Paper Mill Production Problem

Problem Formulation as a Linear Programming Model

- **Introduction to the problem**:

  We need to identify at what minimum cost can we meet the orders of the newspapers companies. The orders (demands) are as follows:

Number of Rolls	Paper Roll width
650 rolls	4ft
2430 rolls	9ft
1470 rolls	19ft

  The paper mill produces 14ft and 21ft rolls and the cost of production is as follows:

  Cost per roll of 14ft wide roll = £635
  Cost per roll of 21ft wide roll = £1130

- **Cutting Scheme**: Since the paper mill produces 14ft and 21ft rolls, we also need to identify the various possible cutting schemes to meet the demand. The possible cutting schemes are as follows:

  (A) For 14 feet wide roll, we can obtain:

Rolls Obtained	Wastage
9ft (1 roll) + 4ft (1 roll)	1ft wide roll
4ft (3 rolls)	2ft wide roll

  (B) For 21 feet wide roll, we can obtain:

Rolls Obtained	Wastage
19ft (1 roll)	2ft wide roll
9ft (1 roll) + 4ft (3 rolls)	No wastage
9ft (2 rolls)	3ft wide roll
4ft (5 rolls)	1ft wide roll

- **Decision Variables**: As evident from above, we have 6 possible ways of cutting the 14ft and 21ft wide rolls. We need to calculate the quantities of 14ft and 21ft rolls required for each cutting scheme. Thus, we will have 6 decision variables which are the unknown entities in the problem.

The 6 decision variables are $P_{14}^{9,4}$, $P_{14}^{4,4,4}$, P_{21}^{19}, $P_{21}^{9,4,4,4}$, $P_{21}^{9,9}$, $P_{21}^{4,4,4,4,4}$
where,

- $P_{14}^{9,4}$: Paper roll of 14ft wide cut into one 9ft wide roll and one 4ft wide roll
- $P_{14}^{4,4,4}$: Paper roll of 14ft wide cut into three 4ft wide rolls
- P_{21}^{19}: Paper roll of 21ft wide cut into one 19ft wide roll
- $P_{21}^{9,4,4,4}$: Paper roll of 21ft wide cut into one 9ft wide roll and three 4ft wide roll
- $P_{21}^{9,9}$: Paper roll of 21ft wide cut into two 9ft wide rolls
- $P_{21}^{4,4,4,4,4}$: Paper roll of 21ft wide cut into five 4ft wide rolls

Hence, the total production of each kind of roll is as follows:

$$\begin{aligned}\text{Production of 4ft rolls} &= \sum(\text{No. of 4ft rolls obtained from all cutting schemes}) \\ &= P_{14}^{9,4} + 3P_{14}^{4,4,4} + 3P_{21}^{9,4,4,4} + 5P_{21}^{4,4,4,4,4}\end{aligned}$$

$$\begin{aligned}\text{Production of 9ft rolls} &= \sum(\text{No. of 9ft rolls obtained from all cutting schemes}) \\ &= P_{14}^{9,4} + P_{21}^{9,4,4,4} + 2P_{21}^{9,9}\end{aligned}$$

$$\begin{aligned}\text{Production of 19ft rolls} &= \sum(\text{No. of 19ft rolls obtained from all cutting schemes}) \\ &= P_{21}^{19}\end{aligned}$$

- **Objective Function**:

  Our objective is to minimize the total cost of production. We also need to define the objective function in terms of decision variables. The cost calculation is as follows:

$$\begin{aligned}\text{Total Cost} &= \text{No. of 14ft wide roll}(P_{14}) \times \text{Cost per roll of 14ft wide roll} \\ &\quad + \text{No. of 21ft wide roll}(P_{21}) \times \text{Cost per roll of 21ft wide roll} \\ &= 635P_{14} + 1130P_{21}\end{aligned}$$

  Thus the objective function can be written as: **Minimize (£635 P_{14} + £1130 P_{21})**
  where,

  - P_{21}: Total no. of 21ft wide paper roll required for all possible cutting schemes.
  - P_{14}: Total no. of 14ft wide paper roll required for all possible cutting schemes.

