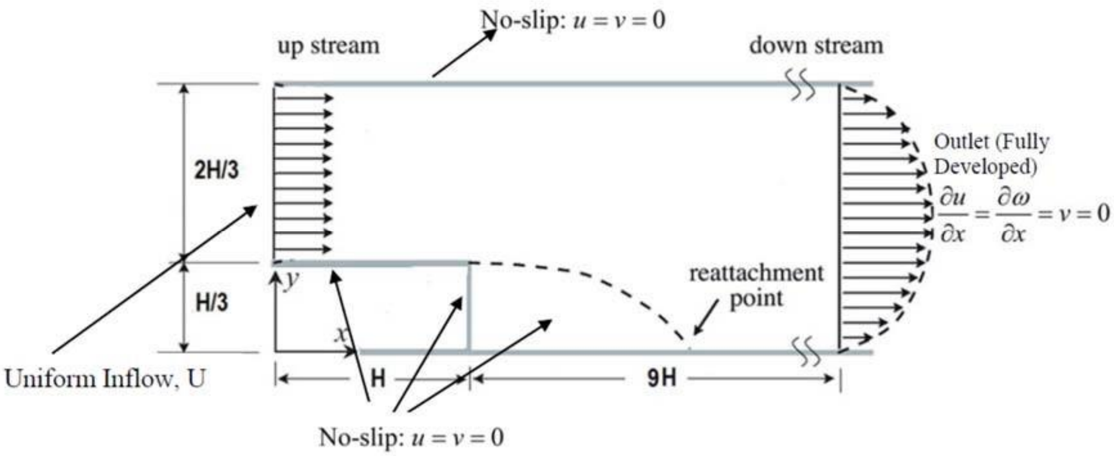
Simulation of 2D Viscous Flow in a Channel Using Stream Function-Vorticity Method.

NAME-SHAHID AFROZ

Roll-231050082



PROBLEM DEFINATION AND BOUNDRY CONDITIONS

The given problem is a backstep channel flow. And the governing equations for the above problem is given as

Primitive form :-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{— (i)}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{— (ii)}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad \text{— (iii)}$$

$\psi - \omega$ form :-

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \qquad \text{— (A)}$$

$$\omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \qquad \text{— (B)}$$

Non- dimentionalising.

$$u^* = \frac{u}{U}, \quad v^* = \frac{v}{U}, \quad x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}$$

$$\psi^* = \frac{\psi}{UL}, \quad \omega^* = \frac{\omega L}{U}$$

Substituting in eqn (A) and eqn (B) we get

$$\frac{\partial^2 \psi^*}{\partial x^{*2}} + \frac{\partial^2 \psi^*}{\partial y^{*2}} = -\omega^* \qquad u^* = \frac{\partial \psi^*}{\partial y^*}, \quad v^* = -\frac{\partial \psi^*}{\partial x^*}$$

$$u^*\frac{\partial \omega^*}{\partial x^*} + v^*\frac{\partial \omega^*}{\partial y^*} = \frac{1}{Re}\left(\frac{\partial^2 \omega^*}{\partial x^{*2}} + \frac{\partial^2 \omega^*}{\partial y^{*2}}\right)$$

For further solving we are going to use these non- dimentionless equation

Discritizaing the above equation we get

$$\rightarrow \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2} = -\omega_{i,j}$$

$$\beta = \frac{\Delta x}{\Delta y}$$

$$\rightarrow \psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j} + \beta^2(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}) = -\omega_{i,j}(\Delta x)^2$$

$$\rightarrow \psi_{i,j} = \frac{\psi_{i+1,j} + \psi_{i-1,j} + \beta^2(\psi_{i,j+1} + \psi_{i,j-1}) + (\Delta x)^2\omega_{i,j}}{2(1+\beta^2)}$$

Vorticity

$$u_{i,j}\frac{\omega_{i+1,j} - \omega_{i-1,j}}{2\Delta x} + v_{i,j}\frac{\omega_{i,j+1} - \omega_{i,j-1}}{2\Delta y} = \frac{1}{Re}\left(\frac{\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}}{(\Delta x)^2}\right) + \frac{1}{Re}\left(\frac{\omega_{i,j+1} - 2\omega_{i,j} + \omega_{i,j-1}}{(\Delta y)^2}\right)$$

