A New Fast Solution Method For Solving k-Tridiagonal Toeplitz Systems

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Abstract

The objective of this paper is to propose a novel, efficient solution method for the solution of linear systems whose coefficient matrix is of a k-Tridiagonal Toeplitz type. This novel solution method utilizes the Fast Block Diagonalization Method introduced in [1], and subsequently solves each subsystem with the aid of the fast solver (Fast Core Method) in [2].

Keywords k-Tridiagonal Toeplitz Systems · Fast Solver · Fast Block Diagonalization · Tridiagonal Toeplitz Matrix

1 Introduction

Consider the following nonsingular linear system of equation

$$T_n^{(k)} x = b \tag{1}$$

where $T_n^{(k)}$ is an $n \times n$ Toeplitz matrix of the form

$$T_n^{(k)} = \begin{bmatrix} \alpha & & & \beta & & & \\ & \alpha & & & \beta & & \\ & & \alpha & & & \ddots & \\ \gamma & & & \ddots & & & \beta \\ & \gamma & & & \ddots & & \\ & & \ddots & & & \alpha & \\ & & & \gamma & & & \alpha \end{bmatrix} \tag{2}$$

This is referred to as a k-tridiagonal Toeplitz matrix. According to the relevant literature, other types of similar structures might be classified as k-tridiagonal; however, in this paper, we focus on matrices of the form (2). These types of structured matrices have been observed in various scientific problems. The unique structure of these matrices can be leveraged to develop a rapid method for solving linear systems, as illustrated in equation (1).

2 Fast Block Diagonalization of k-Tridiagonal Matrices

As it's been demonstrated in [1], k-tridiagonal matrices can be turn into block diagonal matrices by multiplying a permutation matrix to both sides of them. Let A be a k-tridiagonal matrix of the form

$$A = \begin{bmatrix} d_1 & & & & a_1 & & & \\ & d_2 & & & & a_2 & & \\ & & \ddots & & & & \ddots & \\ & & & d_{n-k} & & & & a_{n-k} \\ b_{k+1} & & & & \ddots & & & \\ & b_{k+2} & & & & \ddots & & \\ & & \ddots & & & & d_{n-1} & \\ & & & b_n & & & & d_n \end{bmatrix} \tag{3}$$

For example $T_7^{(3)}$ is given as follows:

$$A_7^{(3)} = \begin{bmatrix} d_1 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & a_3 & 0 \\ b_4 & 0 & 0 & d_4 & 0 & 0 & a_4 \\ 0 & b_5 & 0 & 0 & d_5 & 0 & 0 \\ 0 & 0 & b_6 & 0 & 0 & d_6 & 0 \\ 0 & 0 & 0 & b_7 & 0 & 0 & d_7 \end{bmatrix} \tag{4}$$

Let $\bar{r}$ be the following equivalent class of the set $\mathbb{N}_n := \{\mathbf{i}\}_{1\leq \mathbf{i} \leq n}$:

$$\bar{r} := \{\mathbf{i} \in \mathbb{N}_n | \mathbf{i} \equiv r(modk)\}. \tag{5}$$

Then we have:

$$\mathbb{N}_n = \bigcup_{r\in\{1,2,\ldots,k-1\}} \bar{r}. \tag{6}$$

The number of elements in each equivalent class $\bar{r}$ is denoted by $|\bar{r}|$. Let $P_{\bar{r}}$ be an $n \times |\bar{r}|$ matrix such that the columns are different each other and each column is the ith unit vector e_i, where $i \in \bar{r}$. Then, from (6) we can construct a permutation matrix P by using $P_{\bar{0}}, P_{\bar{1}}, ..., P_{\bar{k-1}}$, e.g.

$$P = [P_{\bar{0}}, P_{\bar{1}}, ..., P_{\bar{k-1}}]. \tag{7}$$

Using the permutation matrix (6), we have a block diagonalization of the original matrix $A_n^{(k)}$ of the form:

$$P^T A_n^{(k)} P = T_0 \oplus T_1 \oplus ... \oplus T_{k-1}, \tag{8}$$

where the symbol $\oplus$ denotes the direct sum of matrices. Each T_i is a tridiagonal matrix of the order $|\bar{i}| \times |\bar{i}|$.

