Johns Hopkins Engineering

625.464 Computational Statistics

Bootstrap Confidence Intervals

Module 9 Lecture 9C

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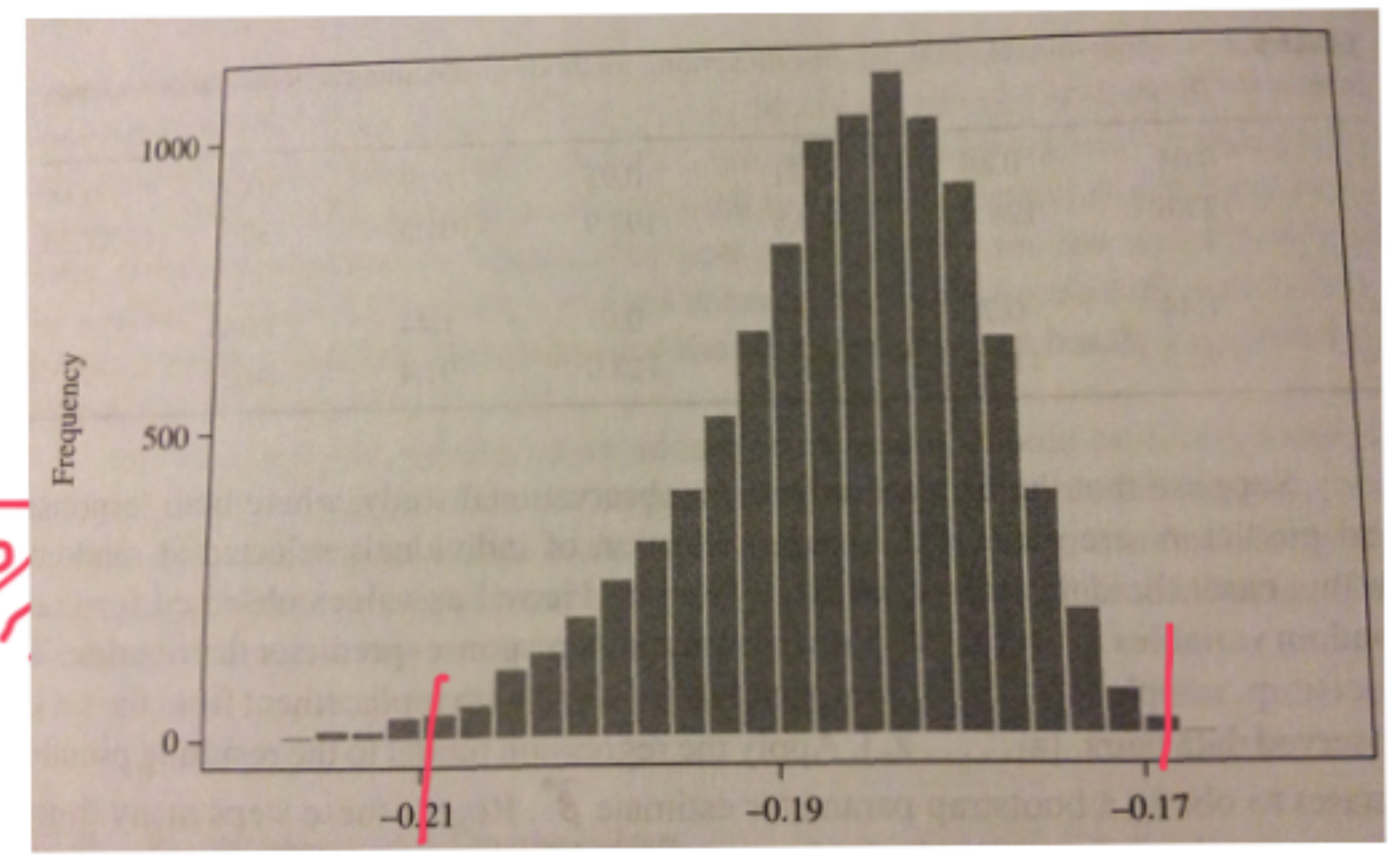
Bootstrap Confidence Intervals - The Percentile Method

Percentile Method → read percentiles off the histogram of $\hat{\Theta}^*$ values produced by Bootstrapping.

$\mathcal{X}^*$	$\widehat{\theta}^*$	$P^*[\widehat{\theta}^*]$
1 1 1	3/3	1/27
1 1 2	4/3	3/27
1 2 2	5/3	3/27
2 2 2	6/3	1/27
1 1 6	8/3	3/27
1 2 6	9/3	6/27
2 2 6	10/3	3/27
1 6 6	13/3	3/27
2 6 6	14/3	3/27
6 6 6	18/3	1/27

25/27



95% confidence interval for Θ is $(-.21, -.17)$

Justification for the Percentile Method

Consider a strictly increasing continuous transformation ϕ and a distribution function H that is continuous and symmetric $(H(z) = 1 - H(z))$ with the property that

$$\circledast \quad P\left[h_{\alpha/2} \leq \phi(\hat{\theta}) - \phi(\theta) \leq h_{1-\alpha/2}\right] = 1-\alpha$$

where h_α is the α-quantile of H

Percentile Method

- Generate B Bootstrap samples and B values $\hat{\theta}^*$.
- Then the bootstrap estimate of a $1-\alpha$ confidence interval is based on $((1-\alpha/2)100\text{th})$ & $(\alpha/2\ 100\text{th})$, empirical percentiles.

