

Perturbations of Operator Algebras

Oral Exam Presentation

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- We focus on the article in Math Annalen titled “Subalgebras of a Finite Algebra.”

General Setting (von Neumann algebras)

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 - $*$ -algebra means we have addition, multiplication, scalar multiplication, and an involution $(*)$ satisfying certain properties.
 - The closure condition is equivalent to $L = L''$, where

$$L' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx \ \forall y \in L\}$$

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- On $\mathcal{B}(\mathcal{H})$, we have the usual norm
$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

General Setting (trace)

- We require that L has a trace $\tau : L \rightarrow \mathbb{C}$ (so τ is linear and $\tau(xy) = \tau(yx) \ \forall x, y \in L$) that is:
 - positive: $\tau(x^*x) \geq 0 \ \forall x \in L$
 - faithful: $\tau(x^*x) = 0 \Rightarrow x = 0$
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- A von Neumann algebra with a trace satisfying these properties is called finite (not to be confused with finite dimensional).

General Setting (in finite dimensions)

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$$L = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$$

with trace

$$\tau = c_1 \tau_{M_{k_1}} + \cdots + c_n \tau_{M_{k_n}},$$

where $\tau_{M_k} = \frac{1}{k} \text{Tr}_{M_k}$ and $c_1 + \cdots + c_n = 1$.

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- These can be thought of as tuples of matrices $(A_{k_1}, \dots, A_{k_n})$ or block matrices

$$\begin{pmatrix} A_{k_1} & & 0 \\ & \ddots & \\ 0 & & A_{k_n} \end{pmatrix}$$

General Setting (factors)

Sometimes we will also need L to be a factor, meaning that $Z(L) = \mathbb{C} \cdot I$ ($Z(L) = L' \cap L$).

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- If L is finite dimensional (in which case $L \cong M_k(\mathbb{C})$), we say L is a factor of type I_k .
- If L is infinite dimensional, we say L is a factor of type II_1 .

General Setting (what it means to be “close”)

In these papers, Christensen studies subalgebras (or rather, pairs of subalgebras) $M, N \subset L$ that are “close” in the sense that

$$\forall x \in M_1, \exists y \in N \text{ such that } \|x - y\|_2 < \delta \quad (*)$$

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- M_1 denotes the unit ball of M .
- $\|\cdot\|_2$ denotes the norm induced by the trace:
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- $\overline{M}^{\|\cdot\|_2}$ is the GNS construction and is denoted by $L^2(M, \tau)$.

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We write $M \overset{\delta}{\subset} N$ when $(*)$ occurs.

Important Remark

For some of these proofs, Christensen introduces what will later be known as Jones' basic construction, from Jones' fields medal paper (83).

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Theorem I

We prove two results, then combine them. Recall that (L, τ) is a finite von Neumann algebra.

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Theorem (I)

If $M, N \subset L$ with M finite dimensional, N a II_1 factor, and $M \overset{\delta}{\subset} N$ for some $\delta < 1/\sqrt{2}$, then there exist $\Phi : M \rightarrow N$, Φ an isomorphism from M to a von Neumann subalgebra $\Phi(M)$ of N , and a fixed constant c such that for all $x \in M_1$

$$\|\Phi(x) - x\|_2 \leq c\delta^{1/2}$$

($c = 1050$ should work).

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*If $M, N \subset L$ with M finite dimensional, N a II_1 factor, and $M \overset{\delta}{\subset} N$ for some $\delta < 1/\sqrt{2}$, then there exist $\Phi : M \rightarrow N$, Φ an **isomorphism** from M to a von Neumann subalgebra $\Phi(M)$ of N , and a fixed constant c such that for all $x \in M_1$*

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Theorem II

Theorem (II)

If $M \subset L$ is a von Neumann subalgebra of L and $\Phi : M \rightarrow L$ is a $$ -homomorphism such that $\|\Phi(x) - x\|_2 \leq t$ for all $x \in M_1$, for some $0 < t < 1$, then for all $x \in M$*

$$\Phi(x)q = v^*xv,$$

where $v \in L$ is a partial isometry with

- $q := v^*v \in \Phi(M)'$
- $r := vv^* \in M'$
- $\|1 - v\|_2 \leq 2t, \|1 - r\|_2 \leq t, \|1 - q\|_2 \leq t.$

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v being a partial isometry means that v^*v, vv^* are projections.

Theorem II'

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*Under the hypotheses of II, if M is a I_k factor, then there is a unitary u in L such that $\Phi(x) = u^*xu$ for all $x \in M$ and $\|1 - u\|_2 \leq 3t$.*

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u being unitary means that $u^*u = uu^* = 1$. We write $\mathcal{U}(L)$ for the set of unitaries in L . Any element of L can be written as a linear combination of unitaries.

