AI1103-Assignment 7

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Download all python codes from

https://github.com/SHASHANK−1−ALL/AI1103−ASSIGNMENT−7/blob/main/Assignment7.py

and latex-tikz codes from

https://github.com/SHASHANK−1−ALL/AI1103−ASSIGNMENT−7/blob/main/Assignment7.tex

Question

(X,Y) follows bivariate normal distribution $N_2(0,0,1,1,\rho)$, -1 $< \rho <$ 1. Then,

1) X+Y and X-Y are uncorrelated only if $\rho = 0$
2) X+Y and X-Y are uncorrelated only if $\rho < 0$
3) X+Y and X-Y are uncorrelated only if $\rho > 0$
4) X+Y and X-Y are uncorrelated for all values of ρ

Solution

Given that

$$\mathbf{M} = \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right] \tag{0.0.1}$$

Here, Mean matrix of X and Y is:

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.0.2}$$

Covariance matrix of X and Y is:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{0.0.3}$$

Now X+Y and X-Y can be written as:

$$X + Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{A}^\top \mathbf{M} \tag{0.0.4}$$

$$X - Y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}^\top \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{B}^\top \mathbf{M} \tag{0.0.5}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.0.6}$$

and

$$\mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.0.7}$$

Defining Covariance in terms of expectation value:

$$Cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)] \tag{0.0.8}$$
$$= E[(X - 0)(Y - 0)] \tag{0.0.9}$$
$$= E(XY) \tag{0.0.10}$$

$$Cov(X + Y, X - Y) = \mathbf{A}^\top \Sigma \mathbf{B} \tag{0.0.11}$$
$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.0.12}$$
$$= \begin{pmatrix} 1+\rho \\ 1+\rho \end{pmatrix}^\top \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.0.13}$$
$$= (1 + \rho) - 1(1 + \rho) \tag{0.0.14}$$
$$= 0 \tag{0.0.15}$$

Note that

$$Var(X + Y) = Cov(X + Y, X + Y) \tag{0.0.16}$$
$$Var(X - Y) = Cov(X - Y, X - Y) \tag{0.0.17}$$

Hence,

$$Var(X + Y) = \mathbf{A}^\top \Sigma \mathbf{A} \tag{0.0.18}$$
$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.0.19}$$
$$= \begin{pmatrix} 1+\rho \\ 1+\rho \end{pmatrix}^\top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.0.20}$$
$$= 1 + \rho + 1 + \rho \tag{0.0.21}$$
$$= 2 + 2\rho \neq 0 \tag{0.0.22}$$

$$Var(X - Y) = \mathbf{B}^\top \Sigma \mathbf{B} \tag{0.0.23}$$
$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.0.24}$$
$$= \begin{pmatrix} 1-\rho \\ \rho-1 \end{pmatrix}^\top \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.0.25}$$
$$= 1 - \rho - \rho + 1 \tag{0.0.26}$$
$$= 2 - 2\rho \neq 0 \tag{0.0.27}$$

So correlation coefficient is:

$$\rho(X+Y, X-Y) = \frac{Cov(X+Y, X-Y)}{\sqrt{var(X+Y) \times var(X-Y)}} = 0 \tag{0.0.28}$$

$\implies$ X+Y and X-Y are uncorrelated irrespective of value of ρ where $\rho \in (-1, 1)$.

$\therefore$ The correct answer is **option 4**.