}{C} q(t) = -\frac{I}{\omega C} \cos \omega t$$

we have

$$v(t) = I[(\omega L - \frac{1}{\omega C}) \cos \omega t + R \sin \omega t]$$

Now define

$$\tan \phi = \left(\frac{(\omega L - 1/\omega C)}{R} \right)$$

so

$$\cos \phi = \frac{R}{[(\omega L - 1/\omega C)^2 + R^2]^{1/2}} = \frac{R}{Z}$$

Then

$$v(t) = IR\left(\frac{\sin \phi}{\cos \phi} \cos \omega t + \sin \omega t \right)$$

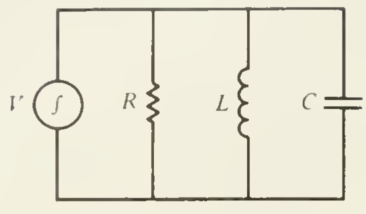
$$= \frac{IR}{\cos \phi} (\sin \phi \cos \omega t + \cos \phi \sin \omega t)$$

$$= IZ \sin (\omega t + \phi).$$

The totally algebraic solution, of course, gives the same result as that obtained by vector addition but requires inspired guesses at a few spots (like the definition of tan ϕ) and in general seems less straightforward.

Example 6

Suppose the components in Example 3 are placed in a parallel connection, with the same voltage source operating at 5000 Hz. What is the maximum value of the current?



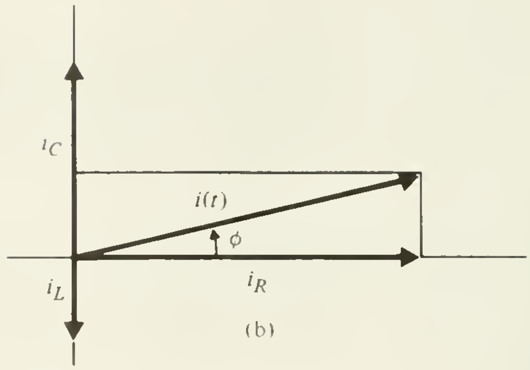


Figure 34-4

Solution:

Referring to Fig. 34-4a, use R = 100 Ω, L = 11 mH, and C = 0.21 μF with $\omega = 3.141 \times 10^4$ rad·s^{-1}. In a parallel circuit, the potential difference across all the elements shown is the same but the current drawn from the source is the sum of the currents through R, L, and C, according to Kirchhoff's law. The currents (but not the voltages) have phase differences and must be added vectorially. Since v(t) is common to all the elements, we can plot i_R, the current through the resistor, along the + x axis in Fig. 34-4b. Since the current leads the voltage for a capacitor, we plot i_C along the + y axis and i_L (since the voltage leads the current in an inductor) along the − y axis. The magnitudes of these currents are:

$$i_R = \frac{V}{R} = \frac{50 \text{ V}}{100\ \Omega} = 0.5 \text{ A}$$

$$i_C = \frac{V}{X_C} = \frac{50 \text{ V}}{152\ \Omega} = 0.329 \text{ A}$$

$$i_L = \frac{V}{X_L} = \frac{50 \text{ V}}{346\ \Omega} = 0.144 \text{ A.}$$

A vector plot would look like Fig. 34-4b.

The maximum current is found from the vector sum to be:

$$I = [(0.5)^2 + (0.329 - 0.144)^2]^{1/2} \text{ A}$$

$$I = 0.533 \text{ A}$$

If we write the voltage as $v(t) = V \sin \omega t$, then the source current is

$$i(t) = (0.533 \text{ A}) \sin (\omega t + \phi)$$

where now

$$\tan \phi = \frac{(1/X_C - 1/X_L)}{(1/R)} \ .$$

Please note that this bears no simple resemblance to the phase difference obtained for the series R-L-C circuit.

The time average power in this case is ($\cos \phi = .938$):

$$\langle P \rangle = (50 \text{ V})(0.533 \text{ A})(1/2)(0.938) = 12.5 \text{ watts}$$

It is instructive to note that the 'on resonance' condition is the same for the parallel circuit as it was for the series circuit, namely $X_C = X_L$. The current drawn from a constant voltage source is a minimum here. If we use however a constant current source, then at resonance, the voltage across the parallel combination will be maximum and the power will be maximum, as in the series resonant circuit using a constant voltage source.

Example 7

A transformer connected to a 110 volt line delivers 10 volts to the secondary circuit. If the power drawn from the primary circuit is 220 watts, what is the equivalent resistance R_s of the secondary circuit?

Solution:

Referring to Fig. 34-5, since the flux per turn linked by the primary and secondary circuits is the same,

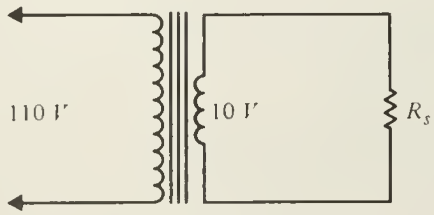


Figure 34-5

$$\frac{z_p}{N_p} = \frac{V_s}{N_s}$$

and we can solve for the turns ratio

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{110 \text{ V}}{10 \text{ V}} = 11.$$

Ignoring any power losses in the transformer, we have $V_pI_p = V_sI_s$, so the secondary current is calculable.

$$I_s = \frac{220\ \text{watts}}{10\ \text{volts}} = 22\ \text{A}.$$

The equivalent resistance of the secondary is

$$R_s = \frac{V_s}{I_s} = \frac{10\ \text{V}}{22\ \text{A}} = 0.455\ \Omega.$$

The effective resistance in the primary circuit is

$$R_p = \frac{V_p}{I_p} = \frac{110\ \text{V}}{2\ \text{A}} = 55\ \Omega.$$

Note that

$$\frac{R_p}{R_s} = (\frac{N_p}{N_s})^2 = 121.$$

QUIZ

1. A series R-L-C circuit carries current

$$I = (1.25\ \text{A}) \sin 2\pi(60)t$$

with $R = 60\ \Omega$, $C = 30\ \mu F$, and $L = 0.15$ H.

(a) Calculate the capacitive reactance.
(b) Calculate the inductive reactance.
(c) Calculate the impedance.
(d) Calculate the power factor.
(e) Calculate the time average power.
(f) Calculate the resonant frequency.

Answer: (a) 88.4 Ω
(b) 56.6 Ω
(c) 67.9 Ω
(d) 0.884
(e) 46.9 W
(f) $f_o = 75.0\ s^{-1}$.

2. A series circuit has a resistance of 50 Ω and a power factor of 0.80 when the frequency is 80 Hz. The voltage leads the current.

(a) Calculate the impedance of this circuit.

(b) Calculate the magnitude of the inductor or capacitor needed to resonate the above circuit assuming the element is placed in series.

Answer: (a) 62.5 Ω

(b) A capacitor C = 53.1 μF placed in series with the above circuit will produce resonance.

3. A 500 Ω resistor, a 5 H inductor, and an 11.1 μF capacitor are connected in <u>parallel</u>. Calculate the impedance of this parallel combination when ω is equal to 300 $\mathrm{rad \cdot s^{-1}}$.

Answer: Z = 300 Ω

4. A lossless transformer with 1000 turns on the primary winding carries a current of magnitude 0.25 A in the primary circuit. If there are 125 turns on the secondary winding, calculate the magnitude of the secondary current.

Answer: I_s = 2.0 A

35
ELECTROMAGNETIC WAVES

OBJECTIVES

In this chapter, your objectives are to:

Calculate the speed of propagation of electromagnetic waves in vacuum and in various materials.

Calculate the Poynting vector **S** and its time average value given **E** and **B**.

Apply the definite relationship between E and B in an electromagnetic wave to calculate their values given the power per unit area (S) in the wave.

REVIEW

Material presented since the introduction to electrostatics is collected and summarized in this chapter. The entire content of all the electromagnetic theory studied previously is contained in four equations known as Maxwell's equations. While it is convenient to incorporate so many diverse phenomena in a small number of equations, the mathematical complexity of these equations is so high that they may not significantly raise the level of understanding of electromagnetic phenomena for every student. A parallel situation occurred when we were able to describe all of classical mechanics in three laws of motion, Newton's laws. Just writing down these laws did not guarantee that we could solve the practical problems met in our study of mechanics.

In this chapter, Maxwell's equations are used to demonstrate the plausibility of wave-like excitations of the electromagnetic field with the following properties:

(1) these waves do not need a material medium for their propagation;
(2) they propagate in a given medium at a fixed and calculable speed;
(3) the waves are transverse waves; the electric and magnetic field vectors are in a plane perpendicular to the direction of propagation;
(4) these waves are capable of transporting energy and momentum.

One of the triumphs of Maxwell's equations is that they permit calculation of the wave velocity from first principles. To obtain this monumental result, the spatial variation of the magnetic field (through the H vector) must be related to the time variation of the electric field. This can be done with Ampere's law when it is modified to include the displacement current that arises from a time changing electric field. Next the spatial variation of the electric field must be related to the time variation of the magnetic field. Faraday's law can be used for this. Then, since the order of taking the derivatives with respect to space and time is immaterial, it is possible to show that both the electric and magnetic fields obey a differential equation that describes wave propagation. From the wave equation it is possible to conclude that the wave speed w satisfies the equation:

$$w^2 = \frac{1}{\mu\varepsilon} = \frac{1}{K_m\mu_o} \; \frac{1}{K\varepsilon_o}$$

In a perfect vacuum, $K_m = 1$, and $K = 1$ so that the speed there is c, with

$$c^2 = \frac{1}{\mu_o\varepsilon_o}$$

Using the previous values for μ_o and ε_o, it is found that the speed of electromagnetic waves in a vacuum is 3×10^8 $m \cdot s^{-1}$. This is, of course, the speed of light, light being one form of electromagnetic waves.

In the demonstration of the above results, it is found that the electric and magnetic field vectors in the wave are related with E = cB. Since $B = \mu_o H$ in vacuum, we find that E/H = 377 ohms. This numerical ratio can be very useful in calculations.

The energy density obtained for the combined electric and magnetic fields enables us to examine the energy content of electromagnetic waves. By calculating the product of the energy density and the volume to obtain the energy and then dividing by the cross-sectional area and the time, it is shown that a new vector, **S**, the Poynting vector, defined by the equation

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

is equal in magnitude to the power per unit area carried by the wave. The direction of **S** is the direction of wave propagation. One very important example of a Poynting vector is the 'solar constant' appropriately labeled S also. The solar constant has a magnitude of about 1.5 $kw \cdot m^{-2}$ and is a measure of the average power radiated from the sun incident on the earth. A calculation of E and H in sunlight using the known value of the solar constant is given in Example 4. A calculation of the Poynting vector for a wire carrying current is given in Example 5, where the power dissipation per unit length is obtained in two ways.

In addition to transporting energy, electromagnetic waves have momentum, an intriguing concept since they carry no mass. One way of looking at this is to make a generalization of the following nature: the momentum is to be defined as the derivative of the kinetic energy with respect to the speed. This generalization is illustrated in Example 6. Since electromagnetic waves carry momentum and force is equal to the rate of change of momentum, these waves can exert force or give rise to pressures on objects in their path.

As with any wave disturbance, superposition (adding E or B fields) of waves with the same frequency traveling in opposite directions can lead to standing waves. As shown in the text, nodes appear in the pattern for standing waves at distances directly related to the wavelength. Since the maximum distance between nodes is the separation between conducting planes (in the treatment of the text), this leads to a discrete set of frequencies called 'normal modes'. If the frequency is known, the wave speed can be measured to the accuracy of the product of the distance (between nodes) and the frequency measurement.

EXAMPLES AND SOLUTIONS

Example 1

A light wave with frequency 5×10^{14} Hz is incident from air on a material with dielectric constant 7.5 and relative permeability 1.25. The frequency of the light wave is unchanged as it passes from air into the other medium. Calculate the wavelength of the light wave in air and in the material.

Solution:

The dielectric constant and relative permeability of air are so close to unity that we will take the speed of light in air to be the same as the speed of light in vacuum, namely $c = 3.00 \times 10^8\ \mathrm{m \cdot s^{-1}}$. The wavelength in air is then

$$\lambda_a = \frac{c}{f} = \frac{3 \times 10^8\ \mathrm{m \cdot s^{-1}}}{5 \times 10^{14}\ \mathrm{s^{-1}}}$$

$$= 6 \times 10^{-7}\ \mathrm{m} = 600\ \mathrm{nm}$$

In the other material, the wave speed w is

$$w = \frac{1}{(\varepsilon\mu)^{1/2}}$$

$$= \frac{c}{(KK_m)^{1/2}}$$

$$= \frac{3 \times 10^8\ \mathrm{m \cdot s^{-1}}}{[(7.5)(1.25)]^{1/2}} = 0.980 \times 10^8\ \mathrm{m \cdot s^{-1}}.$$

The corresponding wavelength is

$$\lambda_m = \frac{w}{f} = \frac{0.980 \times 10^8 \ m \cdot s^{-1}}{5 \times 10^{14} \ s^{-1}}$$

$$= 1.96 \times 10^{-7} \ m = 196 \ nm.$$

Example 2

A plane electromagnetic wave is propagating in air (free space) in the + x direction with

$$\mathbf{H}(x,t) = H_o \sin 2\pi(ft - \frac{x}{\lambda}) \ \mathbf{j}.$$

Calculate (a) **E**(x,t) and (b) the instantaneous value of the Poynting vector, **S**. In this expression for the wave, f is the frequency and λ is the wavelength.

Solution:

(a) **E** must point in the z direction since **E** is perpendicular to **B** (or **H**) and they both are perpendicular to the direction of propagation. As the direction of propagation is given by the direction of **S**,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

then **E** must be

$$\mathbf{E} = E_o \sin 2\pi(ft - \frac{x}{\lambda}) \ (-\mathbf{k})$$

The amplitude E_o is calculable from H_o as

$$E_o = (377 \ \Omega)H_o$$

The final form is

$$\mathbf{E}(x,t) = -(377 \ \Omega)H_o \sin 2\pi(ft - \frac{x}{\lambda}) \ \mathbf{k}$$

(b) The Poynting vector **S** is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = (-\mathbf{k} \times \mathbf{j}) \frac{H_o^2}{wK\varepsilon_o} \sin^2 2\pi(ft - \frac{x}{\lambda})$$

Thus S points in the + x direction and its time average value, S, is equal to:

$$|S| = \frac{1}{2}\ \frac{H_o{}^2}{wK\varepsilon_o}$$

Example 3

A source of electromagnetic waves with power 10^7 radiates uniformly in all directions. Calculate the amplitude of the electric field vector for these waves

(a) at a distance of 100 m from the source
(b) at a distance of 1 km from the source.

Solution:

The number given for the power represents the time average power of the source. To find the power per unit area at a distance R from the isotropic source construct a spherical surface of radius R about the source. The surface area is then

$$A = 4\pi R^2$$

The power per unit area for R = 100 m is then

$$\frac{P}{A} = \frac{10^7\ \mathrm{W}}{4\pi \times 10^4\ \mathrm{m}^2} = 79.6\ \mathrm{W{\cdot}m^{-2}}$$

This power per unit area is equal to the time average value of the Poynting vector, $\bar{S}$:

$$\bar{S} = \frac{1}{2}\,EH = 79.6\ \mathrm{W{\cdot}m^{-2}}$$

Using

$$\frac{E}{H} = 377\ \Omega$$

to replace H, we can solve for E

$$\frac{1}{2}\ \frac{E^2}{377\ \Omega} = 79.6\ \mathrm{W{\cdot}m^{-2}}$$

$$E^2 = 6.00 \times 10^4\ (\mathrm{W{\cdot}m^{-2}{\cdot}\Omega})$$

$$E = 245\ \mathrm{V{\cdot}m^{-1}}$$

(b) At a distance of 1 km form the source, the power per unit area is

$$S = \frac{P}{A} = \frac{10^7\ \mathrm{W}}{4\pi \times 10^6\ \mathrm{m}^2} = 0.796\ \mathrm{W \cdot m^{-2}}$$

Solving for E as above, we have

$$\frac{1}{2}\frac{E^2}{377\ \Omega} = 0.796\ \mathrm{W \cdot m^{-2}}$$

$$E^2 = 6.00 \times 10^2\ (\mathrm{W \cdot m^{-2} \cdot \Omega})$$

$$E = 24.5\ \mathrm{V \cdot m^{-1}}$$

The amplitude E is proportional to R^{-1} where R is the distance from the source.

Example 4

The time average power from the sun falling on the earth can be represented by the solar constant, S, which is approximately equal to 1.4 $\mathrm{kw \cdot m^{-2}}$. What are the maximum values of E and H due to this electromagnetic radiation?

Solution:

The Poynting vector magnitude, when **E** and **H** are perpendicular, is equal to EH. If E and H are sinusoidal functions of time, the time average power, according to Eq. 35-24 is:

$$S_{AV} = \frac{1}{2} EH$$

To simplify the numerical calculation, recall that E/H = 377 Ω yielding

$$S_{AV} = \frac{1}{2} E\left(\frac{E}{H}\right)\left(\frac{H}{E}\right)H = \frac{1}{2}(377\ \Omega)\ E\frac{H}{E}\ H$$

$$= \frac{1}{2}(377\ \Omega)H^2$$

Solving numerically we have

$$H = \left(\frac{2 \times 1.4 \times 10^3}{377} \right)^{1/2} = 2.73\ A \cdot m^{-1}$$

$$E = (377\ \Omega)H = 1.03 \times 10^3\ V \cdot m^{-1}$$

Example 5

A wire of radius a, cross sectional area A, and resistivity ρ carries a current I.

(a) Find the magnetic intensity, H, at the surface of this wire.
(b) Find the electric field strength from Ohm's law.
(c) Calculate the Poynting vector magnitude and direction.
(d) Calculate the power crossing the surface area of a cylinder of height L and radius a using the Poynting vector and show that it is equal to the usual expression for power dissipation in a conductor.

Solution:

(a) From Ampere's law we have that the field, B, at the surface of the wire is:

$$B \cdot 2\pi a = \mu_0 I$$

Since $B = \mu_0 H$ when there is no magnetization, we have $H = I/2\pi a$, with a direction given by the right hand rule as shown in Fig. 35-1.

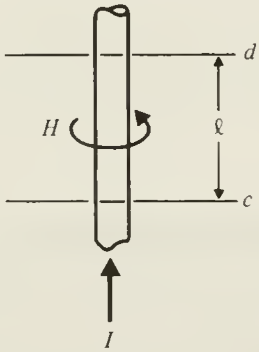


Figure 35-1

(b) The potential difference between c and d, V_{cd} is given by:

$$\int_c^d \mathbf{E} \cdot d\mathbf{L} = I(\text{resistance between c and d})$$

For this uniform wire, E is constant so we have

$$EL = I \rho L/A \text{ so } E = \rho(I/A)$$

The direction of E is the same as the direction of I here.

(c) Since E points along the wire and H is tangent, E x H points radially inward at all points on the surface and the magnitude of E x H is just EH.

$$S = EH = \rho \frac{I}{A} \quad \frac{I}{2\pi a}$$

(d) The power (P) crossing the surface for the wire of length L is given by multiplying the constant value of S by the side area of the cylinder of height L and radius a, $2\pi aL$.

$$P = 2\pi aL\ S = \frac{\rho L}{A} I^2$$

Since $\rho L/A$ is the resistance of this length of wire, it is seen that the power calculated this way is also equal to I^2R but here it is taken from the electromagnetic field and flows radially inward!

Example 6

(a) Generalize the definition of momentum to make momentum equal to the derivative of the kinetic energy with respect to speed and then calculate the momentum Δp in an infinitesimal volume of space.
(b) Using Newton's laws, find the radiation pressure, P.

Solution:

(a) Consider the infinitesimal volume ΔV to have cross-sectional area A and length $c\Delta t$ where c is the speed of light. The energy density, u, is given by:

$$u = \frac{1}{2} \varepsilon_o E^2 + \frac{1}{2} \mu_o H^2 = \varepsilon_o E^2$$

Regarding this as kinetic energy, we have for the kinetic energy in the volume ΔV

$$\Delta E_K = \varepsilon_o E^2 A c \Delta t$$

Let's check the new definition of momentum by returning to classical mechanics. There,

$$E_K = \frac{1}{2} mv^2 \text{ so, } \frac{d}{dv} E_K = mv$$

which was just our previous result. Try this new definition on the kinetic energy in the radiation field:

$$\Delta p = \frac{d}{dc} (\Delta E_K) = \frac{d}{dc} (\varepsilon_o E^2 Ac \; \Delta t)$$

Thus we have

$$\Delta p = \varepsilon_o E^2 A \Delta t$$

which is the momentum contained in the volume element.

(b) From Newton's law we write the force, F, as $F = \Delta p/\Delta t$.

$$F = \frac{\Delta p}{\Delta t} = \varepsilon_o E^2 A$$

The pressure, P, is the force per unit area (F/A) so we have for the pressure

$P = \varepsilon_o E^2 = u$, the energy density in the field.

QUIZ

1. An electromagnetic wave is characterized by an electric field vector

$$\mathbf{E} = (1.6 \text{ V}\cdot\text{m}^{-1}) \sin 2\pi(ft + \frac{z}{\lambda}) \; (\mathbf{i}).$$

Find the wave form that describes the magnetic intensity H.

Answer:

$$\mathbf{H} = (4.24 \times 10^{-3} \text{ A}\cdot\text{m}^{-1}) \sin 2\pi(ft + \frac{z}{\lambda}) \; (-\mathbf{j})$$

2. At a distance of 15 km from a source that radiates uniformly in all directions, the electric field amplitude is found to be 125 $V \cdot m^{-1}$. Calculate
   (a) the magnetic intensity amplitude H,
   (b) the time average value of the Poynting vector,
   (c) the time average power radiated by the source.

Answer: (a) $H = 0.332\ A \cdot m^{-1}$
(b) $\bar{S} = 20.7\ W \cdot m^{-2}$
(c) $\bar{P} = 5.86 \times 10^{10}\ W$.

3. The antenna of a radio station radiates equally in all directions. The total power of the transmitter is 50,000 W. The power per unit area is equal to the magnitude of the Poynting vector, S. Calculate the value of S, in watts per square meter, at a distance of 100 km from the station.

Answer: $S = 3.99 \times 10^{-7}\ W \cdot m^{-2}$

4. In a plane wave propagating in the +x direction, the Poynting vector has the magnitude $1.508 \times 10^{-3}\ W \cdot m^{-2}$. The magnetic intensity vector (H) has a magnitude of 2×10^{-3} S.I. units and points in the +y direction. Give the magnitude and direction of the electric field vector, E, for this plane wave.

Answer: 0.754 $V \cdot m^{-1}$ in the −z direction

36
THE NATURE AND PROPAGATION OF LIGHT

OBJECTIVES

In this chapter your objectives are to:

Identify that part of the electromagnetic spectrum that stimulates the retina of the eye as light.

Describe light waves as transverse waves with E and B fields perpendicular to each other and to the direction of propagation.

Review the historic experiments that measured the velocity of light.

Formulate the laws of reflection and refraction.

Apply the laws of reflection and refraction to a variety of problems such as total internal reflection.

Identify several of the different methods of producing polarized light waves, such as absorption, reflection, scattering, and birefringence.

Distinguish between linear, circular, and elliptical polarizations and make calculations concerned with devices such as quarter wave plates.

Calculate the reduction in intensity obtained when polarizing plates are stacked with various relative orientations.

REVIEW

Whether light should be thought of as a wave-like disturbance or a stream of particles has been a topic of considerable interest for nearly 400 years. Sir Isaac Newton believed that light consisted of particles and explained many known optical phenomena based on his theory. Unfortunately his theory required that the speed of light in a material medium be larger than the speed in vacuum. As we saw in the last chapter, the reverse is true: the speed of light is highest in a vacuum. (The space between the earth and the sun is not a perfect vacuum but it is a pretty good one.) While for the most part, light behaves as a wave, the particle-like nature of light is observed in atomic, nuclear, and high energy physics.

Because of its very large magnitude ($c = 3 \times 10^8\ m \cdot s^{-1}$), the speed of light proved difficult to measure on the earth by ordinary means. Roemer, an astronomer, made the first measurement with any reasonable accuracy but he did not actually claim that he had measured c. It was not until 1850 that a value for c was obtained using terrestial measurements. This method, due to Fizeau, is illustrated in Example 1.

Since 'light' is that portion of the electromagnetic spectrum that affects the retina of the eye, it is important to know what part of the spectrum we are dealing with. Our maximum sensitivity occurs at about a wavelength of 550 nm (where a nanometer is 10^{-9} m) with most eyes being reasonably sensitive from 400 nm (violet) to 700 nm (red).