$$\beta = \frac{\Delta x}{\Delta y}$$

multiply both side by $(\Delta x)^2$ and Re we get

$$-\left(1 - \frac{u_{i,j}\Delta x\, Re}{2}\right)\omega_{i+1,j} - \left(1 + \frac{u_{i,j}\Delta x\, Re}{2}\right)\omega_{i-1,j} - \left(1 - \frac{v_{i,j}\Delta y\, Re}{2}\right)\times \beta^2\omega_{i,j+1}$$

$$-\left(1 + \frac{v_{i,j}\Delta y\, Re}{2}\right)\beta^2\omega_{i,j-1} = -2(1+\beta^2)\omega_{i,j}$$

$$\omega_{i,j} = \frac{1}{2(1+\beta^2)}\left(\left(1 - \frac{u_{i,j}\Delta x\, Re}{2}\right)\omega_{i+1,j} + \left(1 + \frac{u_{i,j}\Delta x\, Re}{2}\right)\omega_{i-1,j} + \left(1 - \frac{v_{i,j}\Delta y\, Re}{2}\right)\beta^2\omega_{i,j+1} + \left(1 + \frac{v_{i,j}\Delta y\, Re}{2}\right)\beta^2\omega_{i,j-1}\right)$$

Now put $u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}$

$$v_{i,j} = -\frac{(\psi_{i+1,j} - \psi_{i-1,j})}{2\Delta x}$$

Simulation of 2D Viscous Flow in a Channel Using Stream Function-Vorticity Method.

$$\omega_{i,j} = \frac{1}{2(1+\beta^2)}\Big[\left(1 - \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}\frac{\Delta x \, Re}{2}\right)\omega_{i+1,j} +$$

$$\left(1 + \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}\frac{\Delta x \, Re}{2}\right)\omega_{i-1,j} +$$

$$\left(1 + \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}\frac{\Delta y \, Re}{2}\right)\beta^2 \omega_{i,j+1} +$$

$$\left(1 - \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}\frac{\Delta y \, Re}{2}\right)\beta^2 \omega_{i,j-1}\Big]$$

Put $\beta = \frac{\Delta x}{\Delta y}$

$$\omega_{i,j} = \frac{1}{2(1+\beta^2)}\Big[\left\{1 - (\psi_{i,j+1} - \psi_{i,j-1})\frac{\beta \cdot Re}{4}\right\}\omega_{i+1,j}$$

$$+ \left\{1 + (\psi_{i,j+1} - \psi_{i,j-1})\frac{\beta \cdot Re}{4}\right\}\omega_{i-1,j}$$

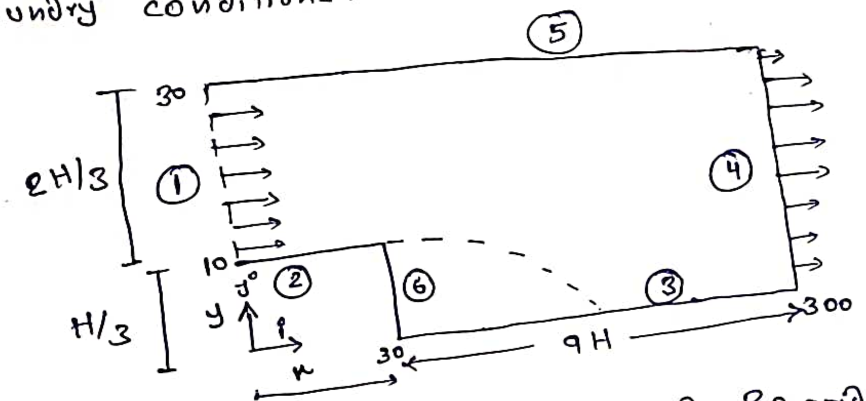
$$+ \left\{1 + (\psi_{i+1,j} - \psi_{i-1,j})\frac{Re}{4 \times \beta}\right\}\omega_{i,j+1}$$

$$+ \left\{1 - (\psi_{i+1,j} - \psi_{i-1,j})\frac{Re}{4 \times \beta}\right\}\beta^2 \omega_{i,j-1}\Big]$$

Boundary conditions for the above stream function and vorticity function for back step flow is given below

Simulation of 2D Viscous Flow in a Channel Using Stream Function-Vorticity Method.

