Last time: For M v.N. alg, $p,q \in P(M)$, we say $p \sim q$ if $\exists v \in M$ s.t. $v^*v = p$, $vv^* = q$. $p(H) \underset{v^*}{\overset{v}{\rightleftarrows}} q(H)$

We may also write $p \overset{v}{\sim} q$.

Example: $M = \mathbb{C} \oplus \mathbb{C} \subset B(\mathbb{C}^2) = M_2(\mathbb{C})$, $M = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x,y \in \mathbb{C} \right\}$.

$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M$ but $p \not\sim q$ in M as $p = vv^*$, $q = v^*v$ w/ $v \in M \Rightarrow v, v^*$ commute $\Rightarrow p = q$.

However, $p \sim q$ in $M_2(\mathbb{C})$, as $p = vv^*$, $q = v^*v$ w/ $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Similar to putting bijections together to make a larger bijection

Prop: If $(p_i)_{i \in I}, (q_i)_{i \in I} \in P(M)$ w/ both (p_i) and (q_i) mutually orthogonal, and $p_i \sim q_i$ $\forall i$, then $\sum_{i \in I} p_i \sim \sum_{i \in I} q_i$.

Remark: $\sum p_i = \bigvee p_i = \lim^{s.o.} \sum_{\text{finite}} p_i \in M$. *Limit exists b/c projections are mutually orthogonal

$p_1 \overset{v_1}{\longrightarrow} q_1$

$p_2 \overset{v_2}{\longrightarrow} q_2$

Proof: Say $p_i \overset{v_i}{\sim} q_i$ (so $v_iv_i^* = q_i$, $v_i^*v_i = p_i$). Then $p = \sum p_i \overset{v}{\sim} q = \sum q_i$ w/ $v = \sum v_i = \lim^{s.o.} \sum_{\text{finite}} v_i$.

Note that $v\xi = v_i\xi$ for $\xi \in p_i(H)$ and $v\xi = 0$ for $\xi \in (\sum p_i)(H)^\perp$. Need to check: $vv^* = q$, $v^*v = p$. $v_j^* v_i = 0$ if $i \neq j$, as v_j^* is 0 on $v_i(H) = q_i(H)$

Let's do the second one: for $\xi \in p_i(H)$ (i fixed), $v^*v\xi = v^*v_i\xi = (\sum_j v_j^*) v_i \xi = \lim^{s.o.} (\sum_{\text{finite}} v_j^*) v_i \xi = v_i^* v_i \xi = p_i \xi = (\sum_i p_i)\xi$. So, $v^*v = \sum p_i$.

"p is equivalent to a subprojection of q" Think about set card. equiv. to a subset

Def: For $p,q \in P(M)$, M v.N. alg., we say $p \precsim q$ if there is $q_1 \leq q$ s.t. $p \sim q_1$ (in M).

no subprojections

Remark: Our prev. example $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $M = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x,y \in M \right\}$ shows that not all projections are comparable: $p \not\precsim q$, $q \not\precsim p$.

$p \in Z(M)$, $p \neq 0,1$, $1-p \in Z(M)$ also.

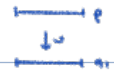
Heading towards: Comparison Thm: If M is a factor, then $\forall p,q \in P(M)$, $p \precsim q$ or $q \precsim p$.

Thm: $\precsim$ has the properties (on $P(M)$, M v.N alg). So, $\precsim$ is a partial order on eq. classes.

(i) $p \precsim p$

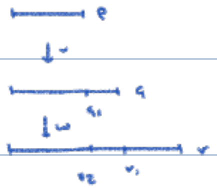
(ii) $p \precsim q$ and $q \sim r \Rightarrow p \precsim r$

(iii) $p \precsim q$ and $q \precsim p \Rightarrow p \sim q$.



$q_1 \neq vpv^* v = v$, not q_1!

Proof: (i) is obvious ($v = p$).

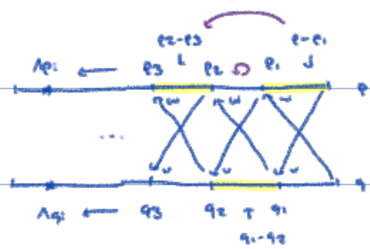


(ii) $p \leq q \Rightarrow p \overset{v}{\sim} q_1 \leq q$ We show $\exists r_2 \leq r_1 (\leq r)$ s.t. $p \overset{wv}{\sim} r_2$. $p \overset{v}{\sim} q_1 \Rightarrow p = v^*v,\ q_1 = vv^* = vpv^*$ $(vpv^* = vv^*vv^* = q_1 q_1 = q_1)$.

$q \leq r \Rightarrow q \overset{w}{\sim} r_1 \leq r$ Define $r_2 = wq_1w^* \leq wqw^* = r_1$ and $p \overset{wv}{\sim} r_2$. Indeed, $(wv)(wv)^* = wvv^*w = wq_1w^* = r_2$,

$(wv)^*(wv) = v^*w^*wv = v^*qv = v^*pv = p$.

as $v^*(q-p)v = 0$ b/c $v^*|_{(q-p)(H)} = 0$



(iii) This is a v.N. version of Cantor-Schröder-Bernstein. Assume that $p \leq q$ and $q \leq p$.

So, $p \overset{v}{\sim} q_1 \leq q$ and $q \overset{w}{\sim} p_1 \leq p$. As $q_1 \leq q$, $q_1 \overset{w}{\sim} p_2 \leq p_1$ w/ $p_2 = wq_1w^*$.

As $p_1 \leq p$ and $p \overset{v}{\sim} q_1$, it follows $p_1 \overset{v}{\sim} q_2 \leq q_1$ w/ $q_2 = vp_1v^*$.

We show $p \sim p_1$ (hence $p \sim p_1 \overset{w^*}{\sim} q$). $p - p_1 \overset{wv}{\sim} p_2 - p_3 \overset{wv}{\sim} p_4 - p_5 \sim \cdots$ and $p = (p - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_4) + \cdots + \Lambda_{i \geq 1} p_i$.

$$
\begin{array}{ccccc}
wv(p-p_1) & p_1 - p_2 & \overset{wv^*}{(p_2-p_3)} & p_3 - p_4 & \\
p_1 = (p_2 - p_3) + & (p_1 - p_2) + & (p_4 - p_5) + & (p_3 - p_4) + \cdots + & \Lambda_{i \geq 1} p_i
\end{array}
$$

Comments on a HW problem

v.N. Ergodic Thm: If $\|T\| \leq 1$ $(T \in B(H))$, then $\dfrac{I + T + T^2 + \cdots + T^{n-1}}{n} \xrightarrow{\text{s.o.}} p$ w/ $p = \text{proj}\{v \in H : Tv = v\}$.

Particular case: When $H = \text{span}\{e_1, \ldots, e_n\}$ and T commutes $\{e_1, \ldots, e_n\}$.