Reflection and refraction of waves are generally easier to picture and understand if we use rays to indicate the direction in which the waves are moving. These rays are perpendicular to the actual wave front (or locus of equal phase). In using such rays is should be noted that:

(1) the incident ray, the reflected ray, the transmitted (refracted) ray and the normal to the interface between two surfaces all lie in the same plane;

(2) the angle of incidence is equal to the angle of reflection (all angles are measured with respect to the normal);

(3) Refraction of rays obeys Snell's law, $n_a \sin \phi_a = n_b \sin \phi_b$, where n_a and n_b are respectively the indices of refraction (measured with respect to vacuum) of media a and b, and ϕ_a, ϕ_b are the angles between the rays and the normal. The index of refraction is a function of the frequency or wavelength. Unless otherwise specified, we will ignore this effect and use an average value.

Snell's law predicts an interesting phenomena known as 'total internal reflection'. For this to occur, light must be traveling from a more dense (higher index of refraction) to a less dense medium (lower index of refraction) as in going from water to air. In this case, there exists a critical angle of incidence ϕ_c for which the refracted ray will be bent parallel to the interface ($\phi = 90^\circ$) and not emerge into the less dense medium. For angles of incidence greater than the critical angle, the rays are totally reflected back into the dense medium. If we designate medium 'a' as the dense medium, the critical angle for total internal reflection is a solution of the equation:

$$(\sin \phi_a)_c = \frac{n_b}{n_a}$$

This is an essential result for the currently interesting field of fiber optics. Of additional interest in this connection is the absorption of light by matter, even high quality optical glass. It is shown in the text that the light intensity decreases exponentially with distance into the material. The absorption coefficient is a material dependent parameter and must be made very small for fiber optic bundles of any significant length, like telephone lines.

By using every point on a wave front as a source of secondary wavelets, Huygens found that he could predict the wave front at a later time from the known front at an earlier time. Using the Huygens' construction, it is possible to show that:

$$\frac{\sin \phi_a}{\sin \phi_b} = \frac{v_a}{v_b}$$

where v_a and v_b are the propagation velocities in media 'a' and 'b'. Since $n_a \sin \phi_a = n_b \sin \phi_b$, then $n_a v_a = n_b v_b$ and we can relate the index of refraction to the propagation velocity. In particular if we choose medium 'a' to be vacuum (free space) so that $n = 1$ and $v = c$, then $n_b = c/v_b$. Since $v_b \leq c$ then $n_b \geq 1$. When a wave passes from one medium to another, while the velocity of propagation does change, the frequency does not. Thus $\lambda_a/v_a = \lambda_b/v_b$ or stated in terms of n's, $\lambda_a n_a = \lambda_b n_b$.

Waves of all wavelengths propagate in the vacuum with the same speed but in a material medium, waves with different wavelengths travel with different speeds. The medium is said to be dispersive or to exhibit dispersion. There is no dispersion in free space. This dispersion can be easily observed by looking at the fan of colors that emerges from a prism illuminated with white light.

The phenomenon of polarization exists only for transverse waves such as electromagnetic waves, waves on a string, etc., but not for longitudinal waves such as sound waves. When a charged particle accelerates, it radiates electromagnetic waves. The electric field vector is proportional to this acceleration vector and parallel to it. Knowledge of the direction of propagation and the electric field vector determines the magnetic field vector. By convention, the direction of polarization is the direction of the electric field vector. Light emitted from a typical source is unpolarized, even though the light emitted in an individual transition from a single charge is polarized. The acceleration vectors of the various charges in the source are randomly oriented so the net effect is an unpolarized beam.

To make the discussion of polarization more quantitative the text introduces the concept of an ideal polarizing filter that passes all light polarized in the direction of the filter's axis. Thus if unpolarized light is incident on such a filter and the resultant light projected on a screen, the intensity of light on the screen is less than it would be if the filter was removed but independent of the orientation of the filter axis. This is due to the fact that the electric field vector has the same magnitude on average for any direction in a plane perpendicular to the direction of propagation for unpolarized light. After the light has passed through this first filter, it is <u>linearly polarized</u> in that its electric field vector now points along the filter axis. It is not correct to think that all other vectors were rejected as the transmitted light would be very weak in that case. Rather it is better to think of each E vector not aligned with the filter axis as being resolved into components parallel and perpendicular to this axis. The parallel components are passed but the perpendicular ones are rejected. The resultant intensity after passing unpolarized light through one ideal filter is just one half the initial intensity of the unpolarized beam. Then if the E vector of polarized light makes an angle of θ with respect to the filter axis, only the component, $E \cos \theta$, is passed through the filter. Since the intensity, I, is proportional to the square of the electric field vector, the transmitted intensity is then:

$$I = I_{max} \cos^2 \theta,$$

where I_{max} is the intensity transmitted when $\theta = 0^\circ$ (but is only one half of the original unpolarized source intensity). See Example 9.

A second filter whose axis is perpendicular to that of the first filter will now pass none of the polarized light incident on it. This is clear from the above expression for the intensity as now we have made $\theta = 90^\circ$ so the cosine vanishes. It is quite interesting that under these conditions of complete extinction, introduction of a third filter in between the first two crossed filters will cause light to be transmitted unless its axis accidentally coincides with that of one of the other filters. This point is illustrated in Example 10.

Another method of producing polarized light is by reflection. The explanation of this phenomenon is contained in a detailed solution of Maxwell's equations at an interface. We will only state without proof that it happens. Futhermore, when the angle between the reflected beam and the refracted ray is exactly 90°, the reflected beam will be perfectly linearly polarized. This fact enables one to derive Brewster's law, namely

$$\tan \phi_p = \frac{n'}{n}$$

where n' is the refractive index of the medium containing the incident and reflected rays and n is the refractive index of the medium containing the refracted ray. Light incident at the angle ϕ_p will be totally polarized upon reflection, with the direction of polarization perpendicular to the plane containing the incident, reflected and refracted rays. This effect is examined in Examples 11 and 12.

Polarization of light by double refraction is also possible. This means that the material exhibits two different indices of refraction depending on the direction of the E vector. For this to occur the refracting medium (crystal) must have highly anisotropic properties. If the direction of propagation is taken as the z axis, such a crystal basically sorts the incoming unpolarized wave and sends the component of **E** along the y axis on a different path from the x component. Such a crystal can be used to produce linearly polarized light by blocking out one ray or it can produce circularly polarized light by recombining the two beams after they have gone through a specific distance. See Example 13 for an illustration of this point.

Light scattered from small particles suspended in in a liquid (or in air) is linearly polarized when viewed at right angles to the incident light. To understand this effect. think of the scattered light as radiation that was absorbed from the incident beam and then re-radiated. Since light cannot be polarized in the direction of propagation, the scattered light viewed at 90° with respect to the incident beam is linearly polarized with its direction of polarization perpendicular to the plane containing the incident and scattered beams. The geometry pertinent to this effect is given in Example 14.

EXAMPLES AND SOLUTIONS

Example 1

In the experiment performed by Fizeau using a toothed wheel, suppose the distance between the wheel and the mirror is 10 km. Calculate the angular velocity needed so that a light ray passing through the center of one tooth and then reflected by the mirror just passes through the center of the next tooth. There are 60 teeth in the rotating wheel.

Solution:

We must calculate two travel times and then equate them. First we calculate the time for the light beam to pass through tooth one and then return to the rotating wheel. The total distance is 20 km so $t = 20 \times 10^3$ m/c. The angular distance between centers of teeth is $360^\circ/60 = 6^\circ$. Converting this to radians, we have the angle the wheel must turn through as $\theta = \pi/30$ rad. Since $\theta = \omega t$ for constant angular velocity, then $t = \theta/\omega$. Thus

$$\frac{\theta}{\omega} = \frac{20 \times 10^3 \text{ m}}{c}$$

Solving for ω,

$$\omega = \frac{(3 \times 10^8 \text{ m}\cdot\text{s}^{-1})\pi/30}{20 \times 10^3 \text{ m}} = 500\pi \text{ rad}\cdot\text{s}^{-1}$$

This corresponds to a frequency of 250 Hz. Note that at all integral multiples of this frequency, we would still get an image of our light source.

Example 2

For the light ray shown in Fig. 36-1, the angle of incidence is 60°. The same ray on leaving the flat glass on the other side is displaced a distance $d = 0.80 \times 10^{-2}$ m from the spot where it entered the glass. Calculate the index of refraction for the glass plate of thickness 1.2×10^{-2} m.

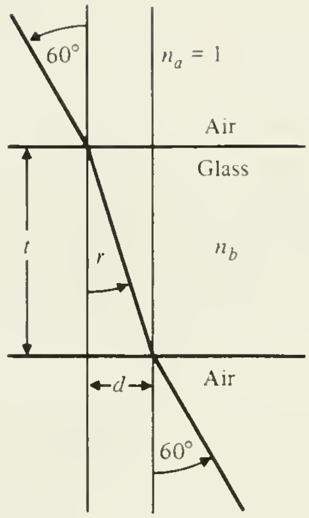


Figure 36-1

Solution:

From Snell's law,

$$n_a \sin 60^\circ = n_b \sin r$$

Here $n_a = 1$, so $n_b = \sin 60^\circ/\sin r$. From Fig. 36-1, $\tan r = d/t$ and both d and t are given so r can be calculated. Since $\sin r = d/(d^2 + t^2)^{1/2} = 0.5547$,

$$n_b = \frac{0.8660}{0.5547} = 1.561$$

Example 3

A coin rests on the bottom of a shallow pond of depth 1 m. Taking the index of refraction of water to be n = 4/3, find the 'apparent depth' of the coin when viewed normal to the air-water interface. See Fig. 36-2.

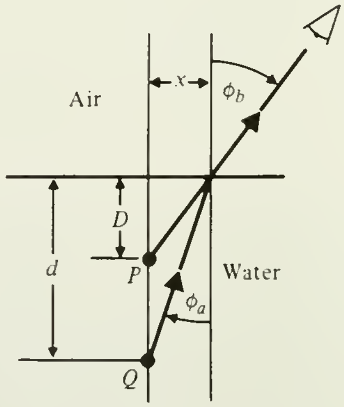


Figure 36-2

Solution:

To obtain the answer for the above question, we must solve another problem first. A ray coming from the coin is refracted at the surface as shown, but the eye traces the ray back along a straight line and one thinks the ray originated at point P (apparent depth D), not at point Q (depth d). Using Snell's law, $\sin \phi_b = n_a \sin \phi_a$, taking $n_b = 1$. From the geometry of the figure, we have

$$\sin \phi_a = \frac{x}{(x^2 + d^2)^{1/2}} \quad \text{and} \quad \sin \phi_b = \frac{x}{(x^2 + D^2)^{1/2}}$$

Thus

$$\frac{n_a x}{(x^2 + d^2)^{1/2}} = \frac{x}{(x^2 + D^2)^{1/2}} \quad .$$

By squaring both sides we obtain

$$n_a^2(x^2 + D^2) = x^2 + d^2.$$

This is the desired equation that relates the true depth (d) to the apparent depth (D). To apply this equation to 'normal' viewing (from above) we take the limit as x goes to zero obtaining $n_a D = d$. Numerically we have

$$D = \frac{d}{n_a} = \frac{3}{4} \text{ m.}$$

Example 4

A point light source 2 m below the surface of water produces a circular pattern of light when viewed from above. Taking the index of refraction of water to be n = 4/3, calculate the radius of this circle.

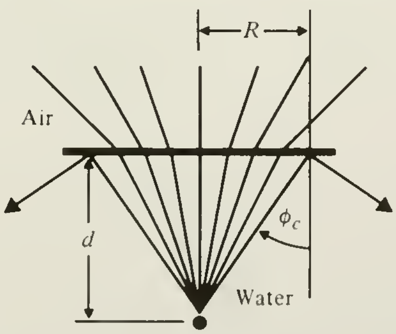


Figure 36-3

Solution:

Light rays from the point source which approach the water-air interface with an angle of incidence less than the critical angle will be partially transmitted and partially reflected so we will see light from them. Those rays approaching at an angle of incidence equal to or greater than the critical angle will be totally internally reflected. Since there is symmetry in this problem about the perpendicular line from the point source to the interface, the pattern seen from above will be a circle (not uniformly illuminated). Referring to Fig. 36-3, the critical angle ϕ_c is a solution of the equation $n_a \sin \phi_a = n_b \sin \phi_b$ with $\sin \phi_a$ set equal to 1 if $n_a < n_b$. In this case $n_a = 1$ so $(\sin \phi_b)_c = 1/n_b$. From the geometry of the figure

$$(\sin \phi_b)_c = R/(R^2 + d^2)^{1/2}$$

so we equate the two expressions for $(\sin \phi_b)_c$ to obtain:

$$\frac{R}{(R^2 + d^2)^{1/2}} = \frac{1}{n_b} \qquad \text{or} \qquad \frac{Rn_b}{(R^2 + d^2)^{1/2}} = 1$$

Solving this for the unknown R, since $(Rn_b)^2 = R^2 + d^2$, we have

$$\frac{R}{d} = \frac{1}{(n_b^2 - 1)^{1/2}} = 1.134 \qquad\qquad R = 2.27 \text{ m.}$$

Example 5

Suppose a camera held underwater is pointed straight up toward the water-air interface (a fish eye view). What part of the horizon will appear in a photo taken in this manner?

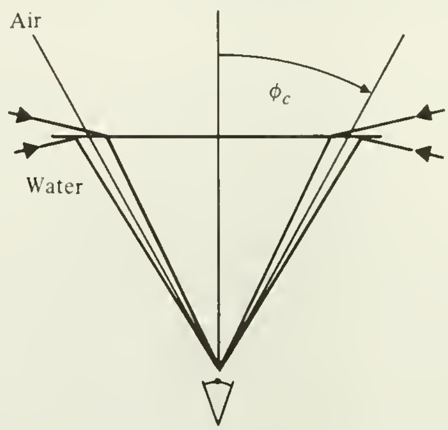


Figure 36-4

Solution:

Referring to Fig. 36-4, some rays near the critical ray for total internal reflection are shown. The critical angle $\phi_c = \sin^{-1}(3/4)$ if we take n for water to be 4/3. This angle is $d_c = 48.59^{\circ}$. Thus the entire horizon will be visible but the hemispherical image will be contained in a distorted way in a cone of angle ϕ_c as shown in Fig. 36-4.

Example 6

Suppose a flat glass plate ($n_b = 1.561$) rests on a layer of water ($n = 4/3$). What angle of incidence in air ($n = 1$) will just give total internal reflection at the glass-water surface?

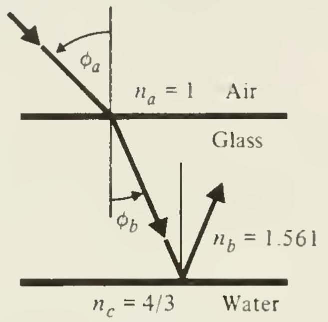


Figure 36-5

Solution:

Referring to Fig. 36-5, the angle of incidence at the glass-water interface is ϕ_b, the angle of refraction at the air-glass interface. For total reflection, we have

$$\sin \phi_b = \frac{n_c}{n_b}$$

Also, $\sin \phi_a = n_b \sin \phi_b$, so $\sin \phi_a = n_c$ for total reflection. This would make the sine greater than unity so this condition is impossible. We can conclude that there is no angle of incidence in air that will produce total internal reflection at the glass-water interface.

Example 7

For the light ray incident on the prism as shown in Fig. 36-6, calculate the difference in angle between the path of the emerging ray and the incident ray. For $A = 60^\circ$, $n = 1.50$, and an angle of incidence of 30°, how large is the deviation?

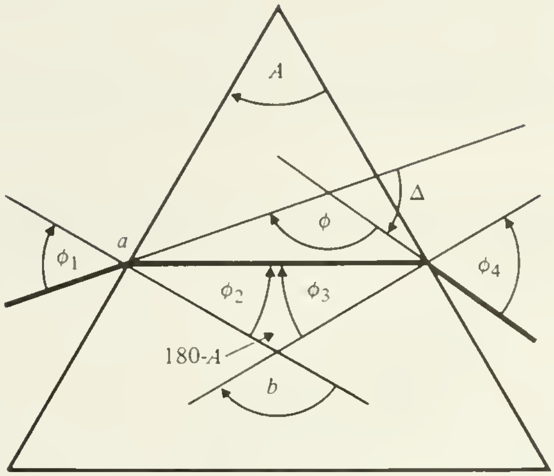


Figure 36-6

Solution:

If the prism angle is A, the normals intersect at an angle of $180^\circ - A$. For the polygon abcd the sum of angles must be 360° so we have

$$\phi_1 + \phi_4 + 180^\circ - A + \Delta = 360^\circ$$

where $\phi + \Delta = 180^\circ$. Δ is the desired deviation in the directions of the emerging beams. Thus

$$\phi_1 + \phi_4 = A + \Delta.$$

Also we have

$$\phi_2 - \phi_3 + 180^\circ - A = 180^\circ \text{ or } \phi_2 + \phi_3 = A.$$

Snell's law gives the following relations:

$$\sin \phi_1 = n \sin \phi_2$$

$$n \sin \phi_3 = \sin \phi_4$$

Given n, A, and ϕ_1, we can find ϕ_2, ϕ_3, and ϕ_4 and then Δ. In this case

$$A = 60^\circ \text{ and } \phi_1 = 30^\circ.$$

Numerically we have

$$1.5 \sin(60^\circ - \phi_3) = 0.5 \qquad \text{so } \phi_3 = 40.53^\circ$$

$$1.5 \sin(40.53^\circ) = \sin \phi_4 \quad \text{so } \phi_4 = 77.10^\circ$$

Thus

$$\Delta = 30^\circ + 77.10^\circ - 60^\circ = 47.10^\circ$$

Since a prism like this disperses the spectrum, measurements of Δ for the various colors allow you to determine the index of refraction for the various wavelengths (at least roughly).

Example 8

For the prism geometry studied in the previous example, let the prism angle A be small and let ϕ_1, ϕ_2, ϕ_3, and ϕ_4 be small enough so that the small angle approximation can be used for Snell's law. Show that the angular deviation, Δ, is independent of the angle of incidence ϕ_1 (an important result for the theory of thin lenses).

Solution:

Summarizing the important results:

(a) $\phi_1 + \phi_4 = A + \Delta$

(b) $\phi_2 + \phi_3 = A$ (so if ϕ_2 and ϕ_3 are small, so is A)

(c) $\sin \phi_1 = n \sin \phi_2$

(d) $n \sin \phi_3 = \sin \phi_4$

Using the small angle approximation for (c) and (d) gives $\phi_1 = n\phi_2$ and $n\phi_3 = \phi_4$. If we add these two equations,

$$\phi_1 + \phi_4 = n\phi_2 + n\phi_3 = n(\phi_2 + \phi_3).$$

From (b) we conclude that $\phi_1 + \phi_4 = nA$ so substituting back into (a) we have:

$$nA = A + \Delta.$$

Thus

$$\Delta = (n - 1)A.$$

The angle through which a ray is bent is thus the same for all rays in this approximation.

Example 9

A beam of unpolarized light with intensity I_0 is incident on a perfect polarizing filter. How is the intensity of the transmitted beam, I, related to I_0?

Solution:

The unpolarized beam can be thought of as a collection of electric field vectors, E, that make all possible angles, θ_i, with respect to any given axis (such as that of the filter). The transmitted component will be $E \cos \theta_i$ and the corresponding intensity will be proportional to $E^2 \cos^2\theta_i$ where we must take the average value of $\cos^2 \theta_i$. Since any value of θ_i between 0 and 2π is equally likely, this average value is 1/2. Thus $I = 1/2\ I_0$. This is a physically reasonable result indicating that the ideal filter absorbs half the unpolarized energy falling on it.

Example 10

Two ideal polarizing filters are arranged so that the light intensity passing the second filter is zero (i.e. their axes are perpendicular). A third such filter is introduced <u>between the first two</u>. What is the total transmitted light intensity as a function of the orientation of this third filter?

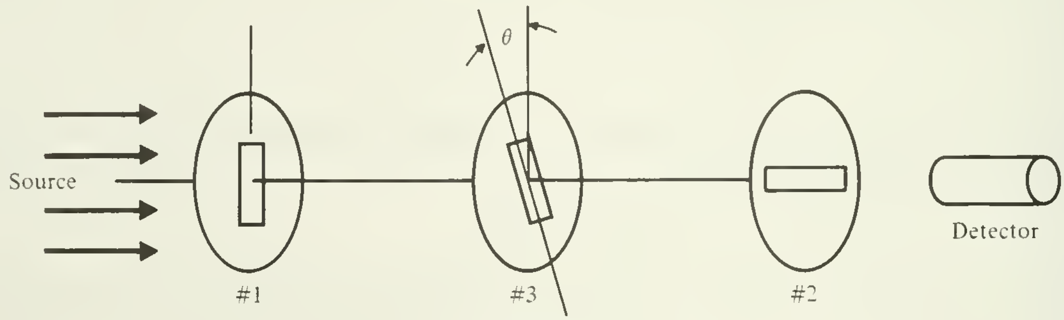


Figure 36-7

Solution:

Refer to Fig. 36-7. Let θ be the angle between the axes of filters 1 and 3. If E_0 is the magnitude of the electric field vector that passes through filter 1, then the component passed through filter 3 is $E_0 \cos \theta$. The component of this vector passed through filter 2 is now

$$(E_0 \cos \theta) \cos(90^\circ - \theta) = E_0 \cos \theta \sin \theta.$$

The intensity at the detector is then:

$$I \propto E_0^2 \cos^2 \theta \sin^2 \theta$$

Since the maximum transmitted intensity (all axes in same direction) is proportional to E_o^2, we can write:

$$I = I_{max} \cos^2 \theta \sin^2 \theta$$

$$= \frac{I_{max}}{4} \sin^2(2\theta)$$

This obviously has its maximum when $\theta = 45^\circ$. The third filter has served to rotate the direction of polarization.

Example 11

What is the relationship between Brewster's angle, θ_p, and the critical angle for total internal reflection?

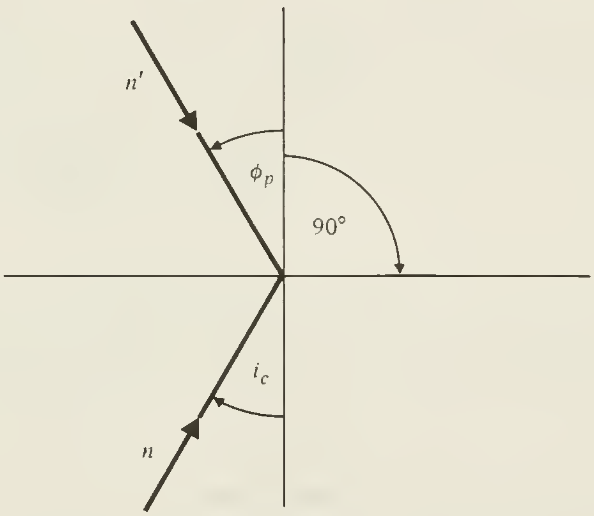


Figure 36-8a

Solution:

For critical internal reflection to be possible at the interface shown in Fig. 36-8a, n must be greater than n' and the critical angle i_c is obtained from:

$$n \sin i_c = n'$$

For the reflected beam in the medium with index n', totally polarized light results when the angle between the reflected and refracted beams is 90°. Thus

$$n' \sin \phi_p = n \sin \phi = n \sin(90^\circ - \phi_p) = n \cos \phi_p.$$

Dividing the first expression by the last yields:

$$\tan \phi_p = \frac{n}{n'}$$

Notice that it is not necessary for $n' < n$ for this effect. However if n and n' are such that critical internal reflection is possible, then

$$(\tan \phi_p)(\sin i_c) = 1.$$

In Fig. 36-8b, note that if the angle of incidence is equal to ϕ_p so that the reflected beam, R', is totally polarized, then the beam R_2 is also totally polarized as it makes a 90° angle with the transmitted beam I' at the second interface.

