Prime Numbers and Analytic Number Theory

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1. Introduction to Prime Numbers

Prime numbers are often taught to students early on, as they have a relatively natural and easy to understand definition. Their importance, however, is hard to overstate. Primes are useful for studying the integers, as the Fundamental Theorem of Arithmetic states that we can look at any positive integer greater than one as a product of primes. In addition to this, their importance in areas like cryptography gives primes other practical uses. As such, prime numbers have been and continue to be an important topic of study in number theory, and the study of their properties leads to some truly interesting results.

There are many interesting areas that one can study when delving into the prime numbers. For example, one can study the distribution of primes, primality-testing, or factorization of integers into primes, to name just a few areas. We will focus mainly on the distribution of primes. As mentioned above, prime numbers are tied to the integers, and the study of primes typically involves a good deal of algebra, as one might expect. The study of the distribution of prime numbers, however, is often looked at through the lens of analytic number theory, which is a branch of number theory that utilizes tools from analysis. Some astounding results, like the Prime Number Theorem, have been proven using analytic number theory. We will try to look at the distribution of primes by using analytic number theory, although a detailed look into what we will be discussing would probably require a more thorough background in this area.

Before getting more in depth with the distribution of primes, we will try to motivate our study by looking at some interesting behavior of primes. Before this, we will start at the very beginning with the definition of a prime number. Much of what we will be looking at has been adapted from *Number Theory* by George Andrews [1].

Definition 1.1.
*An integer $p > 1$ is said to be **prime** if the only positive divisors of p are 1 and p itself.*

Note that 1 is not a prime number, and a positive integer greater than 1 that is not prime is called composite. One of the powerful aspects of primes, as mentioned above, is that integers can be written as a product of powers of primes. This product is known as a prime factorization, and it is unique up to the ordering of the multiplicands. The result itself is important enough to be known as the Fundamental Theorem of Arithmetic.

Theorem 1.1 (The Fundamental Theorem of Arithmetic)**.**
Each integer $n > 1$ has a prime-power factorization

$$n = p_1^{e_1} \cdots p_k^{e_k},$$

where $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k$ are positive integers; this factorization is unique, apart from permutations of the factors.

Now, while it is not entirely obvious from the definition, one might expect from intuition that there are infinitely many primes. Otherwise, we should be able to simply list all possible prime numbers. Euclid managed to prove that there are infinitely many primes around the year 300 BCE. The proof uses the idea that if there were finitely many primes, we could list them all out and reach a contradiction.

Theorem 1.2.
There are infinitely many primes.

Proof. In order to obtain a contradiction, suppose there are only a finite number of primes. Let $p_1, p_2, \ldots, p_n$ be these primes. We will consider the integer $m := p_1 p_2 \cdots p_n + 1$.

Using the Fundamental Theorem of Arithmetic, the integer m must have a prime factorization from our list of primes $p_1, \ldots, p_n$. This implies that at least one of the primes must divide m. Let p_i be this prime. Since p_i is one of the primes in the product $p_1 p_2 \cdots p_n$, it follows that p_i divides $p_1 p_2 \cdots p_n$. Since p_i divides both m and $p_1 p_2 \cdots p_n$, we know that p_i also divides the difference $m - p_1 p_2 \cdots p_n$.

Since $m - p_1 p_2 \cdots p_n = 1$, this means that $p_i | 1$. However, since p_i is a prime number, it must be greater than 1. Hence, we know that p_i cannot divide 1, a contradiction. Therefore, there must be infinitely many primes. $\square$

Now we know that there exist infinitely many primes. Since we are interested in the distribution of primes, it could be interesting to look at the gaps between prime numbers. That is, we could consider the differences between consecutive prime numbers. We can actually show that we can find an arbitrarily large gap between consecutive prime numbers, as this is another well known fact about primes.

Theorem 1.3.
There are arbitrarily large gaps between consecutive primes.

Proof. It will be enough to find an arbitrarily large sequence of consecutive, positive, composite integers, since any such sequence must be surrounded by two consecutive primes.

Suppose we have some integer $N \geq 1$. Consider the sequence of integers

$$(N+1)! + 2, (N+1)! + 3, \ldots, (N+1)! + N + 1.$$

Note that 2 divides both $(N+1)!$ and 2, so $(N+1)! + 2$ cannot be prime. Similarly, 3 divides both $(N+1)!$ and 3, so $(N+1)! + 3$ cannot be prime. For any $i \in \{2, \ldots, N+1\}$, we have that i divides both $(N+1)!$ and i, so $(N+1)! + i$ cannot be prime. Therefore, each integer in this sequence is a composite number. Since these integers are all consecutive, this is indeed the sequence that we want. Hence, there are arbitrarily large gaps between consecutive primes. $\square$

Somewhat contrary to this, there seem to exist infinitely many primes that have a gap of only two before the next prime. Some examples of these include 3 and 5, 5 and 7, 11 and

13, and so on. These primes are called twin primes, and the result that there are infinitely many is known as the Twin Prime Conjecture. While the Twin Prime Conjecture has not yet been proven, the evidence behind it seems to lean towards that it is true. In fact, this conjecture is currently an open problem in number theory that has been around since 1846. While the Twin Prime Conjecture has not been proven, results with a similar idea have been shown to be true. In 2013, Yitang Zhang proved that there exists some integer $N < 7 \cdot 10^7$ such that there are infinitely many primes that differ by N [4]. While seventy million might seem like a number too large to be useful, it was a significant advancement towards actually proving the Twin Prime Conjecture. Furthermore, this result spurred others to improve the methods used by Zhang, and the number N was able to be reduced to 246 [3].

So, we have seen that the distribution of primes has some interesting properties. We know that we can find a gap as large as we like between two primes, but there also seem to exist infinitely many primes with a gap of only two. We saw that the proof regarding arbitrarily large gaps was relatively simple, but the Twin Prime Conjecture has yet to be proven after centuries of work. This is often a trend when working with prime numbers, as simple sounding problems may end up being incredibly hard to solve. In some of these situations, using tools from analysis can help us to make progress. We can now explore some more interesting facts about primes by looking at some different types of primes.

2. Special Types of Primes

It is worth noting that there are special types of primes that themselves deserve some attention. These primes are often studied for some special properties that they have. We can take Mersenne primes, for example. Mersenne primes have a connection with perfect numbers, which were also studied by Euclid, and with large prime numbers.

2.1. Mersenne Primes

Definition 2.1.
A Mersenne Prime is a prime number of the form $2^n - 1$ for some integer n.

It can be shown that if n is a composite integer, then $2^n - 1$ is composite. Indeed, if n is composite, then it can be written as $n = ab$ for some integers $a, b > 1$. Then we have that

$$2^n - 1 = 2^{ab} - 1 = (2^a - 1) \cdot \left(1 + 2^a + 2^{2a} + \cdots + 2^{(b-1)a}\right).$$

This shows that $2^n - 1$ is composite. Therefore, we must actually have n be prime in order for $2^n - 1$ to be prime. The current largest prime number that we know of is a Mersenne prime [5]. This number is $2^{82589933} - 1$ and has 24,862,048 digits in base 10. In fact, 9 out of the 10 largest prime numbers that we know of are Mersenne primes. The usefulness of

Mersenne primes in finding large prime numbers has actually prompted volunteers to start the Great Internet Mersenne Prime Search, which uses volunteers' computers to passively search for large primes. Despite finding such large Mersenne primes, it is not known whether there are infinitely many Mersenne primes.

2.2. Dirichlet's Theorem

Moving away from Mersenne primes, we can think of the positive integers as a sequence, and we can study the primes as a part of this sequence. This is what we have been doing so far. We can generalize this idea to study primes in any sequence of integers. Such a sequence is an example of an arithmetic progression, which is simply any sequence of numbers where the difference between consecutive terms is constant. So, we already know that there are infinitely many primes in the natural sequence of positive integers. In 1837, Dirichlet proved a powerful result concerning primes in arithmetic progressions. We will discuss the theorem itself more later, but we can find interesting examples in special cases of this general theorem. For example, Dirichlet's result implies that there are infinitely many primes of the form $4q+1$ and $4q+3$ for some integer q. We will prove that there exist infinitely many primes of the form $6q+5$, and it also true that there exist infinitely many primes of the form $6q+1$. In order to do this, we will use a similar idea to what Euclid used to prove that there are infinitely many primes. The proof below is adapted from a proof in *Elementary Number Theory* by Jones and Jones, where they show that there are infinitely many primes of the form $4q+3$ [2]. The proof itself is similar to what Euclid did to show that there are infinitely many primes and, as such, is well-known.

Theorem 2.1.
There exist infinitely many primes p such that $p = 6q+5$ for some integer q.

Proof. By contradiction, suppose there are only finitely many primes of the form $6q+5$. Let $p_1, p_2, \ldots, p_n$ be these primes. Furthermore, let $m := 6p_1 \cdots p_n - 1$. Note that $m = 6q+5$ where $q = p_1 \cdots p_n - 1$.

Since $m = 6q+5$, we must have that m is odd. Then for any prime p that divides m, we must have p also be odd. The only options for p are then the forms $6q+1$, $6q+3$, and $6q+5$, as adding any even number to $6q$ would mean p would have to be even and any other odd numbers would be equivalent to one of these listed already. We must have at least one prime that divides m be of the form $6q+5$, since m is of the form $6q+5$.

