Q: Mentioned solid body under the state of plane strain in the x-y plane. On a part of the boundary (C_1), the displacement components u and v are specified while on the remaining part of the boundary (C_2), the stress vector components t_x and t_y are specified. There are 9 unknowns as functions of (x,y) coordinates: (i) two displacement components u and v, (ii) three components of infinitesimal strain tensor: ε_{xx}, $\varepsilon_{yy,}$ and $\varepsilon_{xy,}$ and (iii) four stress components: σ_{xx}, σ_{yy}, $\sigma_{xy,}$ and σ_{zz} the 9 equations governing these unknowns are:

Strain-Displacement Relations (3 equations):

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Stress-Strain Relations (4 equations):

$$\sigma_{xx} = (\lambda+2\mu)\, \varepsilon_{xx} + \lambda\varepsilon_{yy};$$

$$\sigma_{yy} = (\lambda+2\mu)\, \varepsilon_{yy} + \lambda\varepsilon_{xx};$$

$$\sigma_{xy} = 2\mu\, \varepsilon_{xy};$$

$$\sigma_{xx} = \lambda\, \varepsilon_{xx} + \lambda\varepsilon_{yy};$$

Equilibrium Equations (2 equations):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0;$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0;$$

Equilibrium Equations in terms of the Displacement Components:

Combining the stress-strain and strain-displacement relations, we get the stress-displacement relations. Substitution of the stress-displacement relations gives the equilibrium equations in terms of the displacement components. Then, u and v are governed by the following boundary value problem consisting of a partial differential equation (PDE) and boundary conditions (BC) on C_1, and C_2:

PDE:

$$(\lambda + \mu) \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu \nabla^2 u + b_x = 0$$

$$(\lambda + \mu) \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu \nabla^2 v + b_y = 0$$

BC:

Essential: $u = u^*$, $v = v^*$ on C_1;

Natural : $t_x = t_x^*$, $t_y = t_y^*$ on C_2;

Where, $\begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$

Given (Q4 in Computer Assignment - 2):

$b_x = 0$

$b_y = 0$

$t_x^* = 3(t_y^*)^2$

A: We can calculate the Variational Functional of the above BVP as follows,

$$\mathbf{I(u,v)} = \int_D \left[\boldsymbol{U} - \left(\boldsymbol{b_x u} + \boldsymbol{b_y v}\right)\right]\boldsymbol{dxdy} - \int_{C_2} (\boldsymbol{t_x^* u} \ + \boldsymbol{t_x^* v})\boldsymbol{ds}$$

$$= \int_D \left[\frac{1}{2}\{\sigma_{xx}\}^{T}\{\varepsilon_{xx}\} - \{b\}^{T}\{u\}\right] dxdy - \int_{C_2}\{t_x^*\}\{u\}ds$$

$$U = \frac{1}{2}(\sigma_{xx}\ \varepsilon_{xx} + \sigma_{yy}\ \varepsilon_{yy} + 2\sigma_{xy}\ \varepsilon_{xy}) = \frac{1}{2}\{\sigma_{xx}\ \}^{T}\{\varepsilon_{xx}\ \}$$

The array form of the functional equation M1 becomes:

$$\{\sigma\} = [C]\{\varepsilon\}$$

Where the **elasticity matrix [C]** is given by

$$[C] = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1}{2}(1-2v) \end{bmatrix}$$

$$[C] = \frac{1}{\lambda + 2\mu} \begin{bmatrix} 4\mu(\lambda+\mu) & 2\lambda\mu & 0 \\ 2\lambda\mu & 4\mu(\lambda+\mu) & 0 \\ 0 & 0 & \mu(\lambda + 2\mu) \end{bmatrix} = \frac{E}{1 - v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-v) \end{bmatrix}$$

Note that $[C]$ is symmetric. Therefore, substitution of the array form of the stress-strain relation (equation M.6) in (M.5) leads to

$$\boldsymbol{I(u,v)} = \int_D \left[\frac{\mathbf{1}}{\mathbf{2}}\{\boldsymbol{\varepsilon}\}^{\boldsymbol{T}}[\boldsymbol{C}]\{\boldsymbol{\varepsilon}\}\right]\boldsymbol{dxdy} - \underbrace{\int_{C_2} \{\boldsymbol{t}^*\}^{\boldsymbol{T}}\{\boldsymbol{u}\}\boldsymbol{ds}.}_{\text{Right Hand side Vector}}$$