As an example results can be seen on matrix $A_7^{(3)}$ in (4) as follow:

$$\bar{0} = \{3, 6\} \quad , \quad \bar{1} = \{1, 4, 7\} \quad , \quad \bar{2} = \{2, 5\}$$

therefore

$$P = [e_3, e_6, e_1, e_4, e_7, e_2, e_5],$$

then,

$$P^T A_7^{(3)} P = T_0 \oplus T_1 \oplus T_2,$$

where

$$T_0 = \begin{bmatrix} d_3 & a_3 \\ b_6 & d_6 \end{bmatrix} \quad , \quad T_1 = \begin{bmatrix} d_1 & a_1 & 0 \\ b_4 & d_4 & a_4 \\ 0 & b_7 & d_7 \end{bmatrix} \quad , T_2 = \begin{bmatrix} d_2 & a_2 \\ b_5 & d_5 \end{bmatrix}$$

thus, the block diagonal form of the $A_7^{(3)}$ whould be

$$P^T A_7^{(3)} P = \begin{bmatrix} d_3 & a_3 & 0 & 0 & 0 & 0 & 0 \\ b_6 & d_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 & a_1 & 0 & 0 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 & 0 & 0 \\ 0 & 0 & 0 & b_7 & d_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_2 & a_2 \\ 0 & 0 & 0 & 0 & 0 & b_5 & d_5 \end{bmatrix} \tag{9}$$

3 New Solution Method

As it is obvious from submatrices in (9), every block of $P^T A_n^{(k)} P$ is a tridiagonal matrix (for all $|\bar{i}| \geq 3$). Also it is not hard to see that, the result of block diagonalization on an k-tridiagonal Toeplitz matrix gives a block diagonal matrix which all its blocks are tridiagonal Toeplitz matrices. That's been said, we would have

$$P^T T_n^{(k)} P = T_0 \oplus T_1 \oplus ... \oplus T_{k-1} \tag{10}$$

where $T_n^{(k)}$ is a k-tridiagonal Toeplitz matrix like (2), and every T_i is a $|\bar{i}| \times |\bar{i}|$ tridiagonal Toeplitz matrix of the form

$$T_i = \begin{bmatrix} \alpha & \beta & & & \\ \gamma & \alpha & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma & \alpha & \beta \\ & & & \gamma & \alpha \end{bmatrix}_{|\bar{i}| \times |\bar{i}|} \tag{11}$$

Also, since matrix P is a permutation matrix created by rearanging the identity matrix, therefore, P is a orthonormal matrix i.e.

$$P^T P = P P^T = I_n$$

Thus, equation (1) can be rearenged as

$$P^T T_n^{(k)} P P^T x = P^T b \tag{12}$$

Equation (12) can also be represented as

$$\begin{bmatrix} T_0 & & & \\ & T_1 & & \\ & & \ddots & \\ & & & T_{k-1} \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \vdots \\ \hat{x}_{k-1} \end{bmatrix} = \begin{bmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \vdots \\ \hat{b}_{k-1} \end{bmatrix} \tag{13}$$

or,

$$T_i \hat{x}_i = \hat{b}_i \quad , \quad i = 1, ..., k-1 \tag{14}$$

where $\hat{x}_i$'s and $\hat{b}_i$'s are $|\bar{i}| \times 1$ vectors, and T_i's are as explained in (11). Therefore, solving (1) is equivalent to solving $k-1$ systems in (14). Interesting property of systems in (14) that makes them more suitable than the original system (1) is that all systems in (14) are tridiagonal Toeplitz systems which can be solved with the fast method introduced in [2](fast Core Method). Therefore, new solution method can be represented as below algorithm as follow:

