Johns Hopkins Engineering

625.464 Computational Statistics

Markov Chain Theory 2

Module 5 Lecture 5D

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Necessary Definitions

A state to which the chain returns with probability 1 is called a <u>recurrent state.</u>

If the expected time until a recurrence is finite then it is called <u>nonnull.</u>

If any state j can be reached from any state i in an finite number of steps, the chain is said to be <u>irreducible</u>.

$\exists\ m > 0 \text{ s.t. } P[X^{(m+n)} = j \mid X^{(m)} = i] > 0$

Let d(i) be the GC Divisor of all integers n such that $p_{ii}^{(n)} > 0$.

More Necessary Definitions and a BIG Result

If d(i) =1, the state i is said to be aperiodic. Otherwise it is periodic.

If d(i)=1 for all i, then the Markov Chain is said to be aperiodic.

If a Markov Chain is irreducible, aperiodic, and all states are nonnull and recurrent, then the Markov Chain is said to be ergodic.

Steady State Results

$\{X_n\}$ $[p_{ij}] = P$ $\quad P^{(m)} = [p_{ij}^{(m)}] = P^m$

$$\pi_i^{(t)} = P[X^{(t)} = i]$$

$$\pi^{(t)} = [\pi_0^{(t)} \; \pi_1^{(t)} \cdots]$$

Main Result

If a Markov Chain w/ transition matrix P is ergodic, then the stationary distribution ($\pi = \pi P$) is unique and limiting.

$$\lim_{n \to \infty} P\left[X^{(t+n)} = j \mid X^{(t)} = i\right] = \pi_j \quad \forall i$$

$$P^{(m)} = P^m \xrightarrow{m \to \infty} \text{all of rows } \pi$$

$$\pi = \pi P \qquad \pi e = 1 \qquad \pi \geq 0$$

$$e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

How does this relate to the Monte Carlo Method?

If $\{X^{(t)}\}$ are realizations from a Markov Chain w/ steady state dist π then for all functions h

$$\frac{1}{n}\sum_{i=1}^{n} h(x^{(t)}) \longrightarrow E_{\pi}\{h(X)\}$$

as $n \to \infty$