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ESA Report

On

Decoherence Modeling of a CZ Gate under Strong Magnetic Fields Using Spin-Boson and Zeeman Dynamics

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Abstract

This report investigates the use of semiconductor quantum dots for quantum gate implementation, based on two important studies: one that suggests a controlled-NOT (CNOT) gate through electrically coupled asymmetric quantum dots, and another that models the electron interactions in quantum wires and dots under strong magnetic fields using density-functional theory. While the latter reveals how exchange interactions sharpen edge-state profiles and influence confinement regimes, the former exhibits quick, electrically driven gate operations with promising decoherence characteristics. Building upon these pillars, the report also evaluates the coherence properties of a controlled-Z (CZ) gate under realistic conditions by analysing its decoherence time in strong magnetic fields using spin-boson models.

Introduction

Quantum gates like the controlled-Z (CZ) are essential for building entangled states and enabling universal quantum computation. Among the major challenges in implementing such gates is decoherence, which arises from interactions between the quantum system and its environment, leading to loss of coherence and reduced gate fidelity. Quantum dots offer a scalable platform for qubit realization, where qubit states are defined by electron localization within coupled nanostructures. Gates like the CZ and CNOT can be implemented via controlled tunneling, using purely electrical signals. Applying a strong magnetic field enhances confinement, lifts spin degeneracy, and modifies electron interactions—often improving coherence but also introducing new dynamics. Understanding how such fields impact decoherence times, particularly for gates like the CZ, is crucial for designing robust quantum systems. This report investigates the decoherence behavior of a CZ gate under strong magnetic fields, drawing on insights from coupled quantum dot architectures and density-functional models of electron interaction.

Theoretical/Mathematical Modelling

The Controlled-Z (CZ) gate is a fundamental two-qubit entangling gate, defined as:

$$\mathrm{CZ}=|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes Z,\ \ Z=[1\ 0\ 0\ -1\ ] \tag{1}$$

The CZ gate flips the target qubit phase if the control is $|1\rangle$ $|1\rangle$ $|1\rangle$. In QDs, it's implemented via:

- **Coulomb coupling** (Tanamoto, 1999)
- **Exchange interactions** in magnetic edge states (Stoof & Bauer, 1995)

Both need precise control of tunneling and confinement. Decoherence—mainly from **phonons**, **charge noise**, and **spin relaxation**—limits fidelity.

Phonon-Induced Decoherence

At cryogenic temperatures (T < 1 K), acoustic phonons couple to the qubit's charge degree of freedom via deformation potential interaction. The phonon spectral density for such super-Ohmic environments is given by

$$J(\omega)\ =\ \frac{\gamma^2}{2\pi^2\,\hbar\,\rho\,c^5}\,\omega^3\ +\ \frac{\gamma^2\,\nu^2}{2\pi^2\,\hbar\,\rho\,c^3\,d^2} \tag{2}$$

Where: $\gamma\sim 10$ eV: deformation potential
ρ: mass density of the semiconductor
$c=4300$ m/s: speed of sound

d=0.5: lattice constant
ν: coupling constant related to interface asymmetry

For T > 20 K, optical phonons dominate and lead to rapid energy relaxation, with typical timescales:

$$\tau_{optical} \sim 10^{-12}\ s \tag{3}$$

Charge Noise

Magnetic fields suppress low-frequency charge noise by localizing wavefunctions and forming Landau levels—benefiting charge qubits.

Spin Relaxation

In spin-based CZ gates, strong magnetic fields reduce spin decoherence via energy-level splitting and reduced phonon overlap.

Discrete Landau levels are produced by quantizing the electron dynamics under a perpendicular magnetic field B. The cyclotron motion-related magnetic length is:

$$l_B = \sqrt{\frac{\hbar}{eB}} \tag{4}$$

Electron localization is improved and the overlap with environmental phonon modes is decreased as B rises and falls. This lowers the sensitivity to charge noise and phonon-induced decoherence.