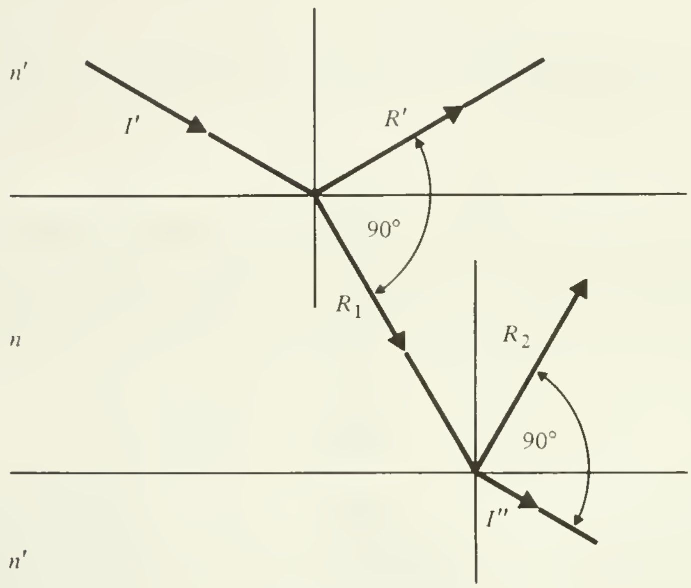


Figure 36-8b

Example 12

In the previous example
(a) suppose $n' = 1.00$ and $n = 1.50$ (glass)
(b) suppose $n' = 1.00$ and $n = 4/3$ (water)
(c) suppose $n' = 1.50$ and $n = 4/3$ (water)
and calculate the Brewster angle ϕ_p.

Solution:

This involves only a simple substitution in the equation

$$\tan \phi_p = \frac{n}{n'}$$

(a) $\tan \phi_p = 1.50$ so $\phi_p = 56.3^{\circ}$

(b) $\tan \phi_p = 4/3$ so $\phi_p = 53.1^{\circ}$

(c) $\tan \phi_p = 8/9$ so $\phi_p = 41.6^{\circ}$

Thus the reflection-polarization can occur at the interface of any two optical media. The ratio n/n' determines whether the Brewster angle is greater or less than 45°.

Example 13

Consider two waves of the same frequency $\omega = 2\pi c/\lambda$ traveling in the + z direction with amplitudes A_1 and A_2 given by,

$$\mathbf{E}_1 = \mathbf{i}A_1 \sin(kz - \omega t)$$

$$\mathbf{E}_2 = \mathbf{j}A_2 \sin(kz - \omega t + \phi)$$

so that ϕ gives the relative phase between E_1 and E_2. Specify A_1, A_2, ϕ for
(a) a linearly polarized wave,
(b) a circularly polarized wave, and
(c) an elliptically polarized wave.

Solution:

(a) If $\phi = 0$ we have a linearly polarized wave, where $E_1/A_1 = E_2/A_2$ so that a plot of E_1 versus E_2 is a straight line through the origin. If $A_1 = A_2$, the direction of polarization is at 45° with respect to our arbitrarily chosen x and y axes so that by rotating the coordinates to a new system x', y' we can have a wave polarized along x' or one polarized along y'.

(b) If $\phi = \pm\ \pi/2$ we have circularly polarized light if $A_1 = A_2$. In this case $E_1{}^2 + E_2{}^2 = A_1{}^2$ so that a plot of E_1 versus E_2 gives a circle of radius A_1.

(c) If $\phi = \pm\ \pi/2$ but $A_1 \neq A_2$ we have elliptically polarized light since $E_1/A_1 = E_2/A_2 = 1$. A plot of E_1 versus E_2 is an ellipse. If the two waves of light are recombined in a quarter wave plate, the emerging single beam in circularly polarized.

Example 14

A beam of light (initially unpolarized) is passed through an ideal polarizing filter P_1 and then traverses a cell containing small particles suspended in water as shown in Fig. 36-9. Light scattered from these small particles is observed at right angles to the incident beam after passing it through a second ideal filter P_2. Originally the filters are oriented so that the 'brightness of the field' as seen by an observer is maximum.

(a) Filter P_2 is rotated through 90°. What does the observer see?

(b) Filter P_1 is now rotated through 90°. Is the field bright or dark?

(c) Filter P_2 is now rotated to its original position. Is the field bright or dark?

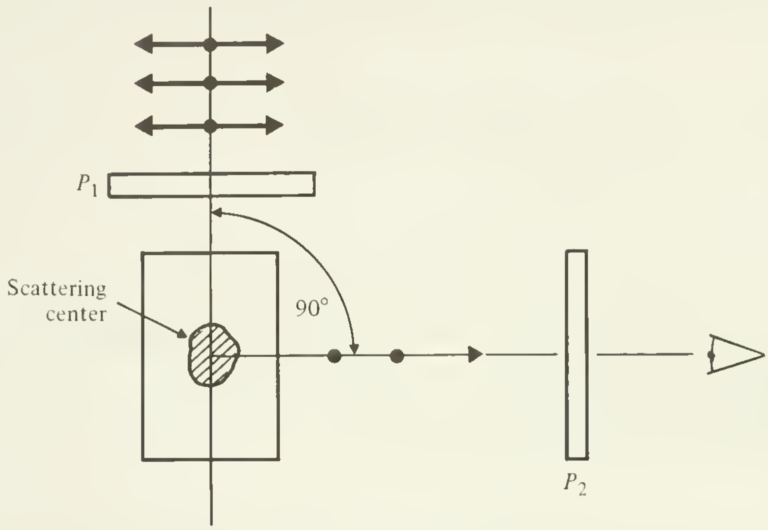


Figure 36-9

Solution:

This is like a detective story. To have maximum brightness initially, P_2 must transmit waves with E vectors perpendicular to the plane of the paper because the scattered beam at 90° contains only those components. Clearly P_1 must transmit those waves as well, or the scattered light intensity at 90° could not be maximum.

(a) If P_2 is rotated through 90°, the observer will see a dark field.

(b) If P_1 is rotated through 90°, it absorbs all the waves with E vectors perpendicular to the plane of the paper. The field remains dark.

(c) If P_2 is now rotated back to its original position, the field remains dark because P_1 is not passing any waves with E vectors perpendicular to the plane of the paper.

QUIZ

1. The angle of incidence in medium 1 is 45° and the angle of refraction in medium 2 is 30°. The index of refraction of medium 2 is 1.510. Calculate the index of refraction in medium 1.

Answer: $n_1 = 1.068$

2. A liquid of index of refraction n floats on top of water of index of refraction $n_w = 1.33$. Light rays in water that make an angle of incidence of 60° or larger are totally reflected. Calculate n.

Answer: $n = 1.152$

3. Unpolarized light of intensity I_o strikes a combination of three ideal polarizing filters. The first and third polarizers are 'crossed' so that their axes are perpendicular. The intensity of light transmitted by this combination is $0.1152\ I_o$. Calculate the angle θ between the axes of the first and third filters.

Answer: $\theta = 36.9^\circ$ or 53.1°

4. Choose the correct statement.

Light polarized by scattering from small particles in suspension in a liquid

(a) is completely polarized only when viewed along the original beam

(b) is never completely polarized

(c) is completely polarized when viewed at 90° with respect to the incident wave, with the electric field vector parallel to the incident beam

(d) is completely polarized when viewed at 90° with respect to the incident wave, with the electric field vector perpendicular to the plane of the incident wave and the scattered wave

(e) none of these.

Answer: Statement (d) is correct.

37
IMAGES FORMED BY A SINGLE SURFACE

OBJECTIVES

In this chapter, the images formed by reflection or refraction at a single surface (either plane or spherical) are studied. Your objectives are to:

Locate the image by ray construction.

Calculate the position of the image using general expressions developed from geometry and Snell's law.

Characterize the image as erect or inverted and find its relative size.

Apply sign conventions for object and image distances as well as those for the radius of curvature of the surface.

REVIEW

In this chapter, reflection and refraction at a single surface are studied in preparation for a treatment of lenses and more complicated optical instruments that involve multiple surfaces. A single sign convention, suitable for both reflection and refraction, is introduced. Since only plane and spherical surfaces are considered (with results for the plane surface derivable from those for the spherical surface by letting the radius approach infinity), only signs for three distances will be needed. These are (1) the object distance, s; (2) the image distance, s'; and (3) the radius of curvature, R. All of these distances are measured from the intersection of the reflecting or refracting surface with the optic axis, a reference line along which we position the object and image. The sign conventions are the following:

(a) The object distance (s) is positive if it is on the same side of the surface as the incoming light;

(b) The image distance (s') is positive if it is on the same side of the surface as the outgoing light;

(c) The radius of curvature (R) is positive if the center of curvature, C, is on the same side of the surface as the outgoing light. This is the same convention as that used for s'.

In this chapter, all of the object distances, s, are positive, basically by construction, since there is only one reflecting or refracting surface to deal with. In the more complicated systems studied in the next chapter, image formation depends on treating the image formed by surface 1 as the object for surface 2, etc., until all surfaces have been treated. Here the object distance for surfaces 2, 3, etc. are frequently negative. Simple numerical examples of both concave (positive R) and convex (negative R) spherical mirrors are found in Examples 1 and 2. The general relationship between object distance and image distance is shown graphically in Example 3. The results for small distances (s) may seem a little perplexing at first but at such small distances ($s \ll R$), the curvature of the surface is unimportant and the results closely resemble those for a plane (flat) mirror. Note that a concave mirror can produce an image that is either magnified or reduced and either inverted or erect, but the convex mirror can produce only a reduced, erect image of a real object.

There are two ways of defining a focal length or focal point. In one method, the object is imagined to be located at infinity and the corresponding image distance is called the second focal length (f'). In the other method, one seeks the position of the object that would give an image at infinity. The object distance here is called the first focal length (f). For spherical mirrors $f = f' = R/2$ where R is positive for a concave mirror but negative for a convex mirror. For a single spherical refracting surface, f is not equal to f'. This is demonstrated in Problem 37-26, SZY.

Two types of magnification are introduced, a lateral magnification (m) which is equal to the ratio of the vertical height of the image (y') to that of the object and a longitudinal magnification (m') which is equal to the ratio of the differential change in image position (s') to a corresponding differential change in object position (s). The values of m and m' for all the surfaces encountered in this chapter are summarized in the text in Table 37-1. Negative values for m always indicate that the image is inverted with respect to the object whereas a positive m accompanies an erect image.

All derivations in this chapter make use of the small angle (or paraxial) approximation at some stage. Only rays near the optic axis can be used (hence the small angles) as the geometry of the spherical reflecting (refracting) surface is such that the various rays that can be constructed to locate the image do not all intersect in the <u>same point</u> for an object of finite size. This imperfect imaging is called 'spherical aberration.'

It is desirable to construct a 'ray diagram' for each optics problem involving lenses, mirrors or combinations of these. For spherical surfaces, a ray through C (the center of curvature) is always undeviated as the angle of incidence is zero. Rays parallel to the optic axis are either reflected (or refracted) through a focal point or they are reflected (or refracted) in such a way that they appear to originate from a focal point. The rules for forming principal rays are summarized for mirrors in the text, SZY.

Ray tracing to locate the image for a concave spherical refracting surface is illustrated in Example 4. There also the general expression developed in the text for such a surface is applied to calculate the image position and the lateral magnification. Use of this general formula to treat a plane refracting surface is illustrated in Example 5 where the plane surface is realized by taking the limit as R approaches infinity.

Finally, for diversion, a derivation of the formula for refraction from a spherical surface (concave surface with negative radius of curvature) is given in Example 6. Only plane geometry and no special sign conventions are used. This is done to emphasize that all we are doing in this chapter is applying plane geometry, Snell's law and the law of reflection in the small angle approximation to locate the image due to a given object.

EXAMPLES AND SOLUTIONS

Example 1

A concave spherical mirror with R = 0.8 m produces an image of an object located 2 m from the mirror. Where is the image located and what is its relative size?

Solution:

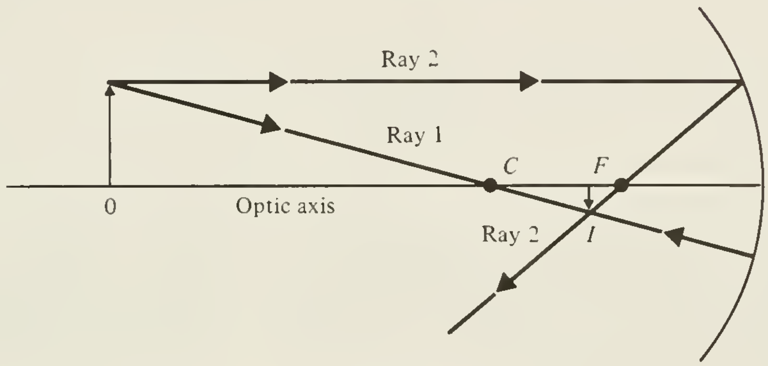


Figure 37-1

Construction of a ray diagram is shown in Fig. 37-1. Ray 1 passes through the center of curvature and is undeviated. Ray 2, parallel to the axis, is reflected through the focal point F located at R/2. The intersection of these two rays locates the image, I. Using the formula for a mirror,

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R}$$

we note that s, the object distance, is positive since it is on the same side of the mirror as incoming light. The image distance, s', will also be positive since the image is on the same side of the mirror as outgoing (or reflected) light. Furthermore, since C is on the same side of the mirror as outgoing light, R is positive. Numerically we have

$$\frac{1}{s'} = \frac{2}{0.8\ \text{m}} - \frac{1}{2\ \text{m}} \quad \text{or } s' = 0.50\ \text{m}$$

Since the lateral magnification, m, is equal to $-s'/s$, then $m = -1/4$. The image height is only 1/4 of the object height and the minus sign indicates that the image is inverted.

Example 2

A convex spherical mirror with R = 0.8 m produces an image of an object 2 m from the mirror. Where is the image located and what is its relative size?

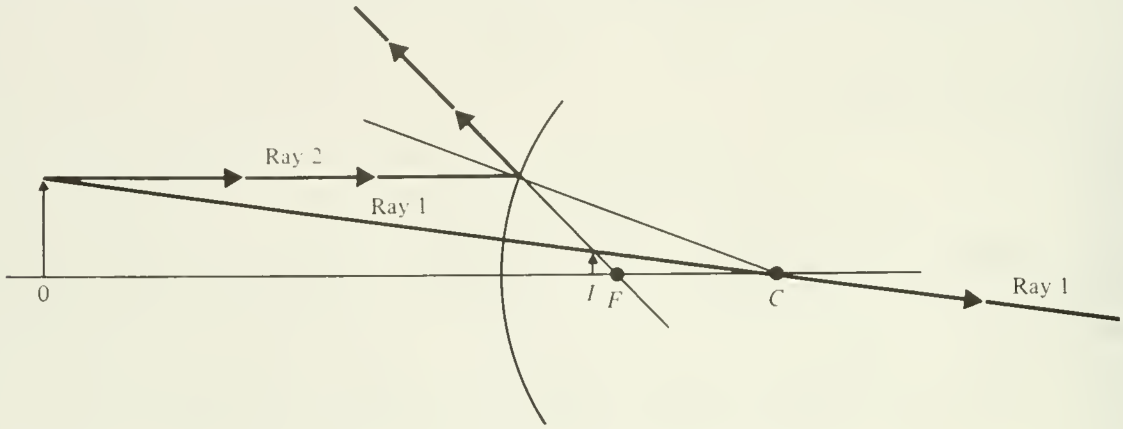


Figure 37-2

Solution:

Referring to Fig. 37-2, we see again that Ray 1 passes through C and is undeviated. Ray - is reflected and appears to originate at F. The small, erect image I is shown at the intersection of rays 1 and 2. Since C is not on the same side of the mirror as outgoing light, R is negative. From the figure, since the image is also not on the same side as outgoing light, we expect s' to be negative. Numerically we have

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} \quad \text{gives} \quad \frac{1}{2\ m} + \frac{1}{s'} = \frac{2}{-(0.8\ m)}$$

Solving for s' yields

$$s' = -\frac{1}{3}\ m$$

The lateral magnification is m = −s'/s = + 1/6, the (+) sign indicating that the image is erect.

Example 3

Since both concave and convex spherical mirrors obey the same equation, construct a graph of object distance versus image distance for each type of mirror.

Solution:

(a) Concave mirror.

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R}$$

where R is positive. See Fig. 37-3a.

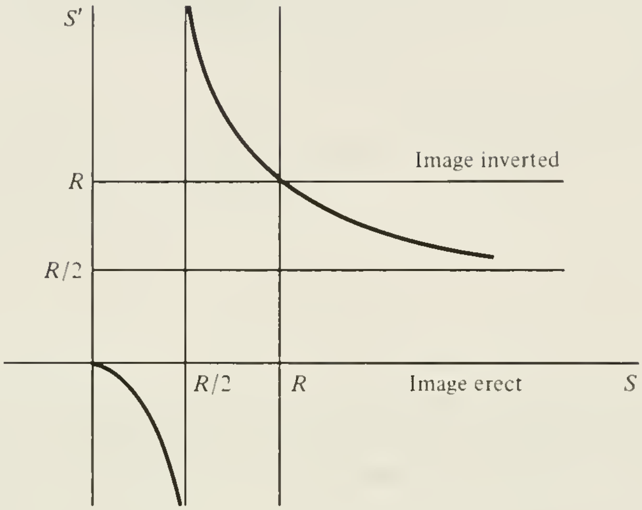


Figure 37-3a

(b) Convex mirror. If we let R stand for a positive number, |R|, then

$$\frac{1}{s} + \frac{1}{s'} = -\frac{2}{|R|}$$

See Fig. 37-3b.

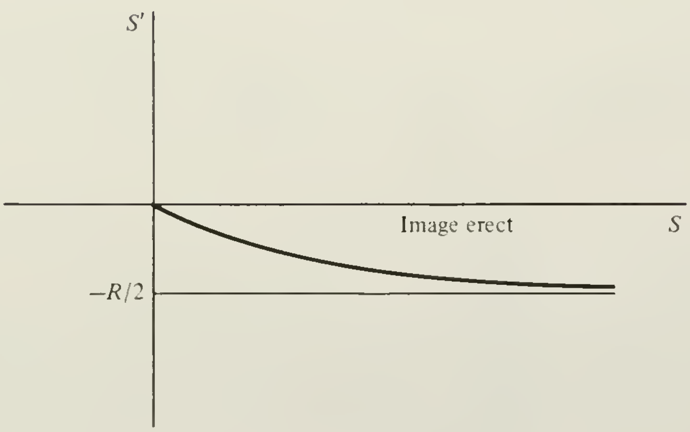


Figure 37-3b

Example 4

A concave spherical ($|R|$ = 0.8 m) refracting surface (n' = 1.50) forms an image of an object placed 1.6 m away in air. Locate the image and calculate the lateral magnification.

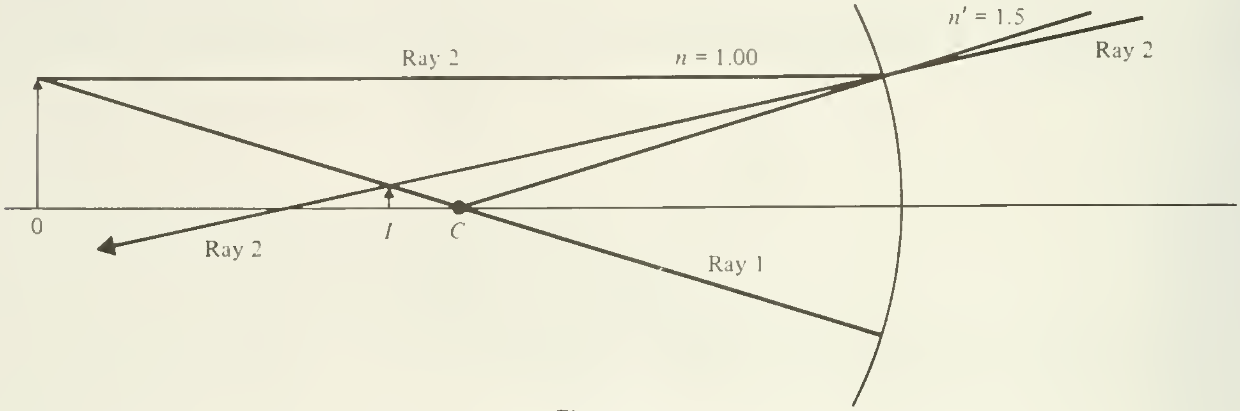


Figure 37-4

Solution:

Refer to Fig. 37-4 where the ray diagram is shown with ray 1 passing through C and ray 2 refracted according to Snell's law. The intersection of ray 1 with the extension of ray 2 locates the image, I. The general formula from the text,

$$\frac{n}{s} + \frac{n'}{s'} = \frac{n' - n}{R}$$

is used here with s = 1.6 m, n = 1.00, n' = 1.50, and R = − 0.8 m. R is negative here because C is <u>not</u> on the same side of the surface as the outgoing light. Thus we have

$$\frac{1.5}{s'} = \frac{1.5 - 1.0}{(-\ 0.8\ \text{m})} - \frac{1.0}{1.6\ \text{m}}$$

Solving for s' gives s' = 1.2 m. The lateral magnification, m, is equal to $-ns'/sn'$ so m = + 0.50, the positive sign again indicating that the image is erect. Note that the distance IC is 0.4 m and OC is 0.8 m. since the triangles are similar, we have y'/y = 0.4/0.8 = 0.5 as before.

Example 5

Use the expression developed for refraction at a spherical surface to obtain the corresponding result for a plane refracting surface. For the spherical surface note that

$$\frac{n}{s} + \frac{n'}{s'} = \frac{n' - n}{R}$$

where n is the index of refraction of the medium where the object is located, n' is the index of refraction of the other medium and R is the radius of curvature of the surface.

Solution:

For a plane surface, the radius of curvature is ∞, so we have

$$\frac{n}{s} + \frac{n'}{s'} = 0$$

Thus for a plane surface the result is

$$\frac{s'}{s} = -\frac{n'}{n}$$

as given in the text (SZY).

It is important to observe the correct sign conventions here. If s is positive, then s' is always negative but it can be either larger or smaller than s depending on the ratio n'/n. To apply this formula to find the 'apparent depth' of an object in water when we view it in air, regard s' as the apparent depth, s as the true depth, n' = 1 and n = 4/3. See Example 3, Chapter 38. The negative sign occurs because the image is not on the same side of the surface as the <u>outgoing</u> light.

Example 6

For an object as shown in Fig. 37-5, locate the image due to refraction at the spherical surface. Assume n'< n for convenience. Calculate both the longitudinal and lateral magnifications.

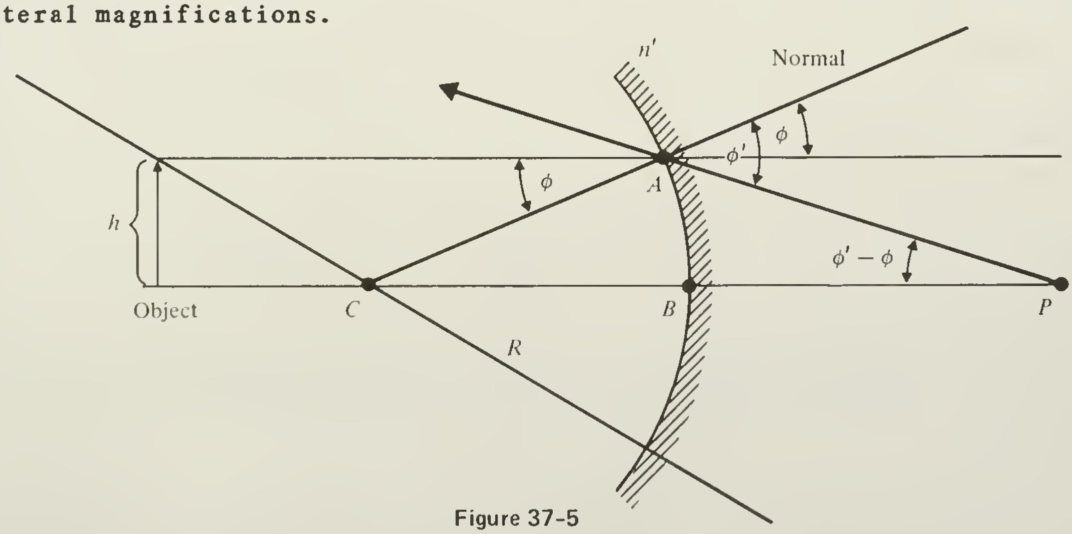


Figure 37-5

Solution:

This problem can be easily solved by using plane geometry and Snell's law. We will use an xy coordinate system with origin at point B. The ray from the tip of the object to A is refracted at the surface and passes through P. Designate the height of the object by h. An equation that describes the straight line through points P and A is

$$y - h = - \tan(\phi' - \phi)x$$

The ray passing from the tip of the object through the center of curvature, C, is undeviated and an equation describing that line is:

$$y = -\left(\frac{h}{s - R} \right)(x + R)$$

where s is the distance from the object to B so that $h/(s - R)$ is the slope of the line through C.

