# Johns Hopkins Engineering

## 625.464 Computational Statistics

### Introduction to Function Estimation
### Pointwise and Global Properties of Estimators

Module 10 Lecture 10B

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# Pointwise Properties of Function Estimators

Assume $\hat{f}$ estimates $f : D \to \mathbb{R}$

① <u>Bias</u>

The bias of <u>$\hat{f}$ at $x$</u> is given by

$$\text{Bias}(\hat{f}(x)) = E[\hat{f}(x)] - f(x)$$

If $\text{Bias}(\hat{f}(x)) = 0$, $\Rightarrow$ $\hat{f}$ is unbiased <u>at $x$</u>

If $\hat{f}$ is unbiased at every point $x$ in $D$, then we say that the estimator is pointwise unbiased.

# Pointwise Properties of Functions Estimators

2. <u>Variance</u>

The variance of $\hat{f}$ <u>at x</u> is given by

$$V[\hat{f}(x)] = E\left[\left(\hat{f}(x) - E[\hat{f}(x)]\right)^2\right]$$

3. <u>Mean Squared Error (MSE)</u>

The MSE of $\hat{f}$ at $x$ is

$$MSE[\hat{f}(x)] = E\left[\left(\hat{f}(x) - f(x)\right)^2\right]$$

$$= V\left(\hat{f}(x)\right) + \left(Bias\left(\hat{f}(x)\right)\right)^2$$

# Pointwise Properties of Functions Estimators

## 4) Mean Absolute Error (MAE)

the MAE of $\hat{f}$ at $x$ is given by

$$MAE[\hat{f}(x)] = E(|\hat{f}(x) - f(x)|)$$

## 5) Consistensy

$\hat{f}$ is said to be pointwise consistent if

$$E[\hat{f}(x)] \longrightarrow f(x)$$

for each $x$ as $n \rightarrow \infty$

# Global Properties of Estimators of Functions

- statistical properties of $\hat{f}$ over all of domain D of f.
- written w/o indication on x
- often integration of pointwise prop or defined in terms of norms.

The main tool for comparing $\hat{f}$ & f is

$$\|\hat{f}-f\|_p = \left(\int_D |\hat{f}(x)-f(x)|^p dx\right)^{1/p}$$

Need

① $\hat{f}$ defined over all of D ② integral exists

# Global Properties of Estimators of Functions

- $L_1$ : Integrated absolute error (IAE)

$$IAE(\hat{f}) = \int_D |\hat{f}(x) - f(x)| dx$$

- $L_2$ : Integrated squared error (ISE)

$$ISE(\hat{f}) = \int_D (\hat{f}(x) - f(x))^2 dx$$

$L_\infty$ : Sup absolute error (SAE)

$$SAE(\hat{f}) = \sup_{x \in D} |\hat{f}(x) - f(x)|$$

# Global Properties of Estimators of Functions

Bias

Integrated Absolute Bias (IAB)

$$IAB(\hat{f}) = \int_D \left| E(\hat{f}(x)) - f(x) \right| dx$$

Integrated Squared Bias (ISB)

$$ISB(\hat{f}) = \int_D \left( E(\hat{f}(x)) - f(x) \right)^2 dx$$

If $\hat{f}$ is unbiased then

$$IAB(\hat{f}) = ISB(\hat{f}) = 0$$

$Bias(\hat{f}(x)) = 0$ almost everywhere.

# Global Properties of Estimators of Functions

Variance

Integrated Variance (IV)

$$IV(\hat{f}) = \int_D V(\hat{f}(x)) = \int_D E\left[\left(\hat{f}(x) - E[\hat{f}(x)]\right)^2\right] dx$$

Integrated Mean Squared Error (IMSE)

$$IMSE(\hat{f}) = \int_D E(\hat{f}(x) - f(x))^2 dx$$

$$= IV(\hat{f}) + ISB(\hat{f})$$

$$= E\left[\int_D (\hat{f}(x) - f(x))^2\right] = E[ISE(\hat{f})] = MISE(\hat{f})$$

# Global Properties of Estimators of Functions

Integrated Mean Absolute Error (IMAE)

$$\text{IMAE}(\hat{f}) = \int_D E\left(|\hat{f}(x) - f(x)|\right.$$

$$= E\left[\int_D |\hat{f}(x) - f(x)|\, dx\right] = \text{MIAE}(\hat{f})$$

- can also extend the idea of consistency. — See Gentle Ch-10.