Johns Hopkins Engineering

625.464 Computational Statistics

Importance Sampling Theory

Module 5 Lecture 5A

JOHNS HOPKINS
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Variance Reduction Techniques

MC estimator for

$$\mu = \int h(x) f(x)\,dx$$

is

$$\hat{\mu}_{MC} = \frac{1}{n} \sum_{i=1}^{n} h(X_i)$$

where $X_1, \ldots X_n$ are sampled from f.

Importance Sampling
A Motivating Example

$p \sim \frac{1}{6}$

n rolls $\longrightarrow \frac{n}{6}$ 1's

var $\frac{5}{36} \cdot n$

coef of var $= \frac{\sqrt{var(x)}}{E(x)} = 5\%$

2000 times

Importance Sampling
A Motivating Example

Conf of var 50%

1, 1, 1, 4, 5, 6 $\quad p \sim \frac{1}{2} \quad$ 400

Problem: We are no longer sampling from the target distribution of a fair die.

Solution: Weight each roll of 1 by $\frac{1}{3}$.

$$Y_i = \begin{cases} 1/3 & \text{if } 1 \\ 0 & \text{o.w.} \end{cases}$$

$E[Y_i]\ \frac{1}{3} \cdot \frac{1}{2} + 0 = \frac{1}{6}$

$Var(y_i) = E[y_i^2] - E[y_i]^2$

$= \frac{1}{9} \cdot \frac{1}{2} - \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

Why does it work?

importance sampling distribution

importance weighting

Importance Sampling - More Formally

$$\mu = \int h(x) f(x)\,dx = \int h(x) \frac{f(x)}{g(x)} \cdot g(x)\,dx$$

$E[h(x)]$

$X_1, \dots X_n \text{ iid} \sim g$

imp. samp function

$$\hat{\mu}_{IS} = \frac{1}{n} \sum_{i=1}^{n} h(x_i)\, w^*(x_i)$$

$$w^*(x_i) = \frac{f(x_i)}{g(x_i)}$$

Importance Sampling Comments

① $$E[w^*(x)] = \int w^*(x)\, g(x)\,dx = \int \frac{f(x)}{g(x)} g(x)\,dx$$

$$= \int f(x)\,dx = 1$$

② $$E[\hat{\mu}_{IS}] = \frac{1}{n}\sum_{i=1}^{n} E[h(x_i)\, w^*(x_i)] =$$

$$= \frac{1}{n}\sum_{i=1}^{n} \overset{\mu}{E[h(x_i)]}\; \overset{1}{E[w^*(x_i)]} = \frac{1}{n} n\mu = \mu$$

③ $$Var[\hat{\mu}_{IS}] = \frac{1}{n^2}\sum_{i=1}^{n} var(h(x_i)\, w^*(x_i))$$

$$= \frac{1}{n} var(h(x)\, w(x))$$

reduce

$g \Leftarrow$

Choice of distribution g

- want $\frac{f(x)}{g(x)}$ to be bounded
- ~~good~~ for g to have heavier tails than f
- in practice we would like g to be nearly proportional to $|h(x)f(x)|$ so that

$$\frac{|h(x)f(x)|}{g(x)} \text{ is nearly constant}$$

Unstandardized Weights

$$w^*(x) = \frac{f(x)}{g(x)}$$