Johns Hopkins Engineering

625.464 Computational Statistics

Spline Estimation

Module 10 Lecture 10E

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Splines

Consider subdividing D & use polynomials of smaller degree

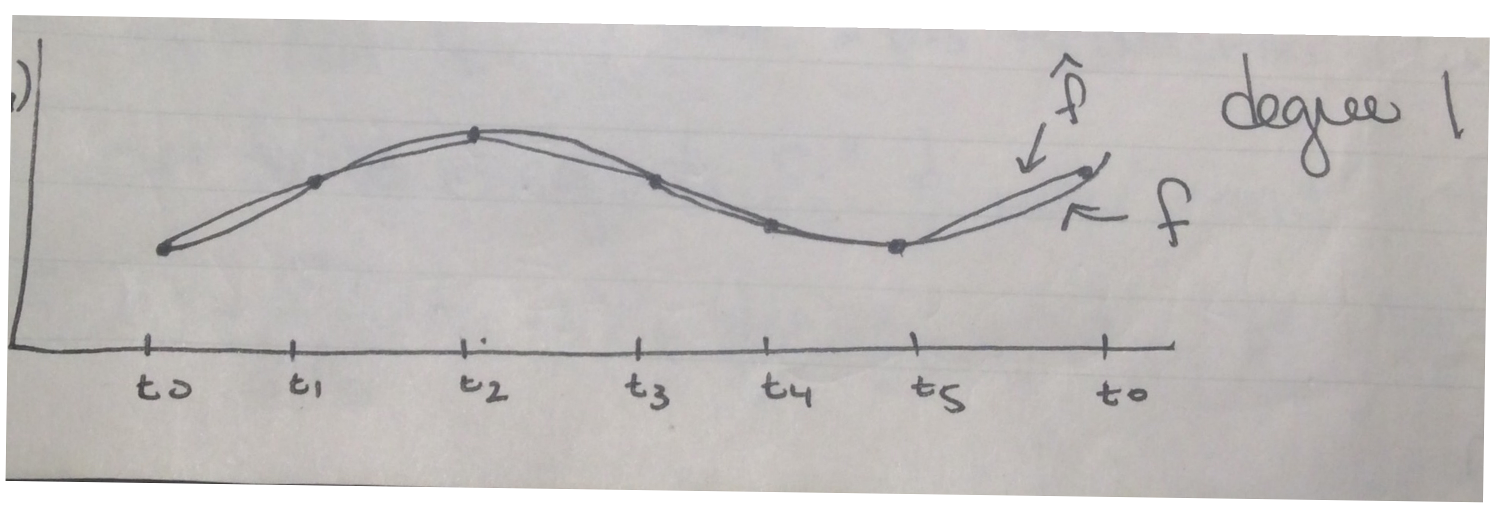
- $\hat{f}(x)$ is sum of piecewise poly
- even w/ low deg, can get good approx.
- force smoothness by using cont. cond.

Spline Approx

Spline Functions

Def: Suppose $n+1$ points $t_0 < t_1 < \cdots < t_n$ subdivide our interval. Then a spline function of degree $k \geq 0$ having knots $t_0, \ldots, t_n$ is a function S such that

1. On each interval $[t_{i-1}, t_i)$, S is a polynomial of degree $\leq k$
2. S has con't $(k-1)$st derivatives on $[t_0, t_n]$



$$S(x) = \begin{cases} a_0 x + b_0 & [t_0, t_1] \\ a_1 x + b_1 \\ \vdots \\ a_{n-1} x + b_{n-1} & [t_{n-1}, t_n] \end{cases}$$

Cubic Splines

Suppose we have values

x	t_0	t_1	$\cdots$	t_n
y	y_0	y_1	$\cdots$	y_n

Then the cubic spline on $[t_0, t_n]$ will have the form

$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

where $\forall i$

$S_i(x)$ is a poly of degree ≤ 3 &

1. $S_{i-1}(t_i) = y_i = S_i(t_i)$
2. $S'_{i-1}(t_i) = S'_i(t_i)$
3. $S''_{i-1}(t_i) = S''_i(t_i) = z_i$

The Cubic Spline Equation

What do we know

- $S_i(x)$ has degree ≤ 3
- $S_i''(t_i) = z_i$ & $S_i''(t_{i+1}) = z_{i+1}$

$\Rightarrow S_i(x)$ has degree $= 3$.

$\Rightarrow S_i''(x)$ must be a linear function

$$\therefore \quad S_i''(x) = \frac{z_i}{h_i}(t_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - t_i)$$

$h_i = t_{i+1} - t_i$

The Cubic Spline Equation

We can twice integrate $S_i''(x)$ and use cond ① $[S_i(t_i) = y_i \;\&\; S_i(t_{i+1}) = y_{i+1}]$ to obtain

$$S_i(x) = \frac{z_i}{6h_i}(t_{i+1}-x)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3$$
$$+\left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right)(x-t_i) + \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)(t_{i+1}-x)$$

$h_i = t_{i+1} - t_i$

If we knew $z_0, \dots, z_n$ we would now $S(x)$.

The Cubic Spline Equation

cond 2 $\quad S_{i-1}'(t_i) = S_i'(t_i) \quad \forall\, i = 1, \ldots, n-1$

implies that

$$h_{i-1} z_{i-1} + 2(h_i + h_{i-1}) z_i + h_i z_{i+1}$$
$$= \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1})$$

$n-1$ eq.

$n+1$ unknowns

$z_0, \ldots, z_n$

The Natural Cubic Spline

Often $z_0 = z_n = 0$ called the natural cubic spline

$$\begin{bmatrix} u_1 & h_1 & 0 & \cdots & & \\ h_1 & u_2 & h_2 & & & \\ 0 & h_2 & u_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-3} & u_{n-2} & h_{n-2} \\ & & & & h_{n-2} & u_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix}$$

where

$$h_i = t_{i+1} - t_i \qquad u_i = 2(h_i + h_{i+1})$$

$$v_i = b_i - b_{i-1} \qquad b_i = \frac{6}{h_i}(y_{i+1} - y_i)$$

General Comments On Spline Usage

- truncated power functions
- B splines

when to put the knots?

① Interpolating points – Each point is a knot & $S(x)$ goes through them

② smoothing – if points subject to error, the splines are eval. at each abscissa (x) & fitted to the ordinate (least squares)