Johns Hopkins Engineering

625.464 Computational Statistics

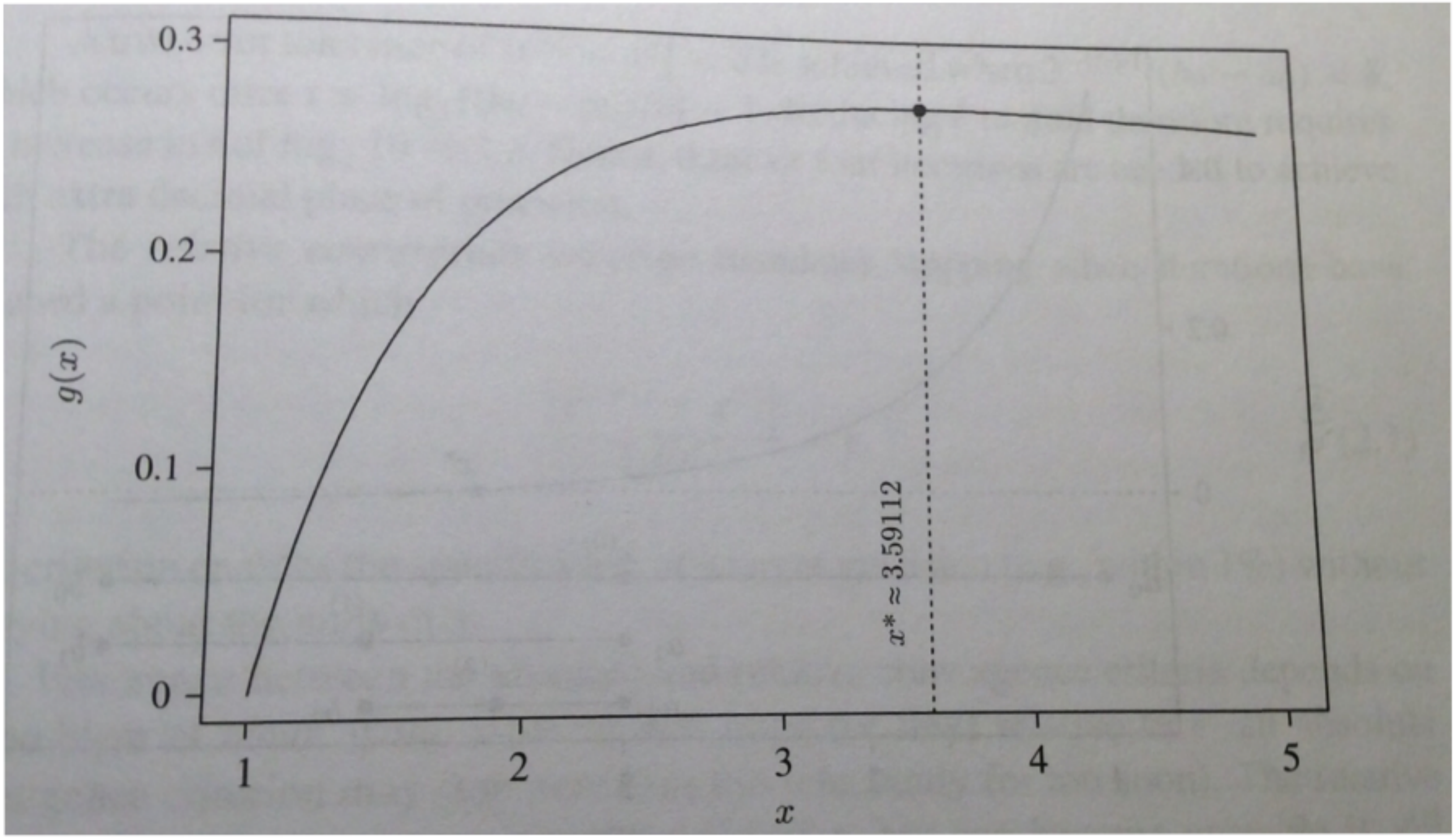
The Bisection Method

Module 2 Lecture 2B

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

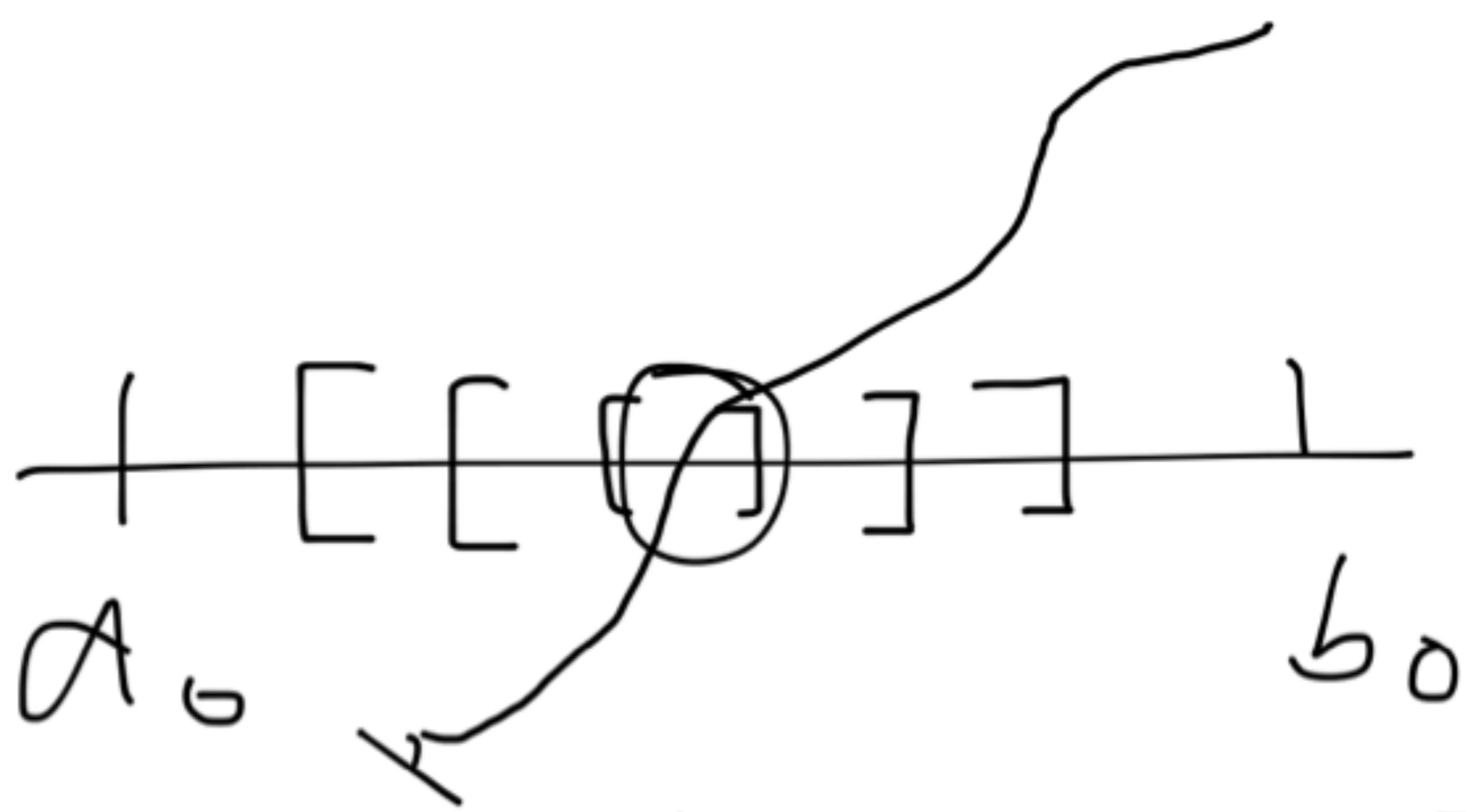
Univariate Optimization Problems