If we let p be a prime that divides m that is of the form $6q+5$, then p must equal p_i for some $i \in \{1, \ldots, n\}$. Then p must also divide $6p_1 \cdots p_n$, since p is included in the product. However, since p divides both m and $6p_1 \cdots p_n$, we must have that p divides their difference. This is impossible, since $m - 6p_1 \cdots p_n = -1$ and $p > 1$. Therefore, there must exist infinitely many primes of the form $6q+5$. □

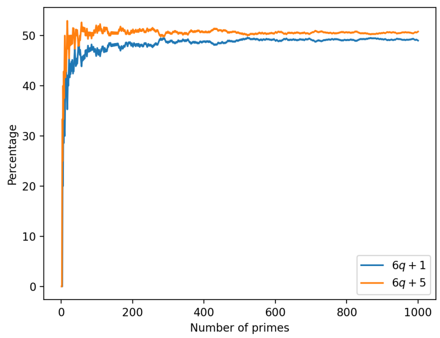

Figure 1: Group Percentages for 6

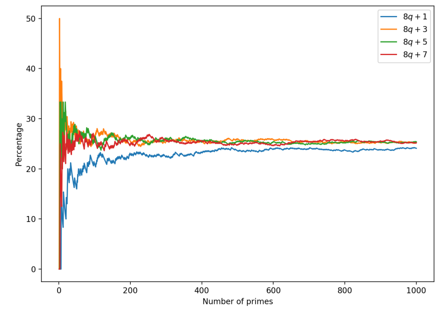

Figure 2: Group Percentages for 8

As mentioned, we can use a similar argument to show that there exist infinitely many primes of the form $4q+3$. The proof for $4q+1$ is actually more difficult to show than these two cases and requires a different approach. The above proof does not necessarily work for the case $4q+1$, as we do not necessarily have to have a prime of the form $4q+1$ that divides the quantity m that we create in the proof.

It can be interesting to look at how many primes fall into these two categories as we traverse through the primes. Naively, one might expect that roughly fifty percent of primes would be of the form $6q+1$ and the other rough fifty percent to be of the form $6q+5$. In Figure 1, we can see that the percentages do seem to approach fifty percent. We can see a similar result if we were to plot the percentages of primes of the form $4q+1$ and $4q+3$.

Interestingly enough, if we look at the percentages of primes of the form $8q+1$ and $8q+7$, we can see that they seem only to approach about twenty-five percent. However, if we also take into account primes of the form $8q+3$ and $8q+5$, we see where the other rough fifty percent of primes are going. We can see a plot of this in Figure 2. Note that each of the numbers that we are adding to $8q$ are relatively prime to 8 (meaning that their greatest common divisor is 1). We can see this is also true for $4q$ and $6q$. If we continue looking into this, it seems to be the case that there are infinitely many primes of the form $mq+n$ if m and n are relatively prime (of course, m must also be positive). This is actually the result proven by Dirichlet that we mentioned earlier. It is known as Dirichlet's theorem.

Theorem 2.2 (Dirichlet's Theorem).
If $m > 0$ and $\gcd(m,n) = 1$, then there are infinitely many primes in the arithmetic progression $mq+n$, where $q = 0, 1, 2, \ldots$

This theorem is also an important result in the realm of analytic number theory, as it is one of the first results utilizing the subject in a rigorous way. As such, the proof of the general theorem is significantly more difficult than the specific cases we have looked at. Furthermore, we can also note that the percentage of primes that fall into each category seems to be 100 divided by the number of positive integers less than m that are relatively prime to m (the denominator is often denoted as $\phi(m)$ and is the value of the so-called Euler's totient function

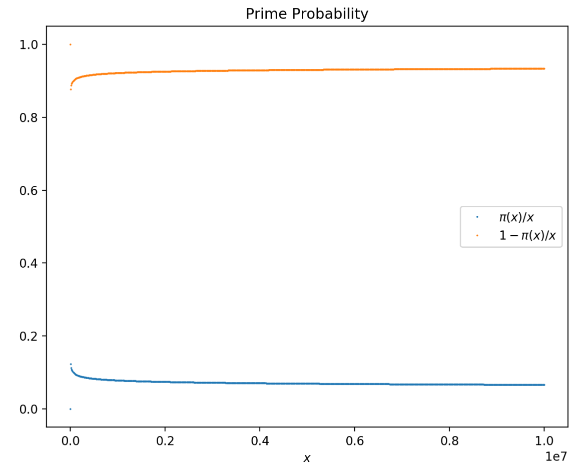

Figure 3: Prime Probability

at m). This is due to the fact that there are $\phi(m)$ distinct n that give distinct forms $mq+n$, and this can also be proven to be true. We will revisit Dirichlet's theorem in section 4. There we will use tools from analytic number theory to show that there exist infinitely many primes of the forms $6q+1$ and $6q+5$.

3. The Prime Counting Function

In order to further study the distribution of primes, we now introduce the prime counting function. We will use it to study some very important results concerning the distribution of prime numbers, including the Prime Number Theorem and Chebyshev's theorem. We will prove the latter of these two theorems and use it to discuss another meaningful result regarding primes called Bertrand's Postulate. Each of these results will give us some powerful new ways to look at the distribution of primes.

3.1. $\pi(x)$ and the Prime Number Theorem

Definition 3.1.
For any real number $x > 0$, let $\pi(x)$ denote the number of primes $p \leq x$. The function $\pi(x)$ is called the Prime Counting Function.

Since this function counts the number of primes up to a certain value, we can use it to ascertain information about the distribution of primes. For example, if we look at the ratio $\pi(x)/x$ for any real number x, we can get a rough idea of the probability that a positive integer less than or equal to x is prime. We know that each positive integer greater than 1 is either prime or composite. Furthermore, by studying the integers, it seems that there are more composite numbers than primes. In fact, as we look at larger and larger integers, the primes seem to be spaced farther and farther apart. Intuitively this should make sense, as larger numbers have more options for factors than smaller numbers. Since there seem to be more composite numbers than prime numbers, we would expect that this probability ratio would get smaller as x got bigger. In fact, it can be shown that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0.$$

We will discuss further how this can be proven soon. A plot of $\pi(x)/x$ and $1 - \pi(x)/x$ can be seen in Figure 3. Note that $1 - \pi(x)/x$ is not quite the probability that a number is composite, as 1 itself is neither prime nor composite. However, we can still see the kind of behavior that we expect, although $\pi(x)/x$ approaches 0 quite slowly (note that the x-axis goes all the way up to 10^7).

The Prime Number Theorem states a result relating to the end behavior of $\pi(x)$ that is even more powerful than this idea about probability. Gauss, among others, was able to arrive at the statement of the Prime Number Theorem around the early 1800s by observing that values of $\pi(x)$ were well approximated, in a sense, by a function involving $x/\log(x)$ (here we take $\log(x)$ to be the logarithm with base e). It took nearly another century for the theorem to actually be proven, at which time Jacques Hadamard and Charles Jean de la Poussin both proved the theorem independently. Before stating the Prime Number Theorem, we introduce a new notation.

Definition 3.2.
Suppose that f, g are two real-valued functions. We say that f is asymptotic to g if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

We write that $f \sim g$ if f is asymptotic to g.

We now state the Prime Number Theorem.

Theorem 3.1 (The Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\log x}$$

Although this result may look simple, it again is usually proven using some heavy-duty machinery from analytic number theory. We can see $\pi(x)$ and $x/\log(x)$ plotted separately in Figure 4, and their ratio is plotted by itself in Figure 5. Looking at them separately, we can see that the two functions do seem to have similar end behavior. Looking at their ratio, we can see that it is indeed approaching one, albeit slowly. As you can imagine, it is probably easier to look at a plot of thousands of values, as we are doing now, in order to see the behavior that we expect. However, the Prime Number Theorem was conjectured and

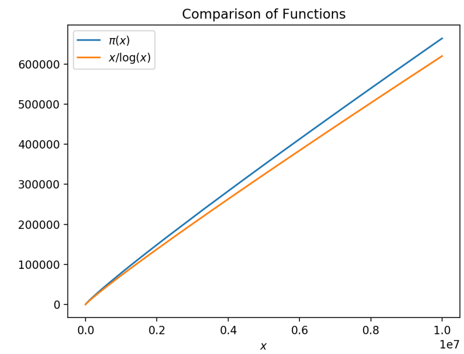

Figure 4: Comparison of Functions

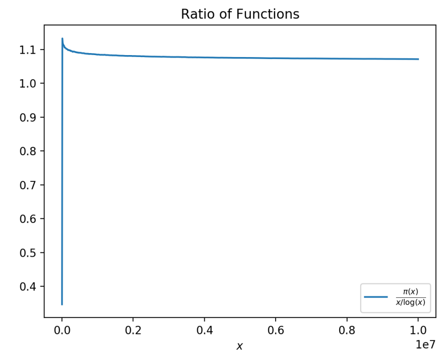

Figure 5: Ratio of Functions

proven centuries ago without the use of a computer, which makes the discovery and proof even more astounding.

We can use the Prime Number Theorem to study $\pi(x)$ in a different way. From this theorem, we can expect that $\pi(x)$ has some behavior similar to $x/\log(x)$. As an example, this gives us a relatively easy way to see that there are infinitely many primes, without going through the proof by Euclid. If there were only finitely many primes, then we know that $\pi(x)$ would plateau to a constant for all x sufficiently large. That is, after counting all the primes, the function $\pi(x)$ would no longer increase. However, we can use simple calculus to see that $x/\log(x)$ continues to increase. As such, the ratio of these two functions would also continue to change (with $\pi(x)$ as the numerator, it would continue to decrease). Therefore, if there were only finitely many primes, then we could not have the limit from the Prime Number Theorem approaching 1. Indeed, the numerator would approach some constant, while the denominator would approach infinity. Thus, the ratio would approach 0, instead of 1. For another example, we could reconsider the probability $\pi(x)/x$ that we mentioned earlier. We said that as x approaches infinity, the ratio $\pi(x)/x$ approaches 0. Now that we know that $\pi(x)$ has similar end behavior to $x/\log(x)$, we can see that $\pi(x)/x$ must have similar end behavior to

$$\frac{x/\log(x)}{x} = \frac{1}{\log(x)}.$$

Thus, we have

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = \lim \frac{x/\log(x)}{x} = \lim \frac{1}{\log(x)} = 0.$$

This is one way to see see that this idea of probability of picking a prime number approaches 0 as we look at bigger and bigger numbers.

