German Research Center for Artificial Intelligence

UNIVARIATE REAL CALCULUS

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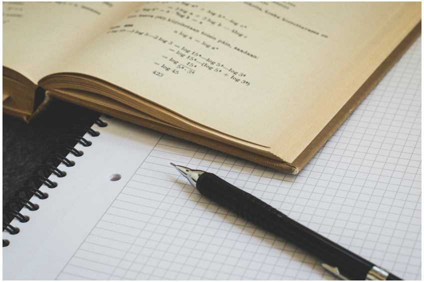
October 13, 2023 – Bremen, Deutschland



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General setup

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General information

- Our goal is to have a common language and understanding of the relevant content.
- The crash course shall provide you an opportunity to identify knowledge gaps.
- If something is too fast, feel free to pause the video.

Table of contents

Functions

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Content

- Definition as binary relation
- Function related notations and definitions
- Visualization as graph

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Definition: Binary relation

If A, B are sets, then $R \subseteq A \times B$ is called a *binary relation* over A and B.

Examples

- For $A := \{bicycle, plant, ball, cup\}$ and $B := \{Arlo, Valen\}$, "is owned by" can describe a relation over $A \times B$.
- $<$ is a relation over $\mathbb{N} \times \mathbb{N}$ with, e.g., $(1, 2) \in < \quad :\Leftrightarrow 1 < 2$.
- Each function f is a *left-total*, *right-unique* relation.

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Totalness

- A relation $R \subseteq A \times B$ is called *(left-)total*, if
$$\forall a \in A : \exists r \in R, b \in B : r = (a, b)$$
- A relation $R \subseteq A \times B$ is called *right-total* or *surjective*, if
$$\forall b \in B : \exists r \in R, a \in A : r = (a, b)$$

Uniqueness

- A relation $R \subseteq A \times B$ is called *right-unique* or *functional*, if
$$\forall r_1 = (a_1, b_1), r_2 = (a_2, b_2) \in R : a_1 = a_2 \Rightarrow b_1 = b_2$$
- A relation $R \subseteq A \times B$ is called *left-unique* or *injective*, if
$$\forall r_1 = (a_1, b_1), r_2 = (a_2, b_2) \in R : b_1 = b_2 \Rightarrow a_1 = a_2$$

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Definition: Function

A left-total, right-unique relation $f \subseteq A \times B$ is called a *function*. This is notated as $f : A \to B$.

Short definitions & notations

- The set A is called the *domain of f* and the set B is called the *codomain of f*.
- A function may also be called a *mapping*.
- For $(a, b) \in f$ we call $f(a) := b$ the *image of a under f*.
- $f : a \mapsto f(a)$ reads as *a maps to $f(a)$*.
- For $U \subseteq A$, the set $f[U] := \{f(a)|a \in U\}$ is called *the image of U under f*.
- A surjective and injective function is called a *bijective function* or *bijection*.

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Definition: Composition

Let $f : A \to B$ and $g : B \to C$ be functions.Then $g \circ f : a \mapsto g(f(a))$ defines a function $g \circ f : A \to C$. The function $g \circ f$ is called the *composition of g and f*.

Definition: Inverse function

Let $f : A \to B$ be bijective, then there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A : A \to A$ and $f \circ g = \mathrm{id}_B : B \to B$. This function g is called the *inverse of f* and denoted by $f^{-1} := g$.

Definition: Inverse image / preimage

Let $f : A \to B$ be a function and $V \subseteq B$. Then the set $f^{-1}[V] := \{a \in A | f(a) \in V\}$ is called the *inverse image of V under f* or the *preimage of V under f*.

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Definition: Continuous

A function between metric spaces $f : A \to B$ is called *continuous*, if the following implication holds:

$$V \subseteq B \text{ open} \Rightarrow f^{-1}[V] \subseteq A \text{ open}$$

Definition: Types of functions by domain

- A function $f : A \to B$ is called *univariate*, if $A \subseteq \mathbb{R}$.
- A function $f : A \to B$ is called *multivariate*, if $A \subseteq \mathbb{R}^m$ for some natural $m > 1$.