100 samples
90%
5^{th} 95^{th}

- In practice we obtain
$(t^*_{(\alpha/2)}, t^*_{(1-\alpha/2)})$ where t^*_π is the $[\pi^{th}]$ order statistic of our B sized bootstrap samples $\hat{\theta}^*$.

Justification for the Percentile Method

$$1-\alpha \approx P^*\left[h_{\alpha/2} \le \phi(\hat\theta^*) - \phi(\hat\theta) \le h_{1-\alpha/2}\right]$$

$$= P^*\left[h_{\alpha/2} + \phi(\hat\theta) \le \phi(\hat\theta^*) \le h_{1-\alpha/2} + \phi(\hat\theta)\right]$$

$$= P^*\left[\phi^{-1}(h_{\alpha/2} + \phi(\hat\theta)) \le \hat\theta^* \le \phi^{-1}(h_{1-\alpha/2} + \phi(\hat\theta))\right]$$

we know the α quantiles ε_α of the emp. dist of $\hat\theta^*$

ie we know $P^*\left[\varepsilon_{\alpha/2} \le \hat\theta^* \le \varepsilon_{1-\alpha/2}\right] = 1-\alpha$

Justification for the Percentile Method

Equate

$$\phi^{-1}\left(h_{\alpha/2} + \phi(\hat{\theta})\right) \approx \varepsilon_{\alpha/2}$$

$$\phi^{-1}\left(h_{1-\alpha/2} + \phi(\hat{\theta})\right) \approx \varepsilon_{1-\alpha/2}$$

Original $\ast$

$$1-\alpha = P\left[h_{\alpha/2} \le \phi(\hat{\theta}) - \phi(\theta) \le h_{1-\alpha/2}\right]$$

$$= P\left[h_{\alpha/2} - \phi(\hat{\theta}) \le -\phi(\theta) \le h_{1-\alpha/2} - \phi(\hat{\theta})\right]$$

$$= P\left[-h_{1-\alpha/2} + \phi(\hat{\theta}) \le \phi(\theta) \le -h_{\alpha/2} + \phi(\hat{\theta})\right]$$

$$= P\left[\phi^{-1}\left(\overset{h_{\alpha/2}}{\cancel{-h_{1-\alpha/2}}} + \phi(\hat{\theta})\right) \le \theta \le \phi^{-1}\left(\overset{h_{1-\alpha/2}}{\cancel{-h_{\alpha/2}}} + \phi(\hat{\theta})\right)\right]$$

$$= \boxed{P\left[\varepsilon_{\alpha/2} \le \underline{\theta} \le \varepsilon_{1-\alpha/2}\right] = 1-\alpha}$$

Pivoting and the Bootstrap t

The bootstrapped statistic should be pivotal, ie it's distribution should not depend on Θ...

~~Ex~~ If g is our standard var. stabilizing transformation, then $g(\hat{\Theta})$ is pivotal.

The Bootstrap t

$\theta = T(F) \quad \hat{\theta} = T(\hat{F})$

est. var of $(\hat{\theta})$ $\quad V(\hat{\theta}) = V(\hat{F})$

$$R(X, F) = \frac{T(\hat{F}) - T(F)}{\sqrt{V(\hat{F})}} = \frac{\hat{\theta} - \theta}{\sqrt{V(\hat{F})}}$$

bootstrap $R(X, F)$ by $R(X^*, \hat{F})$.

The Bootstrap t

Let $\hat{G}$ be the dist $R(X, F)$ and let $\hat{G}^*$ be the dist of $R(X^*, \hat{F})$

$$1-\alpha = P\left[\xi_{\alpha/2}(\hat{G}) \le R(X, F) \le \xi_{1-\alpha/2}(\hat{G})\right]$$

$$= P\left[\xi_{\alpha/2}(\hat{G}) \le \frac{\hat{\theta}-\theta}{\sqrt{v(\hat{F})}} \le \xi_{1-\alpha/2}(\hat{G})\right]$$

$$= P\left[\hat{\theta} - \sqrt{v(\hat{F})}\, \xi_{1-\alpha/2}\cancel{(\hat{G})}\,\hat{G}^* \le \theta \le \hat{\theta} - \sqrt{v(\hat{F})}\, \xi_{\alpha/2}\cancel{(\hat{G})}\,\hat{G}^*\right]$$

$\hat{G}$ roughly equal to $\hat{G}^*$

Bootstrap t CI

$$\left(\hat{\theta} - \sqrt{v(\hat{F})}\, \xi_{1-\alpha/2}(\hat{G}^*),\ \hat{\theta} - \sqrt{v(\hat{F})}\, \xi_{\alpha/2}(\hat{G}^*)\right)$$