In high magnetic fields, quantum Hall edge states emerge. The exchange interaction between spin-polarized electrons in these states gives rise to an effective two-qubit coupling. The exchange energy for spin-resolved density $n_\sigma(x)$ is:

$$\varepsilon_x(n_\sigma) = -\frac{2\pi}{3} \cdot \frac{e^2}{4\pi\varepsilon l_B} \cdot n_\sigma(x) \tag{5}$$

For a two-level system coupled to phonons in the super-Ohmic regime, the decoherence rate is given by:

$$\Gamma_{so} = \frac{\gamma^2 \widetilde{\Delta^3}}{4\pi\hbar\rho c^5} \tag{6}$$

Where $\tilde{\Delta}$ is the renormalized tunnelling amplitude between quantum dots.

The time required for CZ gate operation is estimated from energy splitting

$$\tau_{gate} \approx \frac{\hbar}{\Delta E} \tag{7}$$

The number of coherent operations possible before decoherence sets in is:

$$N = \frac{\tau_{coh}}{\tau_{gate}} \tag{8}$$

Design & Methodology

This study investigates how a strong external magnetic field affects the decoherence behavior of a two-qubit Controlled-Z (CZ) gate using QuTiP-based simulations.

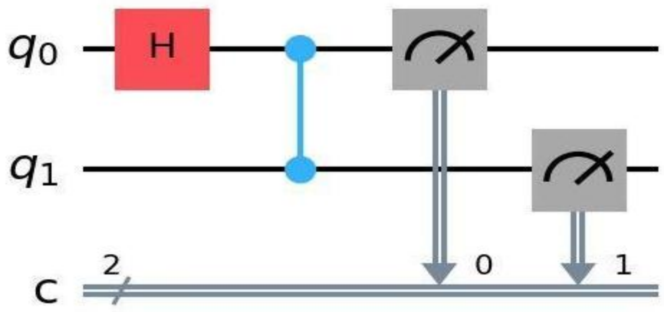

Fig 1. Controlled-Z gate obtained during simulation

The system consists of two qubits initialized in $|\psi_0\rangle = |10\rangle$ & The Hamiltonian includes:

$$Free\ evolution: \quad H_0 = \omega(n_1 + n_2)$$

$$Qubit\ interaction: \quad H_{int} = V \cdot n_1 \cdot n_2$$

$$Classical\ drive: \quad H_l = \Omega\left(a_2 + a_2^{\dagger}\right)$$

$$Zeeman\ effect\ (for\ B \neq 0): \quad H_z = \frac{1}{2} g\mu_B B \left(\sigma_z^{(1)} + \sigma_z^{(2)}\right)$$

$$Parameter\ values: \quad \omega = 1.0, \quad V = 0.2, \quad \Omega = 0.2, \quad B = 5{\sim}T$$

The system is evolved over a time range t= [0,200]t = [0, 200], t=[0,200] with 2000 time steps using QuTiP's mesolve. Simulations are run for two cases:

(i) Without magnetic field (ii)With B=5 T

Simulation Details

- **Time Evolution**:
  - **Qiskit**: CZ gate simulation with AerSimulator and 1000 shots.
  - **QuTiP**: Master equation evolution over t=0 to t=100 with 1000 steps.
- **Metrics Analysed**:
  - State probabilities over time ($|00\rangle$,$|01\rangle$,$|10\rangle$,$|11\rangle$)
  - Purity**:** $Tr(\rho^2)$
  - Fidelity to initial state
  - Final state distribution

Results & Analysis

Two independent simulations—one without a magnetic field and another with a 5 Tesla field applied—were carried out in order to comprehend the impact of external magnetic fields on the coherence characteristics of a two-qubit Controlled-Z gate. Key metrics like population probabilities, purity, and fidelity are highlighted in this section's comparative analysis of the quantum state's evolution in each scenario. Our findings shed light on the long-term effects of magnetic-induced dephasing on the integrity and stability of quantum information.