  $P_{21} = P_{21}^{19} + P_{21}^{9,4,4,4} + P_{21}^{9,9} + P_{21}^{4,4,4,4,4}$

  $P_{14} = P_{14}^{9,4} + P_{14}^{4,4,4}$

- **Constraints** :

  There are 3 constraints that deal with meeting specific demand conditions and other constraint restricts the decision variables from taking a negative value. The constraints are defined as follows :

(A) Constraint on production of 4ft wide paper roll to meet the demand of 650:

$$P_{14}^{9,4} + 3P_{14}^{4,4,4} + 3P_{21}^{9,4,4,4} + 5P_{21}^{4,4,4,4,4} \geq 650$$

(B) Constraint on production of 9ft wide paper roll to meet the demand of 2430:

$$P_{14}^{9,4} + P_{21}^{9,4,4,4} + 2P_{21}^{9,9} \geq 2430$$

(C) Constraint on production of 19ft wide paper roll to meet the demand of 1470:

$$P_{21}^{19} \geq 1470$$

(D) Non-negativity constraints i.e. no decision variable can take negative value:

$$P_{14}^{9,4},\ P_{14}^{4,4,4}, P_{21}^{19},\ P_{21}^{9,4,4,4},\ P_{21}^{9,9},\ P_{21}^{4,4,4,4,4} \geq 0$$

Discussion on Optimal Solution

The optimum solution is obtained on solving the LP model using solver in excel. The decision variables values obtained on solving the model are as follows (Refer Appendix A):

$P_{14}^{9,4} = 650,\ P_{14}^{4,4,4} = 0,\ P_{21}^{19} = 1470,\ P_{21}^{9,4,4,4} = 0,\ P_{21}^{9,9} = 890,\ P_{21}^{4,4,4,4,4} = 0$

Hence,

$\mathbf{P_{21}} = P_{21}^{19} + P_{21}^{9,4,4,4} + P_{21}^{9,9} + P_{21}^{4,4,4,4,4} = 1470 + 0 + 890 + 0 = \mathbf{2360}$

$\mathbf{P_{14}} = P_{14}^{9,4} + P_{14}^{4,4,4} = 650 + 0 = \mathbf{650}$

- Thus the **optimal solution** is to produce 650 rolls of 14ft wide paper roll and 2360 rolls of 21ft wide paper roll.
- The cutting scheme can be identified from the values of decision variables. Therefore, the **cutting scheme** to be followed is:

  (A) $P_{14}^{9,4} = 650$

  i.e Each of 650 paper rolls of 14ft wide roll should be cut into one 9ft wide roll and one 4ft wide roll, creating 650 rolls of 9ft wide and 650 rolls of 4ft wide.

  (B) $P_{21}^{19} = 1470$

  i.e Each of 1470 paper rolls of 21ft wide roll should be cut into one 19ft wide roll, creating 1470 rolls of 19ft wide.

  (C) $P_{21}^{9,9} = 890$

  i.e Each of 890 paper rolls of 21ft wide roll should be cut into two 9ft wide rolls, creating 1780 rolls of 9ft wide.

- The **total cost** of all the rolls combined is £3,079,550 which is the minimum cost of production to meet the demands of paper rolls.

$$\begin{aligned}\text{Total optimal cost} &= \text{number of 14ft wide paper rolls} \times \text{Cost per each 14ft roll} \\ &\quad + \text{number of 21ft wide paper rolls} \times \text{Cost per each 21ft roll} \\ &= 650 \times (£635) + 2360 \times (£1130) \\ &= £3,079,550\end{aligned}$$

- Waste Generated equivalent to 1ft wide roll are as follows:

  (A) For $P_{14}^{9,4} = 650$, the waste generated is 650 (=650x1) rolls of 1ft wide
  (B) $P_{21}^{19} = 1470$, the waste generated is 2940 (=1470x2) rolls of 1ft wide
  (C) $P_{21}^{9,9} = 890$, the waste generated is 2670 (=890x3) rolls of 1ft wide

  Hence, **Total Waste** = 650 + 2940 + 2670 = 6260 rolls of 1ft wide.