If we eliminate y from these two equations, we obtain the value of x for which these two lines intersect. This is where the image is formed, o $x = s'$ in the notation of the text. To evaluate $\tan(\phi' - \phi)$ we can use Snell's law and the small angle approximation,

$$n \sin \phi = n' \sin \phi' \qquad \text{or} \qquad n\phi = n'\phi'$$

to find that

$$\tan(\phi' - \phi) = \phi' - \phi$$

is approximately equal to

$$\left(\frac{n}{n'} - 1\right)$$

The value of ϕ here is h/R, so we have

$$\tan(\phi' - \phi) = \frac{h}{R} \quad \frac{(n - n')}{n}$$

Solving for x we have

$$x\, Rn - s(n - n') = - Rn's$$

Rearranging this result,

$$\frac{Rn - s(n - n')}{Rs} = - \frac{n'}{x}$$

yields

$$\frac{n}{s} + \frac{n'}{x} = \frac{n - n'}{R}$$

This is the same as the general formula in the text if we set $x = s'$ and remember that R is negative for this particular surface.

As a specific application, take $n' = 1$ and position the object at the center of curvature ($s = R$). Then the result is

$$\frac{n}{R} + \frac{1}{x} = \frac{n}{R} - \frac{1}{R}$$

so

$$x = - R.$$

The image of this object is located just at the position of the object. The lateral magnification in this case is:

$$m = \frac{y}{h}$$

where y is found by substituting $x = - R$ into the equation

$$y - h = -(x) \tan(\phi' - \phi).$$

Here we find that $y = nh$ so $m = n$. The image is erect and n times larger (vertically) than the object.

To obtain the longitudinal magnification, return to the expression

$$\frac{n}{s} + \frac{n'}{s'} = \frac{n - n'}{R}$$

and calculate ds'/ds. Note that

$$\frac{n}{s^2} ds + \frac{n'}{(s')^2} ds' = 0 \qquad \text{so } \frac{ds'}{ds} = - \frac{n}{n'} \frac{s'^2}{s^2}$$

Since $s'^2 = s^2$ in this application and $n' = 1$, the longitudinal magnification is equal to the lateral magnification. This is not generally true.

QUIZ

1. A spherical mirror with radius of curvature of 0.4 m forms an image of a 2 cm high object located 1.2 m from the vertex of the mirror.

(a) Calculate the position and height of the image if the mirror is concave.

(b) Calculate the position and height of the image if the mirror is convex.

Answer: (a) $s' = 0.24$ m, $y' = -0.40$ cm
(b) $s' = -0.171$ m, $y' = 0.286$ cm

2. A glass rod ($n_g = 1.55$) is submerged in water ($n_w = 1.33$). The end of the rod is spherical (convex) with radius of curvature $R = +0.20$ m.

(a) How far from the vertex should an object be placed (in water) to produce an image at infinity?

(b) Where will the image of an infinitely distant object in water be located?

Answer: (a) $s = 1.21$ m
(b) $s' = 1.41$ m (inside the glass)

3. An object in water ($n_w = 4/3$) is located 1 m to the left of a concave spherical refracting surface with index of refraction $n_g = 1.50$. The magnitude of the radius of curvature is 2 m. Using the small angle approximation, locate the position of the image.

Answer: The image is 1.059 m to the left of the refracting surface.

4. A hemispherical piece of glass (with refractive index of 1.50) is placed over an object on a flat surface. The small object is at the center of the circular bottom surface of radius $R = 20$ cm. Use the small angle approximation to locate the image of the object.

Answer: The image is located at the same position as the object.

38
LENSES AND OPTICAL INSTRUMENTS

OBJECTIVES

In the previous chapter image formation due to one refracting (or reflecting) surface was discussed. In this chapter images formed by two or more refracting surfaces are treated. Your objectives are to:

Calculate the focal length of a lens made from a pair of refracting spherical surfaces.

Locate the images graphically by ray tracing.

Calculate the image distance and the lateral and longitudinal magnification given the object distance and the focal length.

Apply simple lens formulas to calculate the properties of optical instruments such as magnifiers, cameras, and telescopes.

REVIEW

The basic result from the last chapter, relating the image and object distances to the respective indices of refraction and the radius of curvature of a spherical refracting surface, is used here to obtain the focal length of a thin lens consisting of two such spherical refracting surfaces. The procedure employed is to find the image due to the first surface and then use that image as the object for the second surface. The image formed by the second surface is the 'final' image for this thin lens. If the focal length is found from the lensmaker's formula (or known otherwise), the object distance (s), image distance (s') and focal length (f) are related (in air or free space) by the formula

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

and the two spherical refracting surfaces that make up the lens are thus treated as a whole.

Lenses with a positive value of f are known as converging lenses while those with negative values of f are called diverging lenses. Two definitions of the focal point are given and these are illustrated in Example 1 where the principal rays used to locate the image graphically are also shown for both converging and diverging lenses.

The lateral magnification, m, defined as the ratio of image height to object height, is shown to be equal to $-s'/s$. If m is a negative number, the image is inverted but if m is positive, the image is erect. The longitudinal magnification, m', is shown to be equal to $-m^2$. Applications of these general ideas to image formation by converging and diverging lenses are given in Examples 2 and 3. The general relationship between s and s' for both converging and diverging lenses is shown graphically in Example 4.

Two thin lenses in contact with each other can be treated as we did the two spherical refracting surfaces in order to derive the lensmaker's formula. This is pursued in Example 5 where it is shown that if the individual lenses have focal lengths of f_1 and f_2, the two together, considered as a single system, have a focal length, f, found from:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

The discussion of optical instruments begins with an analysis of the human eye as a lens system. The important point to remember here is that the eye can focus on objects that are no closer than 25 cm from the lens. This distance is important for the definition of angular magnification (M), which is the ratio of the angle subtended by the image at the eye when viewed through the optical system to the angle subtended at the eye by the object when viewed in the 'most favorable' manner. If possible, this means that you view the object from a distance of 25 cm but if this is not possible (due to geographical limitations) then you view the object where it is (i.e., the moon must be viewed in its natural habitat). This angular magnification of a 'magnifier', a simple converging lens, is calculated in Example 6.

The camera, projector, compound microscope and telescope are discussed as examples of lens combinations that perform a specific function. A problem encountered with converting a simple or astronomical telescope (where the image is inverted) into a terrestrial telescope (where the image is erect) is worked out in Example 7.

EXAMPLES AND SOLUTIONS

Example 1

Sketch the three principal rays for both a converging ($f > 0$) and diverging ($f < 0$) lens and show the first and second focal points.

Solution:

(a) The object point for which the image is at infinity is called the first focal point (F). See Fig. 38-1a for its principal ray.

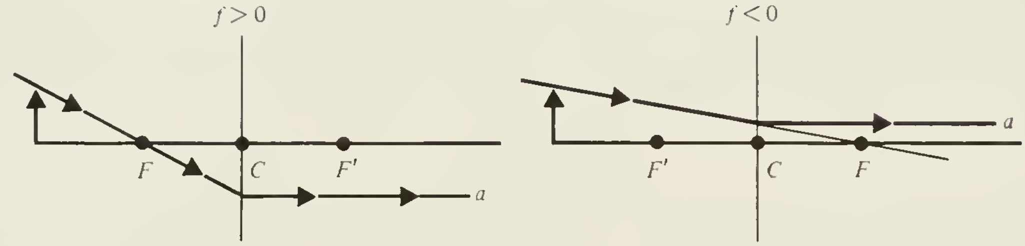


Figure 38-1a

(b) The image point for an infinitely distant object is called the second focal point (F'). See Fig. 38-1b for its principal ray.

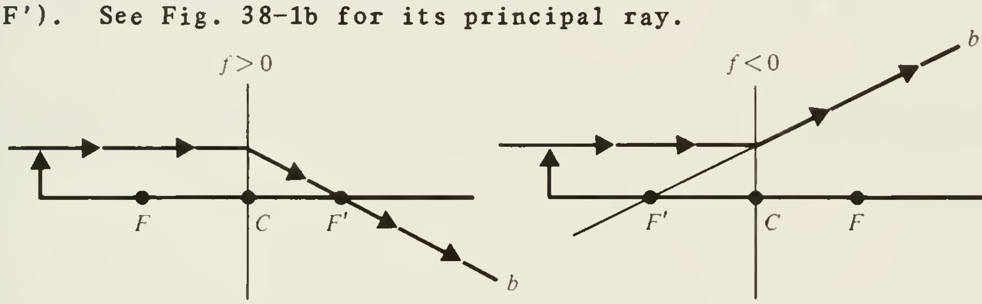


Figure 38-1b

Thus F and F' are located on opposite sides for converging and diverging lenses. For a given <u>thin</u> lens, F and F' are numerically the same.
(c) The ray through the center of a thin lens is deviated by a negligible amount (which we take to be zero) and hence is the same for either a converging or diverging lens. See Fig. 38-1c for this principal ray.

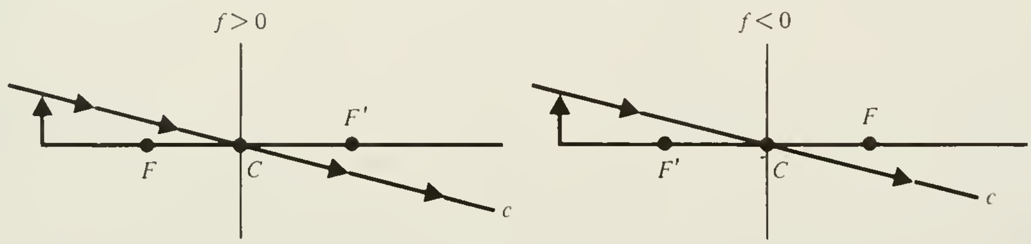


Figure 38-1c

(d) We combine these three principal rays to locate the image due to a real object. The rays are labeled a, b and c, following the previous diagrams. The intersection of these rays locates the tip of the image (I). See Fig. 38-1d.

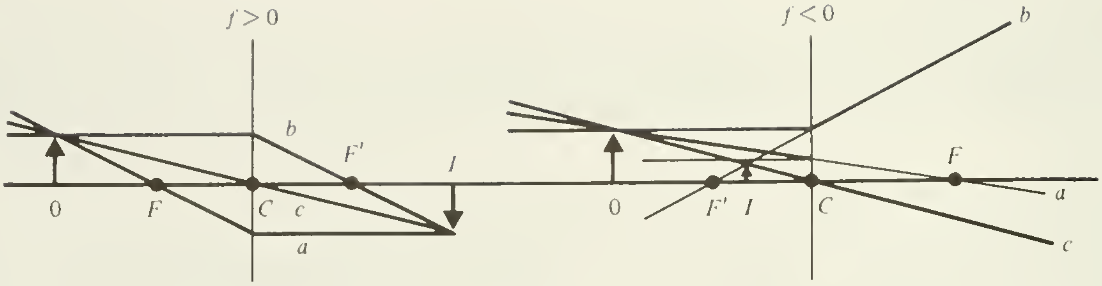


Figure 38-1d

Note: Rays a and b must be projected back to locate I.

Example 2

Graph the image distance (s') as a function of object distance (s) for both a converging (positive focal length) and a diverging (negative focal length) lens.

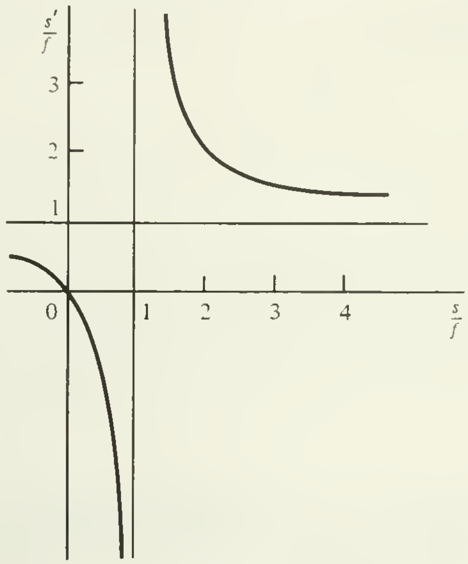


Figure 38-2a

Solution:

(a) For a converging lens, see Fig. 38-2a, we plot the equation

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

Note: To make a 'universal' graph of such a function, multiply by f so that

$$\frac{f}{s} + \frac{f}{s'} = 1$$

and then measure s and s' in units of f.

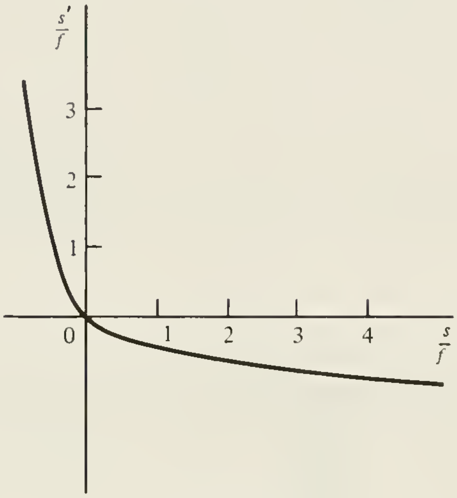


Figure 38-2b

(b) For a diverging lens, see Fig. 38-2b, we plot the equation

$$\frac{|f|}{s} + \frac{|f|}{s} = -1$$

Example 3

For glass with n = 1.50, use the lensmaker's formula to find the focal lengths for lenses with:

(a) $R_1 = 5$ cm, $R_2 = -20$ cm
(b) $R_1 = 5$ cm, $R_2 = \infty$
(c) $R_1 = 5$ cm, $R_2 = 10$ cm

After finding the focal lengths, calculate the image position and lateral magnification for an object placed + 10 cm from the lens.

Solution:

The lensmaker's formula is

$$\frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

(a) $R_1 = 5$ cm and $R_2 = -20$ cm, so for f in centimeters,

$$\frac{1}{f} = (0.5)\left(\frac{1}{5} + \frac{1}{20}\right) = \frac{1}{8} \quad \text{or} \quad f = 8 \text{ cm}.$$

To locate the image due to an object at 10 cm, (all dimensions in cm)

$$\frac{1}{10} + \frac{1}{s'} = \frac{1}{8} \quad \text{or} \quad s' = +40 \text{ cm}.$$

The lateral magnification is $m = y'/y = s'/s = -4$ here. The negative sign indicates the image is inverted. A sketch of this lens would look like Fig. 38-3.

Figure 38-3

(b) $R_1 = 5$ cm and $R_2 = \infty$ (this means that the back surface is flat) and thus we have

$$\frac{1}{f} = (0.5)\left(\frac{1}{5} - 0\right) \quad \text{so } f = +10 \text{ cm}.$$

In this case, the object is located at the focal point so the image is at infinity and the magnification is infinite.

(c) $R_1 = 5$ cm and $R_2 = 10$ cm (both are positive so the lens is crescent shaped)

$$\frac{1}{f} = (0.5)\left(\frac{1}{5} - \frac{1}{10}\right) = \frac{1}{20} \quad \text{so } f = 20 \text{ cm}.$$

To locate the image write:

$$\frac{1}{10} + \frac{1}{s'} = \frac{1}{20} \quad \text{so } s' = -20 \text{ cm}.$$

This image is located on the same side of the lens as the object. The lateral magnification is $m = s'/s = +2$, so the image is enlarged and erect (+ sign).

Example 4

For glass with $n = 1.50$, use the lensmaker's formula to find the focal lengths for lenses with:

(a) $R_1 = -5$ cm, $R_2 = 20$ cm
(b) $R_1 = -5$ cm, $R_2 = \infty$
(c) $R_1 = -5$ cm, $R_2 = -10$ cm

Calculate the image positions and lateral magnifications for an object placed + 10 cm from each of the above lenses.

Solution:

The values of all the R's above are identical to those of the previous example but the signs have all been changed to convert those converging lenses into diverging lenses. Thus each value for f will have the same magnitude but the signs will be negative here.

(a) $f = -8$ cm $\quad \dfrac{1}{s'} = -\dfrac{1}{8} - \dfrac{1}{10}$

Solving for s' gives $s' = -40/9$ cm and $m = +4/9$, and the image is reduced and erect. The image is located on the same side of the lens as the object.

(b) $f = -10$ cm. Here the object is placed at the wrong focal point to produce an image at infinity and the image distance is found from

$$\frac{1}{s'} = -\frac{1}{10} - \frac{1}{10} = -\frac{1}{5}$$

or $s = -5$ cm. Thus $m = +0.5$ so the image is on the same side of the lens as the object, erect and reduced in size.

(c) $f = -20$ cm $\quad \dfrac{1}{s'} = -\dfrac{1}{20} - \dfrac{1}{10} = -\dfrac{3}{20}$

Solving for s' gives $s' = -20/3$ cm so that $m = +2/3$. Again the image is on the same side of the lens as the object, reduced and erect. Consult the graph of s' versus s in Example 2 for a diverging lens and you will see that the image is always on the same side as the object (negative s'), erect (negative s') and reduced ($-s' < s$).

Example 5

Suppose two thin lenses with focal lengths f_1 and f_2 are placed in contact with each other. Obtain an expression for the focal length f of this lens combination.

Solution:

This problem is a good example of the systematic method used to solve optics problems when more than one reflecting or refracting surface is involved.

For lens (1) locate an object at s_1. The image distance s_1' is calculable from:

$$\frac{1}{s_1'} = \frac{1}{f_1} - \frac{1}{s_1}$$

<u>This image of lens 1 is now regarded as the object for the second lens</u>. Note that if s_1' is a positive number in the above expression, the image would be on the same side of the lens combination as outgoing light from lens 1. Thus as an object for lens 2, it would give a <u>negative</u> object distance (s_2). If however s_1' were negative this would correspond to a positive object distance for the second lens. Thus $s_2 = - s_1'$ is correct for both possibilities, and we have

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2}$$

becomes

$$-\left(\frac{1}{f_1} - \frac{1}{s_1}\right) + \frac{1}{s_2'} = \frac{1}{f_2}$$

Rearranging

$$\frac{1}{s_1} + \frac{1}{s_2'} = \frac{1}{f_1} + \frac{1}{f_-}$$

nnterpreting this expression, we note s_1 is the distance of the object from the lens combination, and s_2' is the distance of the image from the lens combination, so the focal length of the combination (f) satisfies the equation

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

where now we write s_1 as simply s (the object distance for the combination) and s_2' as s' (the image distance for the combination). Thus we have

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

To illustrate this approach, consider the specific case where $f_1 = +3$ cm and $f_2 = -5$ cm. Take an object initially 4 cm from the lens combination and locate the final image. The calculation is simple as $s = +4$ cm and f (in cm) is obtained from:

$$\frac{1}{f} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

Thus we have $f = +7.5$ cm, with for s' given by

$$\frac{1}{s'} = \frac{1}{7.5} - \frac{1}{4}$$

or $s' = -8.57$ cm.

Location of this image graphically is shown in Fig. 38-4a, b and c, where the image I_1 of the first lens is treated as the 'object' for the second lens in order to find the final image.

We will symbolize the thin lens combination by a vertical solid line. First we imagine just lens 1 to be located on that line and locate the image due to 1. See Fig. 38-4a.

We now imagine lens 2 to be on the vertical dashed line and treat I_1 as the object (note the object distance would be negative) for it. See Fig. 38-4b. For the combination see Fig. 38-4c.

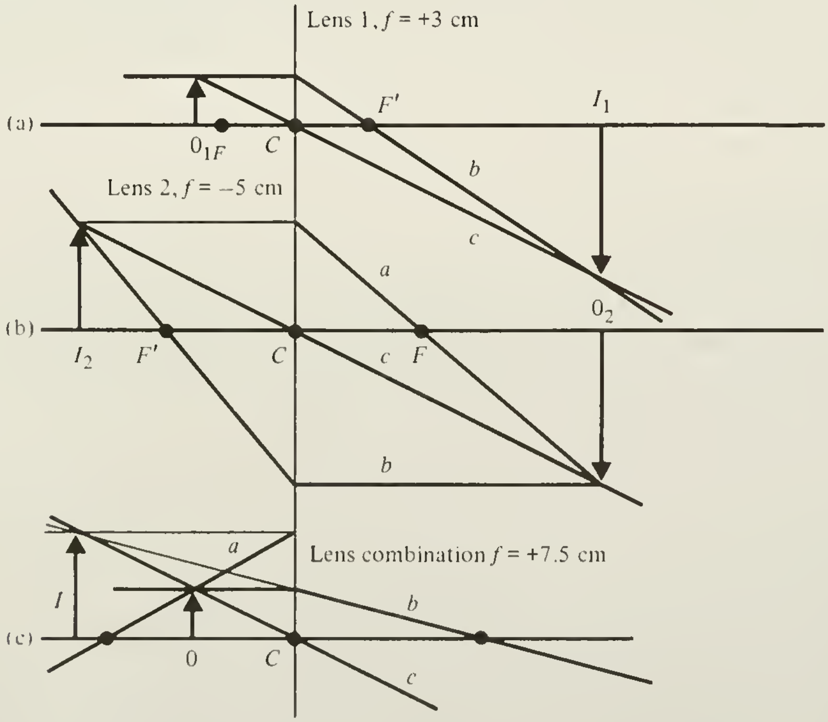


Figure 38-4

Example 6

Suppose a magnifying glass is held so that the image produced is a distance D from the eye. Calculate the angular magnification (M) under these conditions.

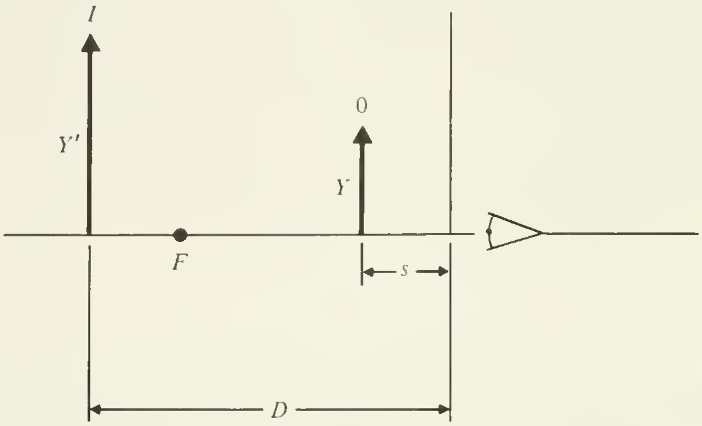


Figure 38-5

Solution:

A sketch of this magnifier is given in Fig. 38-5. The object (O) is located inside F so the image (I) is erect, enlarged, and virtual. The distance D is equal to $-s'$. If we view the object directly, at 25 cm away from the lens of the eye, then the angle subtended at the eye, u, is equal to

$$u = Y/25$$

where Y, Y', f and D are in cm. If we place our eye very close to the lens, the angle subtended at the eye by the image, u', is approximately equal to:

$$u' = Y'/D$$

Thus $M = u'/u$ is given by

$$M = \frac{Y'}{Y}\left(\frac{25}{D}\right)$$

The ratio Y'/Y is equal to $-s'/s$, so to calculate it write:

$$\frac{1}{s} = \frac{1}{f} - \frac{1}{s'} = \frac{1}{f} + \frac{1}{D} \qquad \text{since } -s' = D.$$

Then

$$\frac{-s'}{s} = - s'\left(\frac{1}{f} + \frac{1}{D} \right) = D\left(\frac{1}{f} + \frac{1}{D} \right)$$

This gives for $M = \left(\frac{-s'}{s} \right)\left(\frac{25}{D} \right)$ the value:

$$M = D\left(\frac{1}{f} + \frac{1}{D} \right)\left(\frac{25}{D} \right) = \frac{25}{f} + \frac{25}{D}$$

The expression in the text was derived by assuming the image is located at infinity. The normal eye can focus on an image located as close as 25 cm, but no closer. Thus the angular magnification (subject to being able to focus the eye) is maximum for D = 25 cm and equal to 1 + 25/f in that case. If D is set equal to infinity, the above expression gives the same result as in the text.

Example 7

In a simple telescope, the image is inverted with respect to the object. Suppose a third lens is used to invert the image of the objective lens without enlarging or reducing the image. How much longer must this telescope be?

Solution:

The third lens is called the erecting lens. Let f be the focal length of this lens. To have no magnification from this lens, s′ and s must have the same magnitude. To invert the image, they must have the same sign, so s = s′. Thus

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \quad \text{since } s = s' \quad \text{gives } s = 2f = s'.$$

If this erecting lens is added between the objective lens and the eyepiece, the image of the objective lens must fall a distance s from the third lens and the eyepiece must view the (now erect) image at a distance s′ from this lens∝ Therefore the telescope must be s + s′ = 4f longer than it was originally. This added length is occasionally undesirable so that prism binoculars are used instead.