Theorem III

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Theorem (III)

*If $M \overset{\delta}{\subset} N$ with $M \text{ } l_k$, $N \text{ } II_1$, and $\delta < 10^{-6}$, then there is a unitary u in L such that $u^*Mu \subset N$ and $\|1 - u\|_2 \leq 450\delta^{1/2}$.*

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*If $M \overset{\delta}{\subset} N$ with $M \perp_k$, $N \perp_1$, and $\delta < 10^{-6}$, then there is a unitary u in L such that $u^*Mu \subset N$ and $\|1 - u\|_2 \leq 450\delta^{1/2}$.*

Proof idea: I gives us the map Φ and II' tells us that $\Phi(x) = u^*xu$.

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Proof of II (step 1)

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- Step 1: Come up with an intertwiner: $k \in L$ with $k\Phi(u) = uk$.

$$\|\Phi(x) - x\|_2 \leq t \quad \forall x \in M_1$$

$$\Rightarrow \|\Phi(u) - u\|_2 \leq t \quad \forall u \in \mathcal{U}(M)$$

$$\Rightarrow \|u^*\Phi(u) - 1\|_2 \leq t \quad \forall u \in \mathcal{U}(M) \quad (\|uz\|_2 = \|zu\|_2 = \|z\|_2)$$

Proof of II (step 1)

Consider

$$K = \overline{\left\{ \sum_{i=1}^n c_i u_i^* \Phi(u_i) : n \geq 1, u_i \in \mathcal{U}(M), c_i > 0, \sum_{i=1}^n c_i = 1 \right\}}^{\|\cdot\|_2}.$$

K is a convex closed set in the Hilbert space $L^2(L, \tau)$, so it has a unique element k of minimal norm $\|\cdot\|_2$.

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Proposition

If $C \subset L$ is a bounded set (in regular norm $\| \cdot \|$), then $\overline{C}^{\text{s.o.}}$ (which is part of L , as $L = \overline{L}^{\text{s.o.}}$) is the same as $\overline{C}^{\| \cdot \|_2}$ (in $L^2(L, \tau)$).

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Since

$$\|c_1 u_1^* \Phi(u_1) + \cdots + c_n u_n^* \Phi(u_n)\| \leq c_1 + \cdots + c_n = 1,$$

it follows that $K \subset L$. In particular, $k \in L$.

Proof of II (step 1)

Note that for all $u \in \mathcal{U}(M)$, we have $u^*K\Phi(u) \subset K$. Indeed, for all $u_0 \in \mathcal{U}(M)$:

$$u^*(u_0^*\Phi(u_0))\Phi(u) = (u_0u)^*\Phi(u_0u) \in K.$$

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In particular, $u^*k\Phi(u) \in K$. Also, $\|u^*k\Phi(u)\|_2 = \|k\|_2$.

By uniqueness, it follows that $u^*k\Phi(u) = k$, or $k\Phi(u) = uk$.

Proof of II (step 1)

$k\Phi(u) = uk$ doesn't say anything if $k = 0$. However,

$$\|u^*\Phi(u) - 1\|_2 \leq t < 1 \Rightarrow \|k_0 - 1\|_2 \leq t < 1$$

for all $k_0 \in K$, so $\|k - 1\|_2 \leq t < 1$. Hence, $k \neq 0$.

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- Step 2: From k , we collect its “unitary” part.

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Consider the polar decomposition $k = va$. Here

$$a = |k| = (k^*k)^{1/2}$$

and v is a partial isometry, with

$$\ker k = \ker v = \ker a.$$

Since v is a partial isometry, $q := v^*v$ and $r := vv^*$ are projections.

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We have $k = va$. It is known that v, a are still in L , since $k \in L$ and L is a von Neumann algebra.

Also,

$$q = v^*v = s_r(k) = s(k^*k)$$

and

$$r = vv^* = s_\ell(k) = s(kk^*).$$

s_ℓ, s_r are the projections onto $\overline{\text{Im } k}$, $(\ker k)^\perp$ respectively and are called the left and right support. $s = s_\ell = s_r$ for self-adjoint operators and is just called the support. s, s_ℓ, s_r are also in L .

Proof of II (step 2)

From $\|k - 1\|_2 \leq t$, it can be shown that

$$\|1 - v\|_2 \leq 2t, \quad \|1 - q\|_2 \leq t, \quad \|1 - r\|_2 \leq t$$

(this follows from a technical lemma in the paper, which we skip).