  The total waste generated is nearly equivalent to 447 (=6260/14) rolls of 14ft wide paper roll or 298 (=6260/21) rolls of 21ft wide paper roll.

Problem Re-formulation as a Goal Programming Model

The problem now focuses on producing cheapest production plan that first minimizes waste and potential overproduction. Hence, we have 2 goals, which are as follows:

(a) Goal 1 : Minimize Waste and Overproduction

(b) Goal 2 : Minimize Cost

Minimizing waste and overproduction is the first priority and minimizing cost is the next priority. Hence, we will use ranked goals approach to solve this Goal Programming Problem, in which the goal with highest priority/rank is solved first and then we solve the goal with next priority/rank. The two Goals are ranked as follows:

1) Rank R_1 : Goal 1

2) Rank R_2 : Goal 2

- **Deviation Variables** :

  We will define 2 deviation variables for each goal. Hence, the 4 deviation variables for the 2 goals are: $d^-_{W/O}$, $d^+_{W/O}$, d^-_{CO}, d^+_{CO}

  where,

  - $d^-_{W/O}$: amount by which total waste and overproduction is underachieved
  - $d^+_{W/O}$: amount by which total waste and overproduction is overachieved
  - d^-_{CO} : amount by which production cost is underachieved
  - d^+_{CO} : amount by which production cost is overachieved

- **Goals** :

  We express the 2 goals mathematically using these deviation variables as explained below:

  (A) **Goal 1 : Minimize Waste and Overproduction**

  I. Waste :

  $$\text{Total Waste} = P_{14}^{9,4} + 2P_{14}^{4,4,4} + 2P_{21}^{19} + 3P_{21}^{9,9} + P_{21}^{4,4,4,4,4}$$

  II. Overproduction :

  $$\text{Overproduced 4ft roll} = P_{14}^{9,4} + 3P_{14}^{4,4,4} + 3P_{21}^{9,4,4,4} + 5P_{21}^{4,4,4,4,4} - 650$$
  $$\text{Overproduced 9ft roll} = P_{14}^{9,4} + P_{21}^{9,4,4,4} + 2P_{21}^{9,9} - 2430$$
  $$\text{Overproduced 19ft roll} = P_{21}^{19} - 1470$$

  $$\begin{aligned}\text{Total overproduction} &= 4 \times (\text{Overproduced 4ft roll}) + 9 \times (\text{Overproduced 9ft roll}) \\ &\quad + 19 \times (\text{Overproduced 19ft roll}) \\ &= 4 \times (P_{14}^{9,4} + 3P_{14}^{4,4,4} + 3P_{21}^{9,4,4,4} + 5P_{21}^{4,4,4,4,4} - 650) \\ &\quad + 9 \times (P_{14}^{9,4} + P_{21}^{9,4,4,4} + 2P_{21}^{9,9} - 2430) + 19 \times (P_{21}^{19} - 1470)\end{aligned}$$

  Hence, Goal 1 is as follows :

  Total waste + Total Overproduction + $d_{W/O}^{-} - d_{W/O}^{+} = 0$

  i.e.,

  $$P_{14}^{9,4} + 2P_{14}^{4,4,4} + 2P_{21}^{19} + 3P_{21}^{9,9} + P_{21}^{4,4,4,4,4} + 4 \text{ x } (P_{14}^{9,4} + 3P_{14}^{4,4,4} + 3P_{21}^{9,4,4,4} + 5P_{21}^{4,4,4,4,4} - 650)$$
  $$+ 9 \text{ x } (P_{14}^{9,4} + P_{21}^{9,4,4,4} + 2P_{21}^{9,9} - 2430) + 19 \text{ x } (P_{21}^{19} - 1470) + d_{W/O}^{-} - d_{W/O}^{+} = 0$$