QUIZ

1. An object of height 3 cm is located + 20 cm from a thin lens with index of refraction 1.55. Calculate the position of the image and its height if:

(a) $R_1 = +10$ cm and $R_2 = -20$ cm
(b) $R_1 = -10$ cm and $R_2 = +20$ cm

Answer: (a) $s' = +30.8$ cm and $y' = -4.62$ cm
(b) $s' = -7.55$ cm and $y' = +1.13$ cm

2. Two thin lenses with $f_1 = +15$ cm and $f_2 = +25$ cm are located 5 cm apart. Calculate the location of the final image and the lateral magnification.

Answer: The image is 14.13 cm from the second lens (19.13 cm from the first lens) and the magnification is + 0.65.

3. A real object is placed 8 cm in front of a converging lens with $f_1 = +6$ cm. A second diverging lens with $f^2 = -4$ cm is placed 4 cm behind the first lens. Find the location of final image with respect to the second thin lens.

Answer: The image is located 5 cm in front of lens 2 (1 cm in front of lens 1).

4. An object of height 1 cm is viewed by the unaided eye, 25 cm away from the eye. A magnifying glass of focal length $f = +10$ cm is used to view the object. If the final image, after magnification, is also 25 cm away from the eye, calculate the angular magnification in the small angle approximation.

Answer: 3.5

39 INTERFERENCE AND DIFFRACTION

OBJECTIVES

In this chapter waves of the same frequency, possessing a constant phase relationship, will produce an interference pattern when they overlap in space. This pattern has alternate light and dark regions called fringes. Your objectives are to:

Define the concept of coherence as a constant phase relationship.

Obtain and apply the relationship between phase difference and optical path length.

Incorporate the effect of phase shifts introduced by certain reflections.

Calculate the main features of various interference and diffraction patterns.

Calculate the limits that diffraction places on the ultimate resolution of optical instruments.

Huygens' principle, implying that each wavefront is a source of secondary wavelets, is central to the arguments of this chapter. The two critical derivations for the intensities of the patterns observed for two slit interference or single slit diffraction are obtained by adding phasors (see Chapters 11 and 36) and then squaring the resultant vector to obtain the intensity.

REVIEW

Electromagnetic waves of the same frequency can interfere with each other if they overlap spatially. This interference can be of either the constructive or destructive type. In constructive interference, the total intensity is larger than the sum of the individual intensities from the various sources, whereas in destructive interference the resultant intensity is less than the sum of the individual intensities from the various sources. Although the interference pattern always exists in principle, in practice it may not be observable or detectable. In order for this pattern to be detectable (by ordinary means such as the human eye or photographic film) the phase difference between the two (or more) sources contributing to the pattern must not change appreciably in the time interval over which the pattern is observed. In this case, the sources are said to be coherent. Thus, if conventional methods of detection are used to observe the interference between two sources, the time over which the sources are coherent must be larger than the time constant that characterizes the detector (about 0.1 s for the eye and 10^{-3} s for fast film). Details illustrating the above qualitative statements are given in Example 1, where the geometry of Young's experiment is used. This material should be regarded as supplementary to the treatment of coherence and interference given in the text.

The central region in a Young's experiment is bright, because the path differences and hence the overall phase difference is zero. The interference pattern known as 'Newton's Rings', however, has a central dark region just where the path difference, $r_1 - r_2$, is zero. Similarily in the interference phenomena observed with Lloyd's mirror the fringe observed where the actual path length is zero is dark. (One of the two beams, however, suffers a glancing reflection from a mirror surface.) To explain these phenomena, it is necessary to introduce the idea that a phase shift of π radians occurs for some types of reflections but not for others. Maxwell's equations, applied to the problem of reflection at an interface, give the desired solution but we will just state the result without proof. If the electromagnetic wave strikes an interface with index of refraction higher than that of the medium in which it was traveling, then the reflected wave suffers a phase change of π radians with respect to the incident wave. If the wave strikes an interface with lower index of refraction than the medium in which it was traveling, no phase change upon reflection occurs. This very important rule is illustrated in Example 3 and then used in Example 2 to obtain the interference pattern known as Newton's Rings.

To summarize the essential ideas for our treatment of interference:

(1) The problem of making two spatially separated sources of electromagnetic waves coherent over long time intervals is solved by using one source and somehow deriving two (or more) beams from it that in turn are used for the sources to set up the interference pattern.

(2) For two 'sources' of the above type with equal amplitudes, the light intensity at all points in the pattern depends only on the relative phase of these two sources. Included in this relative phase are differences in optical path length (real path length multiplied by index of refraction) and phase changes suffered upon certain reflections.

(3) When this total phase difference is π, 3π, 5π, etc., the waves interfere destructively and the intensity is zero. When this phase difference is 0, 2π, 4π, etc., the waves interfere constructively and the intensity is four times larger (for two sources) than the intensity due to one source alone. The average intensity over the entire pattern is just twice that expected from a single source.

(4) The critical equation for understanding all of these results is

$$I = I_o \cos^2(\delta/2)$$

where δ is the total difference in relative phases. This equation was derived in the text by finding the vector resulting from the addition of two vectors with the same amplitude but shifted in relative phase by the amount δ (phasors). This is a very important result. To calculate δ for our two sources at a given point we need to know the difference in optical path lengths of the two beams in getting to the point plus the accumulation of all the π phase shifts upon reflection. These notions are discussed in Example 2.

The coherence problem encountered with interference does not occur with diffraction effects which result from interference between the various parts of the same slit. If parallel light from a single slit with width comparable to the wavelength of light is collected by a converging lens and focused on a screen, the central region is bright but dark fringes appear with regularity as you move in either direction away from the central maximum. The angle at which these dark fringes appear can be predicted exactly by the following argument:

(1) The wavefront emerging from the single slit when viewed at an angle θ with respect to the normal can be thought of as being divided into N equal zones or regions.

(2) Each region can be characterized by an electric field vector of the same magnitude as that of all the other zones but differing in phase by an amount $\Delta\phi$ (which depends on the point of observation) with the electric field vectors representing the adjacent zones (i.e. each of these electric field vectors is a phasor.)

(3) The resultant electric field vector at a given point of observation is obtained by summing these N phasors.

(4) The intensity at the point of observation is proportional to the square of the resultant electric field.

To do the calculation exactly, the number of zones, N, is made infinitely large so that the sum of the phasors becomes the chord of an arc of a circle. The resultant electric field vector is

$$E_T = S\,\frac{\sin(\delta/2)}{(\delta/2)}$$

where S is the arc length and represents the sum of all the phasors when they have no phase difference between individual phasors. The intensity when viewed normally ($\sin\theta = 0$) is I_o and is proportional to S^2. Thus the intensity at any viewing angle, θ, $I(\theta)$ is given by

$$I(\theta) = I_o\,\frac{\sin^2(\delta/2)}{(\delta/2)^2}$$

where δ is total the phase difference between the top and bottom of the slit. This phase difference is $\delta = 2\pi/\lambda$ (path difference). The path difference between top and bottom of the slit is a $\sin\theta$ where 'a' is the slit width and θ is the viewing angle. From the above expression for the intensity, it is seen that the sine function has zeroes when the argument is an integral multiple of

π. The condition for dark fringes (zero intensity) is then:

$$\frac{\delta}{2} = n\pi \quad \text{where } n = 1, 2, 3, \text{ etc.}$$

Substituting the previous value for $\delta = 2\pi/\lambda(a \sin \theta)$, one has for the various values of ϕ where diffraction minima occur, θ_n, the following equation:

$$\frac{1}{2}\left(\frac{2\pi}{\lambda}\right) a \sin \theta_n = n\pi$$

or

$$\sin \theta_n = n\left(\frac{\lambda}{a}\right)$$

If y_n is the distance from the center of the pattern on the screen to the nth minimum and F is the focal length of the lens used, then in the small angle approximation is

$$\sin \theta_n = \frac{Y_n}{F} = n\left(\frac{\lambda}{a}\right)$$

From the above expressions, it is easy to see that the angular width of the central bright spot is double that of any of the other maxima and that the positions of the maxima are only approximately half-way between the dark fringes. The intensities of the maxima fall off very rapidly. See Example 5.

To understand the results from two or more slits in complete generality, one must use notions both from interference and diffraction. The diffraction grating is made by ruling many identical thin slits into a glass or plastic plate. Before the advent of the laser, the diffraction grating was the chief means of obtaining monochromatic light. The 'reinforcement' condition is

$$d \sin \theta = N \lambda \text{ where } N = 1, 2, 3, \text{ etc.},$$

with d the spacing between slits. In this case, the integer N is called the 'order' of the pattern. Frequently two or more orders will overlap, reducing the usefulness of the grating.

Diffraction effects limit the resolving power of most optical instruments. Usually the diffraction effect results from the finite size of some circular aperture. For a circular geometry, the intensity distribution does not follow the function

$$\frac{\sin^2(\delta/2)}{(\delta/2)^2}$$

as it does for rectangular geometry but rather is described by a Bessel function. The nulls of these functions are not evenly spaced. The first null is located such that the radius of the first dark ring, R, is given by

$$R = 1.22\left(\frac{\lambda}{D}\right)F$$

where F is the focal length of the lens used and D is the diameter of the limiting aperture (usually the lens). An objective criterion for deciding when the angle between two sources is resolvable is called the 'Rayleigh criterion'. Using this criterion, the sources are just resolvable when the maximum of one diffraction pattern falls at the minimum of the other. This is illustrated in Example 7.

EXAMPLES AND SOLUTIONS

Example 1

Calculate the separation between the interference fringes obtained in Young's experiment when λ = 550 nm, R = 3 m, and d = 0.22 mm. Refer to Fig. 39-1.

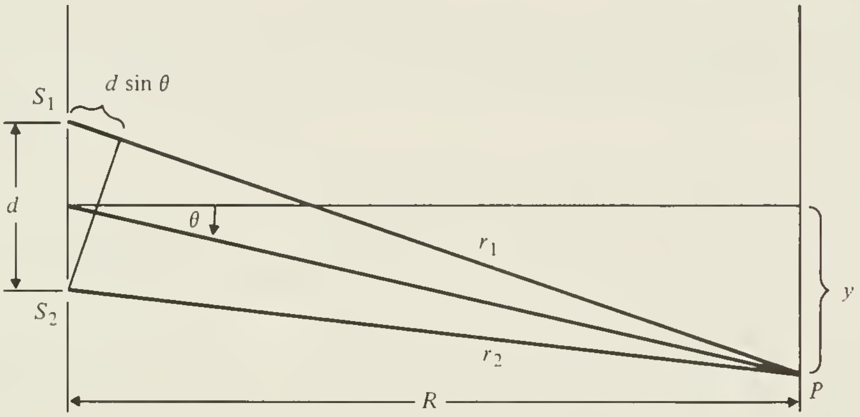


Figure 39-1

Solution:

From Fig. 39-1, the path difference ΔD in the small angle approximation is given by

$$\Delta D = d \sin \theta$$

When this path difference is equal to an integer multiplied by the wavelength, we get <u>constructive interference</u>.

$$\Delta D = n\lambda \qquad n = 0, 1, 2, \ldots \quad \text{(bright fringe)}$$

Conversely when the path difference is equal to a half-integral number of wavelengths we get <u>destructive interference</u>

$$\Delta D = \left(\frac{2n+1}{2}\right)\lambda \qquad n = 0,1,2,\ldots \qquad \text{(dark fringe)}$$

The sine can be approximated by

$$\sin\theta \simeq \tan\theta = (y/R)$$

Thus bright fringes appear for values of y (labeled y_n)

$$y_n = \left(\frac{n\lambda}{d}\right)R$$

Dark fringes appear for values of y_n satisfying

$$y_n = \left(\frac{2n+1}{2}\right)\frac{\lambda}{d}\,R$$

Numerically we have

$$\frac{\lambda}{d}\,R = \frac{(5.50 \times 10^{-7}\ \text{m})(3\ \text{m})}{(0.22 \times 10^{-3}\ \text{m})} = 7.5 \times 10^{-3}\ \text{m} = 7.5\ \text{mm}.$$

The spacing between like fringes is then 7.5 mm.

Example 2

Compare the 'phase shifts' and path differences of rays A, B and C in Fig. 39-2 for normal incidence with that of the incident ray I.

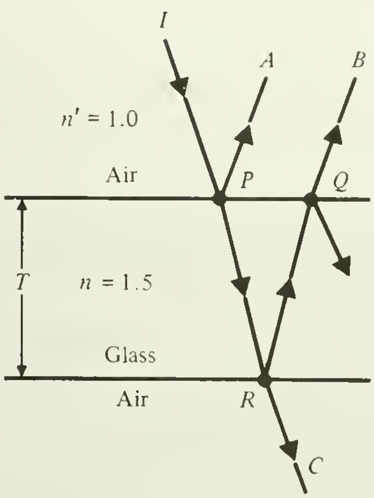


Figure 39-2

Solution:

Here the incident ray, I, as shown in Fig. 39-2, is drawn approaching the surface at a non-zero angle of incidence for convenience in following the various rays. In the end we will let the angle of incidence be zero so points P and Q will coincide. Ray A is the part of the incident beam reflected at P. Since it is coming from air and reflected from glass, at point P. The phase shifts are as follows.

Ray A: As this ray is reflected from a material with index higher than that of the material it was in, it suffers a phase change of π radians.

Ray B: This ray is reflected from a material with lower index than the material it was in so there is no phase shift. Also there is no phase shift upon transmission. This ray has traveled an additional distance (for normal incidence) of 2T <u>in glass</u> compared to the incident ray. The additional <u>optical path</u> is then 2nT where n is the index of refraction of the glass.

Ray C: This ray is transmitted at both the top and bottom surfaces and hence suffers no phase change. The optical path length through the glass is nT. If this wave was compared to the part of the incident wave that <u>did not</u> pass through the glass, the optical path difference of these two waves would be

$$\Delta D = nT - T = (n - 1)T$$

Example 3

Develop a formula for the radii of both the dark and light rings in the Newton's Rings pattern.

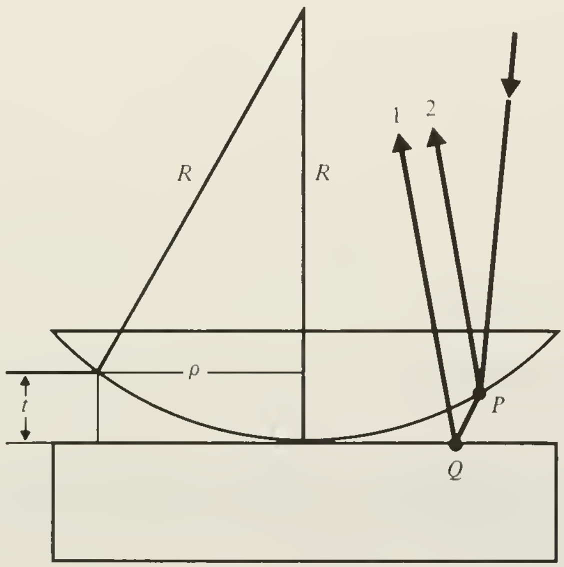


Figure 39-3

Solution:

Referring to Fig. 39-3, the distance P, from the center out to the point where the thickness of the air space is t, is found from

$$R^2 = (R-t)^2 + \rho^2$$

$$= R^2 - 2Rt + t^2 + \rho^2$$

$$(2R - t)t = \rho^2$$

If R >> t then the thickness t is related to ρ by

$$t = \frac{1}{2R}\rho^2$$

Consider now two possible paths for a light ray. Rays along these two paths are coherent because they originate at a single source. These paths are labeled 1 and 2 in the above figure. For path 2, the ray is reflected at point P <u>in glass</u> and hence suffers no phase shift. For ray 1, the reflection occurs at Q <u>in air</u> so there is a phase change of π there. In addition, ray 1 has a longer optical path, by an amount 2t (if viewed normally) where t is the thickness of the air gap at that point. Usually we would say that when the path difference was equal to an integral number of wavelengths we would get constructive interference (and hence bright fringes) but the relative phase change of π now means that when 2t (the path difference between rays 1 and 2) equals an integral number of wavelengths, we will get destructive interference and hence dark fringes.

$$\text{(dark)} \qquad 2t = n\lambda = \frac{2\rho_n^2}{2R} \qquad n = 0, 1, 2, \ldots$$

Thus the radius of the nth dark ring, ρ_n, is equal to

$$\text{(dark)} \qquad \rho_n = (nR\lambda)^{1/2} \qquad n = 0, 1, 2, \ldots$$

where R is the radius of curvature of the lens.

To get constructive interference, we must make the path difference a half-integral number of wavelengths, or

$$\text{(light)} \qquad 2t = \frac{2n-1}{2}\lambda.$$

$$\text{(light)} \qquad \rho_n = \left[\frac{(2n-1)}{2} R\lambda\right]^{1/2} \qquad n = 1, 2, 3, \ldots$$

Note the center is dark since the air thickness is negligible there but the two waves have a relative phase difference of π.

Example 4

A thin layer of water (n_W = 4/3) when viewed normally produces a 'non- glare' optical coating on glass (n_G = 1.5) for light of wavelength 600 nm. What is the minimum thickness of the layer of water?

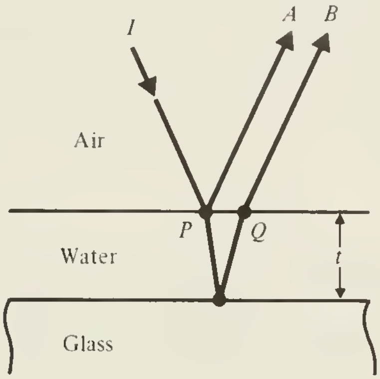


Figure 39-4

Solution:

Referring to Fig. 39-4, ray A suffers a phase change of π upon reflection at P (since it is trying to get into a medium with higher index of refraction). Ray B also suffers a phase change of π when reflected at point R since glass has a higher refractive index than water. To make this 'non-glare' coating, we want rays A and B to interfere destructively. For this to occur, the optical path difference must be equal to a half integral number of wavelengths. The path difference, for normal viewing, is 2t but the <u>optical path difference</u> equals $2tn_W$. For minimum thickness, this difference should be equal to one half wavelength:

$$2tn_W = \lambda/2$$

$$t = \lambda/4n_W$$

Numerically we have

$$t = \frac{3}{4}\left(\frac{600 \times 10^{-9}}{4}\ \text{m}\right) = 112.5\ \text{nm}$$

Example 5

For the single slit diffraction pattern, find the ratio of the intensity at the first maximum to that at the center.

Solution:

When δ is the total phase difference between the wavelets at the two extreme edges of the slit, the intensity is given by

$$I = I_o \frac{\sin^2(\delta/2)}{(\delta/2)^2}$$

where δ is a function of θ. If a is the slit width,

$$\delta(\theta) = \frac{2}{\lambda} a \sin \theta.$$

(a) The first null occurs when $\delta/2 = \pi$ so $\sin \theta_1 = \lambda/a$. For small angles, $\theta_1 = \lambda/a$. There is a symmetrically placed dark fringe on the other side of the central bright spot so the angular width of the central bright spot is $\Delta\theta = 2\lambda/a$. The angular spacing between any other two dark fringes is λ/a.

(b) The first maximum occurs <u>near</u> $\delta/2$ equal to $3\pi/2$. The intensity at $\delta/2 = 3\pi/2$ is equal to:

$$I = I_o \;\; 1/(3\pi/2)^2, \quad (\text{since } \sin 3\pi = 1.)$$

or

$$I = \frac{4}{9\pi^2} I_o \simeq \frac{1}{22} I_o$$

Example 6

A 'replica' diffraction grating has 5276 lines per cm. Derive the intensity distribution due to such a grating and then obtain the positions of the first two maxima away from the central spot for the blue line in the hydrogen spectrum at 486 nm.

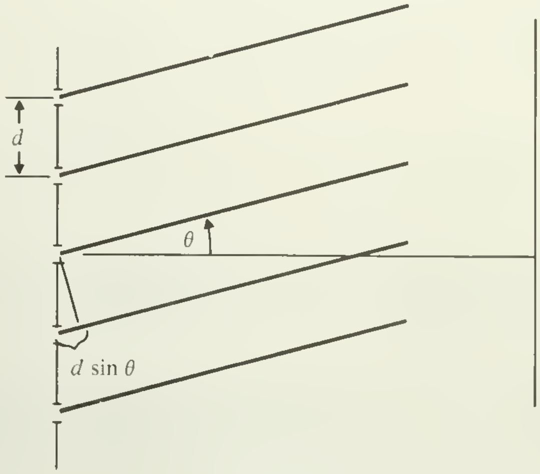


Figure 39-5a

Solution:

Referring to Fig. 39-5a, let d be the spacing between the slits. In cm, d = (1/5276) cm = 1.895×10^{-4} cm. Consider N coherent sources, each one having a phase difference δ with the adjacent sources when viewed at an angle θ with respect to the normal. The path difference between two adjacent slits is d sin θ so one would expect strong reinforcement when $d \sin \theta = n\lambda$. The central region (n = 0) is a superposition of all the wavelengths (unless monochromatic light is used) in the incident beam so the various 'orders' of the pattern correspond to n = 1, 2, 3, etc.

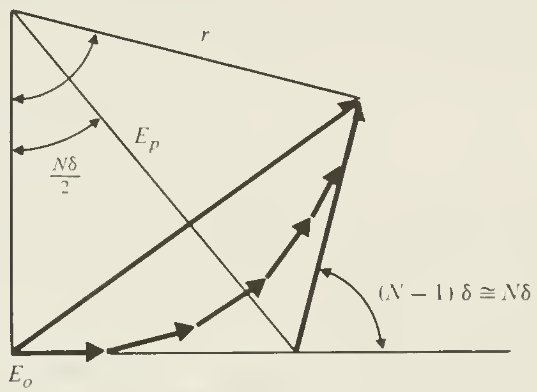


Figure 39-5b

To obtain the intensity distribution due to a finite number, N, of coherent sources, we construct a phasor diagram where δ is now the common phase difference. (See Fig. 39-5b). The total phase difference is $(N - 1)\delta$ which we take to be $N\delta$ for N very large. The resultant electric field vector, E_p is

$$\frac{E_p}{2} = r \sin (N\delta/2)$$

where the radius of the construction circle, r, can be related to the electric field magnitude of a single source, E_o, and the common phase difference, δ, through:

$$r \sin\left(\frac{\delta}{2} \right) = \frac{E_o}{2}$$

This gives

$$E_p = \frac{E_o \sin (N\delta/2)}{\sin (\delta/2)}$$

Squaring to obtain the intensity, we have

$$I = \frac{I_o \sin^2(N\delta/2)}{\sin^2(\delta/2)}$$

where I_o is the intensity due to one source. Notice that for very small δ, (the central bright spot), we can replace sin $\delta/2$ by $\delta/2$ and in that limit the intensity, $I = N^2 I_o$! Without interference we would have expected a value of NI_o but now it has been enhanced considerably since N can be a very large number.

To locate the maxima in the above pattern, we look for nulls in the denominator as this will give intense spikes of intensity with large dark regions in between. The sine function vanishes when the argument is an integral multiple of π so the condition for intensity maxima is:

$$\frac{\delta}{2} = n\pi \qquad n = 1, 2, 3, \text{ etc. (ignoring the central spot)}$$

Since $\delta = (2\pi/\lambda)d \sin \theta$, then $\sin \theta_n = n\lambda/d$ is the condition for maxima, just the condition guessed from the simple picture.

The coherence of these N sources is obtained by using a single light source to illuminate the series of slits. To evaluate the above expression numerically, we use $\lambda = 486$ nm and $d = 1895$ nm. Then $\sin \theta_n = n(0.2564)$ so $\theta_1 = 14.86^\circ$, $\theta_2 = 30.86^\circ$, $\theta_3 = 50.30^\circ$, etc.

Example 7

A binary star at a distance of 100 light years from earth is observed through a telescope using the blue line of hydrogen (486 nm). How large must the diameter of the lens be to see that there are two light sources present if the separation of the two component stars is 5×10^{-3} light years?

Solution:

The Rayleigh criterion is used here. Since the radius of the first dark ring is

$$R = \frac{1.22\ \lambda F}{D},$$

the angle subtended by this ring at the lens will be $\alpha_R = R/F$. For two light sources to be resolvable in the Rayleigh sense, their angular separation must exceed α_R for then the maximum of the second diffraction pattern will fall beyond the minimum in the first diffraction pattern. If the angular separation is less than α_R, the two sources will appear as a 'blur' at the common center and be unresolvable.