Definition: Types of functions by codomain

- A function $f : A \to B$ is called *real-valued*, if $B \subseteq \mathbb{R}$.
- A function $f : A \to B$ is called *vector-valued*, if B is a vector space.

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We can visualize univariate real-valued functions as graphs in 2d-plane.

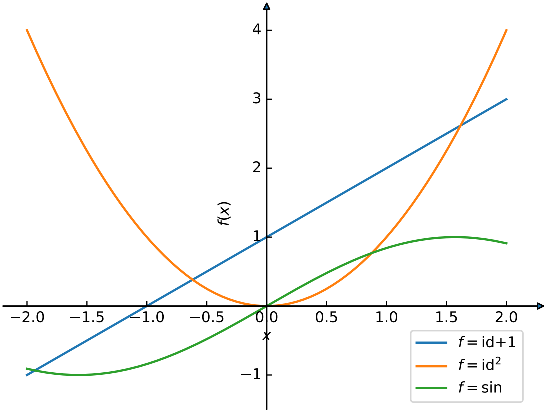

Figure: Graphs as visualizations of the functions $\texttt{id}+1 : x \mapsto x+1$, $\texttt{id}^2 : x \mapsto x^2$ and $\sin : x \mapsto \sin(x)$.

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Extreme values

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Content

- Definitions

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Example function for the section

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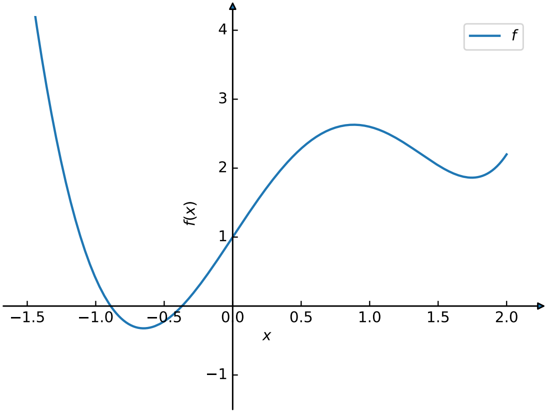

Figure: Graph of the function $f : (-1.5, 2) \to \mathbb{R}, x \mapsto 0.1x^5 + 0.5x^4 - 2x^3 + 3x + 1$ which is used as an example in this section.

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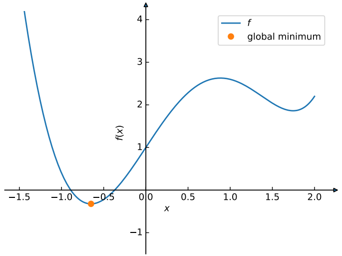
Definition: Global extreme values

Let $U \subseteq \mathbb{R}$ be open, $x_0 \in U$ and $f : U \to \mathbb{R}$.

- If $f(x_0) \leq f(x)$ holds for all $x \in U$, x_0 is called a *global minimum point* and $f(x_0)$ a *global minimum (value)*.
- If $f(x_0) \geq f(x)$ holds for all $x \in U$, x_0 is called a *global maximum point* and $f(x_0)$ a *global maximum (value)*.
- If x_0 is either a global minimum point or global maximum point, it is also called a *global extreme point* and $f(x_0)$ a *global extreme value* or *global extremum*.



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Global extreme values in example

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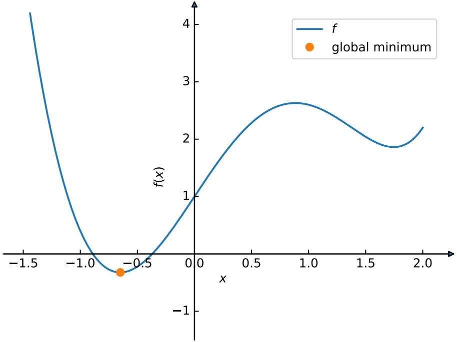

Figure: The example function has a global minimum point at roughly -0.65 but no global maximum point.