Boundry conditions:-



lets take 300 grid in x-direction and 30 grid in y-direction
then $H = 30$ grid point $H/3 = 10$ grid point

Boundry (2) ($j=10$ $i=20-30$)

$u=0$, $v=0$

$$\frac{\partial\psi}{\partial x} = 0 \Rightarrow \frac{\partial^2\psi}{\partial x^2} = 0 \Rightarrow \omega = -\left(\cancel{\frac{\partial^2\psi}{\partial x^2}}^{\,0} + \frac{\partial^2\psi}{\partial y^2}\right)$$

$$\Downarrow \qquad\qquad \omega = -\frac{\partial^2\psi}{\partial y^2}$$

$\partial\psi = 0$

$\psi = c$

let take constan c = 0

$$\boxed{\psi_{i,10} = 0}$$

$$\omega_{i,10} = -\frac{(\psi_{i,11} - 2\psi_{i,10} + \psi_{i,9})}{(\Delta y)^2}$$

also $u=0$

$$\left.\frac{\partial\psi}{\partial y}\right|_{j=10} = 0 \qquad \psi_{i,9} = \psi_{i,11}$$

$$\boxed{\omega_{i,10} = -\frac{2}{(\Delta y)^2}(\psi_{i,11} - \psi_{i,10})}$$

Boundry (6) ($i=30$ $j = 2-9$)

$u=0$, $v=0$

$$\frac{\partial\psi}{\partial y} = 0 \Rightarrow \psi = c = 0 \Rightarrow \boxed{\psi_{30,j} = 0}$$

$$\omega_{30,j} = -\frac{(\psi_{30,j} - 2\psi_{20,j} + \psi_{29,j})}{\Delta y^2}$$

$$\Downarrow$$

$$\frac{\partial^2\psi}{\partial y^2} = 0 \qquad v=0 \Rightarrow \frac{\partial\psi}{\partial x} = 0 \Rightarrow \psi_{31,j} = \psi_{29,j}$$

$$\omega = -\left(\frac{\partial^2\psi}{\partial x^2} + \cancel{\frac{\partial^2\psi}{\partial y^2}}^{\,0}\right) = -\frac{\partial^2\psi}{\partial x^2}$$

$$\boxed{\omega_{30,j} = -\frac{2}{(\Delta x)^2}(\psi_{31,j} - \psi_{30,j})}$$

Simulation of 2D Viscous Flow in a Channel Using Stream Function-Vorticity Method.

Boundary ③ ($j=1$ $\quad i = 30 - \cancel{300}\ 299$)

$v = 0$

$\frac{\partial \psi}{\partial x} = 0 \rightarrow d\psi = 0 \rightarrow \psi = C = 0 \rightarrow \boxed{\psi_{i,1} = 0}$

$\frac{\partial^2 \psi}{\partial x^2} = 0 \rightarrow \omega = -\left(\cancel{\frac{\partial^2 \psi}{\partial x^2}} + \frac{\partial^2 \psi}{\partial y^2}\right) = -\frac{\partial^2 \psi}{\partial y^2}$

0

$$\omega_{i,1} = -\frac{(\psi_{i,2} - 2\psi_{i,1} + \psi_{i,0})}{\Delta y^2}$$

Now $\rightarrow$ $u = 0$ $\quad \frac{d\psi}{\partial y} = 0$ $\quad \psi_{i,2} = \psi_{i,0}$

$$\boxed{\omega_{i,1} = -\frac{2}{\Delta y^2}(\psi_{i,2} - \psi_{i,1})}$$

Boundary ① ($i=1$ $\quad j = 10 - 30$) (inblow)

$u = U$

$\frac{\partial \psi}{\partial y} = u$

$\int \partial \psi = \int U \partial y$

$\psi_{1,j} = \psi_{1,j-1} + U\Delta y$

Non - diminsionilising

$$\boxed{\psi_{1,j} = \psi_{1,j-1} + \frac{3\Delta y}{2H}}$$

at inlet $\quad \frac{dv}{\partial x} = 0$

$\omega = \cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y} = -\frac{\partial \psi}{\partial y^2}$

$$\boxed{\omega_{1,j} = -\frac{1}{(\Delta y)^2}\left[\psi_{1,j+1} - 2\psi_{1,j} + \psi_{1,j-1}\right]}$$

Boundary ④ ($i = 300$ $\quad j = 1-30$) (outblow)

full developed condition.

$\frac{\partial \omega}{\partial x} = 0 \Rightarrow \boxed{\omega_{300,j} = \omega_{299,j}}$

$\frac{\partial v}{\partial x} = 0 \Rightarrow \cancel{\omega}\ \frac{\partial^2 \psi}{\partial x^2} = 0$

$$\boxed{\psi_{300,j} = 2\psi_{299,j} - \psi_{298,j}}$$ $\rightarrow$ backword difference.