In this case, the angular separation of the sources, α, is:

$$\alpha = \frac{5 \times 10^{-3}}{100} = 5 \times 10^{-5} \text{ rad.}$$

Thus if D is chosen such that $\alpha > \alpha_R$ the sources will be resolvable (we can tell that there are two of them). The condition is:

$$5 \times 10^{-5} > \frac{1.22(486 \times 10^{-9} \text{ m})}{D}$$

Thus D must satisfy the inequality:

$$D > 1.19 \times 10^{-2} \text{ m}$$

QUIZ

1. In the geometry of Young's experiment, shown in Fig. 39-1, the two slits are separated by 0.1 mm, the distance to the screen is 0.6 m, and the wavelength of the source used to illuminate the slits is λ = 500 nm. Find the position of the second minimum away from the central maximum.

Answer: $y = 4.5 \times 10^{-3}$ m (symmetric about the central maximum).

2. A lens with an index of refraction n = 1.70 is coated with a film of index n_f = 1.40. (a) Calculate the minimum, non-zero, thickness of the film if a wavelength of 550 nm is to interfere destructively with itself (on reflection) when incident from air on the film at an angle of 0°. (Part of the light is reflected at the air-film interface and part is reflected at the film-glass interface). (b) Calculate the minimum thickness of this film if two wavelengths, λ_1 = 550 nm and λ_2 = 450 nm are to each interfere destructively with itself. This is a non-reflective coating for two wavelengths.

Answer: (a) 9.82 x 10-8 m (b) 8.84×10^{-7} m.

3. A circular aperature of radius $R = 2 \times 10^{-3}$ m is illuminated uniformly by light of wavelength 550 nm. How far from the aperature should one stand in order to see one and only one Fresnel zone?

Answer: 7.27 m

4. Light of wavelength 550 nm strikes a diffraction grating with mean spacing d. The first maximum away from the central bright spot occurs at $\theta = 15.96^\circ$.
(a) At what angle θ should one look to find the second maximum for light of wavelength 500 nm?
(b) Compute the value of the grating spacing d.

Answer: (a) $\theta = 30^\circ$
(b) $d = 2 \times 10^{-6}$ m.

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RELATIVISTIC MECHANICS

OBJECTIVES

The objectives of this chapter are to:

Calculate the space and time differences between events in different inertial reference frames.

Calculate quantities such as momentum and energy relativistically.

Apply the work-energy relationship to the calculation of the velocity change of a particle.

Apply relativistic kinematics and dynamics to a variety of simple problems.

The relationship between space and time (and its implication) will be a substantial test for your physical intuition. Vectors with four (rather than three) components can be used to obtain all of the important results by essentially geometric reasoning.

REVIEW

The principle of relativity states that 'the laws of physics are the same in every inertial frame of reference.' The Galilean (or common sense) transformation from one inertial frame to another, namely,

$$x = x' + ut, \; y' = y, \; z' = z, \text{ and } t' = t,$$

insures that Newton's laws of motion are valid in all inertial frames. Unfortunately the additional requirement that the speed of light be the same for all observers conflicts with the Galilean transformation equations, which predict that the speed in some moving frame, c', would be related to the speed in the laboratory frame, c, by the equation

$$c' = c - u$$

where u is the relative velocity of the two frames (and can be positive or negative). To rectify this problem, the accepted concepts of space and time had to be modified so that a different set of transformation equations (the Lorentz transformation) could be obtained.

In particular, the idea that time intervals would be the same in two reference frames S and S′ in a state of relative motion had to be modified. This can be seen from the famous thought experiment where a point source emits a spherical e.m. wave at $t = t' = 0$ when the origins of S and S′ coincide. The relative motion of S and S′ is along the x, x′ axis. In S at a later time t, this wavefront is given by:

$$x^2 + y^2 + z^2 - c^2t^2 = 0$$

In S′ at a later time t′, we also must have a spherical wave front with:

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

Since the relative motion is along the x,x′ direction, the transverse dimensions y and z are unaffected so $y = y'$ and $z = z'$ just as in the Galilean transformation. However now since the speed of light has the same value, c, in the two frames, setting $t' = t$ would imply that $x = x'$ which is incorrect since the two frames are in relative motion. Thus we can conclude that

$$x^2 - c^2t^2 = x'^2 - c^2t'^2$$

and that since $x \neq x'$, $t \neq t'$. The relativistically correct transformation must couple changes in x to changes in t in order to keep $x^2 - c^2t^2$ the same in all reference frames.

From the relativistically correct Lorentz transformation, it is apparent that two events that are simultaneous in one reference frame are not in another frame in relative motion with respect to the first frame.

The 'time dilation' effect is a really new prediction of Einstein's relativity. The elapsed time t in frame S (which can be thought of as the lab frame) is longer than the elapsed time t′ in the 'moving' frame S′ assuming the elapsed time in S′ is measured by a clock at rest in S′. The 'proper time' is the interval between two points occurring at the same space point.

The Lorentz contraction also is a new prediction of the Einstein relativity. In this effect, the longitudinal dimensions in the moving frame are reduced but the transverse dimensions are unaffected. This predication is easily obtained from the Lorentz transformation.

Two very interesting problems are encountered in Example 4, where the frequency shift called the Doppler effect is calculated for light, and in Example 5, where an age difference for two twins is calculated when one twin takes a long, fast ride. The Doppler formula for light is much simpler than the corresponding formula for sound waves where the actual frequency shift depended on both the source and observer velocities. For light all that matters is the relative velocity of source and observer. The problem with the identical twins is a famous one and called the 'Twin Paradox'. There is still some doubt (not in the minds of most physicists) about this prediction as it has not been directly tested with living subjects. It has been tested and verified with atomic clocks.

The relativistic expressions for linear momentum, p, and energy, E, are different from our previous ones. Linear momentum is given by

$$p = \frac{mv}{[1 - v^2/c^2]^{1/2}}$$

and the energy E is:

$$E = \frac{mc^2}{[1 - v^2/c^2]^{1/2}}$$

The quantity m is called the rest mass and mc^2 is the rest energy, the value of E when $v = 0$. The quantity to be used for the kinetic energy is then $E - mc^2$ and when this quantity is expanded to lowest order in v/c we obtain that $E_k = E - mc^2 \simeq (1/2)mv^2 + \ldots$. By manipulation of the above expression it is shown that:

$$E^2 = p^2c^2 + m^2c^4$$

This is a very important result. Since $(mc^2)^2$ would be the same in any reference frame, then $E^2 - (pc)^2$ must also have the same value in all frames (i.e. it is a relativistic invariant). Also in later chapters we will discuss particles that have no rest mass. For them, $m = 0$, so $E = pc$. The relationship between energy and momentum is illustrated in Example 6.

The constancy of the speed of light implies a very special law for the transformation of relative velocities. This expression, which can be obtained also from the Lorentz transformation is:

$$v' = \frac{v - u}{1 - uv/c^2}$$

Here v' is the velocity of a particle in frame S', v is its velocity in S, and u is the relative velocity of S and S'. This equation is capable of producing some surprising results. This is illustrated in Example 6.

The equivalence of mass and energy implied in the Einstein expression for E has been well tested in particle decay experiments. The amount of energy that can be obtained from conversion of a very small mass is intriguing but sacred conservation laws pertaining to heavy particles such as neutrons and protons limit the amount of energy that can be liberated by conversion of mass into energy. See Example 7.

EXAMPLES AND SOLUTIONS

Example 1

Show that the constancy of the speed of light in all inertial reference systems implies that (a)

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

and (b) if the relative motion of the two frames is along the xx′ axis then

$$x^2 - c^2t^2 = x'^2 - c^2t'^2.$$

Solution:

(a) At time $t = t' = 0$ let the origins of the frames S and S′ coincide. A light wave emitted at the origin at this time spreads out spherically in S and S′. In S the radius of the spherical wavefront is

$$R = (x^2 + y^2 + z^2)^{1/2} = ct$$

In S′ the radius R′ of the spherical wavefront is

$$R' = (x'^2 + y'^2 + z'^2)^{1/2} = ct'$$

Squaring these quantities we have

$$x^2 + y^2 + z^2 - c^2t^2 = 0$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

so that

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

Since this quantity is the same in any two inertial frames S and S′ it is called a relativistic invariant and is very useful in calculations.

(b) For a relative motion parallel to the xx′ axis, the dimensions perpendicular to the motion are unaffected so that

$$y = y'$$

$$z = z'$$

and the above relativistic invariant reduces to

$$x'^2 - c^2t'^2 = x^2 - c^2t^2$$

Example 2

Two events occur simultaneously in frame S where they are separated by 1.28×10^7 m (the diameter of the earth).

(a) Is it possible to find a reference frame S′ where these two events occur at the same space point ?

(b) In a reference frame S′ where the two events are separated by 2.56×10^7 m, calculate the time interval between the events and

(c) the relative velocity of the frames S and S′.

Solution:

(a) It is <u>not</u> possible to find a frame where these two events occur at the same point (or even closer together). This can be seen by using the invariant quantity

$$(\Delta x')^2 - c^2(\Delta t')^2 = (\Delta x)^2 - c^2(\Delta t)^2$$

In S, $\Delta t = 0$ so that

$$(\Delta x')^2 - c^2(\Delta t')^2 = (\Delta x)^2 = \text{a positive real number}$$

If $\Delta x' = 0$ then $\Delta t'$ cannot be a real number. For $\Delta t'$ to be real we must have $\Delta x' > \Delta x$ in this problem.

(b) Writing the invariant for the simultaneous events in S

$$(\Delta x')^2 - c^2(\Delta t')^2 = (\Delta x)^2$$

and substituting

$$\Delta x = 1.28 \times 10^7 \text{ m}$$

$$\Delta x' = 2.56 \times 10^7 \text{ m}$$

we have

$$(\Delta t')^2 = \frac{(2.56 \times 10^7 \text{ m})^2 - (1.28 \times 10^7 \text{ m})^2}{(3 \times 10^8 \text{ m}\cdot\text{s}^{-1})^2}$$

$$\Delta t' = 7.39 \times 10^{-2} \text{ s}$$

The relative velocity of the two frames can be obtained from the Lorentz transformation equation

$$x'_2 - x'_1 = \frac{(x_2 - x_1) - u(t_2 - t_1)}{[1 - (u^2/c^2)]^{1/2}}$$

with $t_2 - t_1 = 0$. Since $x'_2 - x'_1 = 2(x_2 - x_1)$ we have

$$[1 - (\frac{u}{c})^2]^{1/2} = \frac{1}{2}$$

Squaring results in

$$1 - (\frac{u}{c})^2 = \frac{1}{4}$$

Solving for the velocity, we have

$$u = \frac{(3)^{1/2}}{2} c = 0.866 \text{ c} = 2.60 \times 10^8 \text{ m}\cdot\text{s}^{-1}$$

Example 3

An electron is accelerated from rest to a velocity v = 0.9 c by means of a potential difference ΔV. Calculate ΔV using 0.511 MeV for the rest energy of the electron.

Solution:

The energy of the electron is

$$E = \frac{mc^2}{[1 - (v/c)^2]^{1/2}}$$

$$= \frac{0.511 \text{ MeV}}{[1 - (0.9)^2]^{1/2}} = 1.172 \text{ MeV}$$

The increase in kinetic energy is equal to the work done on the particle by the accelerating potential

$$e\Delta V = \Delta K$$

$$= E - mc^2$$

$$= 1.172 \text{ MeV} - 0.511 \text{ MeV}$$

$$= 0.661 \text{ MeV}$$

Solving for ΔV we have

$$\Delta V = 6.61 \times 10^5 \text{ volts} = 661{,}000 \text{ volts}$$

Example 4

If a source of e.m. radiation at rest in S′ produces regular pulses with spacing τ', what is the separation of these pulses in S?

Solution:

Let the first pulse be emitted at $t = t' = 0$ when the origins coincide. If the second pulse is emitted at $t' = \tau'$, then the time interval t in the frame S is equal to:

$$t = \frac{\tau'}{[1 - u^2/c^2]^{1/2}}$$

This pulse is emitted at a value of x equal to ut and so must travel back to the observer at the origin of S taking an additional time equal to x/c. Therefore the time between the two pulses, τ, according to an observer at the origin of S is

$$\tau = t + \frac{x}{c} = t + \frac{ut}{c} = (1 + u/c)t$$

or

$$\tau = \frac{(1 + u/c)}{[1 - u^2/c^2]^{1/2}}\,\tau' = \frac{(1 + u/c)^{1/2}}{(1 - u/c)^{1/2}}\,\tau'$$

If we regard τ and τ' as the respective periods of the sources, then the frequencies f and f′ would be given by:

$$f = \frac{(1 - u/c)^{1/2}}{(1 + u/c)^{1/2}}\,f' \qquad \text{(Doppler Shift)}$$

If the source (in S′) is moving away from the observer (in S), then f is less than f′. The frequency of a receding train's whistle is lowered. If the source is moving toward the observer, then u is negative and $f > f'$.

Known spectral lines from hydrogen and helium seen in the radiation reaching the earth from distant galaxies are shifted in wavelength toward the red part of the visible spectrum. This 'red shift' combined with the previous expression for the Doppler shift is evidence that these galaxies are receding from us and an important component of the 'Big Bang' theory of the universe.

Example 5

If a 'moving clock' runs slower, what will the age difference be between two twins if one stays on the earth while the second makes a round trip to a point in space ten light years from the earth at a speed of 0.95c?

Solution:

The distance, D, traveled is equal to the speed of light multiplied by twenty years. The time taken for this trip according to the twin on earth, T, is:

$$T = \frac{c(20\ \text{years})}{0.95\ c} = 21.05\ \text{years}$$

The time elapsed on the 'moving clock', T', is related to T by the time dilation formula:

$$T = \frac{T'}{[1 - u^2/c^2]^{1/2}}$$

Numerically, we have $[1-u^2/c^2]^{1/2} = 0.312$ so $T' = 6.57$ years. Therefore if the twins were 30 years old when the separation occurred, the twin left on earth is 51 years old at the reunion whereas the space traveller is 36.6 years old.

Example 6

Two electrons traveling in the same direction have energies of 1 Mev and 2 Mev respectively as seen from frame S. Find the velocity of each of these electrons in S and then find the velocity of the most energetic one relative to that of the least energetic one. Use 0.51 Mev for the electron rest energy.

Solution:

Since the energy is

$$E = \frac{mc^2}{[1 - u^2/c^2]^{1/2}}$$

where mc^2 is the rest energy, for electron 1 we have:

$$1\ \text{MeV} = \frac{(0.51\ \text{MeV})}{[1 - u_1/c^2]^{1/2}}$$

Solving numerically for u_1 we have:

$$\frac{u_1}{c} = 0.8602.$$

For electron 2,

$$2 \text{ MeV} = \frac{0.51 \text{ MeV}}{[1 - (u_2/c)^2]^{1/2}}$$

so that

$$\frac{u_2}{c} = 0.9669$$

Since the electrons are travelling in the same direction we have chosen both positive signs for the u's.

To find the velocity of electron 2 with respect to electron 1, we must use the Einstein addition law for relative velocities. The relative velocity is not $u_2 - u_1 = c(.9669 - .8602) = \underline{0.1067\ c.}$

The equation given in the text is:

$$v' = \frac{v - u}{1 - uv/c^2}$$

where we consider the frame S' to be the frame where electron 1 is at rest. Thus we interpret the symbols as:

v = velocity of electron 2 in S = 0.9669 c
u = velocity of S' with respect to S = u_1 = .8602 c
v' = unknown velocity of electron 2 in S'

$$v' = \frac{0.9669\ c - 0.8602\ c}{1 - (.8602)(.9669)} = 0.636\ c$$

Since electron 1 is at rest in S', this is the relative velocity of electron 2 with respect to 1. Note that it is more than a factor six larger than the classically expected result.

Example 7

When the neutron spontaneously decays into a proton, an electron and a neutrino (which is massless), the decay products are observed to have a total kinetic energy of 1.25×10^{-13} J. If the proton mass (M_p) is 1.673×10^{-27}kg and the electron mass (M_e) is 9.110×10^{-31} kg, how large is the neutron mass (M_N)?

Solution:

We equate the rest energy of the neutron to the total energy of the by-products. This energy of the by-products is equal to the sum of the rest energies of the proton and electron (the neutrino is massless) and the kinetic energy of the by-products.

$$M_N c^2 = M_p c^2 + M_e c^2 + E_k$$

or

$$M_N = M_p + M_e + \frac{E_k}{c^2}$$

$$= (1.673 \times 10^{-27} + 9 \times 10^{-31} + 1.39 \times 10^{-30}) \text{ kg}$$

$$= (1673 + 2.3) \times 10^{-30} \text{ kg}$$

$$= 1.675 \times 10^{-27} \text{ kg}$$

Because $M_N > M_p$, the proton cannot spontaneously decay into a neutron (plus a position and neutrino).

QUIZ

1. Two events in frame S occur at the same space point with a time difference of 4 s. In a frame S′ moving relative to S with speed u, the separation in time of the two events is 5 s. Calculate
   (a) the separation between the positions of the two events in S′
   (b) the relative velocity of the two frames.

Answer: (a) 9×10^8 m,
(b) $u = \pm 0.6$ c

2. An electron initially at rest is accelerated through a potential difference of 10^6 V. Calculate the final velocity of the electron.

Answer: $(v/c) = 0.9988$ or $v = 2.997 \times 10^8 \text{ m}\cdot\text{s}^{-1}$

3. The rest mass of the proton is $(1836)m_0$ where m_0 is the electron rest mass. Assuming the above relationship is exact (it isn't), calculate the ratio of the electron velocity to the speed of light that would make the electron 'mass' as large as the proton rest mass.

Answer: $v/c = 0.99999985$

4. The kinetic energy of a beam of electrons is 200 keV. Calculate the mass of the the electrons in the beam, in units of the electron rest mass, m_0.

Answer: $m = 1.391m_0$

41
PHOTONS, ELECTRONS, AND ATOMS

OBJECTIVES

In this chapter your objectives are to:

Calculate the threshold wavelength for light to eject photoelectrons from a surface with a given work function.

Calculate the stopping potential for a given wavelength for photoelectrons.

Calculate the frequencies of absorbed and emitted light given a set of energy levels.

Calculate the wavelength change of an x-ray that is scattered from an electron.

Electromagnetic radiation, previously explained by means of a wave picture, is endowed with some particle-like features, just as particles will be endowed with wave-like features in just a few chapters.

REVIEW

Electromagnetic waves in the radiofrequency range (about 10^8 Hz) can be understood in terms of classical electromagnetism but higher frequency radiation (about 10^{15} Hz), light, produces several puzzles particularly in its interaction with matter which can not be understood in the framework of Maxwell's equations and Newton's laws. Maxwell's equations predict that accelerating charges radiate electromagnetic waves. In the late 1800's it was speculated that light came from a motion of electric charge in individual atoms. The experiments of that era were too crude to confirm or deny this. Some of the puzzles were:

(1) Each element emits a characteristic spectrum when it is heated in a flame or excited electrically (like neon in a glow tube). This spectrum does not contain all possible wavelengths but only certain well defined ones. This suggests that emitted light is related to the characteristics and internal structure of an atom -- an idea that cannot be explained with classical theories.

(2) The photoelectric effect could not be explained classically. For thermionic emission of electrons from a surface where the kinetic energy needed by an electron to escape from the surface is supplied by thermal motions, some electrons escape even from cold surfaces. If light is directed onto the surface, and the frequency of the light is wrong, then no electrons are ejected even though the source intensity is high. If the frequency is properly chosen, electrons can be ejected but they have a maximum kinetic energy that does not depend on the source intensity.

(3) The problems with light extend to higher frequencies (10^{19} Hz) where electromagnetic radiations are called x-rays. The problem here was that the scattered x-rays frequently had a longer wave- length than the original x-ray (which the text likens to a light wave suffering a color change upon reflection).

Although this doesn't exhaust the list of problems, it does suffice to indicate that classical theories didn't have the necessary components for a complete solution. The explanations that were offered on the basis of classical theory were many times more puzzling that the original puzzles.

Einstein suggested that the photoelectric effect could be understood if light had a particle-like character. These light quanta were called photons and their energy was directly related to the frequency of the light wave, f, through

$$E = hf$$

with $h = 6.62 \times 10^{-34}$ J·s being Planck's constant. To eject an electron from a metal, the photon energy would have to be larger than the potential energy holding the electron in the metal (called the work function, ϕ). An electron would not be ejected if it had to wait for several quanta to come along and provide it with enough energy. This explained why even intense sources of light would not eject electrons if the light frequency were too low. Further, the ejected electrons would leave the surface with a kinetic energy

$$\frac{1}{2} mv^2 = hf - \phi$$

If a potential difference, V_o, of suitable polarity was applied, with $eV_o = (1/2)mv^2$ then the electrons could be stopped and no current flow would result. Examples 1 and 2 illustrate aspects of the photoelectric effect.

The line spectra emitted by the elements was explained by the Bohr model of the atom. In this model, picture all of the positive charge as well as most of the mass concentrated in the center or the nucleus and the electron orbiting about this nucleus much like the planets orbit about the sun. The postulates for this model of the hydrogen atom are:

(1) The electron (charge -e) travels about the proton (charge +e) in a circular orbit due to the Coulomb attraction between them.

(2) Not all orbits are allowed. The allowed ones are those for which the angular momentum of the electron about the proton is nh/2 where n is an integer 1,2,3 etc. and $h = 6.62 \times 10^{-34}$J·s is Planck's constant.

(3) While in one of these allowed orbits, the constantly accelerating electron in its circular orbit does not radiate energy, which would cause it to undergo a 'death spiral' into the nucleus. The energy characteristic of the allowed orbits, E_n, is constant in time.

(4) Radiation is emitted by the atom only when the electron makes a transition from one allowed orbit, E_n, to another E_n'. The frequency of the emitted radiation is:

$$hf = E_n - E_n'$$

With these postulates, the radii of the allowed orbit, r_n and the allowed energy levels, E_n, are given by

$$r_n = n^2 r_o$$

$$E_n = -(13.6\ \text{eV})/n^2$$

where $r_o = 0.53 \times 10^{-10}$ m. Application of the above formulas gives the positions of the spectral lines of hydrogen satisfactorily and also indicates many of the gross features of atomic spectra in general. Beyond this however, the Bohr model must be replaced by a model obtained from quantum mechanics as it suffers many shortcomings of its own.

Just as shining light on a metallic surface (in vacuum) could eject electrons, bombarding a surface with electrons can produce electromagnetic radiation. In particular, very short wavelength, high frequency radiation called x-rays can be produced in this manner. Interaction of such x-rays with atoms can lead to ejection of a tightly bound inner core electron. The filling of this vacancy by an outer electron leads to a characteristic line spectrum (that can be used to identify elements) like those observed for simple atoms such as H or He.

The wavelength shift in scattered x-radiation, called the Compton effect, was satisfactorily explained by using the particle-like nature of radiation (the photon picture) and applying the conservation of linear momentum and kinetic energy, (i.e. assume the collision is elastic) to the 'collision' of an electron, at rest initially, with a photon. Since the photon momentum, p, is equal to h/λ, the momentum given to the electron reduces that of the photon — increasing its wavelength to the value λ'. The change in wavelength is also a function of the angle ϕ between the incident x-ray and the emerging one. Explicitly we have:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi)$$

with m the electron mass. Applications of this formula are made in Examples 5 and 6.

EXAMPLES AND SOLUTIONS

Example 1

The work functions of several metals are listed below. Which metals yield photoelectrons when bombarded with light of wavelength 500 nm? For those surfaces where photoemission occurs with the above light source, calculate the stopping potential in volts.

Solution:

Metal	ϕ (in eV)
W	4.5
Ag	4.8
Cs	1.8
Cs on W	1.36

Since only one light source is involved, it is best to calculate the energy of a photon in eV for this source.

$$E = hf = h\frac{c}{\lambda} \qquad \text{so} \qquad E = \frac{(6.62 \times 10^{-34}\ \text{J}\cdot\text{s})(3 \times 10^{8}\text{m}\cdot\text{s}^{-1})}{5 \times 10^{-7}\ \text{m}}$$

$$= 3.97 \times 10^{-19}\ \text{J} = 2.48\ \text{eV}.$$

Thus the visible light source will not produce electrons if it shines on W or Ag. The stopping potential for Cs is 2.48 eV - 1.8 eV or 0.68 V. For Cs on W, the stopping potential is 1.12 Volts.

Example 2

For the metals tungsten (W) and silver (Ag) in Example 1, calculate the threshold wavelength which would just start producing photoelectrons.