We have looked at one plot of the prime counting function so far when we were comparing it to $x/\log(x)$. When we are looking at large values of x, this function may look fairly smooth. However, due to the definition of $\pi(x)$, we know that it is a step function. That is, the function increases in steps of 1 at every new integer that is a prime. This behavior may not be ideal, as it can make studying the function difficult. In calculus, for instance, we are used to studying smooth functions that are continuous and have other nice properties. Again, this is where analytic number theory and the Prime Number Theorem can be useful. Since we know that $\pi(x)$ has some similar behavior to $x/\log(x)$, we can use things like calculus to study $x/\log(x)$ in order to discover new information about $\pi(x)$. As such, the Prime Number Theorem is quite a nice and useful result.

3.2. Chebyshev's Theorem

The Prime Number Theorem itself is not easy to prove, as most of its proofs involve analytic number theory and complex analysis. Thankfully, we can attain some easier and still powerful results regarding the prime counting function. Specifically, we will now look at a result known as Chebyshev's Theorem. Again, while it is not as powerful as the Prime Number Theorem, it gives us a similar idea regarding the end behavior of $\pi(x)$.

Theorem 3.2 (Chebyshev's Theorem)**.**
There exist positive real numbers c_1 and c_2 such that

$$c_1 \frac{x}{\log(x)} < \pi(x) < c_2 \frac{x}{\log(x)},$$

for all $x \geq 2$.

We will include a proof of a slightly more restricted version of this theorem. Namely, we will find $c_1, c_2 \in \mathbb{R}$ that satisfy the above inequality for all $x \geq 8$. Afterwards, we will discuss the relevance of these constants and will work to improve them. Before doing this, we need to establish some results that will be useful in the proof of Chebyshev's theorem. Much of the proof involves setting up inequalities that we will use later. As such, the lemmas that we will state will be useful in setting up these inequalities. We will discuss their importance more as we introduce them. The proof that we will be doing of Chebyshev's theorem has been adapted from Andrews' book.

The first lemma that we will establish concerns the floor function $\lfloor x \rfloor$, which is essentially the function equivalent of "rounding down" to the nearest integer. This function will show up in many inequalities throughout the proof, as we can use it to convert non-integer real numbers into integers.

Lemma 3.1. *If $\lfloor x \rfloor$ denotes the largest integer that does not exceed x, then*

$$0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$$

for all $x \in \mathbb{R}$.

Proof. By using the definition of the floor function $\lfloor x \rfloor$, we can show that

$$2x - 1 < \lfloor 2x \rfloor \leq 2x$$
$$2x - 2 < 2\lfloor x \rfloor \leq 2x$$

for all $x \in \mathbb{R}$. By subtracting the second inequality from the first, we get that

$$-1 < \lfloor 2x \rfloor - 2\lfloor x \rfloor < 2.$$

From the definition of the floor function, we know that $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ must be an integer, and the only integers between -1 and 2 are 0 and 1. Therefore, it follows that

$$0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1.$$

$\square$

The second lemma concerns the exponent of a given prime in the prime factorization of the factorial of a given number, and its proof is less obvious than the proof of the first lemma. What is also less obvious is why we would be interested in this result in the first place. Again, it will be useful in setting up inequalities in the proof of Chebyshev's theorem. Specifically, we will use it to determine the power of a prime in the prime factorization of $\binom{2n}{n}$. We will then bound this power using Lemma 3.1 above, which we will be able to relate to the function $\pi(x)$.

Lemma 3.2. *If p is a prime, then $\displaystyle\sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor$ is the exponent of p appearing in the prime factorization of $n!$.*

Proof. We first consider the number of integers in the set $\{1, \dots, n\}$ that are divisible by p. These are all the multiples of p less than or equal to n, and we can list them all out:

$$p, 2p, 3p, \dots, \left\lfloor \frac{n}{p} \right\rfloor p.$$

Therefore, there are $\left\lfloor \frac{n}{p} \right\rfloor$ integers in the set that are divisible by p. Similarly, of these integers we can repeat the process to find the integers that are divisible by p^2. They are

$$p^2, 2p^2, \dots, \left\lfloor \frac{n}{p^2} \right\rfloor p^2.$$

So, there are $\left\lfloor \frac{n}{p^2} \right\rfloor$ integers that are divisible by p^2. If we repeat the process enough, we will reach an integer j such that none of the numbers in the set $\{1, \dots, n\}$ are divisible by p^j. Once we reach this point, we get that $\frac{n}{p^j} < 1$, since p^j does not divide n. Therefore, applying the floor function gives us

$$\left\lfloor \frac{n}{p^j} \right\rfloor = 0.$$

Furthermore, for any integer k larger than j, we have that

$$\left\lfloor \frac{n}{p^k} \right\rfloor = 0.$$

Since all of the numbers in the set $\{1, \ldots, n\}$ are the divisors of $n!$, we add the number of integers from this set divisible by each power of p to get the exponent of p appearing in the prime factorization of $n!$. This gives the result of the lemma. Note that for high enough powers of p we get no integers that are divisible by the prime power, which is why we are able to continue the sum to infinity. $\square$

The final lemma includes some facts regarding the function $x/\log(x)$. Seeing as Chebyshev's theorem primarily involves this function, its properties will be important. The reason that we have chosen these inequalities in particular may not seem obvious currently, but they will be useful in conjunction with our other inequalities.

Lemma 3.3. *For simplicity, let $f(x) = \frac{x}{\log x}$. Then*

$$\begin{aligned}
&f(x) \text{ is increasing for } x > e,\\
&f(x-2) > \frac{1}{2} f(x) \text{ for } x \geq 4,\\
&f\left(\frac{x+2}{2}\right) < \frac{15}{16} f(x) \text{ for } x \geq 8.
\end{aligned}$$

To prove these, we can simply use calculus and look at properties of logarithms. We will revisit these after the proof of Chebyshev's theorem in order to discover better c_1 and c_2.

Before proving the theorem, we will try to give an idea for the plan we will follow. In the proof we will show one side of the inequality at a time, beginning with the left side. We will first show that

$$2^n \leq \binom{2n}{n} \leq 2n^{\pi(2n)}.$$

Taking logarithms will give us that

$$n \log(2) \leq \pi(2n) \log(2n).$$

From here, we can rearrange and use some clever inequalities with the floor function and Lemma 3.3 to obtain the left side of the theorem. In order to show the right side of the inequality, we will show that

$$n^{\pi(2n)-\pi(n)} < \binom{2n}{n} < 2^{2n}.$$

Again, we can take logarithms of both sides and rearrange to get that

$$\pi(2n) < (2 \log 2) \frac{n}{\log n} + \pi(n).$$

As one last intermediate step, we use induction and the inequality above to prove that

$$\pi(2n) < 32(\log 2) \frac{n}{\log n}$$

for $n > 1$. Finally, we use this inequality and some more clever inequalities with the floor function to get the right side of the inequality. Therefore, we are ready to prove Chebyshev's theorem. The proof will fill in the details not mentioned above.

Proof. We will show that with $c_1 = \log(2)/4$ and $c_2 = 30\log(2)$, Chebyshev's theorem holds for all $x \geq 8$.

We first show the left side of the inequality. Let p be a prime number. We will use Lemma 3.2 to determine the power of p in the prime factorization of $\binom{2n}{n}$. Recall that the lemma applies to the factorial of some integer. We can rewrite our binomial coefficient as

$$\binom{2n}{n} = \frac{(2n)!}{(n!)(n!)}.$$

Then we can apply the lemma to $(2n)!$ and $n!$, as the power of p in the prime factorization of $\binom{2n}{n}$ is the power of p in the prime factorization of $(2n)!$ minus twice the power of p in the prime factorization of $n!$. This comes from the fact that $(2n)!$ appears once in the numerator, while $n!$ appears twice in the denominator. The lemma gives us the exponent of p in the prime factorization of $(2n)!$ as

$$\sum_{j=1}^{\infty} \left\lfloor \frac{2n}{p^j} \right\rfloor.$$

Furthermore, we get the exponent of p in the prime factorization of $n!$ to be

$$\sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

Before combining these to get the exponent in the prime factorization of $\binom{2n}{n}$, we define $r_p := \lfloor \log_p(2n) \rfloor$ for any prime p. Then since $p^{\log_p(2n)} = 2n$ and $\lfloor \log_p(2n) \rfloor \leq \log_p(2n)$, we have that $p^{r_p} \leq 2n$. Furthermore, since $\lfloor \log_p(2n) \rfloor + 1 > \log_p(2n)$, we have that $p^{r_p+1} > 2n$. Therefore, we have that $p^{r_p} \leq 2n < p^{r_p+1}$.