$$g(x) = \frac{\log x}{1+x} \qquad g'(x) = \frac{1 + 1/x - \log x}{(1+x)^2}$$



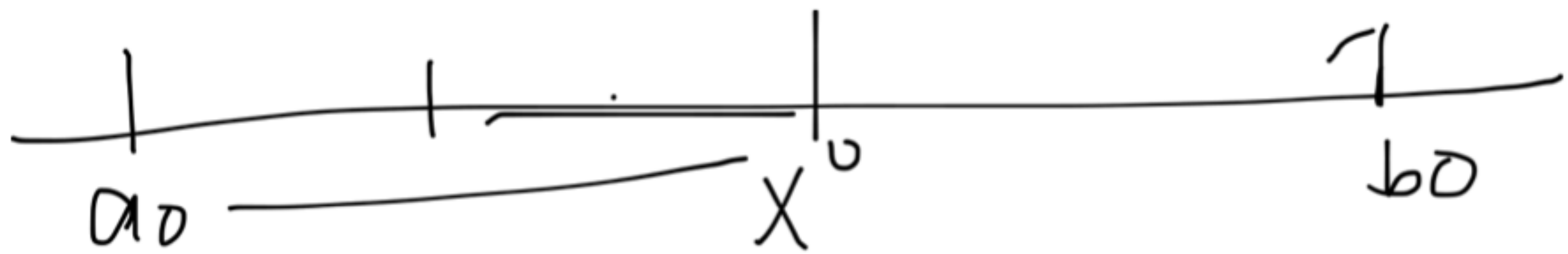
Bisection Method

If $g'(x)$ is continuous on $[a_0, b_0]$ and $g'(a_0)\, g'(b_0) \le 0$, then by IVT there exists and x^* in $[a_0, b_0]$ such that $g'(x^*)=0$ and is a local optimum.