Solution:

The kinetic energy of the photoelectrons is given by

$$hf = \frac{hc}{\lambda} = \frac{1}{2}mv^2 + \phi$$

When the wavelength is the threshold wavelength, the photoelectrons have <u>zero</u> kinetic energy. Thus we have

$$\frac{hc}{\lambda_t} = \phi$$

Solving for λ_t results in

$$\lambda_t = \frac{hc}{\phi}$$

The work functions for W and Ag are 4.5 eV and 4.8 eV respectively. For W this yields

$$\lambda_t = \frac{(6.626 \times 10^{-34}\ \mathrm{J \cdot s})(3 \times 10^{8}\ \mathrm{m \cdot s^{-1}})}{(1.6 \times 10^{-19}\ \mathrm{C})(4.5\ \mathrm{eV})} = 276\ \mathrm{nm}.$$

For Ag, using the previous value and the ratio of the work functions

$$\lambda_t = \left(\frac{4.5}{4.8}\right) 276\ \mathrm{nm} = 259\ \mathrm{nm}.$$

These wavelengths not in the 'visible' region of the spectrum.

Example 3

The stopping potential for photoelectrons ejected from a surface by 375 nm photons is 1.870 volts. Calculate the stopping potential if 600 nm photons are used.

Solution:

The stopping potential is related to the wavelength and the work function by

$$eV_o = \frac{hc}{\lambda} - \phi$$

For a second wavelength λ' the stopping potential would be V_o' where

$$eV_o' = \frac{hc}{\lambda'} - \phi$$

Subtracting these two equations eliminates ϕ

$$eV_o - eV_o' = \frac{hc}{\lambda} - \frac{hc}{\lambda'}$$

Solving for the unknown stopping potential results in

$$V_o' = V_o - \frac{hc}{e} \left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$= 1.870 \text{ V} - \frac{(6.626 \times 10^{-34} \text{ J}\cdot\text{s})(3 \times 10^{8} \text{ m}\cdot\text{s}^{-1})}{(1.6 \times 10^{-19} \text{ C})} \left(\frac{10^{9}}{375 \text{ m}} - \frac{10^{9}}{600 \text{ m}}\right)$$

$$= 1.870 \text{ V} - 1.242 \text{ V}$$

$$= 0.628 \text{ V}.$$

Example 4

An atom has in addition to the ground state energy E_o (taken to be zero) levels $E_1 = 10.20$ eV, $E_2 = 12.09$ eV, and $E_3 = 12.75$ eV. If the atom is excited from its ground state to the state with energy 12.75 eV, calculate the wavelengths of the lines that might exist in the spectrum of this atom.

Solution:

The <u>possible</u> frequencies satisfy the equation

$$hf = E_n - E_m > 0.$$

For transitions out of the state E_3, the atom could go to level E_2, E_1, or E_o, giving frequencies

$$hf_1 = 12.75 \text{ eV} - 12.09 \text{ eV} = \frac{hc}{\lambda_1}$$

$$\lambda_1 = \frac{6.262 \times 10^{-34} \text{ J}\cdot\text{s})(3 \times 10^{8} \text{ m}\cdot\text{s}^{-1})}{(0.64 \text{ eV})(1.6 \times 10^{-19} \text{ C})}$$

$$= 1.88 \times 10^{-6} \text{ m}$$

$$hf_2 = 12.75 \text{ eV} - 10.20 \text{ eV} = \frac{hc}{\lambda_2}$$

$$\lambda_2 = 4.87 \times 10^{-7} \text{ m}.$$

$$hf_3 = 12.75 \text{ eV} - 0 = \frac{hc}{\lambda_3}$$

$$\lambda_3 = 9.73 \times 10^{-8} \text{ m.}$$

Furthermore, from state E_2 the atom could go to states E_1 and E_0:

$$hf_4 = 12.09 \text{ eV} - 10.20 \text{ eV} = \frac{hc}{\lambda_4}$$

$$\lambda_4 = 6.57 \times 10^{-7} \text{ m}$$

and

$$hf_5 = 12.09 \text{ eV} - 0 = \frac{hc}{\lambda_5}$$

$$\lambda_5 = 1.02 \times 10^{-7} \text{ m}$$

Finally from state E_1 the atom could make a transition to the ground state:

$$hf_6 = 10.20 \text{ eV} - 0 = \frac{hc}{\lambda_6}$$

$$\lambda_6 = 1.22 \times 10^{-7} \text{ m.}$$

Example 5

An x-ray of energy 50 KeV strikes an electron initially at rest. The x-ray is scattered through an angle of 90°. Calculate (a) the change in wavelength of the x-ray; (b) the energy of the x-ray after scattering; and (c) the velocity of the electron after scattering.

Solution:

Use the expression from the text relating the original wavelength λ to the wavelength after scattering λ' and the angle of scattering:

$$\lambda' - \lambda = \frac{h}{mc} (1 - \cos \phi)$$

For 90°, the wavelength change becomes

$$\lambda' - \lambda = \frac{h}{mc}$$

$$= \frac{6.626 \times 10^{-34} \text{ J}\cdot\text{s}}{(9.109 \times 10^{-31} \text{ kg})(3 \times 10^{8} \text{ m}\cdot\text{s}^{-1})}$$

$$= 2.42 \times 10^{-12} \text{ m.}$$

The original wavelength can be calculated from the energy

$$hf = (5 \times 10^{4} \text{ V})(1.6 \times 10^{-19} \text{ C}) = \frac{hc}{\lambda}$$

$$\lambda = \frac{3 \times 10^{8} \text{ m}\cdot\text{s}^{-1}}{1.21 \times 10^{19} \text{ s}^{-1}} = 2.48 \times 10^{-11} \text{ m}$$

Solving for λ' results in

$$\lambda' = 2.48 \times 10^{-11} \text{ m} + 2.42 \times 10^{-12} \text{ m}$$

$$= 2.73 \times 10^{-11} \text{ m}$$

(b) The new frequency f' is calculated from

$$f' = \frac{c}{\lambda'} = 1.10 \times 10^{19} \text{ Hz}$$

The x-ray energy after scattering is

$$hf' = (6.626 \times 10^{-34} \text{ J}\cdot\text{s})(1.10 \times 10.^{19} \text{ s}^{-1})$$

$$= 7.29 \times 10^{-15} \text{ J}$$

$$= 45.6 \text{ KeV}$$

(c) The loss in x-ray energy appears as kinetic energy for the electron. Since the kinetic energy increase is much smaller than the electron rest energy, we can use the classical expression for the kinetic energy

$$\frac{1}{2}\, mv^2 = 50 \text{ KeV} - 45.6 \text{ KeV} = 4.44 \text{ KeV}$$

To find the velocity, we can multiply and divide by c^2 to obtain

$$mc^2\left(\frac{v}{c}\right)^2 = 2(4.44 \text{ KeV})$$

Using $mc^2 = 0.511$ MeV $= 511$ KeV we find

$$\left(\frac{v}{c}\right)^2 = \frac{2(4.44 \text{ KeV})}{511 \text{ KeV}} = 1.74 \times 10^{-2}$$

$$v = (0.132)c = 3.96 \times 10^{7} \text{ m}\cdot\text{s}^{-1}$$

Example 6

The explanation of the Compton effect for x-rays just makes use of general properties of photons and conservation laws. Why is this effect easily observable for x-rays but not for visible light?

Solution:

The effect would be observable for visible light but it is a small effect for light whereas it is quite dramatic for x-rays. The Compton formula is:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi)$$

We can rewrite this in terms of the frequencies f and f' as:

$$\frac{1}{f'} - \frac{1}{f} = \frac{h}{mc^2}(1 - \cos\phi)$$

or

$$\frac{f - f'}{f'} = \frac{hf}{mc^2}(1 - \cos\phi)$$

The angular factor $1 - \cos\phi$ can be at most 2, corresponding to the photon being 'back-scattered'. The photon energy is hf and the rest energy of the electron is mc^2 (about 0.5 MeV). For 550 nm light, the energy of the photon is about 2.26eV. Thus $\Delta f = f - f'$ is:

$$\frac{\Delta f}{f'} \lesssim 10^{-5}$$

For x-rays, the frequencies are between 500 and 500,000 times larger than this 'center frequency' for visible light, so the effect is much larger.

QUIZ

1. Light of wavelength 500 nm strikes a metallic surface with a work function of 1.29 eV. Calculate the stopping potential for this wavelength and this surface.

Answer: V_0 = 1.19 volts

2. Given that the wavelength λ' after Compton scattering through an angle ϕ of an x-ray by an electron initially at rest is

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi)$$

(a) Calculate the wavelength after scattering of a 100 KeV x-ray for $\phi = 180°$.
(b) Calculate the momentum given to the electron.
(c) Calculate the kinetic energy of the electron after scattering.

Answer: (a) $\lambda' = 1.73 \times 10^{-11}$ m,
(b) 9.17×10^{-23} kg·m·s^{-1},
(c) non-relativistic result = 2.88×10^{4}eV, relativistic result = 2.82×10^{4} eV.

3. Light of wavelength $\lambda = 247.8$ nm strikes a pure cesium surface. The maximum kinetic energy of the ejected electrons is 3.20 eV. Calculate the work function of cesium.

Answer: The work function is 1.8 eV.

4. The electron in a hydrogen atom drops from an energy level where n = 4 to a level where n = 2. Calculate the energy and momentum of the emitted radiation.

Answer: The energy is 2.55 eV and the momentum is 1.36×10^{-27} kg·m·s^{-1}.

42
QUANTUM MECHANICS

OBJECTIVES

The objectives of this chapter are to:

Calculate the energy levels of the Bohr atom.

Calculate the wavelength associated with a material particle.

Apply the uncertainty relationships to a variety of problems.

Relate the magnetic moment of the electron to its spin angular momentum.

REVIEW

Although Rutherford had suggested a model of the atom that had the positive charge and most of the mass concentrated at the center of the atom with electrons orbiting the nucleus -- much like our solar system -- the deficiencies of this model prevented it from being accepted. The main problem stemmed from the prediction of Maxwell's equations that an accelerating charge must radiate energy at the frequency of the circular revolution. The orbit would collapse due to the loss in energy and the frequency spectrum from an atom would be continuous rather than a sharp line spectrum.

Bohr's postulates are used to obtain the energy levels of a one electron atom. The critical postulate is that the angular momentum must be an integral multiple of $(h/2\pi)$. Incorporation of that one idea into an otherwise classical framework produces a model of the atom with quantized negative energy levels (the zero of potential energy was chosen to coincide with infinite separation between the two charges) which forms the basis of understanding for the observed sharp line spectrum of atoms. Example 1 repeats this calculation for a bound system of an electron and position that forms an atom called positronium. In Example 2, the angular momentum quantization rule is applied to more familiar sized objects and it is shown there that it makes no difference to these ordinary motions.

The deBroglie hypothesis suggests that the allowed orbits should be those containing an integral number of electron wavelengths and gives identically the same results as the original Bohr model. However this was an important step forward in understanding fundamental processes. The association of a wavelength, λ, with a material particle (like an electron), where $\lambda = h/p$, was confirmed in the electron diffraction experiments of Davisson and Germer. The ordinary optical microscope has been duplicated using electrons rather than photons (light) leading to a standard research tool — the electron microscope. In Example 3 these ideas are applied to a particle more massive than the electron, the neutron. Example 4 deals with electron diffraction.

One of the important features of the new theory called 'Quantum Mechanics' is that the deterministic features of Newtonian mechanics are replaced by a statistical interpretation and there exist fundamental uncertainties in basic physical quantities like momentum, position, and time. The Heisenberg uncertainty statements used here are:

$$\Delta p \Delta x = h$$

$$\Delta E \Delta t = h/2\pi$$

These equations are usually stated as inequalities (rather than equalities) and interpreted in the following way: the product of the uncertainty in the x component of the linear momentum, Δp_x, and the uncertainty in the x coordinate, Δx, must be greater than or equal to Planck's constant. This implies that a successively more accurate measurement of a particle's position (reduced Δx) introduces larger and larger uncertainty into the particle's linear momentum. These ideas are extended in Examples 5 and 6.

The uncertainty relationship connecting energy and time is illustrated in the text only to estimate the width, in energy, of a particular energy level if the average lifetime of that level is known. This is illustrated in Example 7.

In Newtonian mechanics, the state of a particle at any time could be specified by giving its velocity and its position at some initial time E. Newton's second law told us how this classical state changes in time. In quantum mechanics, all the information we can have about a particle is contained in its wave function. The absolute square (in many cases the wavefunction is a complex number) of the wavefunction evaluated at a particular point in space gives only the relative probability that the particle will be found at that point. The Schrodinger equation enables you to calculate the wavefunction for a given problem and also enables you to calculate the time dependence of this function.

The quantization of the angular momentum is a prediction of the Schrodinger equation. The condition for quantizing the angular momentum that comes from quantum mechanics permits an 'angular momentum' to have values that are either integral or half-integral multiples of $h/2\pi$. The usual kind of angular momentum, in the Bohr atom for instance, is called orbital angular momentum and has values that are only integral multiples of h. Spin is a property of all the known particles with the electron, proton and neutron (for instance) having a spin of 1/2 while photons, deuterons and mesons have spins that are integral multiples of h. The magnetic moment associated with electron spin is relatively large and leads to nearly all of the observed magnetic properties of bulk matter. Application of a magnetic field to a hydrogen atom causes one of the two spin 1/2 states to lie lower in energy than the other (because of the contribution to the energy of the term $-\mu\cdot B$) but in the absence of an external field these two levels have the same energy.

EXAMPLES AND SOLUTIONS

Example 1

The particle called the positron has the same mass as the electron but opposite electrical charge. A bound state of an electron and a positron is an atom called positronium. Treating this as a Bohr atom, calculate its energy levels.

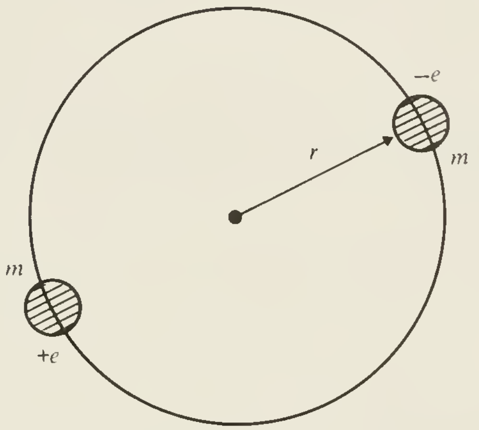

Figure 42-1

Solution:

As a reminder, the electron energy levels in the Bohr atom for Z = 1 are

$$E_n = -\frac{1}{2}\frac{me^4}{(4\pi\varepsilon_o)^2\hbar^2}\frac{1}{n^2} = -\frac{Rch}{n^2}$$

where the product Rch, R being the Rydberg constant, has the numerical value 13.6 eV. Referring to Fig. 42-1, we see that the potential energy of this bound system (zero at $r = \infty$) is:

$$E_p = -\frac{1}{4\pi\varepsilon_o}\left(\frac{e^2}{2r}\right)$$

where 2r is the separation. The kinetic energy is $E_k = (1/2)mv^2 + (1/2)mv^2$ so we have

$$E = E_k + E_p = mv^2 - \frac{1}{4\pi\varepsilon_o}\left(\frac{e^2}{2r}\right)$$

The total angular momentum is the quantized entity:

$$mvr + mvr = n\hbar = 2mvr.$$

The force on the electron that causes it to stay on the circle and provides the centripetal acceleration is the Coulomb force:

$$\frac{1}{4\pi\varepsilon_o}\ \frac{e^2}{(2r)^2} = \frac{mv^2}{r}$$

Using this last equation along with the energy equation, we find

$$E = -\frac{e^2}{4\pi\varepsilon_o}\left(\frac{1}{4r}\right)$$

By squaring the angular momentum equation and comparing with the above expression for v^2, we obtain:

$$r = \frac{n^2\hbar^2 4\pi\varepsilon_o}{me^2}$$

The energy is then:

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_o)^2\hbar^2}\ \frac{1}{n^2} \cdot \frac{1}{2} = -\frac{Rch}{2}\ \frac{1}{n^2}$$

Thus the 'effective' Rydberg constant for positronium is half as large as that for hydrogen resulting in a binding energy for positronium of 6.8 eV.

Example 2

Suppose an uncharged golf ball ($m = 0.1$ kg) teed from a hill went into orbit just at the (uncharged) earth's surface. If its angular momentum is quantized as in the Bohr atom, what is the associated quantum number and what is the spacing between this allowed orbit and the next allowed orbit?

Solution:

Since the objects are uncharged, we will use the gravitational force between them as the dominant force. Then if the mass of the ball is m and M is the earth's mass,

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

then $v^2 = GM/r = g\ R_E^2/r$ since $gR_E^2 = GM$.

Quantizing the angular momentum results in

$$mvr = n\hbar$$

so that $v^2 = (n\hbar/mr)^2$. Equating the two equivalent expressions for v^2 and then solving for the allowed values of r, r_n, we have:

$$r_n = \frac{n^2\hbar^2}{g\ m^2 R_E^2}$$

To find the value of the principle quantum number, n, set $r_n = R_E$ and solve for n,

$$n = \left(\frac{g\ m^2 R_E^3}{\hbar^2} \right)^{1/2}$$

Numerically we have

$$g = 9.8\ \text{m}\cdot\text{s}^{-2}, \qquad m = 0.1\ \text{kg},$$

$$R_E = 6.4 \times 10^6\ \text{m and} \quad \hbar = 1.05 \times 10^{-34}\ \text{J}\cdot\text{s};$$

Thus

$$n \simeq 5 \times 10^{43}\ !$$

Since n is so large, the energy levels and allowed values of r are essentially continuous so to find the next value of r_n, say r_{n+1}, we write:

$$r_{n+1} = r_n + \left(\frac{dr}{dn} \right)\Delta n$$

Here $\Delta n = 1$ and the difference between r_{n+1} and r_n is the very small number,

$$r_{n+1} - r_n = \frac{2n\ \hbar^2}{g\ m^2 R_E^2} = \frac{2\hbar}{m[gR_E]^{1/2}}$$

Notice that:

$$\frac{r_{n+1} - r_n}{R_E} = \frac{2}{n} = 4 \times 10^{-44}$$

Since n is enormous, this fractional difference is totally negligible. It's no surprise that angular momentum quantization was not observed in terrestrial motions.

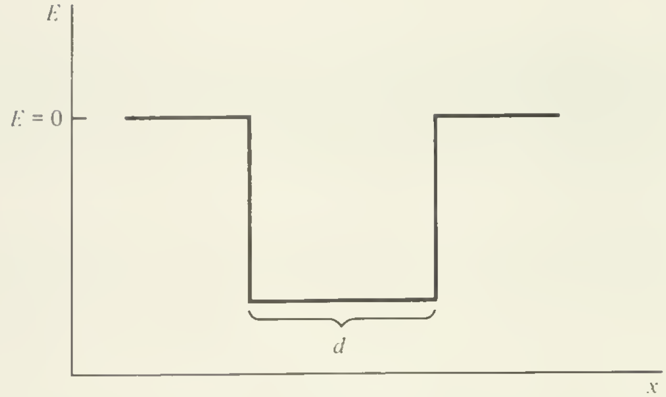

Figure 42-2

Solution:

To confine the particle to this well in one dimension requires that we know its position on the x axis with an uncertainty Δx equal to d. The uncertainty introduced into the x momentum is then:

$$\Delta p_x = \frac{h}{\Delta x} = \frac{h}{d}$$

This causes an uncertainty in the kinetic energy, ΔE_k, equal to:

$$\Delta E_k = \frac{(\Delta p_x)^2}{2m} = \frac{h^2}{2md^2}$$

The total energy, E, of the particle must be less than zero for confinement (otherwise the particle could be anywhere on the x axis).

This condition can be stated as:

$$E = \Delta E_k - V_o < 0$$

since the potential energy is equal to $-V_o$ for this problem. Solving this equation:

$$\frac{h^2}{2md^2} < V_o$$

or

$$V_o d^2 > \frac{h^2}{2m}$$

Thus in one dimension as soon as a region of confinement (d) is specified, a potential energy depth V_o can be calculated that will produce at least one bound state.

Example 6

What value of V_0 would be needed to produce a bound state in a nucleus of width $d = 3 \times 10^{-15}$ m for a proton or neutron (assume their mass to be the same)? Would this value of V_0 confine electrons inside the same well?

Solution:

We will use the inequality from the last example,

$$V_0 d^2 > \frac{h^2}{2m}$$

For d we take the value given above and for m use 1.67×10^{-27} kg (for a proton or neutron). Then

$$V_0 \geq 1.45 \times 10^{-11} \text{ J} = 90 \text{ MeV}$$

Typical nuclear potential energies are about a factor of two lower than the above estimate but the 'confinement' length increases slowly with nuclear mass number.

Since the electron mass is about 1836 times smaller than a proton or neutron mass, the value of V_0 needed to confine an electron in the nucleus would be increased by this factor of nearly two thousand. It is safe to conclude that no electrons are confined within the nucleus.

Example 7

The full width at half maximum intensity for a spectral line characteristic of a pH_2 molecule in an excited rotational energy level is 6×10^9 Hz. What estimate can be made of the lifetime of the molecule in this unstable state?

Solution:

To make this estimate, use the uncertainty relationship

$$(\Delta E)(\Delta t) \geq h/2\pi$$

The value of ΔE to be used is obtained from the frequency spread of the line Δf. Thus

$$\Delta E = h\Delta f$$

Substituting this into the uncertainty relationship yields

$$h(\Delta f)(\Delta t) \geq \frac{h}{2\pi}$$

or

$$\Delta t \geq \frac{1}{2\pi(\Delta f)} = \frac{1\ \text{s}}{2\pi(6 \times 10^9)}$$

$$\Delta t \geq 2.65 \times 10^{-11}\ \text{s}.$$

QUIZ

1. Suppose the charge on the nucleus in the Bohr atom is + Ze rather than + e, where Z is called the atomic number. Calculate the new energy levels E_n of the one electron Bohr atom.

Answer:

$$E_n = -\frac{Z^2}{n^2}(13.6\ \text{eV})$$

2. A particle is confined by a 'square well' potential of depth $-V_o$ and width d = 2×10^{-15} m.
   (a) How large must V_o be in order to confine a proton ?
   (b) How large must V_o be in order to confine an electron ?

Answer: (a) 204 Mev
(b) 375×10^9 eV = 375 GeV.

3. A beam of electrons with kinetic energy of 100 eV strikes a pair of long narrow slits. The slits are separated by 10^{-6} m. Find the spacing between successive maxima in the interference pattern obtained on a screen 2 m from the slit system.

Answer: y = 0.245 mm

4. Assume that initially an electron is localized in a one dimensional region, $\Delta x_o = 3 \times 10^{-10}$ m. (a) Calculate the spread in linear momentum for this electron. (b) Calculate the corresponding spread in velocity. (c) From part (b), calculate the spread in position after 0.2 s.

Answer: (a) $\Delta p = 2.2 \times 10^{-24}\ \text{kg}\cdot\text{m}\cdot\text{s}^{-1}$
(b) $\Delta v = 2.42 \times 10^6\ \text{m}\cdot\text{s}^{-1}$
(c) $\Delta x = (\Delta v)t = 483$ km.

43

ATOMS, MOLECULES, AND SOLIDS

OBJECTIVES

In this chapter your objectives are to:

Apply the Pauli exclusion principle to determine the quantum numbers for atoms in their ground state.

Calculate the rotational energy levels of diatomic molecules and the frequencies emitted in transitions between rotational and vibrational levels.

Compare the various types of solid order.

Relate the electrical conductivity of a metal, insulator, or semi-conductor to the atomic structure of its constituents.

REVIEW

This chapter is mainly descriptive. It uses the idea of discrete energy states in the one-electron atom obtained from solutions of the Schrodinger equation to indicate how more complicated atoms are formed. In this connection, it is important to note that the Bohr model would give the correct energy levels for any one-electron atom independent of the size of the nuclear charge. The nuclear charge results from the protons in the nucleus and the atomic number, Z, counts the number of protons (each of charge + e). Thus to obtain the correct expression for the energy, one only needs to replace the factor e^2 by Ze^2 in the energy (or e^4 by Z^2e^4 etc.)

As more electrons are added to a complicated atom, they have a large effect on the atom's energy levels. The simplest scheme of incorporating the effect of the additional electrons on the energy consists of treating them like a spherically symmetric electron cloud surrounding the nucleus of charge + Ze so that the positive charge is partially neutralized. When this 'central field' calculation is performed, the energy levels of the various atomic electrons depend not only on n, the principle quantum number, but on l, the orbital angular momentum quantum number. If external electric and magnetic fields are present, the energies also depend on the orientation of both the orbital angular momentum and a new quantity, the intrinsic magnetic moment of the electron, thought to arise from the electron's spinning motion. This is known as electron 'spin'.