Proof of II (step 2)

From Step 1, we know $k\Phi(u) = uk$ for all $u \in \mathcal{U}(M)$. Taking adjoints gives $\Phi(u^*)k^* = k^*u^*$. Replacing u^* with u gives $\Phi(u)k^* = k^*u$. So, for all $u \in \mathcal{U}(M)$

$$\begin{cases} k\Phi(u) = uk \\ \Phi(u)k^* = k^*u \end{cases}$$

Proof of II (step 2)

$$\begin{cases} k\Phi(u) = uk \\ \Phi(u)k^* = k^*u \end{cases}$$

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$$\Rightarrow k^*(k\Phi(u)) = k^*(uk) = (k^*u)k = (\Phi(u)k^*)k$$

This shows that k^*k commutes with $\Phi(u)$ for all $u \in \mathcal{U}(M)$, and thus $k^*k \in \Phi(M)'$. So, $a = (k^*k)^{1/2}$ and $q = s(k^*k)$ are in $\Phi(M)'$ also.

Proof of II (step 2)

$$\begin{cases} k\Phi(u) = uk \\ \Phi(u)k^* = k^*u \end{cases}$$

$$\Rightarrow (uk)k^* = (k\Phi(u))k^* = k(\Phi(u)k^*) = k(k^*u)$$

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$$\Rightarrow (uk)k^* = (k\Phi(u))k^* = k(\Phi(u)k^*) = k(k^*u)$$

This shows that kk^* commutes with u for all $u \in \mathcal{U}(M)$, and thus $kk^* \in M'$. So, $r = s(kk^*)$ is in M' also.

Proof of II (step 2)

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$$\Rightarrow \Phi(u)qa = v^*uva \quad (q, a \in \Phi(M)')$$

$$\Rightarrow \Phi(u)q = v^*uv \quad (\mathcal{H} = \overline{a(\mathcal{H})} \oplus \overline{a(\mathcal{H})}^\perp)$$

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It follows that $\Phi(x)q = v^*xv$ for all $x \in M$. QED.

Corollary

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Let (M, τ) be a finite factor. Let $\Phi : M \rightarrow M$ be a $*$ -homomorphism such that $\|\Phi(x) - x\|_2 \leq t < 1$ for all $x \in M_1$ (for $0 < t < 1$). Then there is a unitary $u \in M$ with $\|1 - u\|_2 \leq 2t$ and $\Phi(x) = u^*xu$ for all $x \in M$.

- From II, $\Phi(x)q = v^*xv$ with $q = v^*v$ and $r = vv^*$.

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- $r \in M \cap M' = Z(M) = \mathbb{C} \cdot 1$.

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- $r \in M \cap M' = Z(M) = \mathbb{C} \cdot 1$.
- r is a nonzero projection, so $r = 1$ (r is nonzero since $\|1 - r\|_2 < 1$).

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- Since q is a projection and τ is faithful, we get $q = 1$.

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- Since q is a projection and τ is faithful, we get $q = 1$.
- $v^*v = vv^* = 1$ implies v is a unitary, and $\Phi(x) = v^*xv$.

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- Since q is a projection and τ is faithful, we get $q = 1$.
- $v^*v = vv^* = 1$ implies v is a unitary, and $\Phi(x) = v^*xv$.
- We know $\|1 - v\|_2 \leq 2t$ from II.

Theorem II'

Theorem (II')

*Under the hypotheses of II, if M is a I_k factor, then there is a unitary u in L such that $\Phi(x) = u^*xu$ for all $x \in M$ and $\|1 - u\|_2 \leq 3t$.*

Proof of Π' (idea)

Idea

From Π , there exists $v \in L$ such that $q = v^*v \in \Phi(M)'$, $r = vv^* \in M'$, and $\Phi(x)q = v^*xv$ for all $x \in M$.

Proof of Π' (idea)

Idea

From Π , there exists $v \in L$ such that $q = v^*v \in \Phi(M)'$, $r = vv^* \in M'$, and $\Phi(x)q = v^*xv$ for all $x \in M$.

The map $z \mapsto v^*zv$ sends Mr to $\Phi(M)q$:

$$xr \mapsto v^*xrv = v^*xvv^*v = \Phi(x)q^2 = \Phi(x)q.$$

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We extend this to a map $z \mapsto u^*zu$ ($u \in \mathcal{U}(M)$) that takes M to $\Phi(M)$. First, we get another partial isometry v_1 so that $z \mapsto v_1^*zv_1$ sends $M(1-r)$ to $\Phi(M)(1-q)$, then we let $u = v + v_1$.

Proof of Π' (picture)

$$\begin{array}{ccc} & Mr & \xrightarrow{z \mapsto v^*zv} \Phi(M)q \\ & \swarrow \quad \searrow & \\ M & & \Phi(M) \\ & \nwarrow \quad \nearrow & \\ & M(1-r) & \xrightarrow{z \mapsto v_1^*zv_1} \Phi(M)(1-q) \end{array}$$

Proof of II' (comparing projections)

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The Comparison Theorem

There are projections $x, y \in Z(M)$ such that $px \prec qx$ and $qy \prec py$. In particular, in a factor, either $p \prec q$ or $q \prec p$.