Simulation of 2D Viscous Flow in a Channel Using Stream Function-Vorticity Method.

boundary 5 ($j = 30$ $i = 1 - 299$)

$V = 0$

$$\frac{\partial \psi}{\partial x} = 0 \rightarrow \int_{\psi_1}^{\psi_5} \psi = \int_{0}^{2H/3} u \, dy$$

ψ_5

u

$\psi_1 = 0$

$$\rightarrow \psi_5 = U \frac{2H}{3}$$

Non - dimentionalising we get

$$\rightarrow \psi_5 = \frac{U\,2H/3}{U\,2H/3} = 1 \longrightarrow \boxed{\psi_{i,30} = 1}$$

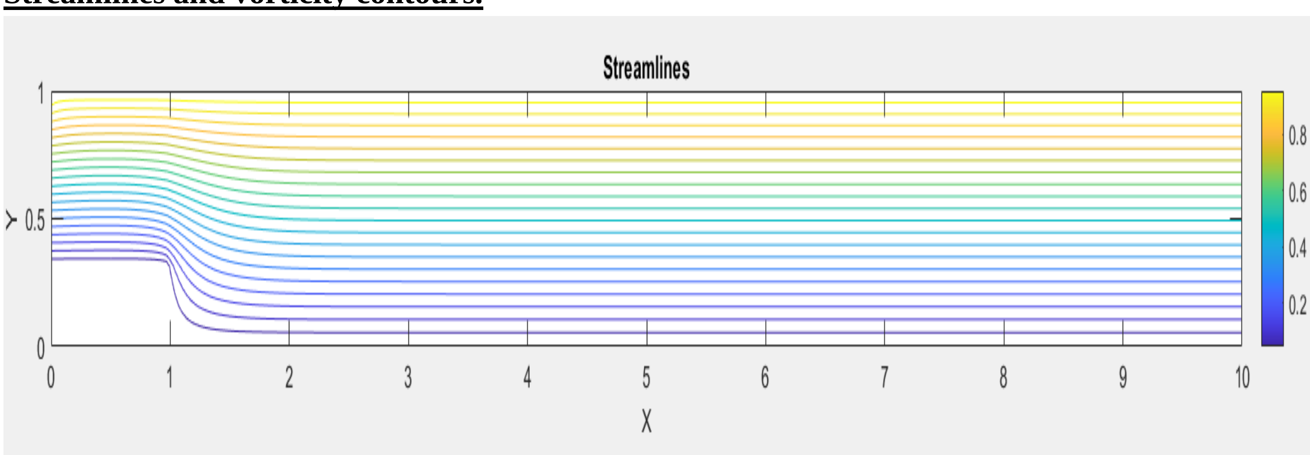
$$\rightarrow \quad \omega = \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{0} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial y^2}$$

$$\omega_{i,30} = -\left[\frac{\psi_{i,31} - 2\psi_{i,30} + \psi_{i,29}}{(\Delta y)^2}\right]$$

$$u = 0; \ \frac{\partial \psi}{\partial y} = 0 \Rightarrow \psi_{i,31} = \psi_{i,29}$$

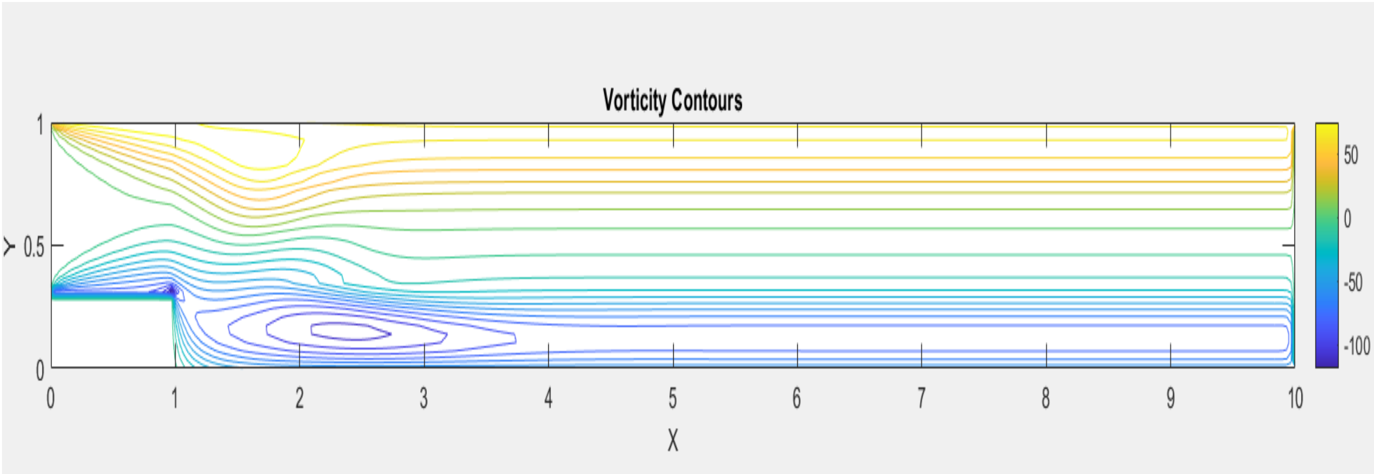
$$\omega_{i,30} = -\frac{2}{(\Delta y)^2}\left[\psi_{i,29} - \psi_{i,30}\right]$$