  (B) **Goal 2 : Minimize cost**

  We equate the cost to the minimum cost of £3,079,550 because we know from part (1) of the assignment that it is the lowest possible cost that can be achieved. Hence, Goal 2 is as follows:
  Total cost + $d_{CO}^{-} - d_{CO}^{+} = 3079550$

  i.e.,

  $$635\ P_{14} + 1130\ P_{21} + d_{\text{CO}}^{-} - d_{\text{CO}}^{+} = 3079550$$

- **Objective Function** :

  $d_{W/O}^{-}$, d_{CO}^{-} can be eliminated from our objective function because we are not concerned about underachievement of any of the goals. Hence, we can formulate our GP model as follows:

  Rank 1 Objective : Min $(-d_{W/O}^{+})$

  Rank 2 Objective : Min $(-d_{\text{CO}}^{+})$

  Minimized Ranked deviations = $R_1(-d_{W/O}^{+}) + R_2(-d_{\text{CO}}^{+})$

- **Decision Variables :**

  The 6 decision variables are same as before which are $P_{14}^{9,4}$, $P_{14}^{4,4,4}$, P_{21}^{19} , $P_{21}^{9,4,4,4}$, $P_{21}^{9,9}$ and $P_{21}^{4,4,4,4,4}$ where,

  - $P_{14}^{9,4}$: Paper roll of 14ft wide cut into one 9ft wide roll and one 4ft wide roll
  - $P_{14}^{4,4,4}$: Paper roll of 14ft wide cut into three 4ft wide rolls
  - P_{21}^{19}: Paper roll of 21ft wide cut into one 19ft wide roll
  - $P_{21}^{9,4,4,4}$: Paper roll of 21ft wide cut into one 9ft wide roll and three 4ft wide roll
  - $P_{21}^{9,9}$: Paper roll of 21ft wide cut into two 9ft wide rolls
  - $P_{21}^{4,4,4,4,4}$: Paper roll of 21ft wide cut into five 4ft wide rolls

- **Constraints:**

  (A) **Goal Constraints** :

  - $P_{14}^{9,4} + 2P_{14}^{4,4,4} + 2P_{21}^{19} + 3P_{21}^{9,9} + P_{21}^{4,4,4,4,4} + 4 \text{ x } (P_{14}^{9,4} + 3P_{14}^{4,4,4} + 3P_{21}^{9,4,4,4} + 5P_{21}^{4,4,4,4,4} - 650)$
    $+ 9 \text{ x } (P_{14}^{9,4} + P_{21}^{9,4,4,4} + 2P_{21}^{9,9} - 2430) + 19 \text{ x } (P_{21}^{19} - 1470) + d_{W/O}^{-} - d_{W/O}^{+} = 0$
  - $635\ P_{14} + 1130\ P_{21} + d_{\text{CO}}^{-} - d_{\text{CO}}^{+} = 3079550$

  (B) **Normal Constraints** :

  - $P_{14}^{9,4} + 3P_{14}^{4,4,4} + 3P_{21}^{9,4,4,4} + 5P_{21}^{4,4,4,4,4} \geq 650$
  - $P_{14}^{9,4} + P_{21}^{9,4,4,4} + 2P_{21}^{9,9} \geq 2430$
  - $P_{21}^{19} \geq 1470$

  (C) **Non-negativity Constraints :**

  - $P_{14}^{9,4}$, $P_{14}^{4,4,4}$, P_{21}^{19} , $P_{21}^{9,4,4,4}$, $P_{21}^{9,9}$, $P_{21}^{4,4,4,4,4}$, $d_{W/O}^{-}, d_{W/O}^{+}$, d_{CO}^{-}, $d_{\text{CO}}^{+} \geq 0$

- **Results** :

  The optimum solution is obtained on solving the GP model using solver in excel (Refer Appendix C).