Proof of II' (claim)

We know that M is a I_k factor, so $M \cong M_k(\mathbb{C})$. Let e_{ij} be the matrix units of M (corresponding to the matrix with a 1 in the ij entry and 0's elsewhere).

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Claim

$e_{11}(1-r) \sim \Phi(e_{11})(1-q)$, i.e. there exists a partial isometry $w \in L$ such that $w^*w = \Phi(e_{11})(1-q)$ and $ww^* = e_{11}(1-r)$.

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Using the same argument we'll see next, it follows that $e_{ii}(1-r)z \prec \Phi(e_{ii})(1-q)z$ for all $1 \leq i \leq k$, which implies that $(1-r)z \prec (1-q)z$.

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However,

$r \sim q \Rightarrow 1-r \sim 1-q \Rightarrow (1-r)z \sim (1-q)z \ \forall z \in Z(L)$
projection. $\Rightarrow \Leftarrow$

Proof of II' (diagram)

We also know that $e_{11} \sim e_{ii}$ for all $1 \leq i \leq k$, as

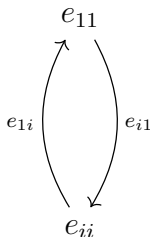
$$e_{11} = e_{1i}e_{i1} = (e_{i1})^*e_{i1}.$$

Proof of II' (diagram)

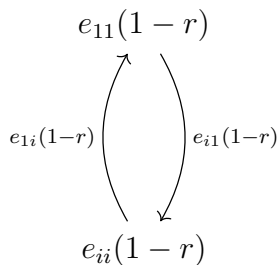
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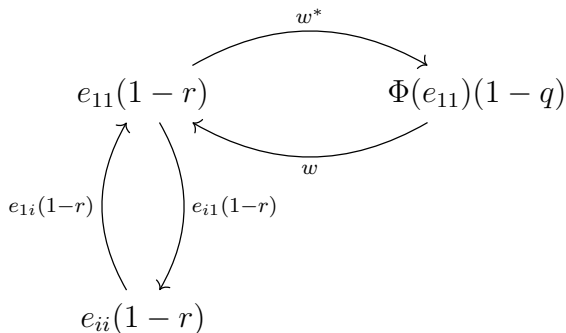
So, we get the diagram



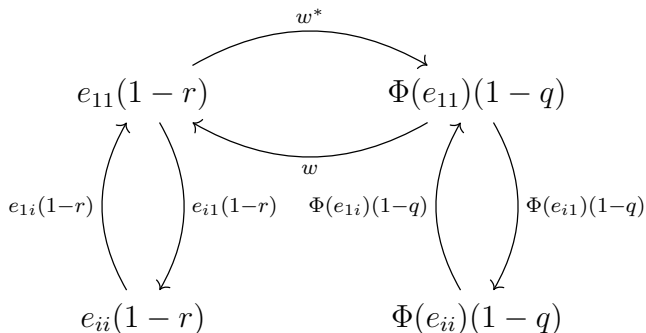
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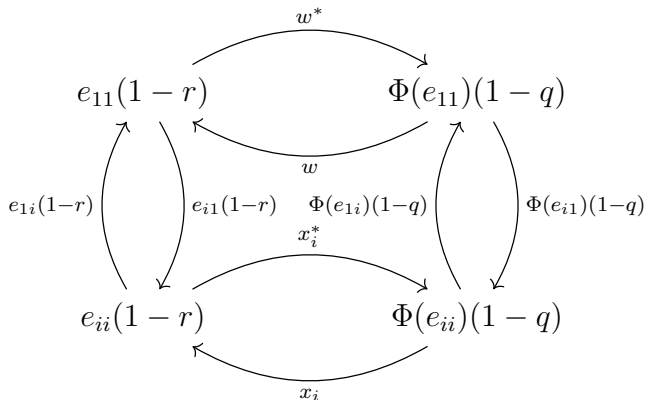
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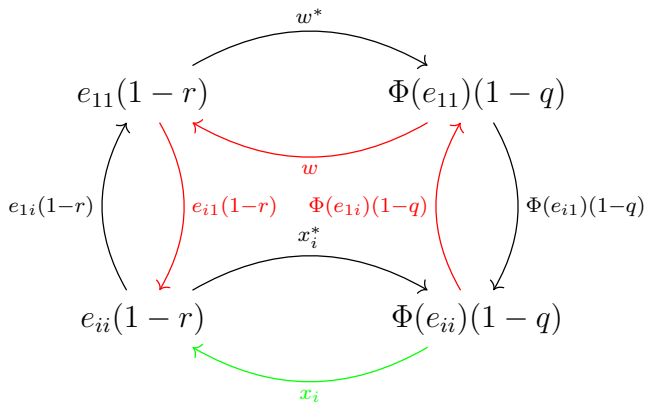
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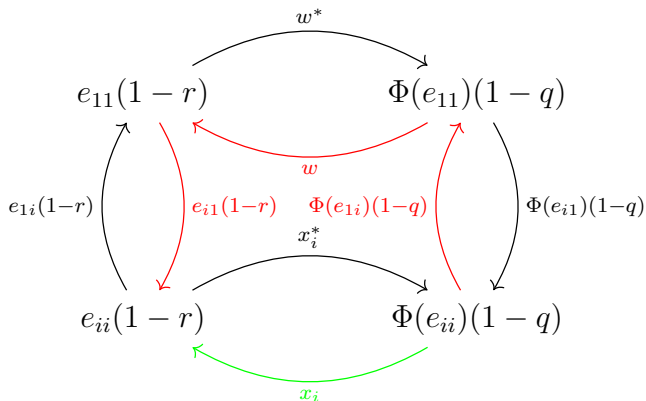
Proof of II' (diagram)