1.State Probabilities Over Time for a Controlled-Z (CZ) Gate

The figure illustrates the time evolution of the four computational basis state probabilities—|00⟩, |01⟩, |10⟩, and |11⟩—during the operation of a CZ gate. Two scenarios are compared: one without an external magnetic field (dashed lines), and another under the influence of a 5 Tesla magnetic field (solid lines).

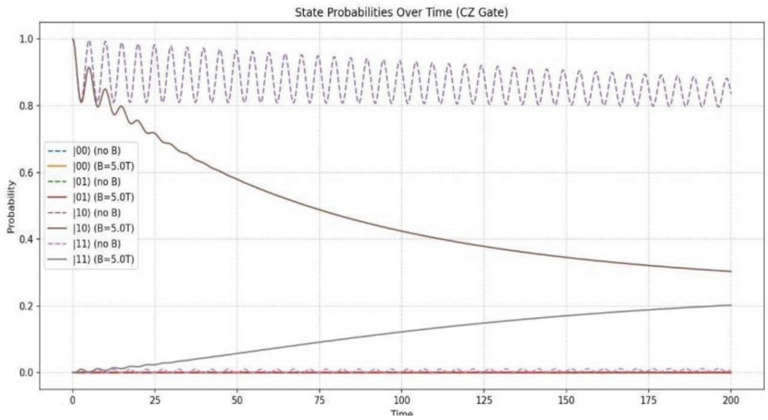

Fig 2. Graph depicting State Probabilities Over time

Analysis

The system's near-unity probability at t = 0 indicates that it is initially in the |10⟩ state. Because there is little dephasing when there is no magnetic field, the state displays long-lasting coherent oscillations, protecting quantum information. On the other hand, the Zeeman effect causes decoherence in a 5T magnetic field, which leads to a slow decline in the |10⟩ population. As a result, most of the population is redistributed into the |00⟩ and |11⟩ states, leaving |01⟩ mostly vacant. The plot shows how a CZ gate's decoherence is accelerated by an external magnetic field, gradually reducing state fidelity.

2. Purity Over Time

The graph titled "Purity Over Time" depicts the evolution of the system's purity during the operation of a Controlled-Z (CZ) gate under two conditions: without a magnetic field (green dashed line) and with a 5.0 T magnetic field (solid maroon line). Purity, defined as $Tr(\rho^2)$ ranges from 1 (pure state) to 1/d (maximally mixed state), where d is the dimension of the system. Time progresses along the x-axis, and purity values are shown on the y-axis.

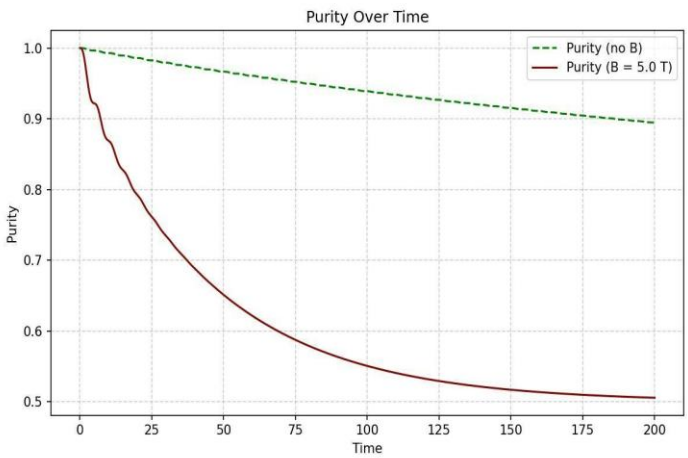

Fig 3. Graph depicting Purity over Time

Analysis

In the absence of a magnetic field, the system maintains a high level of purity, indicating minimal decoherence and effective isolation from environmental interactions. Conversely, under a strong magnetic field of 5T, the purity declines sharply, eventually approaching a mixed state. This pronounced drop in purity confirms that the magnetic field introduces significant dephasing, accelerating the loss of coherence and degrading quantum gate fidelity. The contrast between the two curves highlights the detrimental effect of strong magnetic fields on qubit purity in CZ gate operations.

