Johns Hopkins Engineering

625.464 Computational Statistics

Review of Maximum Likelihood Estimation, the Score Function, and Fisher Information Matrices

Module 2 Lecture 2D

JOHNS HOPKINS
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Maximum Likelihood Estimation Review

$X_1, \dots X_n$, iid $f(x \mid \theta)$

for $\vec{\theta} = (\theta_1, \dots, \theta_p)$

$x_1, \dots, x_n$

$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta)$

$\hat{\theta} \longrightarrow$ maximizes $L(\theta)$

Maximum Likelihood Estimation Review

$$L(\theta)$$

$$\ell(\theta) = \log L(\theta)$$

$$\ell'(\theta) = 0$$

$$\ell'(\theta) = \left(\frac{d\ell(\theta)}{d\theta_1}, \frac{d\ell(\theta)}{d\theta_2}, \dots, \frac{d\ell(\theta)}{d\theta_p} \right) = 0$$

$\rightarrow$ score function $E[\ell'(\theta)] = 0$

Maximum Likelihood Estimation Review

$$N(\mu, \sigma^2)$$

$$\theta = (\mu, \sigma^2)$$

$$f(x|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-1/2\left(\frac{X-\mu}{\sigma^2}\right)^2}$$

$$\ell(\theta|x) = -\frac{1}{2}\ln\left(2\pi\sigma^2\right) - \frac{(x-\mu)^2}{2\sigma^2}$$

Maximum Likelihood Estimation Review

$$\ell(\theta|x) = -\frac{1}{2}\ln\left(2\pi\sigma^2\right) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\ell'(\theta) = \left(\frac{\partial \ell}{\partial \mu}, \frac{\partial \ell}{\partial \sigma^2}\right)$$

$$= \left(\frac{x-\mu}{\sigma^2}, -\frac{1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4}\right)$$

$$E\left[\ell'(\theta)\right] = (0,0)$$

$$E[x] = \mu \qquad E\left[(x-\mu)^2\right] = \sigma^2$$

Fisher Information Matrix

$\ell''(\theta)$ pxp matrix

$$\left[\frac{\partial^2 \ell(\theta)}{\partial\theta_i \partial\theta_j}\right]_{ij}$$

$$I(\theta) = -E[\ell''(\theta)] = E[\ell'(\theta)\ell'(\theta)^T]$$

$N(\mu, \sigma^2)$

Fisher Information Matrix

$$\ell'(\theta) = \left(\frac{x-\mu}{\sigma^2}, -\frac{1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4} \right)$$

$$\ell''(\theta) = \begin{bmatrix} \frac{\partial^2 \ell(\theta)}{\partial \mu^2} & \frac{\partial^2 \ell(\theta)}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \ell(\theta)}{\partial \mu \partial \sigma^2} & \frac{\partial^2 \ell(\theta)}{\partial (\sigma^2)^2} \end{bmatrix}$$

$$= -E\left[\begin{bmatrix} -\frac{1}{\sigma^2} & -\frac{(x-\mu)}{\sigma^4} \\ -\frac{(x-\mu)}{\sigma^4} & \frac{1}{2\sigma^4} - \frac{(x-\mu)^3}{\sigma^6} \end{bmatrix} \right]$$

$N(\mu, \sigma^2)$ Fisher Information Matrix

$$I(\theta) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4 \end{bmatrix}$$