Proof of Π' (obtaining v_1)

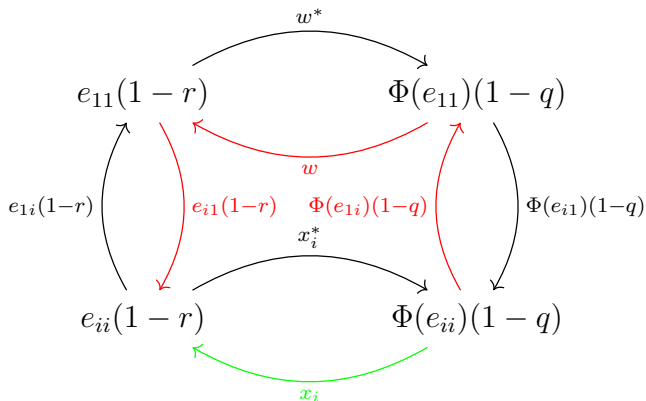


Proof of Π' (obtaining v_1)



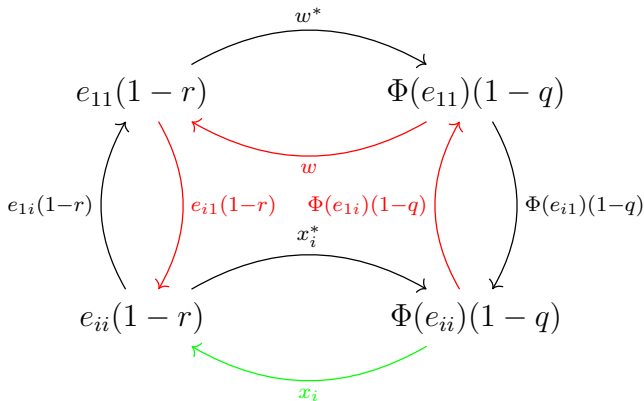
Let $x_i = e_{i1}(1-r)w\Phi(e_{1i})(1-q)$.

Proof of Π' (obtaining v_1)



Let $x_i = e_{i1}(1-r)w\Phi(e_{1i})(1-q)$. Let $v_1 = \sum_{i=1}^k x_i$.

Proof of Π' (obtaining v_1)



Let $x_i = e_{i1}(1-r)w\Phi(e_{1i})(1-q)$. Let $v_1 = \sum_{i=1}^k x_i$.
 v_1 is a partial isometry with $v_1^*v_1 = 1-q$ and $v_1v_1^* = 1-r$.

Proof of II' ($v_1^* v_1 = 1 - q$)

$$\begin{aligned} v_1^* v_1 &= \sum_{j=1}^k x_j^* \sum_{i=1}^k x_i \\ &= \sum_{j=1}^k \Phi(e_{j1})(1-q)w^* e_{1j}(1-r) \sum_{i=1}^k e_{i1}(1-r)w\Phi(e_{1i})(1-q) \\ &= \sum_{i,j=1}^k \Phi(e_{j1})(1-q)w^* e_{1j}(1-r)e_{i1}(1-r)w\Phi(e_{1i})(1-q) \\ &= \sum_{i,j=1}^k \Phi(e_{j1})(1-q)w^*(\delta_{ji}e_{11}(1-r))w\Phi(e_{1i})(1-q) \\ &= \sum_{i,j=1}^k \Phi(e_{j1})(1-q)w^*(\delta_{ji}ww^*)w\Phi(e_{1i})(1-q) \end{aligned}$$

Proof of II' ($v_1^* v_1 = 1 - q$)

$$\begin{aligned} &= \sum_{i=1}^k \Phi(e_{i1})(1-q)w^*w\Phi(e_{1i})(1-q) \\ &= \sum_{i=1}^k \Phi(e_{i1})(1-q)\Phi(e_{11})(1-q)\Phi(e_{1i})(1-q) \\ &= \sum_{i=1}^k \Phi(e_{ii})(1-q) \\ &= (1-q)\Phi\left(\sum_{i=1}^k e_{ii}\right) \\ &= (1-q)\Phi(1) \\ &= 1-q. \end{aligned}$$

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$v_1 v_1^* = 1 - r$ is similar.