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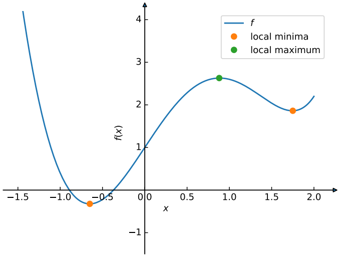
Definition: Local extreme values

Let $U \subseteq \mathbb{R}$ be open, $x_0 \in U$ and $f : U \to \mathbb{R}$.

- If there is an open $U' \subseteq U$ such that $x_0 \in U'$ and $f(x_0) \leq f(x)$ for all $x \in U'$, x is called a *local minimum point* and $f(x_0)$ a *local minimum (value)*.
- If there is an open $U' \subseteq U$ such that $x_0 \in U'$ and $f(x_0) \geq f(x)$ for all $x \in U'$, x is called a *local maximum point* and $f(x_0)$ a *local maximum (value)*.
- If x_0 is either a local minimum point or local maximum point, it is also called a *local extreme point* and $f(x_0)$ a *local extreme value* or *local extremum*.



Local extreme values in example

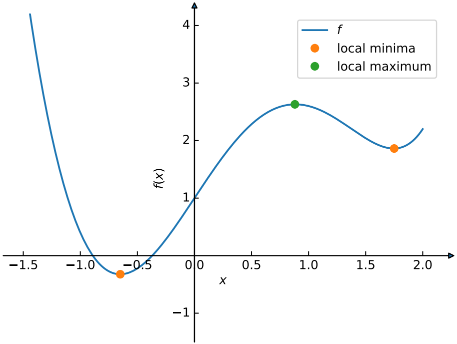


Figure: The global minimum point of the function is also a local minimum point. Additionally there is a local minimum point at roguhly 1.74 and a local maximum point at 0.88.

Derivatives

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Content

- Definition by first principles
- Notations
- Derivation rules
- Identification and classification of extrema

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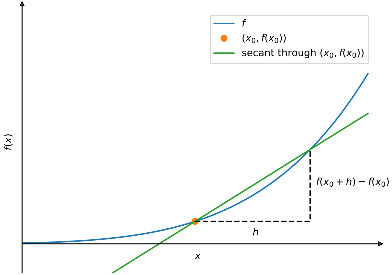
Definition: Derivative by first principles

Let $U \subseteq \mathbb{R}$ be open, $x_0 \in U$ and $f : U \to \mathbb{R}$ be continuous. Then f is said to be *differentiable in* x_0, if the following limit exists and is unique:

$$f'(x_0) := \lim_{h \to 0, h \neq 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If f is differentiable for all $x \in U$, it is said to be differentiable in U and $f' : U \to \mathbb{R}, x \mapsto f'(x)$ is called the *derivative of* f *in* U.



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Alternative notations

- Leibniz notation: $\frac{d}{dx}f := f'$
- Newton notation for $x : t \mapsto x(t)$ and t representing time: $\dot{x} := x'$.

Higher order derivatives

Let $U \subseteq \mathbb{R}$ be open and $f : U \to \mathbb{R}$ be differentiable in U. By setting $f^{(0)} := f$ and $f^{(1)} := f'$, we can inductively define derivatives of higher order:

If $f^{(n)}$ is differentiable in U for some $n \in \mathbb{N}$, we define $f^{(n+1)} := f^{(n)\prime} : U \to \mathbb{R}$.
Leibniz notation of higher order: $\frac{d^n}{dx^n} f := f^{(n)}$.

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Let $U \subseteq \mathbb{R}$ be open, $f, g : U \to \mathbb{R}$ differentiable in $x_0 \in U$.

