625.664 Computational Statistics

Problem Set 6

Associated Reading: Chapter 7: 7.1 - 7.1.2, 7.2 - 7.2.5, 7.3 - 7.3.2

Complete the problems either by hand or using the computer and upload your final document to the Blackboard course site. All final submittals are to be in PDF form. Please document any code used to solve the problems and include it with your submission.

1. (a) Show that if in the Metropolis-Hastings algorithm the proposal distribution is $g(\cdot|x^{(t)}) = f(\cdot|x^{(t)})$, then the Metropolis-Hastings Ratio is always equal to one. (This is a first step to showing that the M-H ratio of a Gibbs sampler is always 1).
   (b) Let (X_1, X_2) be a two-dimensional random variable with joint density $f(x_1, x_2)$. Denote by $f_{X_1|X_2}(x_1|x_2)$ and $f_{X_2|X_1}(x_2|x_1)$ the corresponding conditional densities. Show that
$$f(x_1, x_2) = \frac{f_{X_1|X_2}(x_1|x_2)}{\int \frac{f_{X_1|X_2}(x_1|x_2)}{f_{X_2|X_1}(x_2|x_1)} dx_1}$$

2. (a) Derive the conditional distributions that are required to create a Gibbs sampler for drawing samples from a bivariate normal distribution with $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. (Hint: Look at the class notes.)
   (b) Now assume that $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$ in the above distribution. For $\rho = 0, .1, .2, .3, .4, .5$, generate Gibbs samples with varying lengths of burn-in. Do you think that Gibbs sampling is a viable method for this problem?

3. Posted on the course Blackboard site are the data sets MCdata1.txt and MCdata2.txt each containing Markov chains of length 10000 and 20000 respectively. For each of these Markov chains:
   (a) Plot the Sample Path
   (b) Plot the Cusum diagnostic Plot for the mean
   (c) Plot the Autocorrelation Plot
   (d) Discuss the mixing and convergence of the chain based on those three diagnostics.
   Hint: For the autocorrelation plot, you calculate $R(i)$, the correlation between iterates that are i iterations apart, as follows:
$$R(i) = \frac{C_i}{C_0}$$
   where C_i is the autocovariance function
$$C_i = \frac{1}{n} \sum_{t=1}^{n-i} (X^{(t)} - \bar{X})(X^{(t+i)} - \bar{X})$$

and C_0 is the variance function

$$C_0 = \frac{1}{n}\sum_{t=1}^{n}(X^{(t)} - \bar{X})^2$$

4. Posted on the course Blackboard site is the file MultipleChains.txt. It contains 7 Markov Chains each of length 1000. For each of the following subsets of the chains, calculate B, W and $\sqrt(R)$ as described in Section 7.3.1.2 of the text and discuss your results. Be careful when you load the chains in. You made need to separate them into different files first depending on your method.
   (a) The entire chains with $D = 0$ and $L = 1000$.
   (b) The entire chains with $D = 500$ and $L = 500$.
   (c) The first 500 elements of the chains with $D = 0$ and $L = 500$.
   (d) The first 500 elements of the chains with $D = 250$ and $L = 250$.
   (e) The first 50 elements of the chains with $D = 0$ and $L = 50$
   (f) The first 50 elements of the chains with $D = 25$ and $L = 25$