CSIR-UGC NET-June 2013-Problem(57)

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Pre-requisites

Some important things to understand

1. Meaning of notation $\mathbf{(A,B)} \sim \mathrm{N}_2\mathbf{(b,c,d,e,f)}$
2. **Dot product** in terms of **matrix multiplication**
3. Finding **Covariance by matrix method**

Meaning of notation (A,B) $\sim N_2$(b,c,d,e,f)

N_2 represents a bivariate normal distribution (bivariate since the number in the subscript of N is 2, which also represents the number of variables) with

$$b = E(A) \tag{1}$$
$$c = E(B) \tag{2}$$
$$d = Var(A) \tag{3}$$
$$e = Var(B) \tag{4}$$
$$f = \rho(A, B) \tag{5}$$

Dot product in terms of matrix multiplication

If we have two vectors A and B, then their dot product can be written in matrix form as:

$$A \cdot B = A^{\top} B \tag{6}$$

where A and B are column matrices.

Covariance by matrix method

We can write A and B as:

$$\mathrm{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top \begin{pmatrix} A \\ B \end{pmatrix} = \mathrm{C}^\top \mathrm{E} \tag{7}$$

$$\mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \begin{pmatrix} A \\ B \end{pmatrix} = \mathrm{D}^\top \mathrm{E} \tag{8}$$

Covariance of A and B will be defined as:

$$Cov(A, B) = \mathrm{C}^\top \Sigma \mathrm{D} \tag{9}$$

Similarly it can be shown that:

$$Cov(B, A) = \mathrm{D}^\top \Sigma \mathrm{C} \tag{10}$$

Also,

$$Var(A) = Cov(A, A) = \mathrm{C}^\top \Sigma \mathrm{C} \tag{11}$$

Similarly,

$$Var(B) = Cov(B, B) = \mathrm{D}^\top \Sigma \mathrm{D} \tag{12}$$

where,

$$\mathrm{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

$$\mathrm{D} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

$$\mathrm{E} = \begin{pmatrix} A \\ B \end{pmatrix} \tag{15}$$

Note

Here Σ is the Covariance matrix of A and B always.
This means that even if you are finding Cov(A+B,A-B) , Var(A+B) or Var(A-B), Σ does not change only C and D will vary.

Question

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(X,Y) follows bivariate normal distribution $N_2(0,0,1,1,\rho)$, -1 $< \rho <$ 1. Then,

1. X+Y and X-Y are uncorrelated only if $\rho = 0$
2. X+Y and X-Y are uncorrelated only if $\rho < 0$
3. X+Y and X-Y are uncorrelated only if $\rho > 0$
4. X+Y and X-Y are uncorrelated for all values of ρ

Solution

Given that

$$M = \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right] \tag{16}$$

Here, Mean matrix of X and Y is:

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{17}$$

Covariance matrix of X and Y is:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{18}$$

Solution Contd.

Now X+Y and X-Y can be written as:

$$X + Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\top} \begin{pmatrix} X \\ Y \end{pmatrix} = \mathrm{A}^{\top}\mathrm{M} \tag{19}$$

$$X - Y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}^{\top} \begin{pmatrix} X \\ Y \end{pmatrix} = \mathrm{B}^{\top}\mathrm{M} \tag{20}$$

where

$$\mathrm{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{21}$$

and

$$\mathrm{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{22}$$

Solution Contd.

Defining Covariance in terms of expectation value:

$$
\begin{aligned}
Cov(X, Y) &= E[(X - \mu_x)(Y - \mu_y)] && (23)\\
&= E[(X - 0)(Y - 0)] && (24)\\
&= E(XY) && (25)
\end{aligned}
$$

Finding Covariance of X+Y and X-Y:

$$
\begin{aligned}
Cov(X + Y, X - Y) &= \mathrm{A}^\top \Sigma \mathrm{B} && (26)\\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} && (27)\\
&= \begin{pmatrix} 1 + \rho \\ 1 + \rho \end{pmatrix}^\top \begin{pmatrix} 1 \\ -1 \end{pmatrix} && (28)\\
&= (1 + \rho) - 1(1 + \rho) && (29)\\
&= 0 && (30)
\end{aligned}
$$

Solution Contd.

Note that

$$Var(X+Y) = Cov(X+Y, X+Y) \tag{31}$$
$$Var(X-Y) = Cov(X-Y, X-Y) \tag{32}$$

Hence,

$$Var(X+Y) = \mathrm{A}^\top \Sigma \mathrm{A} \tag{33}$$
$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{34}$$
$$= \begin{pmatrix} 1+\rho \\ 1+\rho \end{pmatrix}^\top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{35}$$
$$= 1+\rho+1+\rho \tag{36}$$
$$= 2+2\rho \neq 0 \tag{37}$$

Solution Contd.

$$\begin{aligned}
Var(X-Y) &= \mathrm{B}^\top \Sigma \mathrm{B} && (38)\\
&= \begin{pmatrix} 1 \\ -1 \end{pmatrix}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} && (39)\\
&= \begin{pmatrix} 1-\rho \\ \rho-1 \end{pmatrix}^\top \begin{pmatrix} 1 \\ -1 \end{pmatrix} && (40)\\
&= 1-\rho-\rho+1 && (41)\\
&= 2-2\rho \neq 0 && (42)
\end{aligned}$$

So correlation coefficient is:

$$\rho(X+Y, X-Y) = \frac{Cov(X+Y, X-Y)}{\sqrt{var(X+Y) \times var(X-Y)}} = 0 \quad (43)$$

Solution Contd.

Hence, X+Y and X-Y are uncorrelated irrespective of value of ρ where $\rho \in (-1, 1)$.
Therefore, the correct answer is **option 4**.