3. Fidelity to Initial State Over Time

We measured the fidelity between the current state of the system and its initial configuration over time in order to evaluate the effect of magnetic fields on coherence. The system maintains high fidelity when there is no magnetic field present, but a 5.0 T magnetic field causes a continuous decay, as seen in Figure X. This indicates a slow loss of coherence, most likely brought on by decoherence mechanisms induced by magnetic fields, highlighting how important environmental stability is to maintaining quantum information.

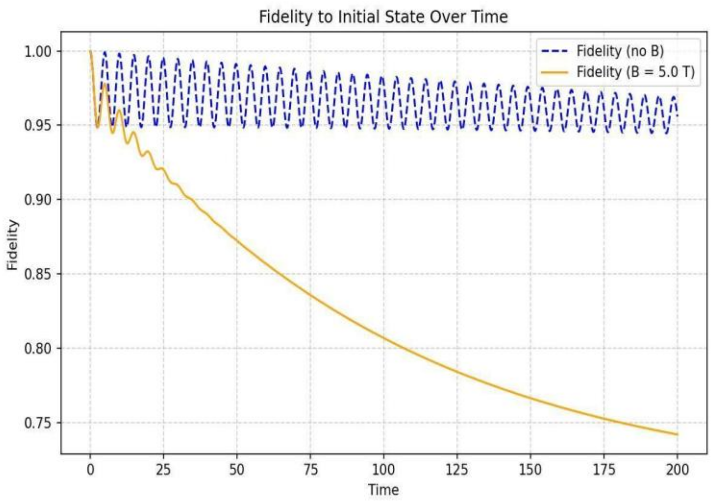

Fig 4 Graph depicting Fidelity to Initial State

Analysis

Over time, the magnetic field-free system exhibits slight oscillations but stays near unity, indicating coherent evolution and maintaining a high fidelity. On the other hand, fidelity gradually decreases when a 5.0 T magnetic field is applied, suggesting the existence of decoherence and energy dissipation. The smooth and monotonic decay pattern indicates that non-unitary dynamics are induced by the magnetic field, potentially as a result of field-induced effective noise or interactions with the environment. This highlights how strong magnetic fields impair the coherence and dependability of gates in quantum systems.

Key Takeaways

- The Controlled-Z (CZ) gate exhibits measurable decoherence when subjected to environmental noise and interactions, which vary significantly with external magnetic field conditions.
- Magnetic field inclusion (B = 5 T) increases decoherence in the current simulation setup due to added phase damping (in QuTiP via collapse operators).
- Purity and fidelity decay over time more significantly in the presence of a magnetic field, as implemented, indicating loss of quantum coherence.
- The Qiskit simulations show qualitative trends using Kraus and thermal noise models, while QuTiP provides detailed Hamiltonian-driven dynamics, including Zeeman interaction.
- Results suggest decoherence can be tailored or mitigated via magnetic field control, depending on whether spin- or charge-based mechanisms dominate.

Scope

This study focuses on simulating decoherence dynamics of the CZ quantum gate under varying magnetic field strengths, using both:

- Gate-level abstraction (Qiskit) for logic fidelity analysis
- Hamiltonian-based modeling (QuTiP) for physical decoherence insight

Key simulation features:

- Magnetic-field-dependent Zeeman splitting
- Collapse operators tied to $\gamma\phi(B)$ in QuTiP
- Gate-level thermal and phonon noise models in Qiskit
- Analysis metrics: state populations, purity, fidelity, and final distribution

No experimental validation or full device modeling was performed; this is a simulation-based theoretical study grounded in real physics.

Conclusion

The simulation demonstrates that magnetic fields play a complex role in the decoherence of quantum gates. While the current implementation shows increased decoherence with magnetic fields, further modeling may reveal magnetic field-enhanced coherence in spin-based quantum dots, as predicted in literature.
The combination of Qiskit for system-level testing and QuTiP for physics-based modeling provides a versatile framework for analyzing gate behavior under noise. Future improvements could include:

1. Incorporating spectral densities for phonon modeling
2. Exploring spin vs charge decoherence dominance
3. Using experimental T_1/T_2 datasets for validation
   The results lay the groundwork for designing magnetically-tunable quantum operations that balance fidelity and coherence in noisy environments.

