DAY-2 — INTRODUCTION TO MATRICES

→ Dr Raja Solanki

MATRICES

Rectangular array of numbers, symbols or expressions arranged in rows & columns.

- Matrix algebra has atleast two advantages:
  - Reduce complicated sys of eqn to simple expression
  - Adaptable to systematic method of mathematical treatment & well suited to computers

$$\begin{array}{c} R \longleftarrow \\ R \longrightarrow \end{array} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\uparrow \quad \uparrow$$
$$c \quad c$$

Property — specified no. of row & colm
→ row x colm → order of matrix, size of matrix

$(3\times3) \rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $(2\times4) \rightarrow \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix}$

$(1\times2) \rightarrow [1 \ -1]$

Bold capital letter — within matrix element → lowercase

$$A_{m\times n} \quad {}_mA^n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{ij} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3j} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & a_{mn} \end{bmatrix}$$

i goes from 1 to m
j goes from 1 to n

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$i < j$

$i = j$

$i > j$

TYPES OF MATRICES :

1. column matrix (vector) — no. of Col. = 1 $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$

2. Row matrix or (vector) — no. of row = 1 $[1 \ 1 \ 6]$ $[0 \ 3 \ 5 \ 2]$
$[a_{11} \ a_{12} \ a_{13} \ \cdots \ a_{1n}]$

3. Rectangular matrix — $n(\text{Row}) \neq n(\text{Col})$, $m \neq n$. $\begin{bmatrix} 1 & 1 \\ -3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

4. Square matrix $(m=n)$ $A_{m\times m} = A_m$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 6 \\ 6 & 6 & 1 \end{bmatrix}$

- main/principle dig of sq. matrix → a_{ij} where $i=j$

5. Digonal matrix — sq matrix — all ele except diag. ele → 0

dag. ele may or may not be 0.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad \begin{bmatrix} a_{ij} = 0, & i \neq j \\ a_{ij} \neq 0, & i = j \end{bmatrix}$$

6. Unit or identity matrix (I) — diag. matrix — ele of main dig = 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} = 0 & , i \neq j \\ a_{ij} = 1 & , i = j \end{bmatrix}$$

7. Null matrix - O — all ele = 0

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad a_{ij} = 0 \ \forall \ i, j$$

8. Triangular matrix — sq. matrix whose above or below the main diag are all '0'

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

lower Δ — $a_{ij} = 0 \ \forall \ i < j$

upper Δ — $a_{ij} = 0 \ \forall \ i > j$

9. Scalar matrix — dig matrix — main diag ele are same
└ single number or constant

$$\begin{bmatrix} a_{ij} = 0 \ \forall \ i \neq j \\ a_{ij} = a \ \forall \ i = j \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Matrix-Operations

Equality of Matrix – equal only when all corresponding ele are equal
∴ size & order – equal as well.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

$A = B$

Some properties of equality:
- If $A = B \Rightarrow B = A \quad \forall A \& B$
- $A = B$ & $B = C \rightarrow A = C$

$\boxed{\text{If } A = B \Rightarrow a_{ij} = b_{ij}}$

Add^n & subs^n of matrices

$C = A \pm B$ — A, B same order

$\boxed{C_{ij} = a_{ij} \pm b_{ij}}$

*) $A+B = B+A$ (commutative)
*) $A+(B+C) = (A+B)+C = A+B+C$ (Associative)
*) $A+0 = 0+A = A$
*) $A+-A = 0$ ($-A \Rightarrow$ composed of $-a_{ij}$ as elements)

Scalar Multiplication of Matrices

$kA = A \cdot k$

properties
- $k(A+B) = kA + kB$
- $(k+g)A = kA + gA$
- $k(AB) = (kA)B = A(kB)$
- $k(gA) = (kg)A$

*** Multiplication of Matrices**
- pdt of two matrices is a matrix

$A_{m \times n} \times B_{n \times p} = AB_{m \times p}$

$n(\text{col of } A) = n(\text{Row} - B)$

$\boxed{IA = A}$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_2b_3 + a_3b_5 & a_1b_2 + a_2b_4 + a_3b_6 \\ a_4b_1 + a_5b_3 + a_6b_5 & a_4b_2 + a_5b_4 + a_6b_6 \end{bmatrix}$$

[Row by Column multiplication]

Assuming that matrices A, B & C are conformable for the operation indicated,
- ① $AI = IA = A$
- ② $A(BC) = (AB)C = ABC$ – (associative law)
- ③ $A(B+C) = AB + AC$ – (first distributive law)
- ④ $(A+B)C = AC + BC$ – (2nd distributive law)

CAUTION!
- $AB \neq BA$ generally
  - may not exist (be conformable)
- $AB = 0$, neither A nor B necessarily $= 0$
- $AB = AC$, $B \neq C$ (not necessarily)

TRANSPOSE OF A MATRIX

$$A_{2\times3} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

$$A^T_{3\times2} = {}_3A^{T\,2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$\boxed{a_{ij} = a^T_{ji} \quad \forall i,j}$$

Properties

- 1> $(A+B)^T = A^T + B^T$
- 2> $(AB)^T = B^T A^T$
- 3> $(kA)^T = kA^T$
- 4> $(A^T)^T = A$

INVERSE OF A MATRIX

$K=7$, inverse of $k = k^{-1} = \frac{1}{k} = \frac{1}{7}$

Division of matrices is not defined $\because$ they may be $AB = AC$ while $B \neq C$

$\therefore$ Hence matrix inversion is used.

$\therefore$ The Inverse of a square matrix, A, if it exists is the unique matrix. A^{-1}, where

$$AA^{-1} = A^{-1}A = I$$

e.g. $A_{2\times2} = {}_2A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

Becoz, $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Properties of the inverse

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$

~~A sq. matrix~~

only nonsingular matrix $\rightarrow$ inverse

Det $\neq 0$

if Det = 0 – singular $\rightarrow$ inverse ✗

Sq. matrix have inverses except when det = 0

Determinant of Matrix

$$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix} = |5 - 12| = -7$$

$A = [A]_{1\times1}$

$|A| = A$

$A_{1\times1} = [9]$

$|A| = 9$

MINORS

└ A → (n×n) matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_a = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - hf$$

COFACTORS

└ $C_{ij} = (-1)^{i+j} M_{ij}$

$$\boxed{\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \end{aligned}}$$

DETERMINANT

└ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11}(a_{33} \cdot a_{22} - a_{32} a_{23}) - a_{12}(a_{33} \cdot a_{21} - a_{31} \cdot a_{23}) + a_{13}(a_{32} \cdot a_{21} - a_{31} \cdot a_{22})$$

ADJOINT MATRICES

└ Transpose of cofactor Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Cofactor matrix i.e of $A = C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$\text{Adjoint} = C^T = \begin{matrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{matrix}$$

$$\boxed{\text{Adj } A = C^T}$$

* $A(\text{adj } A) = (\text{adj } A) A = |A| I$

$A \cdot A^{-1} = I = A^{-1} A$ & $A(\text{adj} A) = (\text{adj } A) A = |A| I$

$$\boxed{A^{-1} = \frac{\text{adj } A}{|A|}}$$