Now, for $j > r_p$, we have that $p^j > 2n$. From this, we can see that $\left\lfloor \frac{2n}{p^j} \right\rfloor = 0$ for $j > r_p$. Therefore, we can stop the sums once we reach r_p, as all other terms after this are 0. This means we can rewrite the exponent of p in the prime factorization of $(2n)!$ as

$$\sum_{j=1}^{r_p} \left\lfloor \frac{2n}{p^j} \right\rfloor.$$

Similarly, we can rewrite the exponent of p in the prime factorization of $n!$ as

$$\sum_{j=1}^{r_p} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

Now, combining these two, we get

$$\sum_{j=1}^{r_p} \left(\left\lfloor \frac{2n}{p^j} \right\rfloor - 2\left\lfloor \frac{n}{p^j} \right\rfloor \right)$$

to be the power of p in the prime factorization of $\binom{2n}{n}$. Using Lemma 3.1, we get that

$$0 \leq \left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor \leq 1$$

for all $j \in \{1, \dots, r_p\}$. We can apply this to each term in the sum to get that

$$0 \leq \sum_{j=1}^{r_p} \left(\left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor \right) \leq \sum_{j=1}^{r_p} 1 = r_p.$$

In order to use this, we let Q_n be the product of all p^{r_p}, where p is a prime not greater than $2n$. For any prime p, we have established that the exponent of p in the prime factorization of $\binom{2n}{n}$ is less than or equal to r_p. Note that the primes contained within the prime factorization of $\binom{2n}{n}$ are less than or equal to $2n$. Hence, these primes make up a subset of the primes contained in Q_n. Therefore, we get that

$$\binom{2n}{n} | Q_n,$$

which implies that

$$\binom{2n}{n} \leq Q_n. \tag{1}$$

Now, for each prime p in the product Q_n, we have that $p^{r_p} \leq 2n$ by the definition of r_p. Furthermore, there are $\pi(2n)$ factors of the form p^{r_p} in the product Q_n. This is due to the fact that Q_n is made up of all the primes not greater than $2n$, of which there are $\pi(2n)$. We can label these primes as $p_1, p_2, \dots, p_{\pi(2n)}$. Combining these results, it follows that

$$Q_n = p_1^{r_{p_1}} \cdot p_2^{r_{p_2}} \cdots p_{\pi(2n)}^{r_{p_{\pi(2n)}}} \leq 2n \cdot 2n \cdots 2n = (2n)^{\pi(2n)}. \tag{2}$$

Combining this with inequality (1) gives us that

$$\binom{2n}{n} \leq (2n)^{\pi(2n)}. \tag{3}$$

Now, we can show that

$$2^n \leq \binom{2n}{n} \tag{4}$$

using a combinatorial argument. Recall that the binomial coefficient $\binom{2n}{n}$ is the number of ways to pick a subset of size n from a set of size $2n$. We can think of this as the number of subsets of the set of integers $\{1, 2, \dots, 2n\}$. Now, we can make a subset of size n from this set by choosing up to n integers to include in the subset from the integers 1 to n and then simply adding numbers in order from n to $2n$ to get our subset to the size of n. For each integer from 1 to n, we can either include or exclude it from the subset, which gives us two possibilities. Since there are n numbers that we are making a choice for, this gives us 2^n possible subsets. Again, note that we have only one choice for each number from n to $2n$. However, note that these are only a subset of all the possible subsets that we could make.

Therefore, the number of these subsets must be less than or equal to the total number of subsets, which gives us what we want.

Combining this with inequality (3) gives us that

$$2^n \leq (2n)^{\pi(2n)}.$$

Taking logarithms of both sides, we get that

$$\log(2^n) \leq \log((2n)^{\pi(2n)}),$$

which we can then use properties of logarithms to obtain the inequality

$$n \log(2) \leq \pi(2n) \log(2n).$$

Dividing both sides by $\log(2n)$ gives us that

$$\log(2) \cdot \frac{n}{\log(2n)} \leq \pi(2n), \tag{5}$$

which we can finally use to obtain the left side of the theorem. To do this, we consider $x \in \mathbb{R}$ such that $x \geq 5$. For convenience, let $f(x) = x/\log(x)$. Then we apply inequality (5) with $n = \left\lfloor \frac{x}{2} \right\rfloor$ to get that

$$\pi\left(2\left\lfloor \frac{x}{2} \right\rfloor\right) \geq \frac{\log 2}{2} \cdot \frac{2\left\lfloor \frac{x}{2} \right\rfloor}{\log\left(2\left\lfloor \frac{x}{2} \right\rfloor\right)} = \frac{\log 2}{2} f\left(2\left\lfloor \frac{x}{2} \right\rfloor\right).$$

Applying parts 1 and 2 of Lemma 3.3 to the above inequality, we get that

$$\frac{\log 2}{2} f\left(2\left\lfloor \frac{x}{2} \right\rfloor\right) > \frac{\log 2}{2} f(x-2) > \frac{\log 2}{4} f(x) = \frac{\log 2}{4} \frac{x}{\log x}.$$

It is easy to see that $2\left\lfloor \frac{x}{2} \right\rfloor \leq x$, and since $\pi(x)$ is an increasing function, we get that $\pi\left(2\left\lfloor \frac{x}{2} \right\rfloor\right) \leq \pi(x)$. This gives us that

$$\frac{\log 2}{4} \frac{x}{\log x} < \pi(x).$$

This is the left side of Chebyshev's theorem, so we will now move on to the right side.

To prove the right side of the theorem, we again consider $\binom{2n}{n}$. Recall that

$$\binom{2n}{n} = \frac{(2n)!}{(n!)(n!)} = \frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1}.$$

If p is a prime in the interval $(n, 2n]$, then p must appear in the numerator once and not appear in the denominator, as $n+1 \leq p \leq 2n$. Therefore, we must have that p divides $\binom{2n}{n}$. Since this is true for any prime in the interval $(n, 2n]$, it is also true for the product of all these primes. Therefore, if we let P_n be the product of all primes in the interval $(n, 2n]$, it follows that

$$P_n | \binom{2n}{n}.$$

Of course, this implies that

$$P_n \leq \binom{2n}{n}. \tag{6}$$

Now, note that there are $\pi(2n) - \pi(n)$ primes in the interval $(n, 2n]$. We will label these primes as $p_1, p_2, \ldots, p_{\pi(2n)-\pi(n)}$. Since each of these primes are greater than n, and since P_n is the product of all these primes, we have that

$$n^{\pi(2n)-\pi(n)} = n \cdot n \cdots n < p_1 \cdot p_2 \cdots p_{\pi(2n)-\pi(n)} = P_n. \tag{7}$$

Combining this with inequality (6), we get that

$$n^{\pi(2n)-\pi(n)} < \binom{2n}{n}. \tag{8}$$

Using the Binomial Theorem, we have that

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}.$$

Note that $\binom{2n}{n}$ is one of the terms included in the sum, and since each of these terms is positive, we have that

$$\binom{2n}{n} < 2^{2n}. \tag{9}$$

Combining this with inequality (8), we get that

$$n^{\pi(2n)-\pi(n)} < 2^{2n}.$$

Taking logarithms of both sides gives us that

$$\log(n^{\pi(2n)-\pi(n)}) < \log(2^{2n}),$$

which we can then use properties of logarithms on to get

$$(\pi(2n) - \pi(n)) \log n < (2n) \log 2.$$

Rearranging this gives us that

$$\pi(2n) < (2 \log 2) \frac{n}{\log n} + \pi(n). \tag{10}$$

We now use this and induction to prove that the inequality

$$\pi(2n) < 32(\log 2) \frac{n}{\log n} \tag{11}$$

holds for all $n > 1$. First, we will show that the inequality is true for $2 \le n \le 8$. By counting the number of primes, we can see that

$$\pi(4) < \pi(6) < \pi(8) \le \pi(10) < \pi(12) < \pi(14) \le \pi(16) = 6.$$

Note that $x/\log(x)$ has a minimum at $x = e$. Knowing this, we evaluate the right side of the inequality that we are trying to prove at e to get

$$32(\log 2)\frac{e}{\log e} > 32 \cdot \frac{1}{2} \cdot 2 = 32.$$

Since $32 > 6$, we have that

$$\pi(2n) < 32(\log 2)\frac{n}{\log n}$$

for all $2 \le n \le 8$.

Now, assume that (11) holds for all $n \le k$, where $k \ge 8$. Again, for convenience let $f(x) = x/\log(x)$. Using inequality (10), we get that

$$\pi(2(k+1)) < (2\log 2)f(k+1) + \pi(k+1).$$

Now, it is easy to see that $k+1 \le 2\left\lfloor \frac{k+2}{2} \right\rfloor$, and since $\pi(x)$ is an increasing function, it follows that $\pi(k+1) \le \pi\left(2\left\lfloor \frac{k+2}{2} \right\rfloor\right)$. Using this with the above inequality gives

$$(2\log 2)f(k+1) + \pi(k+1) \le (2\log 2)f(k+1) + \pi\left(2\left\lfloor \frac{k+2}{2} \right\rfloor\right).$$

Since $\left\lfloor \frac{k+2}{2} \right\rfloor \le k$, we can use the inductive hypothesis with the above inequality to get that

$$(2\log 2)f(k+1) + \pi\left(2\left\lfloor \frac{k+2}{2} \right\rfloor\right) < (2\log 2)f(k+1) + 32(\log 2)f\left(\left\lfloor \frac{k+2}{2} \right\rfloor\right).$$

Since $\left\lfloor \frac{k+2}{2} \right\rfloor \le \frac{k+2}{2}$ and f is increasing for sufficiently large values, we get that

$$(2\log 2)f(k+1) + 32(\log 2)f\left(\left\lfloor \frac{k+2}{2} \right\rfloor\right) \le (2\log 2)f(k+1) + 32(\log 2)f\left(\frac{k+2}{2}\right).$$

Since $k \ge 8$, we can use part 3 of Lemma 3.3 to get that

$$\begin{aligned}(2\log 2)f(k+1) + 32(\log 2)f\left(\frac{k+2}{2}\right) &< (2\log 2)f(k+1) + 32(\log 2)\frac{15}{16}f(k+1)\\ &= (2\log 2)f(k+1) + 30(\log 2)f(k+1)\\ &= 32(\log 2)f(k+1)\\ &= 32(\log 2)\frac{k+1}{\log(k+1)}.\end{aligned}$$

This is what we want to show, so we get that inequality (11) holds for all $n > 1$. We are finally ready to show the right side of the theorem.