$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \cdots \quad x^*$$

Bisection Method

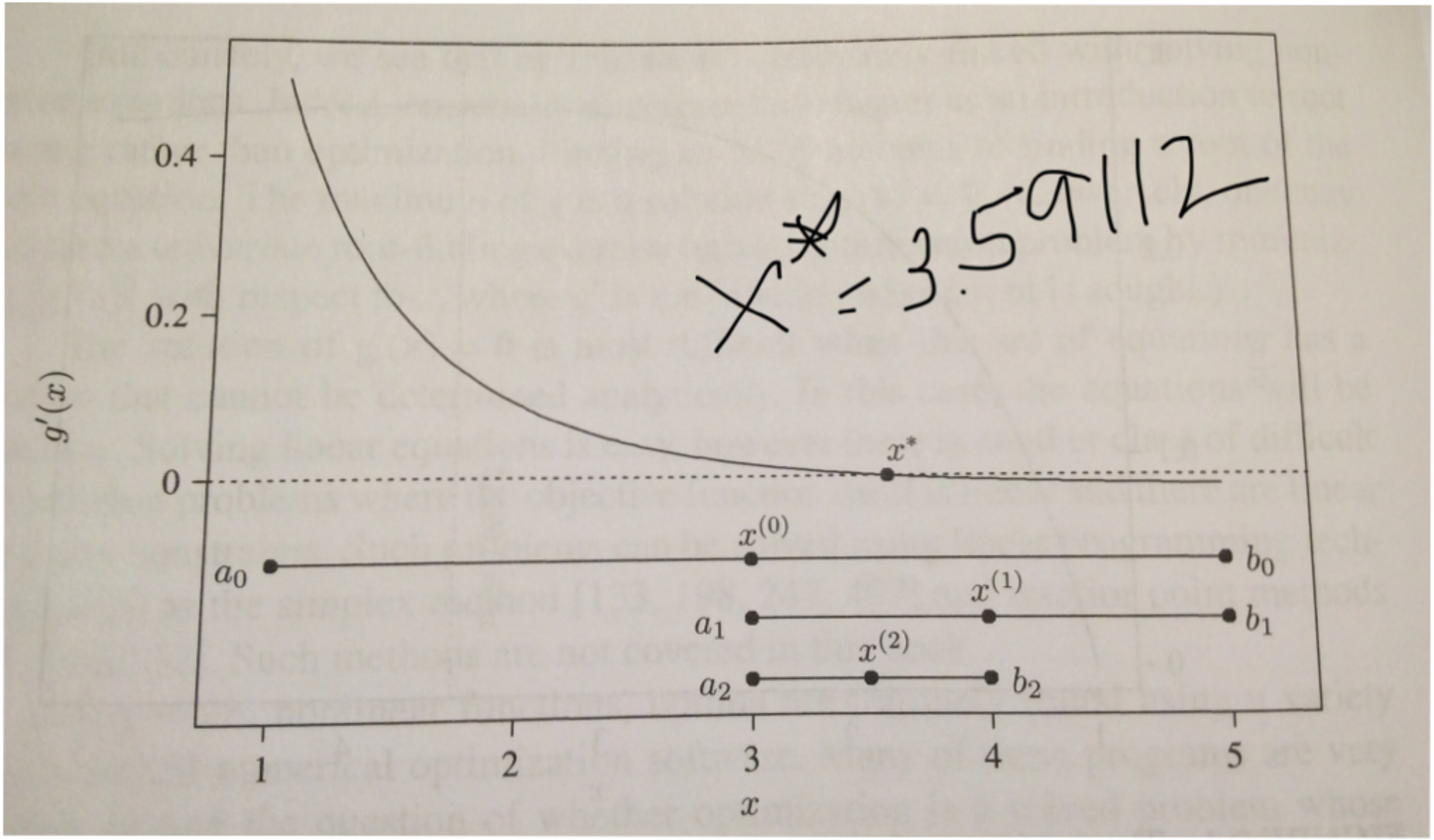


$$x^{(0)} = \frac{(a_0 + b_0)}{2}$$

$$[a_{t+1}, b_{t+1}] = \begin{cases} [a_t, x^{(t)}] & \text{if } g'(a_t)\, g'(x^{(t)}) \leq 0 \\ [x^{(t)}, b_t] & \text{if } g'(a_t)\, g'(x^{(t)}) > 0 \end{cases}$$