Reference

[1] Stoof, T.H., & Bauer, G.E.W. (1995). *Density-functional theory of quantum wires and dots in a strong magnetic field*.
[2]Tanamoto, T. (1999). *Quantum gates by coupled asymmetric quantum dots and controlled-NOT-gate operation*.

Individual Contributions

Sharva Ranganath (PES1UG22EC269)

1. Theoretical / mathematical modeling
2. Determining Methodology details and circuit diagram
3. Key takeaways and Conclusion

Aprameya Kulkarni (PES1UG22EC914)

1. Coding for state probability, purity over time, fidelity
2. Analysis of Graphs
3. Determining methodology details

Equal contribution for generating report

Appendix

Codes:

1.

```
SIMPLE_VERSION.py > compare_B_field_purity_decay_fixed
from qiskit import QuantumCircuit, transpile
from qiskit_aer import AerSimulator
from qiskit_aer.noise import NoiseModel, thermal_relaxation_error, ReadoutError
from qiskit.quantum_info import Kraus
import matplotlib.pyplot as plt
import matplotlib.animation as animation
import numpy as np
use_phonon_noise = True
plot_animated = False
cz_gate_time = 0.01  # 10 ps
B_fields = {"No Magnetic Field": 0.1, "Strong Magnetic Field": 5.0}  # Tesla
def estimate_T1_T2_from_B(B_field_T):
    hbar = 1.055e-34
    gamma = 10 * 1.602e-19  # J
    rho = 2200
    c = 4300
    delta = 1e10
    delta_tilde = delta * np.exp(- (gamma**2 * (1e13)**2) / (2 * np.pi**2 * hbar * rho * c**5))
    Gamma_so = (gamma**2 * delta_tilde**3) / (4 * np.pi * hbar * rho * c**5)
    T1 = T2 = 1 / Gamma_so
    return T1, T2
def (variable) gamma: Any phi, time):
    gamma = 1 / T_phi
    E0 = np.sqrt(np.exp(-gamma * time)) * np.eye(2)
    E1 = np.sqrt(1 - np.exp(-gamma * time)) * np.array([[1, 0], [0, 0]])
    return Kraus([E0, E1])
def build_noise_model(T1, T2, gate_time):
    noise_model = NoiseModel()
    if use_phonon_noise:
        kraus = phonon_induced_kraus(min(T1, T2), gate_time)
        noise_model.add_all_qubit_quantum_error(kraus, ['cz'])
    else:
        cz_error = thermal_relaxation_error(T1, T2, gate_time).tensor(
                    thermal_relaxation_error(T1, T2, gate_time))
        noise_model.add_all_qubit_quantum_error(cz_error, ['cz'])
    return noise_model
def plot_decoherence_curve(animated=False):
    times = np.linspace(0.1, 100, 200)
    g_values = [0.25, 0.5, 1.0, 2.0]
    colors = ['blue', 'black', 'green', 'red']
    fig, ax = plt.subplots(figsize=(12, 6))