Several seemingly ad hoc rules are needed to explain atomic structure. First we need the Pauli exclusion principle which stated for atoms is 'no two electrons can occupy the same quantum state'. Secondly we need to note that the specification of a 'quantum state' actually depends on the problem under consideration but for an atom, four quantum numbers suffice to specify the state; n (the principle quantum number), l (the orbital angular momentum quantum number), m (the projection of the orbital angular momentum quantum number on the z axis), and m_s (the projection of the spin angular momentum on the z axis). Thirdly, the quantum numbers n, l, m, and m_s have only certain ranges of values permitted. As previously stated, n is an integer 1, 2, 3 etc., and whatever value n takes, l must be an integer less than n. For a given value of l, m can take on all values between −l and +l so there are 2l + 1 possible values of m. For m_s the situation is simple; it can only be +1/2 or −1/2 and all it basically does to our discussion is to double the number of allowed states that would have been predicted for a given set of quantum numbers n, l, m, without any other effect. The quantum number m_s is very important for understanding the details of atomic structure and the magnetic properties of atoms, but we will not be concerned with either of these in this chapter. Finally, the atoms are formed by filling the electron levels with lowest energy first. These rules are illustrated in Tables 43-1 and 43-2. See Examples 1 and 2.

The formation of molecules from atoms can be qualitatively understood in terms of the picture of energy levels in atoms given here. The stability of closed shell configurations (since such a shell is of the inert gas form) makes it plausible that atoms with one electron outside such a shell would like to donate this electron to an atom lacking one electron to fill a shell. Such a charge exchange leads to the heteropolar bond -- an extremely favorable arrangement energetically. The homopolar bond requires the sharing of electrons to create the bound state of the molecule. The Pauli exclusion principle leads to very symmetric electron cloud distributions in homopolar bonded molecules.

The spectrum of most molecules is confined to the region of the electromagnetic spectrum called the infrared. Interpretation of these spectra in terms of the spectra expected from a rigid rotating molecule or a molecule with vibrational modes like a simple harmonic oscillator (or a combination of the two) can provide valuable insight into the detailed structure of the molecule. See Examples 3 and 4.

All of the naturally occurring elements except helium form solids at some temperatures without any externally applied pressure. In a crystalline solid, the molecules are organized into spatially repetitive patterns that are called lattices. The actual form of the lattice can be determined by either x-ray diffraction or neutron diffraction techniques as the spacing between planes in the lattice is comparable to the wavelengths of the projectiles used. Information about the collective state of molecules in a solid lattice can be obtained from measurements of the specific heat capacity, the thermal conductivity, the electrical conductivity, etc.

Based on the magnitude of the electrical conductivity, materials were historically classified as metals or insulators. Obviously semiconductors, as the name suggests, fall somewhere in between. Pure or intrinsic semiconductors have the same number of holes as electrons but in doped (impure) semiconductors the number of majority carriers per unit volume is much larger than the corresponding number of minority carriers. However the product of these two carrier densities in the doped semiconductor is equal to the corresponding product in the intrinsic semiconductor. The large disparity in carrier densities makes it relatively simple to construct a diode -- which is really just a one-way valve for electrical conduction. See Example 5.

EXAMPLES AND SOLUTIONS

Example 1

What physical considerations can be used to decide how the energy of an atomic state with given n depends of the orbital angular quantum number L?

Solution:

The energy levels of a purely rotating system are given by the expression:

$$E = \left(\frac{h}{2\pi} \right)^2 \frac{L(L + 1)}{2I} \geq 0$$

where I is the moment of inertia. If the moment of inertia is independent of L, then higher values of L lead to higher contributions to the energy. This positive contribution to the energy would indicate that for fixed n, the lower values of L would correspond to the lower energy states. Thus s orbitals would fill up before p orbitals, which would fill up before d orbitals, etc.

The above 'correction term', while a guide to the insight, cannot be taken too seriously. First of all, a portion of this energy has already been included in the Bohr model. Secondly, when evaluated numerically for an electron in the first Bohr orbit, it gives a very large positive energy (of the order of tens of electron volts).

Example 2

What simple physical considerations can be used to decide how the energy of an atomic orbital with given n and l depends on the projection of l, m, and the spin projection, m_s, if there are no external fields applied?

Solution:

There appear to be none and for simple (low Z) atoms these levels all have about the same energy. In more complicated atoms, relativistic considerations (well beyond our scope) do make distinctions in these energy levels as do externally applied electric and magnetic fields.

Example 3

The homonuclear molecule D_2 is made up of two deuterons (a proton plus a neutron) and two electrons (as in H_2). The spacing between the nuclei in the molecule is approximately 7.5×10^{-11} m. Assuming the masses of the electrons are negligible compared to the nuclear masses, calculate the moment of inertia of this molecule about the center of mass and the rotational energy in electron volts of the ground state and the first two excited states.

Solution:

The moment of inertia is

$$I = \sum m_i r_i^2 .$$

If we denote the mass of the proton by by m_H, the mass of the deuteron is about $2m_H$. If D is the separation, then each value of r is D/2 so that:

$$I = 2m_H\left(\frac{D}{2}\right)^2 + 2m_H\left(\frac{D}{2}\right)^2 = m_H D^2$$

Numerically since $m_H = 1.67 \times 10^{-27}$ kg and $D = 7.5 \times 10^{-11}$ m, we have,

$$I = 9.39 \times 10^{-48}\ \mathrm{kg \cdot m^2}$$

The energy levels are given by:

$$E = \left(\frac{h}{2\pi}\right)^2 \frac{L(L+1)}{2I} = (5.92 \times 10^{-22} J)\ L(L+1)$$

$$= (3.70 \times 10^{-3}\ \mathrm{eV})\ L(L+1)$$

The ground state (state with lowest energy is obviously the state with $L = 0$ so $E_o = 0$. The first excited state has $L = 1$ so $E_1 = 7.40 \times 10^{-3}$ eV. The second excited state has $L = 2$ so $E_2 = 2.22 \times 10^{-2}$ eV.

Example 4

(a) If a molecule with moment of inertia I is induced to make a pure rotational transition from a state L to a state L + 1, what frequency of radiation is needed?
(b) If the same molecule makes a transition from the state L to the state L − 1, what frequency of radiation is emitted?

Solution:

(a) Since the energy is

$$E = \left(\frac{h}{2\pi} \right)^2 \frac{L(L + 1)}{2I}$$

in going from L to L + 1, we must supply energy, hf, equal to:

$$hf = \left(\frac{h}{2\pi} \right)^2 \frac{1}{2I} \left[(L + 1)(L + 2) - L(L + 1)\right]$$

$$= \left(\frac{h}{2\pi} \right)^2 \frac{2(L + 1)}{2I}$$

(b) If the molecule drops down from the level characterized by L to the level L - 1, the emitted radiation frequency, f′, is given by:

$$hf' = \left(\frac{h}{2\pi} \right)^2 \frac{1}{2I} \left[L(L + 1) - (L - 1)L\right]$$

$$= \left(\frac{h}{2\pi} \right)^2 \frac{2L}{2I}$$

Note: the spacings between the possible values of f and f′ will be the same.

Example 5

The current-voltage relationship for a p-n junction is given as:

$$I = I_0(e^{eV/kT} - 1)$$

Obtain an expression for the ratio of the 'forward' resistance to the 'backward' resistance. Evaluate this ratio when the voltage is 0.2 volts and T is 293 K.

Solution:

The 'forward' resistance is that resistance obtained when the battery polarity is such that the current is large.

$$R_F = \frac{V}{I} = \frac{V}{I_0(e^{eV/kT} - 1)}$$

The backward resistance, R_B, is obtained when the above voltage polarity is reversed. The current direction is also reversed so:

$$R_B = \frac{-V}{I} = \frac{-V}{I_o(e^{-eV/kT} - 1)}$$

$$\frac{R_F}{R_B} = -\frac{(e^{-eV/kT} - 1)}{(e^{eV/kT} - 1)}$$

Numerically we have

$$\frac{eV}{kT} = \frac{(1.6 \times 10^{-19})(0.2)}{(1.38 \times 10^{-23})(293)} = 7.91$$

Since $e^{7.91} = 2.74 \times 10^3$, the above ratio is approximately the inverse of this number or $R_F/R_B = 3.6 \times 10^{-4}$.

QUIZ

1. Cu^{29} has one electron in a 4s state while Co^{27} has 2 electrons in a 4s state.
(a) How many 3d electrons are there in Cu^{29} and Co^{27}?
(b) If, in forming a solid from the atoms, the 4s electrons are shared with all the other atoms (forming a sea of conduction electrons), which of the two solids should display strong magnetic properties?

Answer: (a) Cu has 10 (complete shell) and Co has 7 (3 less than a complete shell).
(b) Cobalt is strongly magnetic as 5 of the 3d electron spins point in one direction with 2 pointing in the opposite direction.

2. The separation between the nuclei H and D in the molecule HD is 0.074 nm (as in H_2 or D_2). Calculate
(a) the moment of inertia of this molecule about its center of mass and (b) the frequency of radiation emitted in a pure rotational transition from the L = 1 state to L = 0 state.

Answer: (a) $I = 6.06 \times 10^{-48}$ kg·m^2,
(b) $f = 2.77 \times 10^{12}$ Hz.

3. In some semiconductors at very low temperatures, a one-electron atomic spectrum is observed. Assuming the effective charge on the nucleus is +e,
(a) Calculate the radius of the first Bohr orbit in a material where the relative dielectric constant $\varepsilon = 7$. (b) Calculate the frequency of radiation needed to remove an electron from the first Bohr orbit in the above material and drive it to infinity.

Answer: (a) $r(1) = 3.71 \times 10^{-10}$ m, (b) $f = 6.73 \times 10^{13}$ Hz (in the infrared region).

4. An atom of hydrogen with kinetic energy of 10 eV collides with an H_2 molecule

originally in its lowest vibrational level. The fundamental vibrational frequency of an H_2 molecule is 1.29×10^{14} Hz. If the atom transmits all or part of its kinetic energy to the vibrational energy of the molecule, find the possible vibrational states produced in the collision.

Answer: States up to and including $n = 18$ can be produced.

44

NUCLEAR AND HIGH-ENERGY PHYSICS

OBJECTIVES

In this final chapter, aspects of sub-atomic physics are introduced. Your objectives are to:

Recognise the nucleus as a small core (size about 10^{-14} m) inside the larger atom (size about 10^{-10} m).

Apply tables of nuclear data and atomic data (SZY Table 44-1 and 44-2) to practical problems. Distinguish between mass number (A), atomic number (Z) and neutron number (N) and use the notation $_ZX^A$ where X is chemical name for the nucleus.

Calculate mass defect and binding energies for the nuclei in these tables.

Calculate the kinetic energy released in α, β and γ radioactive decays.

Calculate abundances and activities for a radioactive source given the decay constant or the half-life.

Find the kinetic energy which must be supplied to, or is released in, a nuclear reaction.

Distinguish between nuclear fission and nuclear fusion.

Find the radius of rotation and the angular frequency of a charged particle in a cyclotron.

Recognize that the building blocks of the nucleus--protons and neutrons--may themselves be composed of even more elementary constituents.

REVIEW

The nucleus is composed of Z protons and A-Z neutrons where A is the mass number and Z is the atomic number. The charge on the nucleus is + Ze. There are Z electrons surrounding the nucleus in a neutral atom.

Table 44-2 (SZY) lists the atomic mass, including the atomic electrons, in the unit u = 1.660566 x 10^{-27} kg. The nuclear mass is

$$m({}_ZX^A) = \text{atomic mass} - Zm_e$$

where Z is the number of electrons.

The mass of the constituent nucleons (protons and neutrons) is

$$Zm_p + (A-Z)m_n$$

The mass defect Δ is

$$\Delta = Zm_p + (A-Z)m_n - m({}_ZX^A)$$

and is also called the binding energy. The binding energy per nucleon is Δ/A.

Nuclei can decay in various ways, for example by falling apart into:

(1) another nucleus plus an α particle (α-decay)

(2) another nucleus plus an electron plus a neutrino (β-decay)

(3) another nucleus plus a photon (γ-decay)

The decay is possible only if the kinetic energy release is positive. The kinetic energy released is equal to the difference between the mass of the parent nucleus and the sum of the masses of the decay products. The neutrino is massless. The mass of the α particle, a helium nucleus ${}_2He^4$, and other nuclei may be calculated from Table 44-2 (SZY) by subtracting the mass of the atomic electrons. The stability of a nucleus against α and β decay may be checked in this way to see if the released kinetic energy is positive; if it is not, kinetic energy must be supplied to make the reaction go, and the nucleus is stable against the given decay.

The number of unstable nuclei in a sample declines as

$$N = N_o e^{-\lambda t}$$

where λ is the decay constant, related to the 'half-life' by

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}.$$

Since the activity A is the number of decays per unit time - dN/dt, we have also

$$A = A_o e^{-\lambda t}$$

The orbital radius r of a charged particle of velocity v, charge q and mass in a magnetic field of magnitude B is

$$r = \frac{mv}{Bq}$$

Its angular velocity is

$$\omega = \frac{v}{r} = \frac{Bq}{m}$$

and the non-relativistic kinetic energy is

$$\frac{1}{2} mv^2 = \frac{1}{2} m(\frac{Bqr}{m})^2 = \frac{B^2 q^2 r^2}{2m}$$

In nuclear <u>fission</u> a large nucleus is induced, usually by neutron bombardment, to fall apart into two smaller nuclei. If this reaction itself produces neutrons which in form trigger another fission reaction, a chain reaction is said to occur and the process is sometimes self-sustaining. This is the principle behind a nuclear reactor or a bomb. In both cases energy is released.

Another energy release reaction is the fusion reaction in which two light nuclei (e.g. hydrogen or helium) combine to form a heavier one, with the release of energy. An application of this reaction for power generation is being intensely studied but as yet no practical scheme has been found.

At one time protons, neutrons, electrons, and neutrinos were thought to belong to a select group of elementary and indivisible particles. Further study indicated that protons and neutrons are but two members of a large class of particles (hadrons, including many other baryons and mesons) that are themselves composed of more basic strongly interacting particles called quarks. Electrons and neutrinos are still elementary and indivisible as far as can be experimentally determined, but also are but two members of a large class of particles called leptons. Hadrons undergo strong, electromagnetic, and weak interactions. Leptons do not have strong interactions. All particles are subject to gravitation.

PROBLEM-SOLVING STRATEGIES

Many of the problems of this chapter involve energy unit conversion:

$$1 \text{ eV} = 10^6 \text{ MeV} = 1.6 \times 10^{-19} \text{ J}$$

$$1 \text{ u} = \text{atomic mass unit} = 931.5 \text{ MeV}$$

A curie is a measure of the radioactive decay intensity.

$$1 \text{ curie} = 3.7 \times 10^{10} \text{ decays per second}$$

If a radioactive decay or reaction equation is given, Table 44-2 (SZY) may be used to calculate the mass of each side of the equation, term by term. To balance the energy conservation equation, supply kinetic energy to the appropriate side of the equation. Remember to subtract the atomic electron masses from the atomic masses of Table 44-2 (SZY) if the nuclear mass is desired.

Some problems involve photons or γ rays. Recall that the energy of a photon is

$$E = pc = \frac{hc}{\lambda}$$

EXAMPLES AND SOLUTIONS

Example 1

An alpha particle with 5 MeV kinetic energy makes a head on collision with a silver nucleus. Find the distance of closest approach between the alpha particle and nucleus.

Solution:

If the alpha particle and nucleus stay far enough apart so that nuclear forces are negligible, only the force of coulomb repulsion acts. The conserved energy is

$$E = K = U = \frac{1}{2} mv^2 + \frac{Q_1 Q_2}{4\pi\varepsilon_0 r},$$

where r is the separation. The charges are given by the charge on the α particle, $Q_1 = 2e$ and the charge on the silver nucleus, $Q_2 = 47e$.

When the distant α particle begins its motion at $r = \infty$ the initial energy is

$$E_1 = \frac{1}{2} mV^2 = 5 \text{ MeV} = 5 \times 10^6 \text{ eV}$$

$$= 5 \times 10^6 (1.6 \times 10^{-19} \text{J}) = 8 \times 10^{-13} \text{ J}$$

If the final energy is taken to be the point of closest approach, when the velocity is zero, we have

$$E_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r} = \frac{(2e)47e}{4\pi\varepsilon_0 r}$$

$$= \frac{94e^2}{4\pi\varepsilon_0 r}$$

Since the energy is conserved

$$E_1 = E_2$$

$$8 \times 10^{-13} \text{ J} = \frac{94\ e^2}{4\pi\varepsilon_0 r}$$

$$r = \frac{94\ e^2}{4\pi\varepsilon_0 (8 \times 10^{-13} \text{ J})}$$

$$= \frac{94(1.6 \times 10^{-19} \text{ C})^2}{4\pi(8.9 \times 10^{-12} \text{ C}^2\cdot\text{N}^{-1}\cdot\text{m}^{-2})(8 \times 10^{-13} \text{ J})}$$

$$= 2.7 \times 10^{-14} \text{ m}$$

This is close to the radius of the nucleus, where nuclear (non-coulomb) forces begin to act.

Example 2

Find the mass defect, the total binding energy, and the binding energy per nucleon for N^{14}.

Solution:

Referring to Table 44-2 in SZY the atomic mass of N^{14} is 14.00307 u. The mass of the N^{14} nucleus is obtained by subtracting from this the mass of the seven electrons ($Z = 7$) in the atom

$$m_{N^{14}} = 14.00307\ u - 7(0.000549\ u)$$

$$= 13.99923\ u$$

The mass of the seven protons ($Z = 7$) and seven neutrons ($A-Z = 14 - 7 = 7$) is

$$7\ m_n + 7\ m_p = 7(1.008665 + 1.007276)\ u$$

$$= 14.11159\ u$$

The mass defect is thus

$$7\ m_n + 7\ m_p - m_{N^{14}} = (14.11159 - 13.99923)\ u = 0.124\ u$$

Since 1 u = 931.5 MeV the mass defect or total binding energy is

$$0.124(931.5\ \text{MeV}) = 104.7\ \text{MeV}$$

The binding energy per nucleon is

$$\frac{104.7\ \text{MeV}}{14} = 7.5\ \text{MeV per nucleon.}$$

Example 3

(a) Find the mass of the tritium nucleus $_1H^3$ if its atomic mass is 3.01647 u.
(b) Find the mass of the helium nucleus $_2He^3$ plus an electron at rest.
(c) If a tritium nucleus decays into a helium nucleus plus an electron plus a neutrino, what energy is released as kinetic energy?

Solution:

(a) The mass of the $_1H^3$ nucleus is the atomic mass from Table 44-2 (SZY) less the mass of a single electron,

$$m(_1H^3) = 3.01647\ u - 0.000549\ u$$

$$= 3.01592\ u$$

(b) By Table 44-2 (SZY), atomic helium has a mass of 3.01603 u. By subtracting the mass of the two atomic electrons we find the mass of the nucleus,

$$m(_2He^3) = 3.01603\ u - 2(0.000549\ u)$$

$$= 3.014932 \text{ u}$$

The mass of this nucleus plus the mass of a single electron at rest is

$$m({}_2He^3) + m_e = 3.014932 \text{ u} + 0.000549 \text{ u} = 3.015481 \text{ u}$$

(c) In the decay

$${}_1H^3 = {}_2He^3 + \text{electron} + \text{neutrino}$$

The kinetic energy of the electron and the neutrino is the left over mass or 'mass defect',

$$K(\text{electron} + \text{neutrino}) = m({}_1H^3) - m({}_2He^3) - m_e$$

$$= 3.01592 \text{ u} - 3.015481 \text{ u}$$

$$= 4.4 \times 10^{-4} \text{ u} = 0.41 \text{ MeV}$$

Example 4

The half-life of C^{14} is 5568 years. If C^{14} dating was attempted for a piece of wood believed to be 2000 years old, what is the abundance of C^{14} in the wood compared to wood of the same kind which has been freshly cut from a tree?

Solution:

The number of C^{14} nuclei, N, decreases with time according to

$$N = N_o \exp(-\lambda t)$$

The half-life $T_{1/2}$ is related to λ by

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{5568 \text{ yr}} = 1.24 \times 10^{-4} \text{ yr}^{-1}$$

If N_o is the number of C^{14} (for equal mass) in the new wood, the old wood has the fractional abundance

$$\frac{N}{N_o} = \exp(-\lambda t) = \exp[-(1.24 \times 10^{-4} \text{ yr}^{-1})(2000 \text{ yr})]$$

$$= 0.78$$

Example 5

Co^{60} decays to Co^{59} with a half-life of 5.3 years.

(a) What is the activity, in curies, of a source containing 0.015 g of Co^{60}?

(b) What is the activity of the source 2 years later?

Solution:

(a) The number of Co^{60} atoms initially present is

$$N_o = \frac{0.015 \text{ g}}{60 \text{ g mol}^{-1}} 6.02 \times 10^{23} \text{ mol}^{-1}$$

$$= 1.5 \times 10^{20}$$

The activity is the number of decays per unit time. At the initial time we have

$$\left|\frac{dN}{dt}\right| = \left|\frac{d}{dt}(N_o e^{-\lambda t})\right| = \lambda N_o e^{-\lambda t} = \lambda N_o$$

$$= \frac{0.693}{T_{1/2}} N_o = \frac{0.6936}{5.3 \text{ yr}}(1.5 \times 10^{20}) = 1.96 \times 10^{19} \text{yr}^{-1}$$

$$= \frac{1.96 \times 10^{19}}{(365)(24)(3600) \text{ s}} = 6.2 \times 10^{11} \text{ s}^{-1}$$

Since a curie is 3.7×10^{10} decays s^{-1}, the activity in this unit is

$$\lambda N_o = \frac{6.2 \times 10^{11} \text{ s}^{-1}}{3.7 \times 10^{11} \text{ s}^{-1}} = 16.9 \text{ curie}$$

(b) In two years the activity will be

$$\left|\frac{dN}{dt}\right| = (\lambda N_o)e^{-\lambda t} = (16.9 \text{ curie})e^{-(0.63 \cdot 2 \text{ yr}/5.3 \text{ yr})}$$

$$= 13 \text{ curie}$$

Example 6

Protons in a cyclotron spiral out to a radius of 15 cm. The magnetic field has a magnitude of 1.25 T. (a) Find the frequency of the alternating voltage used to accelerate the protons in the gap. (b) Find the energy of the protons, in MeV.

Solution:

(a) The cyclotron frequency, or angular frequency of rotation of the protons

$$\omega = \frac{Be}{m} = \frac{(1.25\ T)(1.6\ x\ 10^{-19})}{1.67\ x\ 10^{-27}\ kg}$$

$$= 1.2\ x\ 10^{8}\ s^{-1}.$$

The frequency of the alternating voltage is

$$f = \frac{\omega}{2\pi} = 1.91\ x\ 10^{7}\ Hz$$

(b) The kinetic energy of the protons is

$$\frac{1}{2}\ mv^{2} = \frac{1}{2}\ m(\omega r)^{2}$$

$$= \frac{1}{2}\ (1.6\ x\ 10^{-27}\ kg)(1.2\ x\ 10^{8}\ s^{-1}\cdot 0.15\ cm)^{2}$$

$$= \frac{2.6\ x\ 10^{-13}\ J}{1.6\ x\ 10^{-19}\ J\cdot(eV)^{-1}} = 1.6\ MeV$$

QUIZ

1. Suppose that the proton decayed into a positron (particle with mass of the electron but opposite charge) and a photon. Find the kinetic energy released in MeV.

Answer: 938 MeV

2. For how many half-lives would one have to wait before the activity of a radioactive source declines to 1/100 of its original value?

Answer: 3.3

3. In a proposed accelerator (the Superconducting Super Collider, or SSC) protons will achieve momenta of 20 TeV/c = 20×10^{12} eV/c, where c is the velocity of light. If the bending magnets that keep the protons moving in a circle have a magnetic induction of 6.6 Tesla, estimate the radius of curvature of the protons in the magnetic field.

Answer: 10 km

4. A proton (charge +e) is known to be composed of two 'up' quarks and one 'down' quark. A neutron (charge zero) is known to be composed of one 'up' and two 'down' quarks. From this information alone, calculate the charge on the up and down quarks.

Answer: $q_u = 2e/3$, $q_d = -e/s$

Also available to accompany *University Physics, Seventh Edition* by Sears, Zemansky and Young:

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Prepared by A. Lewis Ford, *Texas A&M University*

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