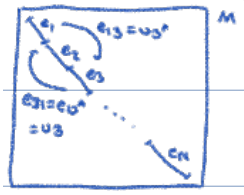
Thm: If M is a factor and it is discrete (i.e. has minimal projections), then $M \cong B(H)$.

Proof: Let's first do the case when M is finite dim. Let $(e_i)_{i\in I}$ be a maximal family of mutually orthogonal minimal projections in M (exist by Zorn's Lemma). We may assume $I=\{1,\dots,N\}$ for some $N\geq 1$. Also, note that $e_1+\dots+e_N=1$. Indeed, if $f=1-(e_1+\dots+e_N)$ is a proj. $\neq 0$, then $e_1 \prec f$ or $f \prec e_1$. If $e_1 \prec f$, then f contains a min. proj. $\sim e_1$, call it e_{N+1}, which contradicts maximality of $\{e_1,\dots,e_N\}$. If $f \prec e_1$, then $f \sim e_1$ and we can use the same argument. Thus, $e_1+\dots+e_N=1$.



$v_1 = e_1$

Since $e_1 \sim e_2 \sim \dots \sim e_N$ (as they are minimal), there exist p.i. $(v_i)_{i\in\{1,\dots,N\}}$ w/ $e_i \overset{v_i}{\sim} e_1$, so $e_i = v_i^* v_i$, $e_1 = v_i v_i^*$.

Let $e_{1i}=v_i$ and $e_{i1}=v_i^*$, and let $e_{ij}=e_{i1}e_{1j}=v_i^* v_j$.

Compare to $M_n(\mathbb{C})$

We show that there is a $*$-morphism from M to $M_n(\mathbb{C})$ taking e_{ij} to E_{ij}, w/ E_{ij} = matrix unit w/ 1 on entry (i,j).

Note $e_j = e_{j1}e_{1j}$, as $e_{j1}e_{1j} = v_j^* v_1 v_1^* v_j = v_j^* e_1 v_j = v_j^* v_j = e_j$. Also, $e_1 x e_j \in e_1 M e_1 = \mathbb{C} e_1$, so $e_{1i} x e_{j1} = x_{ij} e_1$ for some unique $x_{ij} \in \mathbb{C}$.

Let $x \in M$. $x = 1\cdot x \cdot 1 = (\sum_{i=1}^N e_i)\, x\, (\sum_{j=1}^N e_j) = \sum e_i x e_j = \sum e_{i1} e_{1i} x e_{j1} e_{1j} = \sum (x_{ij} e_1) e_{ij} = \sum x_{ij} e_{ij} \in \operatorname{span}\{e_{ij}\}$.

$e_2 \overset{v_2}{\sim} e_1$, $v_2 v_2^* = e_1$,

So, any $x \in M$ can be uniquely written as $x = \sum x_{ij} e_{ij}$, where x_{ij} is the unique scalar w/ $e_{1i} x e_{j1} = x_{ij} e_1$.

E_{21} E_{12}

$v_2^* v_2 = e_2$

So, $\Phi: M \to M_n(\mathbb{C})$, $\Phi(\sum x_{ij} e_{ij}) = \sum x_{ij} E_{ij}$. Φ is a $*$-morphism: it suffices to check $\Phi(e_{ij}^*) = \Phi(e_{ij})^*$ so

$\Phi(e_{ji}) = E_{ij}^* = E_{ji}$. True!

Φ is multiplicative: $\Phi(e_{ij} e_{kl}) \overset{?}{=} \Phi(e_{ij})\Phi(e_{kl}) = E_{ij} \cdot E_{kl} = \delta_{jk} E_{il}$. We show $e_{ij} e_{kl} = \delta_{jk} e_{il}$ (and we are done).

So, $e_{ij} e_{kl} = v_i^* v_j v_k^* v_l = v_i^* \delta_{jk} e_1 v_l = \delta_{jk} v_i^* v_l = \delta_{jk} e_{il}$.

$v_j v_k^* \xi = 0$ if $j \neq k$, as $v_k^* \xi \in \operatorname{Range}(e_k)$

If M is inf. dim., similarly let $(e_i)_{i\in I}$ be maximal as before. Fix $i_0 \in I$, and define $e_j \overset{v_j}{\sim} e_{i_0}$, so $v_j v_j^* = e_{i_0}$, $v_j^* v_j = e_j$. Define $e_{ij} = v_i^* v_j$.

Let H be a Hilbert space w/ ONB $(\xi_i)_{i\in I}$, and let E_{ij} = rank 1 operator taking ξ_j to ξ_i.

Define $\Phi(\sum x_{ij} e_{ij}) = \sum x_{ij} E_{ij}$ for finite sums and extend s.o. Then $M \overset{\Phi}{\cong} B(H)$.

Cor: If M is a fin. dim. factor, then $M \cong M_n(\mathbb{C})$ (M must be discrete).

Cor: If M is any fin. dim. $*$-algebra, then $M \cong M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C}) = \left\{ \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix} : A_i \in M_{n_i}(\mathbb{C}) \right\}$.

Proof: Strong induction on $\dim Z(M)$: If $\dim Z(M)=1$, then M is a factor, so $M=M_n(\mathbb{C})$.

If $\dim Z(M)>1$, then $\exists p\in Z(M)$ proj., $p\neq 0,1$. $M = \begin{pmatrix} \boxed{pMp} & 0 \\ 0 & \boxed{qMq} \end{pmatrix}$ w/ $q=1-p$. $M=1\cdot M\cdot 1=(p+q)M(p+q)=pMp+\overset{0}{\overset{\shortparallel}{pMq}}+\overset{0}{\overset{\shortparallel}{qMp}}+qMq=pMp+qMq=Mp\oplus Mq$.

$\dim Z(Mp)<\dim Z(M)$ and $\dim Z(Mq)<\dim Z(M)$ (if $x\in Z(Mp)$, then it commutes w/ Mq also, as $xq=0$, so $x\in Z(M)$). Done by induction!

Not equivalent to any subprojection; compare to finite sets

Def: $p\in P(M)$ is finite if $\forall q\in P(M)$, $q\leq p$ and $q\sim p\Rightarrow q\sim p$.

Examples: (i) $M=L^\infty([0,1],m)$, $p=\chi_E\neq 0$ ($m(E)>0$). If $q\leq p$, then $q=\chi_F$ w/ $F\subset E$. $q\sim p\Rightarrow q=p$, b/c $u^*u=uu^*$ in Abelian alg. So, all projs. are finite! Think about the trace: trace of any χ_E is finite.

(ii) $M=B(H)$. E_{11} is finite, $E_{11}+E_{22}$ is finite, etc. I is not finite unless H is fin. dim.

If $p\in B(H)$ and $\dim p(H)<\infty$, then p is finite: if $q<p$, then $q(H)\subsetneq p(H)$, hence $\dim q(H)<\dim p(H)$ and $q\not\sim p$.

(iii) $M=R=\overline{\bigotimes}^{\geq 0} M_2(\mathbb{C})$, any proj. is finite. If $q<p$, then $\tau(q)<\tau(p)$, so $q\not\sim p$ ($q\sim p\Rightarrow\tau(q)=\tau(p)$).