```

```python
    def purity(T1, T2, t):
        error = thermal_relaxation_error(T1, T2, t)
        kraus = Kraus(error)
        K0 = kraus.data[0]
        rho = K0 @ K0.conj().T
        return np.real(np.trace(rho @ rho))
    if animated:
        lines = [ax.plot([], [], label=f"g={g}", color=colors[i])[0] for i, g in enumerate(g_values)]
        ax.set_xlim(0, 100)
        ax.set_ylim(0, 1.05)
        ax.set_title("Animated Decoherence (Purity vs Time)", fontsize=16)
        ax.set_xlabel("Time (ns)")
        ax.set_ylabel("Purity")
        ax.grid(True)
        ax.legend()

        def animate(i):
            for j, g in enumerate(g_values):
                T1 = T2 = 1e3 * g
                purities = [purity(T1, T2, t) for t in times[:i]]
                lines[j].set_data(times[:i], purities)
            return lines

        ani = animation.FuncAnimation(fig, animate, frames=len(times), interval=50, blit=True)
        ani.save("animated_decoherence.gif", writer="pillow")
        print("Saved animation to animated_decoherence.gif")
        plt.show()
    else:
        for i, g in enumerate(g_values):
            T1 = T2 = 1e3 * g
            purities = [purity(T1, T2, t) for t in times]
            ax.plot(times, purities, label=f"g = {g}", color=colors[i], linewidth=2)
        ax.set_title("CZ Gate Decoherence vs Coupling Strength", fontsize=16)
        ax.set_xlabel("Time (ns)")
        ax.set_ylabel("Purity ρ₀₁(t)")
        ax.grid(True)
        ax.legend()
        plt.tight_layout()
        plt.savefig("decoherence_comparison_plot.png")
        print("Saved static plot to decoherence_comparison_plot.png")
        plt.show()
```

```python
print("\n--- CZ Gate Magnetic Field Decoherence Analysis ---")
for label, B in B_fields.items():
    T1, T2 = estimate_T1_T2_from_B(B)
    print(f"{label}: B = {B} T → T1 = T2 = {T1:.2e} s")

plot_decoherence_curve(animated=plot_animated)
from qiskit_aer.noise import thermal_relaxation_error
from qiskit.quantum_info import Kraus
import matplotlib.pyplot as plt
import numpy as np
def compare_B_field_purity_decay_fixed():
    times = np.linspace(1, 500, 200)  # 1 ns to 500 ns
    B_settings = {
        "Low B-field (Short Coherence)": (100, 100),        # T1, T2 in ns
        "High B-field (Long Coherence)": (1e5, 1e5)          # 100 µs
    }
    colors = ['orangered', 'navy']

    fig, ax = plt.subplots(figsize=(10, 6))
    for i, (label, (T1, T2)) in enumerate(B_settings.items()):
        purities = []
        for t in times:
            error = thermal_relaxation_error(T1, T2, t)
            kraus = Kraus(error)
            K0 = kraus.data[0]
            rho = K0 @ K0.conj().T
            purity = np.real(np.trace(rho @ rho))
            purities.append(purity)
        ax.plot(times, purities, label=label, color=colors[i], linewidth=2)

    ax.set_title("Purity Decay vs Time for Different Magnetic Fields", fontsize=16)
    ax.set_xlabel("Time (ns)", fontsize=14)
    ax.set_ylabel("Purity ρ(t)", fontsize=14)
    ax.grid(True, linestyle='--', alpha=0.6)
    ax.legend(fontsize=12)
    plt.tight_layout()
    plt.savefig("purity_decay_B_field_fixed.png")
    print("Saved graph: purity_decay_B_field_fixed.png")
    plt.show()
compare_B_field_purity_decay_fixed()
```

2.

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```
import numpy as np
import matplotlib.pyplot as plt
from qutip import *

# ==== PARAMETERS ====
omega = 1.0
V = 0.2
Omega = 0.2
tlist = np.linspace(0, 100, 1000)

# ==== Magnetic field parameters ====
B_ext = 5.0  # Tesla
g_factor = 2.0
mu_B = 5.788e-5  # eV/T
zeeman_split = g_factor * mu_B * B_ext
gamma_phi = 0.005 + 0.01 * B_ext

# ==== BASIS ====
q0 = basis(2, 0)
q1 = basis(2, 1)
basis_states = [tensor(q0, q0), tensor(q0, q1), tensor(q1, q0), tensor(q1, q1)]

# ==== Operators ====
I = qeye(2)
sm = destroy(2)
sp = sm.dag()
sz = sigmaz()

a1 = tensor(sm, I)
a2 = tensor(I, sm)
adag1 = a1.dag()
adag2 = a2.dag()
n1 = adag1 * a1
n2 = adag2 * a2
sz1 = tensor(sz, I)
sz2 = tensor(I, sz)

