Johns Hopkins Engineering

625.464 Computational Statistics

Introduction to the Monte Carlo Method

Module 4 Lecture 4A

JOHNS HOPKINS
WHITING SCHOOL
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Introduction to the Monte Carlo Method

$h : [a,b] \to \mathbb{R}$ $\qquad X \sim U[a,b]$

$$I = \int_a^b h(x)\, dx$$

$$I = \int_a^b (b-a)\, h(x)\, \frac{1}{b-a}\, dx$$

$$= \int_a^b g(x) f(x) = E[g(X)]$$

where $g(x) = (b-a)\, h(x)$; $f(x) = \frac{1}{b-a}$

Introduction to the Monte Carlo Method

$$I = \int_a^b h(x)\,dx = \int_a^b g(x) f(x)\,dx = E[g(x)]$$

$$\hat{I} = \frac{1}{n} \sum_{i=1}^{n} g(x_i)$$

$$x_1, x_2, \ldots x_n \qquad U[a, b]$$

Monte Carlo Integration Example

$$I = \int_1^9 \frac{x}{4\sqrt{x}}\,dx = \int_1^9 \frac{x^{1/2}}{4}\,dx$$

$$I = \left[\frac{1}{6}x^{3/2}\right]\Big|_1^9 = \frac{13}{3} = 4.3\overline{3}$$

$n = 5000$

$$I = \int_1^9 8\left(\frac{x}{4\sqrt{x}}\right)\frac{1}{8}\,dx = \int_1^9 g(x)f(x)\,dx$$

$= E[g(x)]$ where $g(x) = 2\sqrt{x}$ $f(x) = \frac{1}{8}$

Monte Carlo Integration Example

$$I = \int_1^9 \frac{x}{4\sqrt{x}}\,dx = 4.3\overline{3} \longleftarrow \text{exactly}$$

$$\left.\begin{aligned} I &= \int_1^9 g(x)\,f(x)\,dx = E[g(x)] \\ \text{where } & g(x) = 2\sqrt{x} \ \ \&\ \ f(x) = \frac{1}{8} \end{aligned}\right]\ \text{MC Method}$$

Monte Carlo Integration in 3 Steps

1. Decomposition of integrand into 2 functions, 1 of which is a density.

2. Reformulation of the problem as an expected value.

3. Sample from the density & compute the empirical average.

Uses of the Monte Carlo Method

$$\Theta = \int_{R_x} h(x)\,dx = \int_{R_x} g(x) f(x)\,dx$$

$$E[h(x)]$$

$$h(x) = g(x) f(x)$$

$$\int_{R_x} f(x)\,dx = 1 \qquad f(x) > 0 \quad \forall\, x \in R_x.$$

Uses of the Monte Carlo Method.

① Estimate $\mu = E[X] = \int_{R_X} x f(x)\,dx$

$$\hat{\mu}_{mc} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

② Estimate $\mu = E[h(x)] = \int_{R_X} h(x) f(x)\,dx$

$$\hat{\mu}_{mc} = \frac{1}{n}\sum_{i=1}^{n} h(y_i)$$

③ Bayesian inference: Compute post. exp.

$$E[h(\theta)|y] = \int_{\Theta} h(\theta)\,p(\theta|y)\,d\theta$$

$$E[h(\theta)_{mc}|y] = \frac{1}{n}\sum_{i=1}^{n} h(\theta^{(i)})$$

where $\theta^{(i)}$ are drawn from $p(\theta|y)$

What next?

Q: How to generate the random samples from a given dist. $f(x)$?