Proof of II' (obtaining u)

Let $u = v + v_1$. Then u is a unitary. Indeed:

$$\begin{aligned}uu^* &= (v + v_1)(v + v_1)^* \\&= vv^* + v_1v_1^* + vv_1^* + v_1v^* \\&= r + 1 - r + 0 + 0 \\&= 1.\end{aligned}$$

($vv_1^* = 0$, as $\overline{\text{Im } v_1^*} = \text{Im}(1 - q)$ and $v = 0$ here)

u^*u is similar.

Proof of II' ($\Phi(x) = u^*xu$)

Finally,

$$\begin{aligned}u^*e_{nm}u &= v^*e_{nm}v + v_1^*e_{nm}v_1 \\&= \Phi(e_{nm})q + \Phi(e_{nm})(1 - q) \\&= \Phi(e_{nm}).\end{aligned}$$

(the calculation for $v_1^*e_{nm}v_1$ is similar to $v_1^*v_1$)

It follows that $\Phi(x) = u^*xu$ for all $x \in M$.

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It follows that $\Phi(x) = u^*xu$ for all $x \in M$.

Also,

$$\|1 - u\|_2 \leq \|1 - v\|_2 + \|v_1\|_2 \leq 2t + \|1 - q\|_2 \leq 3t.$$

QED.

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The Basic Construction

The following idea of Christensen's will later become known as Jones' basic construction, as it was used ingeniously in Vaughan Jones' papers that led to a fields medal.

Setting

L is a type II_1 von Neumann algebra and N is a von Neumann subalgebra of L . We have $N \subset L \subset \mathcal{B}(\mathcal{H})$ with $\mathcal{H} = L^2(L, \tau)$.

Since τ is faithful, we also have the embedding $L \subset L^2(L, \tau)$.

Note that L acts on \mathcal{H} via left multiplication, as this is the GNS construction. If we denote by ξ or $\hat{1}$ the vector in \mathcal{H} corresponding to 1 in L , then $x \cdot \hat{1} = \hat{x}$ for all $x \in L$ (where \hat{x} denotes the vector corresponding to x via the embedding $L \subset L^2(L, \tau)$).

We will denote by e the projection from $L^2(L, \tau)$ onto $L^2(N, \tau)$ and by $E : L \rightarrow N$ the conditional expectation from L to N , which is just the restriction of e to L . That is to say:

$$e(\hat{x}) = \widehat{E(x)}.$$

Properties of e and E

- E is well-defined (i.e. it takes values in N) and bimodular: $E(axb) = aE(x)b$ for all $a, b \in N$ and $x \in L$.

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- e commutes with N : $ey = ye$ for all $y \in N$. Indeed, for $x \in L$

$$ey\hat{x} = e\widehat{yx} = \widehat{E(yx)} = \widehat{yE(x)} = y\widehat{E(x)} = ye\hat{x}.$$

Properties of e and E (continued)

- For all $x \in L$, $exe = E(x)e$. Indeed, for $z \in L$

$$\begin{aligned} exe\hat{z} &= ex\widehat{E(z)} = \widehat{exE(z)} = \widehat{E(xE(z))} \\ &= \widehat{E(x)E(z)} = E(x)\widehat{E(z)} \\ &= E(x)e\hat{z}. \end{aligned}$$

Similarly, $eLe = Ne$.

The Basic Construction

We define the basic construction to be

$$L_1 = \langle L, e \rangle = (L \cup \{e\})'' \subset \mathcal{B}(\mathcal{H}),$$

the smallest von Neumann algebra of $\mathcal{B}(\mathcal{H})$ (for $\mathcal{H} = L^2(L, \tau)$) containing L and e .

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- $eL_1e = Ne$.
- If N is a factor, then so is L_1 .
- The trace τ of L gives a (faithful, normal, semifinite) trace φ on L_1 with $\varphi(e) = 1$, related to τ by $\varphi(xe) = \tau(x)$ for all $x \in L$.

To be more precise

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- We close in the s.o. topology of $\mathcal{B}(\mathcal{H})$:
 $\overline{L + \text{span } LeL}^{\text{s.o.}} = L_1$.
- The trace is defined by $\varphi(xey) = \tau(xy)$ for all $x, y \in L$ then extended to the closure $\overline{L + \text{span } LeL}^{\text{s.o.}}$ by s.o. continuity.