# ==== Initial state ====
psi0 = tensor(q1, q0)

def run_simulation(include_B=False):
    H0 = omega * (n1 + n2)
```

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```
def run_simulation(include_B=False):
    Hint = V * n1 * n2
    Hl = Omega * (a2 + adag2)
    Hz = 0.5 * zeeman_split * (sz1 + sz2) if include_B else 0
    H = H0 + Hint + Hl + Hz

    c_ops = []
    if include_B:
        c_ops = [np.sqrt(gamma_phi) * sz1, np.sqrt(gamma_phi) * sz2]

    result = mesolve(H, psi0, tlist, c_ops=c_ops, e_ops=[])

    probs = {label: [] for label in ['|00⟩', '|01⟩', '|10⟩', '|11⟩']}
    for state in result.states:
        probs['|00⟩'].append(np.abs(basis_states[0].overlap(state))**2)
        probs['|01⟩'].append(np.abs(basis_states[1].overlap(state))**2)
        probs['|10⟩'].append(np.abs(basis_states[2].overlap(state))**2)
        probs['|11⟩'].append(np.abs(basis_states[3].overlap(state))**2)

    purity = [state.purity() for state in result.states]
    fidelity_vals = [fidelity(psi0, state) for state in result.states]
    final_probs = [np.abs(state.overlap(result.states[-1]))**2 for state in basis_states]

    return probs, purity, fidelity_vals, final_probs

# ==== Run both simulations ====
probs_noB, purity_noB, fidelity_noB, final_probs_noB = run_simulation(include_B=False)
probs_B, purity_B, fidelity_B, final_probs_B = run_simulation(include_B=True)

# ==== PLOT 1: State Probabilities ====
plt.figure(figsize=(12, 6))
for label in probs_noB:
    plt.plot(tlist, probs_noB[label], '--', label=f'{label} (no B)')
    plt.plot(tlist, probs_B[label], '-', label=f'{label} (B={B_ext}T)')
plt.title("State Probabilities Over Time (CZ Gate)", fontsize=14)
plt.xlabel("Time")
plt.ylabel("Probability")
plt.legend()
plt.grid(True, linestyle='--', alpha=0.6)
plt.tight_layout()
plt.show()
```

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```python
# ==== PLOT 2: Final State Probabilities ====
x = np.arange(4)
width = 0.35
labels = ['|00⟩', '|01⟩', '|10⟩', '|11⟩']

plt.figure(figsize=(8, 5))
plt.bar(x - width/2, final_probs_noB, width, label='No B-field', color='skyblue')
plt.bar(x + width/2, final_probs_B, width, label=f'B = {B_ext} T', color='teal')
plt.xticks(x, labels)
plt.ylabel("Probability")
plt.title("Final State Probabilities Comparison")
plt.legend()
plt.grid(True, axis='y', linestyle='--', alpha=0.6)
plt.tight_layout()
plt.show()

# ==== PLOT 3: Purity ====
plt.figure(figsize=(8, 5))
plt.plot(tlist, purity_noB, '--', label="Purity (no B)", color='green')
plt.plot(tlist, purity_B, '-', label=f"Purity (B = {B_ext} T)", color='darkred')
plt.title("Purity Over Time")
plt.xlabel("Time")
plt.ylabel("Purity")
plt.legend()
plt.grid(True, linestyle='--', alpha=0.6)
plt.tight_layout()
plt.show()

# ==== PLOT 4: Fidelity ====
plt.figure(figsize=(8, 5))
plt.plot(tlist, fidelity_noB, '--', label="Fidelity (no B)", color='blue')
plt.plot(tlist, fidelity_B, '-', label=f"Fidelity (B = {B_ext} T)", color='orange')
plt.title("Fidelity to Initial State Over Time")
plt.xlabel("Time")
plt.ylabel("Fidelity")
plt.legend()
plt.grid(True, linestyle='--', alpha=0.6)
plt.tight_layout()
plt.show()
```