$$x^{(t+1)} = \frac{[a_{t+1}, b_{t+1}]}{2}$$

Bisection Method



$g(x) = \frac{\log x}{1+x}$ $g'(x) = \frac{1 + 1/x - \log x}{(1+x)^2}$ $[1, 5]$

When To Stop?

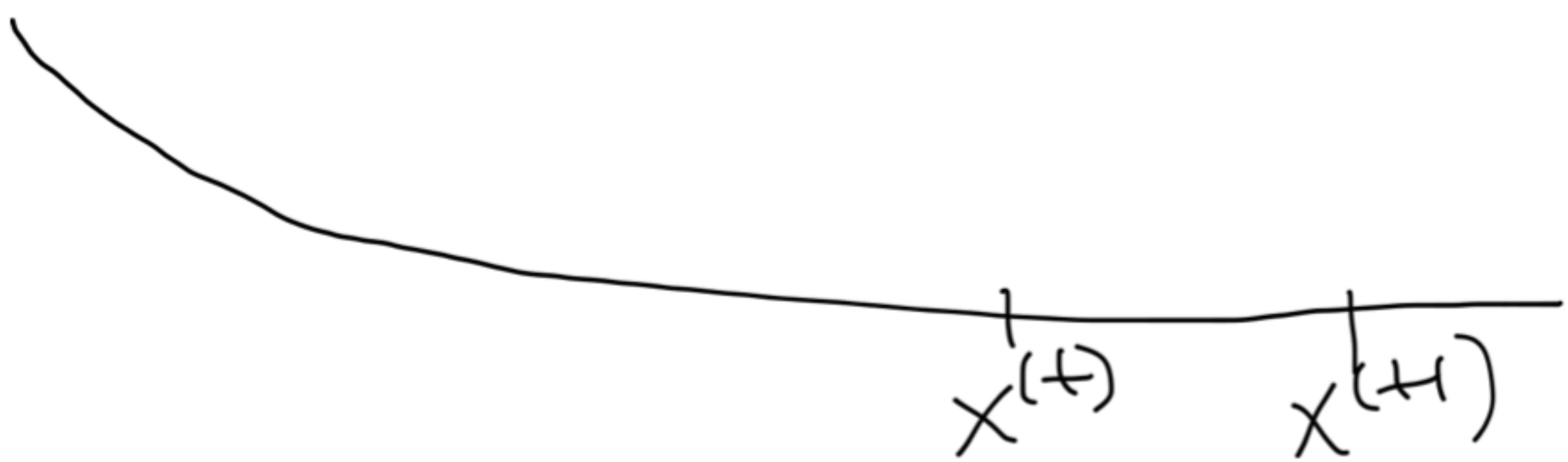
$$x^{(t)} \longrightarrow x^*$$

① Convergence criteria

② Failure rule

Convergence Criteria

$$g'(x^{(t+1)}) \to 0$$



Absolute Convergence

$$\left| x^{(t+1)} - x^{(t)} \right| < \varepsilon \qquad \varepsilon > 0$$

$$b_t - a_t = 2^{-t}(b_0 - a_0)$$

$$|x^{(t)} - x^{*}| < \delta$$

$$|x^{(t+1)} - x^{(t)}| = 2^{-(t+1)}(b_0 - a_0) < \delta$$

$$t > \log_2\left(\frac{b_0 - a_0}{\delta}\right) - 1$$

Bisection Method Will Converge!

$$\lim_{t \to \infty} a_t = \lim_{t \to \infty} b_t = x_\infty$$

$$g'(a_t)\,g'(b_t) \leq 0$$

$$\left[g'(x_\infty)\right]^2 \leq 0$$

$$\Longrightarrow g'(x_\infty) = 0$$

Relative Convergence

$$\frac{|x^{(t+1)} - x^{(t)}|}{|x^{(t)}|} < \varepsilon$$

Failure Rule

1. Stop after N iterations

2. Stop if fail to decrease