This is the array form of the functional. For a typical Lagrangian/serendipity triangular/rectangular element e of nodes p, the approximations for the displacement components u and v :

$$u = \sum_{i=1}^{p} N_i^e u_i^e, \qquad v = \sum_{i=1}^{p} N_i^e v_i^e$$

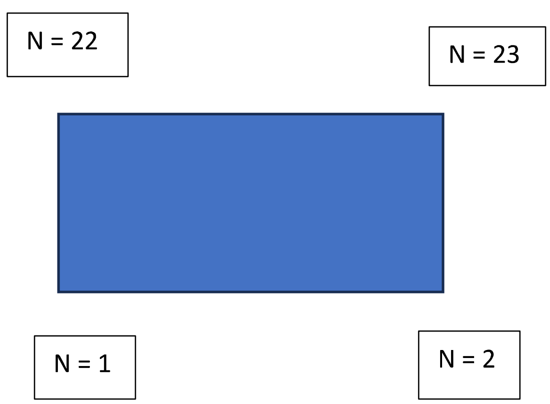
Assembly Relations used:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1^e & 0 & . & . & . & . & N_p^e & 0 \\ 0 & N_1^e & . & . & . & . & 0 & N_p^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ . \\ . \\ . \\ . \\ u_p^e \\ v_p^e \end{Bmatrix}$$

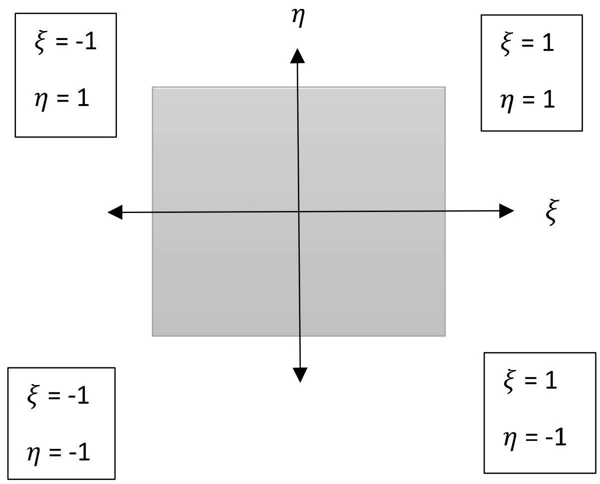
$$\{u\} = [N]^e\{u\}^e$$

Area Element Used:

1. Local element used (Drawing for first element):
   - Here we are taking 20 divisions in X-axis and 10 divisions in Y-axis and the below is the first element and the values in the boxes represent the Node numbers.



2. Element in Natural coordinates (used for calculation):



The stiffness matrix of the plain stress strain problem

$$K^e = \int_D h\, B^T C B\, dxdy$$

Force Vector or Right-hand side vector

$$f^e = \int_D hB^T b dxdy + \int_{C_2} hN^T \hat{t} ds$$

Coordinates Used:

x = $\sum_{i=1}^{4} N_i x_i$

y = $\sum_{i=1}^{4} N_i y_i$

Lagrangian Shape Functions:

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta);$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta);$$

$$N_3(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta);$$

$$N_4(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta);$$

Displacement Interpolation:

u = $\sum_{i=1}^{4} N_i u_i$

v = $\sum_{i=1}^{4} N_i v_i$

Jacobian Matrix used:

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Conversion of x,y into ξ and η

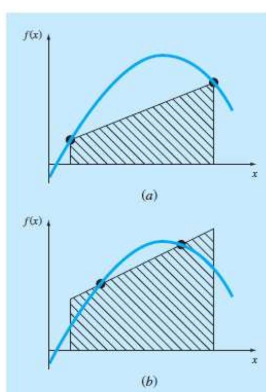
$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

Brief about two point gauss quadrature:

Two-point Gauss quadrature is a numerical integration method that approximates the definite integral of a function over a finite interval using just two sample points. It is a specific instance of Gauss quadrature designed to achieve higher accuracy for quadratic polynomials.

In two-point Gauss quadrature, the sample points and corresponding weights are chosen such that the method exactly integrates polynomials up to degree 3. This means that for any polynomial function of degree up to 3, the two-point Gauss quadrature will provide an exact result. The sample points and weights in two-point Gauss quadrature are typically precomputed and stored in tables. These values are derived using mathematical techniques based on orthogonal polynomials, such as Legendre polynomials, to ensure optimal accuracy.

Due to its simplicity and accuracy for quadratic functions, two-point Gauss quadrature is commonly used in numerical methods and scientific computing applications where integrals need to be computed efficiently and accurately.



(*a*) Graphical depiction of the trapezoidal rule as the area under the straight line joining fixed end points. (*b*) An improved integral estimate obtained by taking the area under the straight line passing through two intermediate points. By positioning these points wisely, the positive and negative errors are balanced, and an improved integral estimate results.