  - The decision and deviation variables values obtained are:
    - $P_{14}^{9,4} = 650$, $P_{14}^{4,4,4} = 0$, $P_{21}^{19} = 1470$, $P_{21}^{9,4,4,4} = 0$, $P_{21}^{9,9} = 890$, $P_{21}^{4,4,4,4,4} = 0$
    - $d_{W/O}^{-} = 0$, $d_{W/O}^{+} = 6260$, $d_{\text{CO}}^{-} = 0$, $d_{\text{CO}}^{+} = 0$
  - The overproduction values obtained are:
    - Overproduction of 4ft wide roll = 0
    - Overproduction of 9ft wide roll = 0
    - Overproduction of 19ft wide roll = 0

Therefore,

(i) Total wastage obtained is as follows:

$$\begin{aligned}\text{Total wastage} &= \text{Overachievement of Waste and Overproduction}(d^{+}_{W/O}) - \text{Total Overproduction}\\ &= 6260 - 0\\ &= 6260 \text{ rolls of 1ft wide}\end{aligned}$$

Thus, the total waste generated is same as before which is nearly equivalent to 447 (=6260/14) rolls of 14ft wide paper roll or 298 (=6260/21) rolls of 21ft wide paper roll.

(ii) The cutting scheme to be followed is also same as before which can be identified from the values of decision variables and is as follows:

(A) $P^{9,4}_{14} = 650$
i.e Each of 650 paper rolls of 14ft wide roll should be cut into one 9ft wide roll and one 4ft wide roll, creating 650 rolls of 9ft wide and 650 rolls of 4ft wide.

(B) $P^{19}_{21} = 1470$
i.e Each of 1470 paper rolls of 21ft wide roll should be cut into one 19ft wide roll, creating 1470 rolls of 19ft wide.

(C) $P^{9,9}_{21} = 890$
i.e Each of 890 paper rolls of 21ft wide roll should be cut into two 9ft wide rolls, creating 1780 rolls of 9ft wide.

(iii) Cost calculation is as follows:

$$\begin{aligned}\text{Total cost} &= \text{number of 14ft wide paper rolls} \times \text{Cost per each 14ft roll}\\ &\quad + \text{number of 21ft wide paper rolls} \times \text{Cost per each 21ft roll}\\ &= 650 \times (£635) + 2360 \times (£1130)\\ &= £3,079,550\end{aligned}$$

- **Conclusion** :

The optimal solution is same as obtained in before, which is to produce 650 rolls of 14ft wide paper roll and 2360 rolls of 21ft wide paper roll. The total cost of all the rolls combined is £3,079,550 which is the cheapest production plan that first minimizes waste and potential overproduction, to meet the demands of paper rolls. The total waste and overproduction is also same as before, which is equal to 6260 rolls of 1ft wide.

Obtaining same results as before shows that changing the cutting scheme and overproducing the rolls to minimize waste generated from the previous cutting scheme is not a good idea. It is because the waste generated due to overproduction itself is far more than the waste generated due to the wastage pieces left by the cutting scheme.