Product rule

The function $f \cdot g : x \mapsto f(x)g(x)$ is differentiable in x_0 with
$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

Quotient rule

If $g(x_0) \neq 0$, then the function $\frac{f}{g} : x \mapsto \frac{f(x)}{g(x)}$ is differentiable in x_0 with
$(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$.

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Let $U, V \subseteq \mathbb{R}$ be open, $g : U \to \mathbb{R}$ differentiable in $x_0 \in U$ with $g[U] \subseteq V$ and $f : V \to \mathbb{R}$ differentiable in $g(x_0)$.

Chain rule

The function $f \circ g : U \to \mathbb{R}$ is differentiable in x_0 with $(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$.

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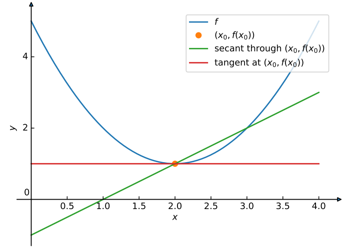
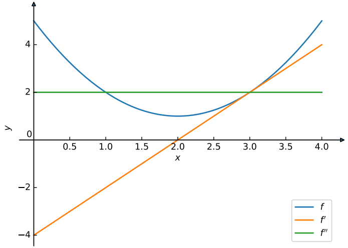
Conditions for extreme values of differentiable functions

Let $U \subseteq \mathbb{R}$ be open, $x \in U$ and $f : U \to \mathbb{R}$ be differentiable in x.

- Then $f'(x) = 0$ is a necessary condition for x being a local extreme point.
- If $f'(x) = 0$, f' is differentiable in x and $f''(x) > 0$, then x is a local minimum point.
- If $f'(x) = 0$, f' is differentiable in x and $f''(x) < 0$, then x is a local maximum point.





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Convexity

Content on convexity

Content

- Definition
- Relation to properties of extrema

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Definition: Convexity

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$.

- If for all $t \in [0, 1]$ and $x, y \in I$ the inequality $f(t \cdot x + (1 - t) \cdot y) \leq t \cdot f(x) + (1 - t) \cdot f(y)$ holds, f is called *convex*.
- If for all $t \in (0, 1)$ and $x \neq y \in I$ the inequality $f(t \cdot x + (1 - t) \cdot y) < t \cdot f(x) + (1 - t) \cdot f(y)$ holds, f is called *strict convex*.

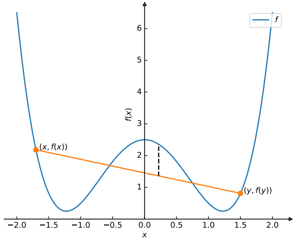

Figure: Example of a **non**-convex function, as the values of the function f exceed the values of the linear interpolation between $(x, f(x))$ and $(y, f(y))$.

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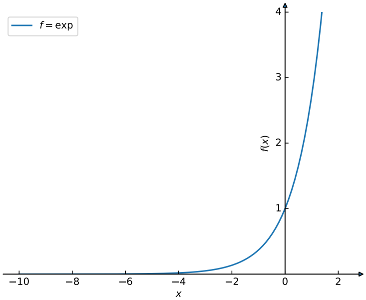
Existence of local minimum point?

Let $f : \mathbb{R} \to \mathbb{R}$ be strict convex. Does f necessarily have an extreme point?

No! Example: $f = \exp$.



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Existence of local minimum point?

Let $f : \mathbb{R} \to \mathbb{R}$ be strict convex. Does f necessarily have an extreme point?

No! Example: $f = \exp$.

Uniqueness and globalness of local minimum points

Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be convex and $x \in I$ a local minimum point of f.

- Then x is a global minimum point of f.
- If f is strict convex, the global minimum point is unique.

Summary

We recapped

- Important definitions regarding functions, including:
  - Extreme values,
  - Derivatives,
  - Convexity.
- Differentiation rules.
- Ways to argue about extreme values using derivatives and convexity.

Thank You!

Feel free to ask questions in the forums!