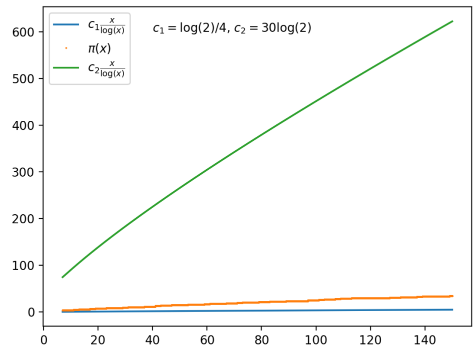

Figure 6: Large Upper Bound

Consider $x \in \mathbb{R}$ such that $x \geq 8$. Note that $x < 2\left\lfloor \frac{x}{2} \right\rfloor + 2$, so $\pi(x) < \pi\left(2\left\lfloor \frac{x}{2} \right\rfloor + 2\right)$. Using inequality (11) with $f(x) = x/\log(x)$, we get that

$$\pi\left(2\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right)\right) < 32(\log 2) f\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right).$$

It is easy to see that $\left\lfloor \frac{x}{2} \right\rfloor + 1 \leq \frac{x+2}{2}$, so we use it and the fact that f is increasing to get that

$$32(\log 2) f\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right) \leq 32(\log 2) f\left(\frac{x+2}{2}\right).$$

Using part 3 of Lemma 3.3 again, we get that

$$\begin{aligned} 32(\log 2) f\left(\frac{x+2}{2}\right) &< 32(\log 2)\frac{15}{16} f(x) \\ &= 30(\log 2) f(x) \\ &= 30(\log 2)\frac{x}{\log x}. \end{aligned}$$

Using the fact that $\pi(x) < \pi\left(2\left\lfloor \frac{x}{2} \right\rfloor + 2\right)$, this shows that

$$\pi(x) < 30(\log 2)\frac{x}{\log x}.$$

This is the right side of the theorem, so we are done. $\square$

While the many different inequalities used in the proof of this theorem can be hard to follow, the proof itself is not incredibly difficult. Furthermore, even though Chebyshev's theorem is not quite as nice as the Prime Number Theorem, it does at least give us a way to bound the prime counting function. This can be useful in its own ways. Similar to the Prime Number Theorem, we can use this theorem to get an idea of the end behavior of $\pi(x)$.

Figure 6 contains a plot of $\pi(x)$ and the upper and lower bounds given in the proof of Chebyshev's theorem. As we can see, the upper bound is especially lenient. In order to improve this, we can consider the behavior of $\pi(x)$ and that of the upper bound. We know that the upper bound has a minimum at e, and we can see that the value of the upper bound at e is approximately 56. However, the value of $\pi(3)$, where 3 is slightly greater than e, is only 2. We know that $\pi(x)$ grows a similar rate to $x/\log(x)$ from the Prime Number Theorem, so if we can make the minimum of the upper bound just enough over the value of $\pi(3)$, then we might get a more strict upper bound. Note that $x/\log(x)$ evaluated at e simply gives us e, so we can multiply the function by the constant $\frac{4}{e}$ to get a value of 4 at the minimum. If we replace c_2 by this value, we do get a new and valid upper bound. We will include a plot of this, but we will not prove it. Instead, we will attempt to modify the above proof in order to get a more strict upper bound than the original, although not quite as strict as $\frac{4}{e}$.

If we consider how we showed the right side of Chebyshev's inequality, we can find a couple of points in the proof at which we can improve our bound. One of the important facts that we used was that $f\left(\frac{x+2}{2}\right) < \frac{15}{16}f(x)$ for all $x \geq 8$, where $f(x) = \frac{x}{\log x}$. If we look at slightly larger x, we can see that $f\left(\frac{x+2}{2}\right) < \frac{3}{4}f(x)$ for all $x \geq 12$. We will use this fact to show that the value $6(\log 2)$ will work for c_2 for all $x \geq 12$. While this is not as great as $\frac{4}{e}$, it is still five times smaller than our original c_2.

Proposition 3.1. *With $c_2 = 6(\log 2)$, we have that*

$$\pi(x) < c_2 \frac{x}{\log(x)}$$

for all $x \geq 8$.

Proof. By induction, we will show that

$$\pi(2n) < 8(\log 2)\frac{n}{\log n} \tag{12}$$

for all $n > 1$. For $2 \leq n \leq 12$, we have that

$$\pi(4) \leq \pi(6) \leq \ldots \leq \pi(24) = 9.$$

Since $\frac{x}{\log x}$ has a minimum at e, we can evaluate the right side of inequality (12) to see that

$$8(\log 2)\frac{n}{\log n} \geq 8(\log 2)e > 11.$$

Since $11 > 9$, it follows that our inequality holds for $2 \leq n \leq 12$.

Now, suppose that (12) holds for $n \leq k$, where $k \geq 12$. Using the same argument as before, we have that

$$\pi(2(k+1)) < (2\log 2)f(k+1) + \pi\left(2\left\lfloor\frac{k+2}{2}\right\rfloor\right).$$

Using the inductive hypothesis, we get that

$$(2\log 2)f(k+1) + \pi\left(2\left\lfloor\frac{k+2}{2}\right\rfloor\right) < (2\log 2)f(k+1) + 8(\log 2)f\left(\left\lfloor\frac{k+2}{2}\right\rfloor\right).$$

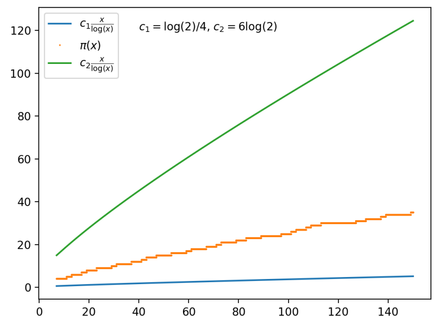

Figure 7: A Better Bound

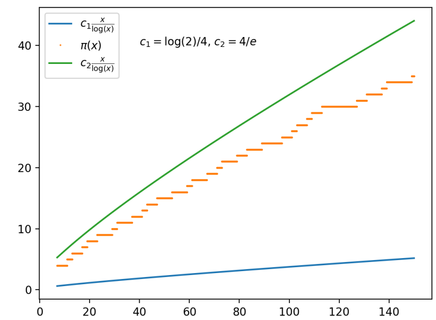

Figure 8: An Even Better Bound

Again, since $\left\lfloor \frac{k+2}{2} \right\rfloor \leq \frac{k+2}{2}$, we get that

$$(2\log 2)f(k+1) + 8(\log 2)f\left(\left\lfloor \frac{k+2}{2} \right\rfloor\right) < (2\log 2)f(k+1) + 8(\log 2)f\left(\frac{k+2}{2}\right).$$

Finally, we use the fact that $k \geq 12$ to get that

$$\begin{aligned}(2\log 2)f(k+1) + 8(\log 2)f\left(\frac{k+2}{2}\right) &< (2\log 2)f(k+1) + 8(\log 2)\frac{3}{4}f(k+1) \\ &= (2\log 2)f(k+1) + 6(\log 2)f(k+1) \\ &= 8(\log 2)f(k+1) \\ &= 8(\log 2)\frac{k+1}{\log(k+1)},\end{aligned}$$

which finishes our induction proof.

We now consider $x \in \mathbb{R}$ such that $x \geq 12$ and use the same argument as in the original proof to get that

$$\begin{aligned}\pi(x) < \pi\left(2\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right)\right) &< 8(\log 2)f\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right) \\ &\leq 8(\log 2)f\left(\frac{x+2}{2}\right) \\ &< 8(\log 2)\frac{3}{4}f(x) \\ &< 6(\log 2)f(x) \\ &= 6(\log 2)\frac{x}{\log x}.\end{aligned}$$

As this is what we wanted to show, the proof is finished. $\square$

We can see a plot of the new bound with $c_2 = 6(\log 2)$ in Figure 7 alongside the plot with $c_2 = \frac{4}{e}$ in Figure 8. Surprisingly, these bounds actually seem to work for all $x \geq 1$, although we have only proven them for larger values of x. When looking to get a better bound we ended up looking at larger values of x. Intuitively, we know that we should be able to get even stricter bounds if we look at larger and larger x. In fact, using what we know from the Prime Number Theorem, we can expect that both c_1 and c_2 approach 1 as x approaches infinity. The bounds themselves do not matter so much, especially for smaller values of x. What is more important is the end behavior of our function, and Chebyshev's theorem at least gives us an idea at that.

3.2.1. Bertrand's Postulate

We will now look at an interesting application of Chebyshev's theorem. The next two propositions that we mention are also found in Andrews' book as exercises. In 1852, Chebyshev proved a result known as Bertrand's Postulate. A restricted version of this is that for all $n > 1$, there exists some prime p such that $n < p < 2n$. We can actually use Chebyshev's theorem to show a similar result. Andrews shows that for x sufficiently large, there exists some prime p such that $x < p < 125x$. We will use our improved bounds to bring this down to $x < p < 25x$.

Proposition 3.2.
For x sufficiently large, there exists some prime p such that $x < p < 25x$.