Algorithm 1 Fast Solution Method

Require: k-tridiagonal Toeplitz matrix $T_n^{(k)}$, Right Hasd Side Vector b

1: Calculate the equivalent calsses in (5).
2: Form permutation matrix P in (7).
3: Multiply matrix P to the original system (1) as in (12) and determine $k-1$ systems in (14).
4: **for** $i = 1, ..., k-1$ **do**
5:     Solve $T_i \hat{x}_i = \hat{b}_i$ with the help of algorithm 3 in [2].
6: **end for**
7: $\bar{x} = [\hat{x}_0, \hat{x}_1, ..., \hat{x}_{k-1}]^T$.
8: **return** $x = P\bar{x}$

4 Numerical Experiments

This section shows the effectiveness of new solution method in compair to calssical LU method.

All the numerical tests were done on an ASUS laptop PC with AMD A12 CPU, 8Gb RAM and by Matlab R2016(b) with a machine precision of 10^{-16}. For convenience, throughout our numerical experiments, we denote the relative residual error of methods by Relative error $= ||b - Ax||/||b||$, and computing time of methods by Time (in seconds). In all examples, LU denotes the LU facorization method with pivoting (if it's necessary). In all tables, the Time is the average value of computing times required by performing the corresponding algorithm 10 times.

Example 1 :

First example is consist of an artificial system of equations that their coefficient matrices are in form of (2) with $\alpha = 5$, $\beta = 1$, $\gamma = 2$.

Table 1: Results for different n's and different k's

k		$n = 2^5$	$n = 2^8$	$n = 2^{10}$
1	**New method time**	**0.0031**	**0.0579**	**5.5683**
	New method error	**1.2660 e-16**	**1.3886 e-16**	**1.3780 e-16**
	LU time	**0.0035**	**0.1850**	**37.1914**
	LU error	**8.7896 e-17**	**1.2241 e-16**	**1.2386 e-16**
$2^{n-1} - 2$	**New method time**	**0.0068**	**0.0594**	**0.2171**
	New method error	**3.1562 e-16**	**3.4896 e-16**	**3.4368 e-16**
	LU time	**0.0036**	**0.1873**	**33.5555**
	LU error	**5.9957 e-17**	**7.6601 e-17**	**7.5353 e-17**
$2^n - 10$	**New method time**	**0.0099**	**0.0232**	**0.4018**
	New method error	**2.4544 e-16**	**1.2974 e-16**	**8.1819 e-17**
	LU time	**0.0034**	**0.1911**	**39.5602**
	LU error	**5.4749 e-17**	**6.5165 e-17**	**5.2028 e-17**

Table (1) results perfectly shows the effectiveness of new solution method in compare to classical LU method (as a representative of all classical direct methods). The k's in the above example are been chosen in a way that first, be the lowest possible, second be very close to the middle, and last be almost close to the end. The first case, $k = 1$, is exactly the tridiagonal Toeplitz system and the results are exactly the results of applying the fast core method in [2]. The other two cases represent the different systems with different numbers of blocks with different block size. Numbers in Table (1) show that the new solution method is very faster than LU method with almost same relative error. It's worth mentioning that, the perfect consistent results in Table (1) is because the systems are diagonally dominant. Applying this method on systems that are not diagonally dominant ends with similar results in time, but as the size of systems getting bigger, both methods start to lose precision.

5 Conclusion

Through this paper, a new approach for solving k-tridiagonal Toeplitz systems been introduced. This approach uses the Block Diagonalization procedure introduced in [1], and creates a block diagonal system that its diagonal blocks are tridiagonal Toeplitz matrices. Then it uses the fast direct method introduced in [2] (fast core method) to solve all subsystems in the block diagonal system.

In future works, main idea would be creating new methods to solve k-tridiagonal Toeplitz systems directly and a through comparison between new methods and the new solution method introduced in this paper.

References

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