A note on φ

Note that φ may not take just finite values; $\varphi(1) = \infty$ is possible. That is to say, L_1 is in general a semifinite algebra, so it may not be finite.

If N is a factor, then L_1 is also a factor, and thus it has a unique trace (up to rescaling). So, φ is the trace of L_1 .

Example

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We can identify $\mathcal{B}(M_2(\mathbb{C}))$ with $M_4(\mathbb{C})$, with $L = M_2(\mathbb{C})$ embedded in it via left multiplication.

Example (continued)

It can help to think of $M_4(\mathbb{C})$ as $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ and $L = M_2(\mathbb{C}) \otimes I \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$.

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In this case, $e \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is $e_{11} \otimes e_{11} + e_{22} \otimes e_{22} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\langle L, e \rangle = M_2(\mathbb{C}) \otimes D_2$.

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and $\langle L, e \rangle = M_2(\mathbb{C}) \otimes D_2$.

Lastly, $\varphi = \frac{1}{2} \text{Tr}_{M_4(\mathbb{C})}$ (so $\varphi(e) = 1$).

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Theorem I

Theorem (I)

If $M, N \subset L$ with M finite dimensional, N a II_1 factor, and $M \overset{\delta}{\subset} N$ for some $\delta < 1/\sqrt{2}$, then there exist $\Phi : M \rightarrow N$, Φ an isomorphism from M to a von Neumann subalgebra $\Phi(M)$ of N , and a fixed constant c such that for all $x \in M_1$

$$\|\Phi(x) - x\|_2 \leq c\delta^{1/2}$$

($c = 1050$ should work).

Proof of I (setup)

We have $M, N \subset L$, M finite dimensional, N II_1 factor,
 $M \overset{\delta}{\subset} N$ with $\delta > 0$ small.

We want to construct $\Phi : M \rightarrow N$ morphism (isomorphism onto $\Phi(M)$) with $\|\Phi(x) - x\|_2 \leq c\delta^{1/2}$ for all $x \in M_1$, for some constant c (specifically $c = 1050$).

Proof of I (idea)

- Construct $L_1 = \langle L, e \rangle$ with e projection from $L^2(L, \tau)$ to $L^2(N, \tau)$. Since N is a factor, L_1 is also. We know e commutes with N , and since $M \overset{\delta}{\subset} N$, we can show that e “almost” commutes with M . In other words, $\|e - u^*eu\|_{2,\varphi}$ is small for $u \in \mathcal{U}(M)$.

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- We can find k of minimal norm $\| \cdot \|_{2,\varphi}$ in $\overline{\text{co}}_M^{\|\cdot\|_{2,\varphi}}(e)$, with $\|e - k\|_{2,\varphi}$ small and $k \in M'$.

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- From k , we construct one of its spectral projections q , with $\|e - q\|_{2,\varphi}$ small and $q \in M'$.
- Since L_1 is a factor with $e, q \in L$, the Comparison Theorem tells us that $q \prec e$ or $q \succ e$.
 - If $q \overset{v}{\prec} e$, we construct $\Phi(x)$ from v^*xv .
 - If $q \succ e$, then $q \geq r \sim e$, and we use r as above ($r \sim e$, so $r \prec e$).

Proof of I (the basic construction)

Consider the basic construction $L_1 = \langle L, e \rangle$, where e is the projection from $L^2(L, \tau)$ to $L^2(N, \tau)$.

- N is a factor, so L_1 is a factor.
- L_1 comes with a semifinite trace φ satisfying $\varphi(e) = 1$.

Proof of I (e almost commutes with M)

We know that e commutes with N , so $ue = eu$ for all $u \in \mathcal{U}(N)$. Hence $u^*eu = e$ for all $u \in \mathcal{U}(N)$.

Proof of I (e almost commutes with M)

We know that e commutes with N , so $ue = eu$ for all $u \in \mathcal{U}(N)$. Hence $u^*eu = e$ for all $u \in \mathcal{U}(N)$.

Since $M \overset{\delta}{\subset} N$, we expect that for $u \in \mathcal{U}(M)$, u^*eu is “close” to e . Indeed:

$$\begin{aligned} \|e - u^*eu\|_{2,\varphi}^2 &= \varphi((e - u^*eu)^*(e - u^*eu)) \\ &= \varphi(e - eu^*eu - u^*eue + u^*eu) \\ &= 2\varphi(e - eu^*eue) \quad (\varphi \text{ trace}) \\ &= 2\varphi(e - E(u^*)E(u)e) \quad (exe = E(x)e) \\ &= 2\varphi((1 - E(u^*)E(u))e) \\ &= 2\tau(1 - E(u^*)E(u)) \quad (\varphi(xe) = \tau(x)) \\ &\leq 2\delta^2. \end{aligned}$$

Proof of I (e almost commutes with M)