Proof. Note that there is a prime between x and $25x$ if and only we have that

$$\pi(25x) - \pi(x) > 0.$$

Using Chebyshev's theorem with $f(x) = x/\log(x)$, $c_1 = \frac{\log 2}{4}$, and $c_2 = 6\log 2$, we get that

$$c_1 f(25x) < \pi(25x) < c_2 f(25x)$$

and

$$-c_2 f(x) < -\pi(x) < -c_1 f(x).$$

Adding these two together, we get that

$$c_1 f(25x) - c_2 f(x) < \pi(25x) - \pi(x) < c_2 f(25x) - c_1 f(x).$$

Therefore, if

$$c_1 f(25x) - c_2 f(x) > 0,$$

then we have that

$$\pi(25x) - \pi(x) > 0.$$

Adding $c_2 f(x)$ to both sides of the first inequality gives us $c_1 f(25x) > c_2 f(x)$. We will show that this inequality is true for large enough x.

In order to show that this inequality is true for large enough x, we consider the ratio

$$\frac{c_1}{c_2} \cdot \frac{f(25x)}{f(x)} = 25 \cdot \frac{c_1}{c_2} \cdot \frac{\log(x)}{\log(25x)}.$$

Using properties of logarithms, we can rewrite this as

$$25 \cdot \frac{c_1}{c_2} \cdot \frac{\log(x)}{\log(x) + \log(25)}.$$

As x approaches infinity, we can see that this approaches $25 \cdot \frac{c_1}{c_2} > 1$. For large enough x, both $c_1 f(25x)$ and $c_2 f(x)$ are positive. Since this ratio approaches a value greater than 1, the numerator must eventually be greater than the denominator. Therefore, for x large enough, $c_1 f(25x) > c_2 f(x)$, which is what we wanted to show. This means that for x large enough, there does indeed exist a prime between x and $25x$. □

Similarly, if we use $c_2 = \frac{4}{e}$, we can go as low as $9x$. Although it would be more difficult to show, we would expect that $c_1 = 0.87$ and $c_2 = 1.26$ work for all $x \geq 7$. Furthermore, we have that $c_1 f(2x) > c_2 f(x)$ for $x \geq 7$. So, for all $x \geq 7$, there exists some prime between x and $2x$. For $1 < x < 7$, we can verify that this is also true by using the primes $2, 3, 5, 7$; that is for any $x \in (1, 7)$, one of these four primes is between x and $2x$. Therefore, there does indeed exist a prime between x and $2x$ for all $x > 1$. The difficulty lies in actually showing that these c_1 and c_2 work.

As another quick exercise, we take a closer look at the integer r_p that we defined when proving Chebyshev's theorem.

Proposition 3.3.
Show that, as a consequence of the definition of r_p, if $r_p \geq 2$, then $p \leq \sqrt{2n}$.

Proof. Suppose that $r_p \geq 2$. By definition of r_p, we have that $p^{r_p} \leq 2n$. Since $r_p \geq 2$ and $p > 1$, as p is prime, we have that

$$0 < p^2 \leq p^{r_p} \leq 2n.$$

Since p^2 and $2n$ are positive, it follows that $p \leq \sqrt{2n}$, so we are done. □

4. Dirichlet's Theorem Revisited

We will now move away from the Prime Number Theorem and Chebyshev's Theorem in order to revisit Dirichlet's Theorem for our final endeavor. We have mentioned multiple times how analytic number theory is important to many of the theorems that we have discussed so far, but we have not really delved too far into the subject. Our goal now will be to try to understand how we could use analytic number theory to prove Dirichlet's Theorem. We will not actually prove the theorem itself, as we will again focus specifically on primes of the form $6q+1$ and $6q+5$. We will prove that there are infinitely many primes of both of these forms (recall that we have only seen a proof for primes of the form $6q+5$ up to this point). Ideally, the proof will give us more understanding of both the tools of analytic number theory and how Dirichlet's theorem works. The proof that we will give is adapted from the book *Not Always Buried Deep* by Paul Pollack, where Pollack proves that there are infinitely many primes of the forms $4q + 1$ and $4q + 3$ [6].

Before discussing the proof, we need to introduce the idea of Big O notation.

Definition 4.1.
If f, g are real valued functions defined on some unbounded subset of the positive real numbers, and g is strictly positive for large enough values, then we say that f is big O of g if $|f(x)| \leq cg(x)$ for some constant c and for all x sufficiently large. We then write that $f = O(g)$.

This notation can be useful when working with the end behavior of functions, as we will be doing. Throughout the proof we will see $O(1)$ used frequently, which can be intuitively understood as a function eventually being bounded by a constant (by eventually, we mean for large enough x). Furthermore, we will also see that big O notation can be used in equations in a natural way. For example, we will also often right that $f = g + O(1)$, which simply means that $f - g = O(1)$. Again, this can be seen as the difference $f - g$ eventually being bounded by a constant, which means that f and g are eventually within a constant of each other.

In order to prove that there are infinitely many primes congruent to 1 and 5 modulo 6, we will actually show a stronger result. Namely, we will show that

Theorem 4.1.

$$\sum_{\substack{p \leq x \\ p \equiv 1 \pmod 6}} \frac{\log p}{p} = \frac{1}{\varphi(6)} \log x + O(1).$$

This sum ranges over all primes less than or equal to x that are congruent to 1 modulo 6. We will show a similar result for primes congruent to 5 modulo 6. Here, $\varphi(6)$ is the number of integers less than or equal to 6 that are relatively prime to 6. Therefore, $\varphi(6) = 2$. This would imply that there are infinitely many primes congruent to 1 modulo 6. To see this, note that $\log x$ approaches infinity as x approaches infinity. This means that the sum also approaches infinity, which would not be possible if there were only finitely many terms. Therefore, we must have infinitely terms, which is what we want.

4.1. Proof of Theorem 4.1

In order to prove this equation, we will establish a connection between

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod m}} \frac{\log p}{p}$$

and

$$\sum_{\substack{n\le x\\ n\equiv a \pmod{m}}} \frac{\Lambda(n)}{n}.$$

This sum extends over all positive integers that are less than or equal to x and congruent to a modulo m. Here, the function Λ is known as the von Mangoldt function and is defined as follows.

Definition 4.2.

$$\Lambda(n) := \begin{cases} \log p & \textit{if } n = p^k \textit{ is a prime power} \\ 0 & \textit{otherwise} \end{cases}.$$

We will show that

$$\sum_{\substack{n\le x\\ n\equiv a \pmod{m}}} \frac{\Lambda(n)}{n} = \sum_{\substack{p\le x\\ p\equiv a \pmod{m}}} \frac{\log p}{p} + O(1).$$

This way we will be able to study the sum involving the von Mangoldt function in order to learn about the sum involving the logarithm, which is the sum that we are interested in. This will be the first part of the proof.

We will define two functions χ and χ_0, which are both related to congruence classes modulo 6. We will use them to prove that

$$\sum_{n\le x} \frac{\chi_0(n)\Lambda(n)}{n} = \log x + O(1)$$

and

$$\sum_{n\le x} \frac{\chi(n)\Lambda(n)}{n} = O(1).$$

These are the second and third parts of the proof. Using these two, we will show that

$$\sum_{\substack{n\le x\\ n\equiv 1 \pmod{6}}} \frac{\Lambda(n)}{n} = \frac{1}{2}\log x + O(1).$$

This can be shown by using the definition of χ and χ_0. We will show a similar result for integers congruent to 5 modulo 6. This will be the fourth part of the proof. Finally, for the fifth part of the proof, we will use the first and fourth parts to show that

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod{6}}} \frac{\log p}{p} = \frac{1}{\varphi(6)}\log x + O(1),$$

again with a similar result for integers congruent to 5 modulo 6.

We will assume some pieces of information for the following proof, which we will include as lemmas. The proofs of these parts can be found in Pollack's book [6]. The first of these will be useful in proving the first part of our proof.

Lemma 4.1 (Equation 3.24 in Pollack's book)**.**

$$\sum_{\substack{p^k \le x \\ k \ge 2}} \frac{\log p}{p^k} = O(1).$$

Here the range of the summation can be interpreted as all prime-powers less than or equal to x, where the power is at least 2.

The next lemma will be used in showing the second part of the proof.

Lemma 4.2 (Equation 3.22 in Pollack's book)**.**

$$\sum_{d \le x} \frac{\Lambda(d)}{d} = \log x + O(1).$$

Following the trend, our third lemma will be used in the third part of the proof.

Lemma 4.3 (Lemma 3.9 in Pollack's book)**.**

$$\sum_{d \mid n} \Lambda(d) = \log n.$$

Here the sum is over the divisors d of n.

Our final lemma will also be used in the third part of the proof.

Lemma 4.4.

$$\sum_{d \le x} \Lambda(d) = O(x).$$

Before showing the first part of the proof, we will also define a completely multiplicative function, as it will be relevant to our proof.

Definition 4.3.
If $f : \mathbb{Z}^+ \to \mathbb{C}$ is a function, f is not identically zero, and $f(ab) = f(a)f(b)$ for all positive integers a, b, then f is called completely multiplicative.

In order to use this definition, we first must define our own completely multiplicative function.

Definition 4.4.
We define the function $\chi : \mathbb{Z} \to \mathbb{Z}$ by

$$\chi(n) = \begin{cases} +1 & \text{if } n = 6k+1 \\ -1 & \text{if } n = 6k+5 \\ 0 & \text{if } \gcd(n, 6) > 1 \end{cases}$$

for all $n \in \mathbb{Z}$.

Note that if n is not of the form $6k+1$ nor of the form $6k+5$, then we must have that $\gcd(n,6) > 1$. Indeed, if n is not of either of these two forms, then it must be of the form $6k+0$, $6k+2$, $6k+3$, or $6k+4$. In any of these cases, it can be seen that n and 6 share a common factor of either 2 or 3. Therefore, $\gcd(n,6) > 1$. Now, we will show that χ is a completely multiplicative function.