Streamlines and vorticity contours.



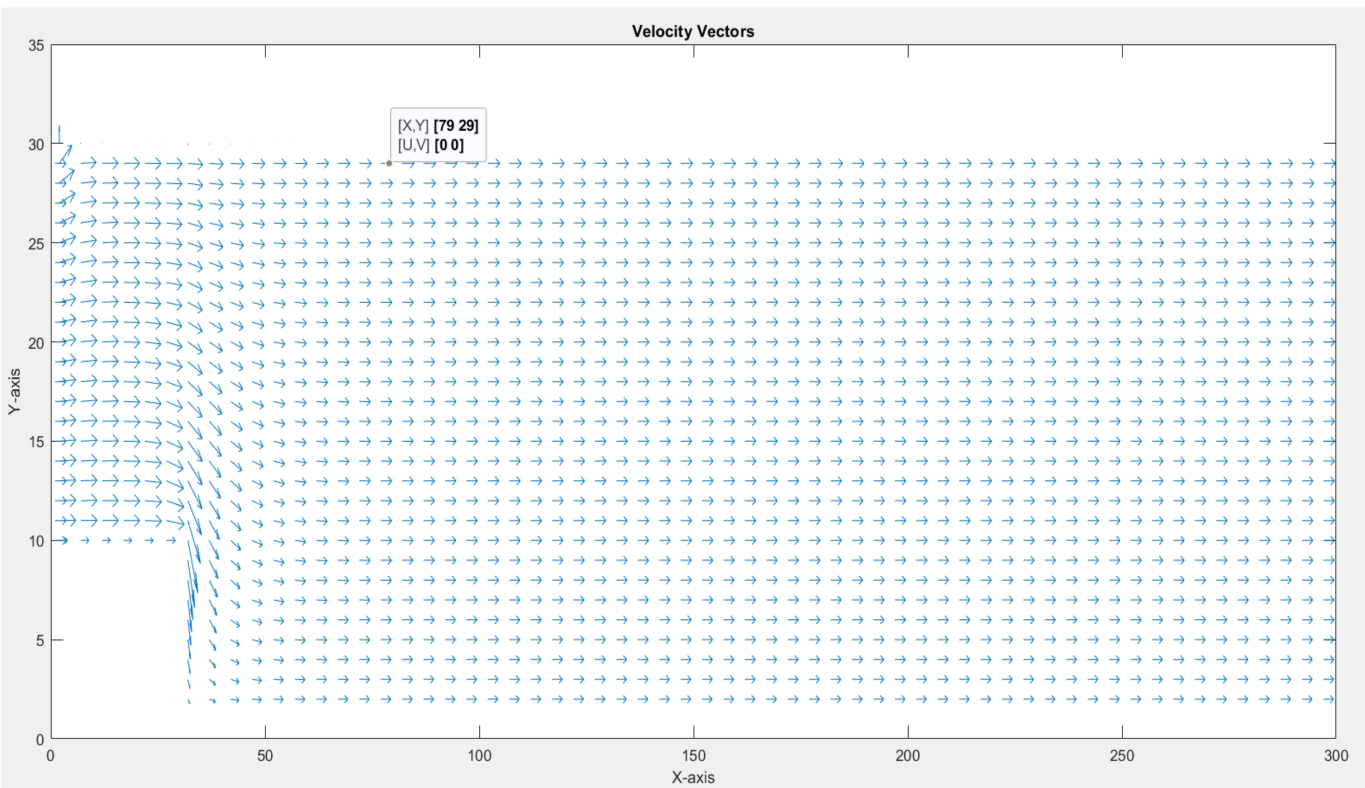
Streamline

Simulation of 2D Viscous Flow in a Channel Using Stream Function-Vorticity Method.



Vorticity contours

Vector plot.



Velocity vector