Appendix

A. Model's Setup in Excel - Linear Programming

	Paper roll 14ft wide cut into 9ft (1no.) and 4ft (1 no.) rolls	Paper roll 14ft wide cut into 4ft (3 no.) rolls	Paper roll 21ft wide cut into 19ft (1 no.) rolls	Paper roll 21ft wide cut into 9ft (1 no.) and 4ft (3 no.) rolls	Paper roll 21ft wide cut into 9ft (2 no.) rolls	Paper roll 21ft wide cut into 4ft (5 no.) rolls			
	$P_{14}^{9,4}$	$P_{14}^{4,4,4}$	P_{21}^{19}	$P_{21}^{9,4,4,4}$	$P_{21}^{9,9}$	$P_{21}^{4,4,4,4,4}$			
No. of Paper Rolls	650	0	1470	0	890	0			
Cost per roll	635	635	1130	1130	1130	1130	3079550	<- Minimize cost (Objective)	
Waste generated per roll (in ft roll)	1	2	2	0	3	1	**6260**	<- Total Waste Generated (ft roll)	
							447	<- Nearly 447 rolls of 14 ft	
							298	<- Nearly 298 rolls of 21 ft	
Constraints									
No. of 4 ft roll	1	3		3		5	650	>=	650
No. of 9 ft roll	1			1	2		2430	>=	2430
No. of 19 ft roll			1				1470	>=	1470
							L.H.S	**Sign**	**R.H.S**

B. Solver Sensitivity Report from Excel

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C4	Paper roll 14 ft wide cut into 9ft (1no.) and 4ft (1 no.) rolls $P_{14}^{9,4}$	650	0	635	118.3333333	70
D4	Paper roll 14 ft wide cut into 4ft (3 no.) rolls $P_{14}^{4,4,4}$	0	425	635	1E+30	425
E4	Paper roll 21 ft wide cut into 19ft (1 no.) rolls P_{21}^{19}	1470	0	1130	1E+30	1130
F4	Paper roll 21ft wide cut into 9ft and 4ft (3 no.) rolls $P_{21}^{9,4,4,4}$	0	355	1130	1E+30	355
G4	Paper roll 21ft wide cut into 9ft (2 no.) rolls $P_{21}^{9,9}$	890	0	1130	140	283.3333333
H4	Paper roll 21 ft wide cut into 4ft (5 no.) rolls $P_{21}^{4,4,4,4,4}$	0	780	1130	1E+30	780

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
I8	No. of 4 ft	650	70	650	1780	650
I9	No. of 9 ft	2430	565	2430	1E+30	1780
I10	No. of 19 ft	1470	1130	1470	1E+30	1470

C. Model's Setup in Excel - Goal Programming

		Paper roll 14ft wide cut into 9ft (1no.) and 4ft (1 no.) rolls	Paper roll 14ft wide cut into 4ft (3 no.) rolls	Paper roll 21ft wide cut into 19ft (1 no.) rolls	Paper roll 21ft wide cut into 9ft (1 no.) and 4ft (3 no.) rolls	Paper roll 21ft wide cut into 9ft (2 no.) rolls	Paper roll 21ft wide cut into 4ft (5 no.) rolls	Amount by which total waste and overproduction is underachieved	Amount by which total waste and overproduction is overachieved	Amount by which production cost is underachieved	Amount by which production cost is overachieved				
		$P_{14}^{9,4}$	$P_{14}^{4,4,4}$	P_{21}^{19}	$P_{21}^{9,4,4,4}$	$P_{21}^{9,9}$	$P_{21}^{4,4,4,4,4}$	$d_{W/O}^{-}$	$d_{W/O}^{+}$	d_{CO}^{-}	d_{CO}^{+}				
	No. of Paper Rolls	650	0	1470	0	890	0	0	6260	8.73115E-11	0	0	<- Objective Function		
Goal Constraint	**Waste generated per roll (in ft roll)**	1	2	2	0	3	1	1	-1			0	=	0	<- Goal 1: Minimize waste and overproduction
Normal Constraint	**No. of 4 ft roll**	1	3		3		5					650	>=	650	
	No. of 9 ft roll	1			1	2						2430	>=	2430	
	No. of 19 ft roll			1								1470	>=	1470	
Goal Constraint	**Cost per roll**	635	635	1130	1130	1130	1130			1	-1	3079550	=	3079550	<- Goal 2: Minimize cost
												L.H.S	**Sign**	**R.H.S**	

Overproduction for 4ft roll	0
Overproduction for 9ft roll	0
Overproduction for 19ft roll	0
Total Overproduction =	0