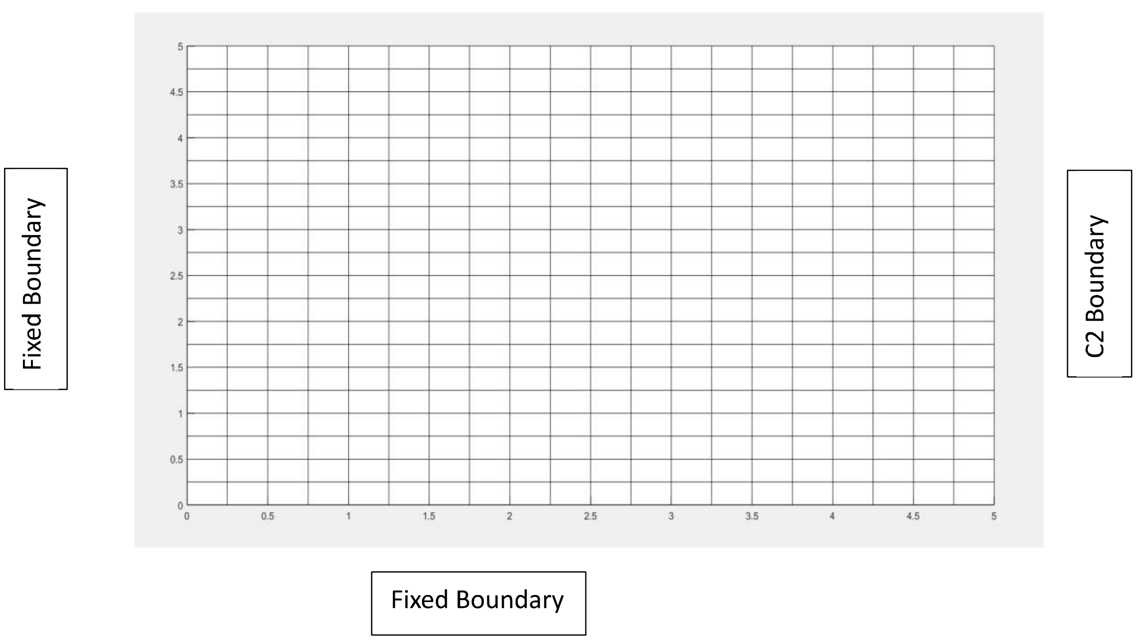
Instructions to run the 2D Assignment Code:

Initially save all subroutines (functions) named below:

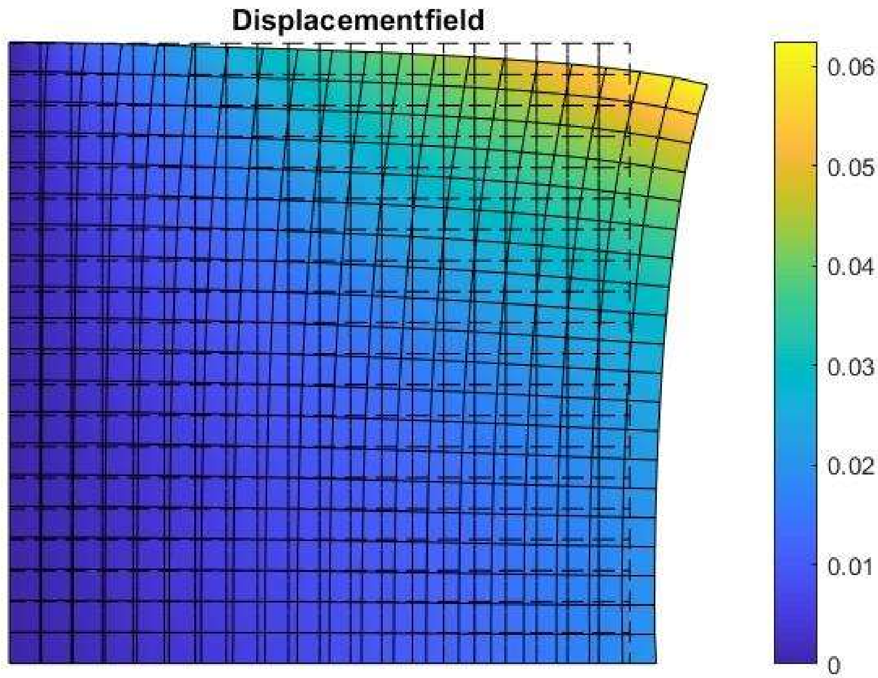
1. main_2d_fem.m
2. formStiffness2D.m
3. drawingField.m
4. drawingMesh.m
5. gaussQuadrature.m
6. Jacobian.m
7. rectangularMesh.m
8. shapeFunctionsQ.m
9. solution.m

Now run the main master code i.e. main_2d_fem.m code to get the result.

Discritization of element:



Result:



X-axis length = 5 units,

Y-axis length = 5 units,

X – axis discretization = 20 elements.,

Y – axis discretization = 20 elements,

t_x = 10000,

E = 200 GPa,

Poisson's ratio = 0.3.