$(A$ $*$-alg, $\varphi) \longmapsto H = \overline{A}^{\|\|_{2,\varphi}}$. $\pi: A \to B(H)$??? A acts on H, how to extend to H? Need norm with properties.

Def: A pre-C^* algebra is any $*$-algebra A with a unit 1 and a norm $\| \|$ satisfying $\|xy\| \le \|x\|\|y\|$ and $\|x^* x\| = \|x\|^2$ $\forall x, y \in A$.

Def: An abstract C^*-algebra is a pre-C^* algebra which is complete in $\| \|$ (or Banach algebra).

Def: A concrete (unital) C^*-algebra is any Banach unital $*$-algebra of $B(H)$.

* can show φ always exists (the GNS below)

Thm: (Gelfand) Any abstract C^*-alg. is a concrete C^*-alg. (on some $B(H)$).

So, abstract = concrete. Note: once you have $A \subset B(H)$, you can do $\overline{A}^{s.o.}$.

Two examples to keep in mind:

(1) $A = C([0,1])$ with $\| \|_\infty$ is an abstract C^*-alg.

really inductive limit, not just union

(2) $D = \bigcup_{k \ge 1} M_{2^k}(\mathbb{C})$ via $x \hookrightarrow \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ with matrix $\| \|$ (note $\|x\| = \left\|\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right\|$ $\forall x \in M_{2^k}(\mathbb{C})$) is a pre-$C^*$-algebra.

$\overline{D}^{\| \|}$ (completion of D in $\| \|$) is an abstract C^*-algebra.

The Gelfand-Naimark-Segal (GNS) Construction

Start with A a (unital) pre-C^*-algebra and $\varphi: A \to \mathbb{C}$ a linear, positive, faithful, norm-cont. map w/ $\varphi(1) = 1$.

Then there is a unique map $\pi: A \to B(H)$, where $H = \overline{A}^{\|\|_{2,\varphi}}$ such that π is a $*$-alg. morphism, π is injective, norm-preserving, and $\pi(a)\hat{b} = \widehat{ab}$ $\forall a, b \in A$ (here $\hat{b}$ denotes b via the embedding $A \hookrightarrow \overline{A}^{\|\|_{2,\varphi}}$).

$A \hookrightarrow H = \overline{A}^{\|\|_{2,\varphi}}$
$*$-alg
φ

$A \overset{\pi}{\hookrightarrow} B(H)$, $\pi(a)\hat{b} = \widehat{ab}$; $\pi(a)$ is defined on a, but need $\overline{A}^{\|\|_{2,\varphi}}$.

Can't extend it unless we know some cont. prop. for π: $a_n \to a \Rightarrow \pi(a_n) \to \pi(a)$?

$\overline{C([0,1])}^{\|\ \|_{2,\varphi}} = L^2([0,1])$. $C([0,1]) \subset B(L^2([0,1]))$

$f \longmapsto \hat{f}$ $\qquad$ $f \longmapsto M_f(g) = \hat{f_g}$

Proof: We may assume A is closed in $\|\ \|$ (if not, replace A by its completion in $\|\ \|$ and extend φ by cont.).

Recall on A we defined $\langle x,y\rangle = \varphi(y^*x)$ $\forall x,y \in A$ and $\|x\|_{2,\varphi} = \varphi(x^*x)^{1/2}$. Let H = completion of the pre-Hilbert space

$(A, \langle\ \rangle) = \overline{A}^{\|\ \|_{2,\varphi}}$. Let $\pi(a)\hat{b} = \widehat{ab}$ $\forall a,b \in A$ (so $\pi(a)$ is a linear map on $A \subset H$). We want to extend $\pi(a)$ to H: for $\xi = \lim_{\substack{b_n \in A}} \hat{b}_n$,

$\pi(a)\xi = \lim_{n\to\infty} \pi(a)\hat{b}_n = \lim_{n\to\infty} \widehat{ab_n}$. To show this is well-defined, we must show: if $\hat{b}_n \to \xi$, $\hat{c}_n \to \xi$ w/ $b_n, c_n \in A$, then

$\lim \widehat{ab_n} = \lim \widehat{ac_n}$. Need: if $\widehat{b_n - c_n} \to 0$, then $\widehat{a(b_n - c_n)} \to 0$. Note: $\|\widehat{ab}\|_2^2 = \varphi(b^*\overset{\text{positive}}{a^*a}b) \le \varphi(b^*(\|a\|^2 \cdot 1)b) = \|a\|^2\varphi(b^*b) = \|a\|^2\|\hat{b}\|_2^2$.

So, indeed if $\widehat{b_n - c_n} \to 0$, then $\widehat{a(b_n - c_n)} \to 0$. So, $\pi(a)$ is well-defined on H, and it is in $B(H)$ as $\overset{\text{as operator}}{\|\pi(a)\|} \le \overset{\text{in } A}{\|a\|}$.

Remark: In an abstract C^*-alg A, we can define $A_+ = \{x^*x : x \in A\}$ and prove properties like: $a^*a \le \|a^*a\| \cdot 1$, $x \le y \Rightarrow b^*xb \le b^*yb$.

Check: π is a $*$-morphism and $\|\pi(a)\| = \|a\|$ (so π is injective).

Example 1: $A = C([0,1])$, $\varphi(f) = \int_0^1 f\,dm$. $H = \overline{A}^{\|\ \|_{2,\varphi}} = L^2([0,1], m)$, $\|f\|_{2,\varphi} = \varphi(f^*f)^{1/2} = (\int |f|^2)^{1/2} = \|f\|_2$.

$\pi(f)(\hat{g}) = \hat{f_g}$ $\forall f,g \in A$, so $\pi(f) = M_f$ = left mult. by f and it extends to $L^2([0,1], m)$. So, $\underset{\text{with } \|\ \|_\infty}{\pi : C([0,1])} \to B(L^2([0,1]))$.

$f \longmapsto M_f$

Remark: If $A \subset B(H)$, we can consider $\overline{A}^{\|\ \|_{s.o.}}$, which is a v.N. alg (since A was $*$-closed, unital).

$\overline{C([0,1])}^{\|\ \|_{B(L^2)}} = L^2([0,1])$.

Example 2: $B = \bigcup_{n \ge 1} M_{2^n}(\mathbb{C})$ yields $\overline{B}^{\|\ \|}$ C^*-alg, $\overline{B}^{s.o.}$ v.N. alg.

different $\qquad$ same

Example 2': $C = \bigcup_{n \ge 1} M_{3^n}(\mathbb{C})$, yields $\overline{C}^{\|\ \|}$ C^*-alg, $\overline{C}^{s.o.}$ v.N. alg.

$x \hookrightarrow \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$

Thm: $\overline{\bigcup M_{2^n}}^{\|\ \|} \ne \overline{\bigcup M_{3^n}}^{\|\ \|}$ (as C^*-algebras, so via a (norm-preserving) $*$-isomorphism).

First, why $\bigcup M_{2^k} \neq \bigcup M_{3^k}$ as $*$-algebras? * 1st K theory, K_0 invariant

$\tau(1)=1$ $\Downarrow$

Let's look, for any A an alg. with a trace τ, at $\{\tau(x) : x \text{ proj. in } A\}$.

On the left, trace should involve powers of 2, whereas on the right powers of 3.