Thus, $\|e - u^*eu\|_{2,\varphi} \leq \sqrt{2}\delta$. For the last step,

$$2\tau(1 - E(u^*)E(u)) \leq 2\delta^2,$$

we used

- $\tau(1 - E(u^*)E(u))^{1/2} = \|E(u) - u\|_{2,\tau}$
- $\|E(u) - u\|_{2,\tau} \leq \delta$, which follows from $M \overset{\delta}{\subset} N$ and the fact that $E(u)$ minimizes the $\|\cdot\|_{2,\tau}$ distance between u and N

Proof of I (closed convex hull)

Consider now the closed convex hull $\overline{\text{co}}_M^{\|\cdot\|_{2,\varphi}}(e) =$

$$\overline{\left\{ \sum_{i=1}^n c_i u_i^* e u_i : n \geq 1, u_i \in \mathcal{U}(M), c_i > 0, \sum_{i=1}^n c_i = 1 \right\}}^{\|\cdot\|_{2,\varphi}},$$

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Also, $k \in \langle M, e \rangle$ (with $\langle M, e \rangle = (M \cup \{e\})''$).

Proof of I (spectral projection)

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$$\gamma = 2^{1/4} \delta^{1/2} (1 - 2^{1/4} \delta^{1/2})^{-1}.$$

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$$\gamma = 2^{1/4} \delta^{1/2} (1 - 2^{1/4} \delta^{1/2})^{-1}.$$

Since $\varphi(e) = 1$, we can show that $|1 - \varphi(q)| \leq \gamma^2$.

Proof of I (The Comparison Theorem)

Since $k \in M' \cap \langle M, e \rangle$, so is q :

$$q \in M' \cap \langle M, e \rangle \subset M' \cap L_1.$$

Since L_1 is a factor and q, e are projections in L_1 , the Comparison Theorem tells us that $q \prec e$ or $e \prec q$.

Proof of I (case 1: $q \prec e$)

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We have

$$\begin{aligned}\|q - q'\|_{2,\varphi} &\leq \|q - e\|_{2,\varphi} + \|e - q'\|_{2,\varphi} \\ &\leq \gamma + \varphi(e - q')^{1/2} \\ &= \gamma + (1 - \varphi(q))^{1/2} \\ &\leq 2\gamma.\end{aligned}$$

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Since $q \sim q'$, there is a partial isometry $v \in L_1$ such that $v^*v = q'$ and $vv^* = q$. Using one of Christensen's lemmas, since $\|q - q'\|_{2,\varphi} \leq 2\gamma$, we can choose v with $\|v - q'\|_{2,\varphi} \leq 12\gamma$.

Proof of I (defining Φ)

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Really, this takes values in Ne , rather than N :

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Thankfully, Ne is isomorphic to N via $n \mapsto ne$ (since e commutes with N). So, $\Phi = \alpha \circ \text{Ad } v^*$

$$M \xrightarrow{\text{Ad } v^*} Ne \xrightarrow{\alpha} N$$

Proof of I (inequality)

Using the inequalities that we have so far, we can show that $\|\Phi(x) - x\|_2 \leq \beta\gamma$ for all $x \in M_1$, for some fixed β ($\beta = 105$ should work) (for δ sufficiently small, $\gamma < 10\delta^{1/2}$).

Proof of I (case 2: $e \prec q$)

If $e \prec q$, then $e \sim r \leq q$ with $r \in M' \cap L_1$ and $\varphi(r) = \varphi(e) = 1$. As above,

$$\begin{aligned}\|e - r\|_{2,\varphi} &\leq \|e - q\|_{2,\varphi} + \|q - r\|_{2,\varphi} \\ &\leq \gamma + (1 - \varphi(q))^{1/2} \\ &\leq 2\gamma.\end{aligned}$$

So, we can use r as in the previous case, instead of q , to construct $\Phi : M \rightarrow N$ ($r \sim e$, so in particular $r \prec e$). QED.

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The Jones Tower

In 1983, Jones introduced the basic construction, along with a notion of index $[M : N]$ (for $N \subset M$).

Under certain conditions, the basic construction can be repeated to obtain a Jones tower:

$$N \subset M \overset{e_0}{\subset} M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} \cdots$$

The Standard Invariant

By intersecting the Jones tower with commutants, we obtain the Standard Invariant:

$$\begin{array}{ccccccc} \mathbb{C} & = & N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \cup \\ & & \mathbb{C} & = & M' \cap M & \subset & M' \cap M_1 & \subset & \cdots \end{array}$$

Commuting Squares

A square of inclusions

$$\begin{array}{ccc} A & \subset & B \\ \cup & & \cup \\ C & \subset & D \end{array}$$

such that $A \oplus C \perp D \oplus C$ is called a commuting square (equivalently, $E_A E_D = E_D E_A = E_C$).

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