Johns Hopkins Engineering

625.464 Computational Statistics

Markov Chain Theory 1

Module 5 Lecture 5C

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Markov Chains

Consider a sequence of r.v. $\{X^{(t)}\}_{t=0,1,2,\dots}$ where each value may equal 1 of an at most countably infinite set of possible values called states

- Let S be the set of all possible values
  - state space
  - $X^{(t)} = j$ indicates that the process is in state j at time t.

Markov Chain

Suppose:

$p_{ij}^{(t)}$ is the probability that the process changes from state i to state j at time $t+1$

If $\forall\ t = 0, 1, \ldots$; $X^{(0)}, X^{(1)}, \ldots X^{(t-1)}$,

and $i, j \in S$

$$p_{ij}^{(t)} = P[X^{(t+1)} = j \mid X^{(0)} = x_0, X^{(1)} = x_1, \ldots, X^{(t)} = i]$$

$$= P[X^{(t+1)} = j \mid X^{(t)} = i]$$

$\{X^{(t)}\}\ t = 0, 1, \ldots$ Markov Chain

P_{ij} and Transitions Matrices

- The $p_{ij}^{(t)}$ are called single step transition probabilities.
- If $p_{ij}^{(t)}$ are independent of t, then the chain is said to be homogeneous.

$$p_{ij}^{(t)} = p_{ij}$$

① $p_{ij} \geq 0 \quad i,j$

② $\sum_j p_{ij} = 1 \; \forall i$

$$P = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots & p_{0j} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1j} \\ \vdots & & & & \vdots \\ p_{i0} & p_{i1} & \cdots & p_{ij} & \cdots \end{bmatrix}$$

stochastic matrix

③ Size $\Rightarrow S$

Markov Chain Example

Consider a seq of Bernoulli Trials p-sucess & q-failure. Let X_n be the # of uninterupted successes that have been completed to time $t=n$.

$$SFSSF \quad X_0=1 \quad X_1=0 \quad X_2=1 \quad X_3=2 \quad X_4=0$$

$$P = \begin{bmatrix} q & p & 0 & \cdots & & & \\ q & 0 & p & 0 & \cdots & & \\ q & 0 & 0 & p & 0 & \cdots & \\ q & 0 & 0 & 0 & p & 0 & \cdots \\ q & & & & & & \end{bmatrix}$$

$\{X_n\}$

$\mathcal{S} = \{0, 1, 2, \cdots\}$

Mth Step Transition Matrix

M: step

$$P_{ij}^{(m)} = P\left[X^{(t+m)} = j \mid X^{(t)} = i\right]$$

$$P^{(m)} = \left[P_{ij}^{(m)}\right]$$

$$P^{(m)} = P^m$$