Proposition 4.1. *The function χ is completely multiplicative.*

Proof. That is, for any $a, b \in \mathbb{Z}$, we must have that $\chi(ab) = \chi(a)\chi(b)$. To see this, we consider the different forms that a and b can take. If $a \equiv 1 \pmod 6$ and $b \equiv 1 \pmod 6$, then we have that

$$ab \equiv 1 \cdot 1 \equiv 1 \pmod 6.$$

Therefore,

$$\chi(ab) = 1 = 1 \cdot 1 = \chi(a)\chi(b).$$

If $a \equiv 1 \pmod 6$ and $b \equiv 5 \pmod 6$, then we have that

$$ab \equiv 1 \cdot 5 \equiv 5 \pmod 6.$$

Therefore,

$$\chi(ab) = -1 = 1 \cdot -1 = \chi(a)\chi(b).$$

The case is similar if $a \equiv 5 \pmod 6$ and $b \equiv 1 \pmod 6$. If $a \equiv 5 \pmod 6$ and $b \equiv 5 \pmod 6$, then

$$ab \equiv 5 \cdot 5 \equiv 25 \equiv 1 \pmod 6.$$

Therefore,

$$\chi(ab) = 1 = -1 \cdot -1 = \chi(a)\chi(b).$$

Finally, if a or b is not relatively prime to 6, then the product of a and b cannot be relatively prime to 6. We again have that

$$\chi(ab) = 0 = \chi(a)\chi(b).$$

Since these are all the possible cases, we have shown that $\chi(ab) = \chi(a)\chi(b)$. Therefore, χ is indeed a completely multiplicative function. $\square$

Since χ is completely multiplicative, we can easily see that the function $\chi(n)/n$ is also completely multiplicative. For the proof that we will do, we will still study the sum

$$\sum_{n \leq x} \frac{\chi(n)}{n},$$

where x is some real number.

4.1.1. Part One

We will now show the first part of our proof, which is that

$$\sum_{\substack{n\le x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{n} = \sum_{\substack{p\le x\\ p\equiv a \pmod m}} \frac{\log p}{p} + O(1).$$

If we let m be any positive integer and a be any integer, then we have that

$$\sum_{\substack{n\le x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{n} = \sum_{\substack{p^k\le x\\ p^k\equiv a \pmod m}} \frac{\log p}{p^k} \tag{13}$$

using the definition of the von Mangoldt function (since $\Lambda(n) = 0$ if n is not a prime power, we are left with only prime powers). Separating the terms with $k = 1$ and $k \ge 2$ into different sums, we get that

$$\sum_{\substack{p^k\le x\\ p^k\equiv a \pmod m}} \frac{\log p}{p^k} = \sum_{\substack{p\le x\\ p\equiv a \pmod m}} \frac{\log p}{p} + \sum_{k\ge 2} \sum_{\substack{p\le x^{1/k}\\ p^k\equiv a \pmod m}} \frac{\log p}{p^k}. \tag{14}$$

We can see that

$$\sum_{k\ge 2} \sum_{\substack{p\le x^{1/k}\\ p^k\equiv a \pmod m}} \frac{\log p}{p^k} \le \sum_{\substack{p^k\le x\\ k\ge 2}} \frac{\log p}{p^k}.$$

Using this and Lemma 4.1, we get that

$$\sum_{k\ge 2} \sum_{\substack{p\le x^{1/k}\\ p^k\equiv a \pmod m}} \frac{\log p}{p^k} = O(1).$$

Combining this with Equation (13) and Equation (14) gives us that

$$\sum_{\substack{n\le x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{n} = \sum_{\substack{p\le x\\ p\equiv a \pmod m}} \frac{\log p}{p} + O(1). \tag{15}$$

Therefore, the difference of these two sums is eventually bounded by a constant. This is important to us because this means that we can use the sum on the left in order to study the sum on the right, which is our ultimate goal. This is the first part of the proof.

4.1.2. Part Two

We now move onto the second part of the proof. We will again focus on $m = 6$. We will also reuse the function χ from earlier, and we will now define a new function which we will call χ_0.

Definition 4.5.
We define χ_0 *by*

$$\chi_0(n) = \begin{cases} 1 & \textit{if } \gcd(n,6) = 1 \\ 0 & \textit{if } \gcd(n,6) \neq 1 \end{cases}$$

for all $n \in \mathbb{Z}$.

We can show that $\chi_0 + \chi$ is 2 when the argument is congruent to 1 modulo 6 and 0 otherwise. Indeed, if $n = 6k+1$ for some $k \in \mathbb{Z}$, then we have that $\chi_0(n) = 1$ and $\chi(n) = 1$. Therefore,

$$(\chi_0 + \chi)(n) = 1 + 1 = 2.$$

If $n = 6k+5$, then we get that $\chi_0(n) = 1$ and $\chi(n) = -1$. Therefore,

$$(\chi_0 + \chi)(n) = 1 - 1 = 0.$$

Finally, if n is congruent to anything else modulo 6, then we know that the gcd of n and 6 is greater than 1. By definition, we have that $\chi_0(n) = 0$ and $\chi(n) = 0$, so of course their sum is also zero. Similarly, we can show that $\chi_0 - \chi$ is 2 when the argument is congruent to 5 modulo 6 and 0 otherwise, using a similar argument to what we did above. From here, we will study the sums

$$\sum_{n \le x} \frac{\chi_0(n)\Lambda(n)}{n} \tag{16}$$

and

$$\sum_{n \le x} \frac{\chi(n)\Lambda(n)}{n}. \tag{17}$$

Starting with the first sum, we have that

$$\sum_{n \le x} \frac{\chi_0(n)\Lambda(n)}{n} = \sum_{n \le x} \frac{\Lambda(n)}{n} - \sum_{\substack{n \le x \\ \gcd(n,6)>1}} \frac{\Lambda(n)}{n},$$

since $\chi_0(n) = 0$ if $\gcd(n,6) > 1$ and $\chi_0(n) = 1$ if $\gcd(n,6) = 1$. Recall that the von Mangoldt function is only nonzero for prime-powers. In the second sum we require that the argument of the von Mangoldt function has a gcd with 6 that is greater than 1. The only prime-powers that have a gcd with 6 greater than 1 are those of the form 2^k and 3^k, since $6 = 2 \cdot 3$. For numbers of the form 2^k, we know that the von Mangoldt function gives us $\log 2$. Similarly, numbers of the form 3^k give $\log 3$. Knowing this, we can split up the second sum into two sums to get that

$$\sum_{n \le x} \frac{\Lambda(n)}{n} - \sum_{\substack{n \le x \\ \gcd(n,6)>1}} \frac{\Lambda(n)}{n} = \sum_{n \le x} \frac{\Lambda(n)}{n} - \sum_{2^k \le x} \frac{\log 2}{2^k} - \sum_{3^k \le x} \frac{\log 3}{3^k}.$$

Note that the final two sums are simply multiples of geometric series, at least as x approaches infinity. Therefore, we know that these sums must converge to some number, which means that they are $O(1)$. Therefore, we have that

$$\sum_{n\le x}\frac{\Lambda(n)}{n}-\sum_{2^k\le x}\frac{\log 2}{2^k}-\sum_{3^k\le x}\frac{\log 3}{3^k}=\sum_{n\le x}\frac{\Lambda(n)}{n}+O(1). \tag{18}$$

We can apply Lemma 4.2 to this equation to get that

$$\sum_{n\le x}\frac{\Lambda(n)}{n}+O(1)=\log x+O(1).$$

Putting this all together, we have shown that

$$\sum_{n\le x}\frac{\chi_0(n)\Lambda(n)}{n}=\log x+O(1), \tag{19}$$

which is the second part of the proof.

4.1.3. Part Three

In order to study the sum in (17) for the third part of our proof, we consider the sum

$$\sum_{n\ge 1}\frac{\chi(n)}{n}=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\cdots$$

We can see that this series is an alternating series with terms decreasing in absolute value. As such, the series must converge, and we will denote its value by L. We have that

$$\begin{aligned} L &= 1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\cdots \\ &= \left(1-\frac{1}{5}\right)+\left(\frac{1}{7}-\frac{1}{11}\right)+\left(\frac{1}{13}-\frac{1}{17}\right)\cdots \\ &> \frac{4}{5} \end{aligned}$$

In particular, this shows that $L\ne 0$. Furthermore, we can see that if we pick out any term in the series above, it will be greater than or equal to the absolute value of the sum of all the terms including and after it. In other words, if we let $x\ge 1$ and N be the smallest integer congruent to 1 or 5 modulo 6 that is greater than x, then we have that

$$\left|\sum_{n>x}\frac{\chi(n)}{n}\right|\le\left|\frac{\chi(N)}{N}\right|.$$

Since $\chi(N)$ is either 1 or -1 by definition, we have that

$$\left|\frac{\chi(N)}{N}\right|=\frac{1}{N}.$$

Furthermore, since $x < N$, we have that

$$\frac{1}{N} < \frac{1}{x}.$$

Putting this together, we have shown that

$$\left| \sum_{n > x} \frac{\chi(n)}{n} \right| < \frac{1}{x}. \tag{20}$$

Before using this, we can use Lemma 4.3 to get that

$$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{n \leq x} \frac{\chi(n)}{n} \sum_{d \mid n} \Lambda(d).$$

For each term n in the outer sum, the inner sum goes over every divisor d of n. We could instead look at this as for every term d in the outer sum and every multiple of d that is at most x in the inner sum. Making this change, we have that

$$\sum_{n \leq x} \frac{\chi(n)}{n} \sum_{d \mid n} \Lambda(d) = \sum_{d \leq x} \Lambda(d) \sum_{\substack{n \leq x \\ d \mid n}} \frac{\chi(n)}{n}.$$

Since each n is a multiple of d, we can think of it as $n = de$ for some e. We can use this to rewrite the above as one sum, which gives us that

$$\sum_{d \leq x} \Lambda(d) \sum_{\substack{n \leq x \\ d \mid n}} \frac{\chi(n)}{n} = \sum_{\substack{n, d \leq x \\ n = de \text{ for some } e}} \frac{\Lambda(d)\chi(n)}{n}.$$

Since $n = de \leq x$, we must have that $d \leq x$. This means we can rewrite the range of the sum, giving us that

$$\sum_{\substack{n, d \leq x \\ n = de \text{ for some } e}} \frac{\Lambda(d)\chi(n)}{n} = \sum_{de \leq x} \frac{\chi(de)\Lambda(d)}{de}.$$

Since χ is completely multiplicative, we have that

$$\sum_{de \leq x} \frac{\chi(de)\Lambda(d)}{de} = \sum_{de \leq x} \frac{\chi(d)\chi(e)\Lambda(d)}{de}.$$

We can rearrange this to get that

$$\sum_{de \leq x} \frac{\chi(d)\chi(e)\Lambda(d)}{de} = \sum_{de \leq x} \frac{\chi(d)\Lambda(d)}{d} \cdot \frac{\chi(e)}{e}.$$

We can now split this into two sums again, where one sum goes over $d \leq x$ and the other goes over $e \leq x/d$. That way we still have $de \leq x$. This gives us that

$$\sum_{de \leq x} \frac{\chi(d)\Lambda(d)}{d} \cdot \frac{\chi(e)}{e} = \sum_{de \leq x} \frac{\chi(d)\Lambda(d)}{d} \sum_{e \leq x/d} \frac{\chi(e)}{e}.$$

Putting what we have shown together, we have that

$$\sum_{de\leq x}\frac{\chi(de)\Lambda(d)}{de}=\sum_{d\leq x}\frac{\chi(d)\Lambda(d)}{d}\sum_{e\leq x/d}\frac{\chi(e)}{e}. \tag{21}$$

Recall that

$$L=\sum_{n\geq 1}\frac{\chi(n)}{n}.$$

We can rewrite this sum as

$$\sum_{n\geq 1}\frac{\chi(n)}{n}=\sum_{n\leq m}\frac{\chi(n)}{n}+\sum_{n>m}\frac{\chi(n)}{n}$$

for some $m\geq 1$. If we let $m=x/d$, then the sum on the left is the inner sum in Equation (21). Therefore, we get that

$$\sum_{e\leq x/d}\frac{\chi(e)}{e}=L-\sum_{e>x/d}\frac{\chi(e)}{e}. \tag{22}$$

Going back and applying inequality (20), we have that

$$\left|\sum_{e>x/d}\frac{\chi(e)}{e}\right|<\frac{1}{x/d}=\frac{d}{x}.$$

This gives us that

$$\sum_{e>x/d}\frac{\chi(e)}{e}=O(d/x).$$

Using this with Equation (22), we get that

$$\sum_{e\leq x/d}\frac{\chi(e)}{e}=L+O(d/x).$$

Combining this with Equation (21), we get that

$$\sum_{d\leq x}\frac{\chi(d)\Lambda(d)}{d}\sum_{e\leq x/d}\frac{\chi(e)}{e}=\sum_{d\leq x}\frac{\chi(d)\Lambda(d)}{d}(L+O(d/x)).$$

Distributing gives us that

$$\sum_{d\leq x}\frac{\chi(d)\Lambda(d)}{d}(L+O(d/x))=L\sum_{d\leq x}\frac{\chi(d)\Lambda(d)}{d}+\sum_{d\leq x}\frac{\chi(d)\Lambda(d)}{d}O(d/x). \tag{23}$$

Using properties of Big O, we have that

$$\left|\sum_{d\leq x}\frac{\chi(d)\Lambda(d)}{d}O(d/x)\right|\leq\sum_{d\leq x}\left|\frac{\chi(d)\Lambda(d)}{d}\right|c\cdot\frac{d}{x}$$

for some constant c and x sufficiently large. By definition, we have that $\chi(d) \leq 1$. Using this, and cancelling out the d in the numerator and denominator, we have that

$$\sum_{d \leq x} \left| \frac{\chi(d)\Lambda(d)}{d} \right| c \cdot \frac{d}{x} \leq \frac{c}{x} \sum_{d \leq x} \Lambda(d).$$

This gives us that

$$\sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} O(d/x) = O\left(\frac{1}{x} \sum_{d \leq x} \Lambda(d) \right).$$

Using this with Equation (23), we get that

$$L \sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} + \sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} O(d/x) = L \sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} \Lambda(d) \right).$$

Using Lemma 4.4 gives us

$$L \sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} \Lambda(d) \right) = L \sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} + O(1).$$

We can see that

$$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = O(1). \tag{24}$$

Indeed, we have that

$$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = \frac{\log 1}{1} - \frac{\log 3}{3} + \frac{\log 5}{5} - \cdots$$

which is an alternating sum with terms that are eventually decreasing. We can see that the terms eventually decrease, since we know that n grows faster than $\log n$ and the fractions are of the form $\log n/n$. We have shown that

$$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = L \sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} + O(1).$$

Combining this with Equation (24), we get that

$$L \sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} = O(1).$$

Since $L \neq 0$, it follows that

$$\sum_{d \leq x} \frac{\chi(d)\Lambda(d)}{d} = O(1), \tag{25}$$

which is the third part of the proof.

4.1.4. Part Four

Therefore, we move onto part four. We know that $\chi_0(n) = 1$ if the gcd of n and 6 is 1, and that χ_0 is 0 otherwise. If we apply this to Equation (19), we get that

$$\sum_{n\leq x} \frac{\chi_0(n)\Lambda(n)}{n} = \sum_{\substack{n\leq x\\ \gcd(n,6)=1}} \frac{\Lambda(n)}{n}.$$

We know that $\gcd(n, 6) = 1$ if and only if n is congruent to 1 or 5 modulo 6. Therefore, we can split this sum to get

$$\sum_{\substack{n\leq x\\ \gcd(n,6)=1}} \frac{\Lambda(n)}{n} = \sum_{\substack{n\leq x\\ n\equiv 1 \pmod 6}} \frac{\Lambda(n)}{n} + \sum_{\substack{n\leq x\\ n\equiv 5 \pmod 6}} \frac{\Lambda(n)}{n}.$$

From Equation (19), we know that

$$\sum_{\substack{n\leq x\\ n\equiv 1 \pmod 6}} \frac{\Lambda(n)}{n} + \sum_{\substack{n\leq x\\ n\equiv 5 \pmod 6}} \frac{\Lambda(n)}{n} = \log x + O(1). \tag{26}$$

In a similar way, we can apply the definition of χ to Equation (25) to get that

$$\sum_{d\leq x} \frac{\chi(d)\Lambda(d)}{d} = \sum_{\substack{n\leq x\\ n\equiv 1 \pmod 6}} \frac{\Lambda(n)}{n} - \sum_{\substack{n\leq x\\ n\equiv 5 \pmod 6}} \frac{\Lambda(n)}{n} = O(1). \tag{27}$$

Adding this to Equation (26) gives us

$$2 \sum_{\substack{n\leq x\\ n\equiv 1 \pmod 6}} \frac{\Lambda(n)}{n} = \log x + O(1),$$

which implies that

$$\sum_{\substack{n\leq x\\ n\equiv 1 \pmod 6}} \frac{\Lambda(n)}{n} = \frac{1}{2}\log x + O(1). \tag{28}$$

Subtracting Equation (27) from Equation (26) gives us that

$$\sum_{\substack{n\leq x\\ n\equiv 5 \pmod 6}} \frac{\Lambda(n)}{n} = \frac{1}{2}\log x + O(1), \tag{29}$$

so we are done with part four.

4.1.5. Part Five

Finally, we finish with part five. In Equation (15) we showed that

$$\sum_{\substack{n\leq x \\ n\equiv a \pmod{m}}} \frac{\Lambda(n)}{n} = \sum_{\substack{p\leq x \\ p\equiv a \pmod{m}}} \frac{\log p}{p} + O(1).$$

Applying this to Equation (28) and Equation (29) gives us that

$$\sum_{\substack{p\leq x \\ p\equiv 1 \pmod{6}}} \frac{\log p}{p} = \frac{1}{2}\log x + O(1)$$

and

$$\sum_{\substack{p\leq x \\ p\equiv 5 \pmod{6}}} \frac{\log p}{p} = \frac{1}{2}\log x + O(1).$$

Since $\varphi(6) = 2$, this is in fact what we wanted to show, so we are done. We have shown that there are infinitely many primes congruent to 1 and 5 modulo 6.

5. Conclusion

Within a fairly short time frame, we were able to start with the simple definition of a prime number and end up discussing some relatively complex topics from analytic number theory. Immediately, we were able to see how fundamentally important prime numbers are to number theory through the Fundamental Theorem of Arithmetic, and as we attempted to delve more into the world of analytic number theory, we got brief glimpses at the beautiful Prime Number Theorem and Dirichlet's Theorem. Ideally, this shows how the theory of analysis can be utilized to better understand these prime numbers. If nothing else, this goes to show how deep and difficult the study of prime numbers can be.

To expand upon the work seen here, one could look to generalize the proof given in Section 4 to try to prove the rest of Dirichlet's Theorem. Furthermore, more work could be done to improve the constants in Chebyshev's Theorem, which could also possibly help with Bertrand's Postulate. The books by Pollack, Andrews, and Jones and Jones could all be great resources for these problems, as they delve into the tools of analytic number theory with much more detail than what is seen here.

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