Johns Hopkins Engineering

625.464 Computational Statistics

Importance Sampling
An Example

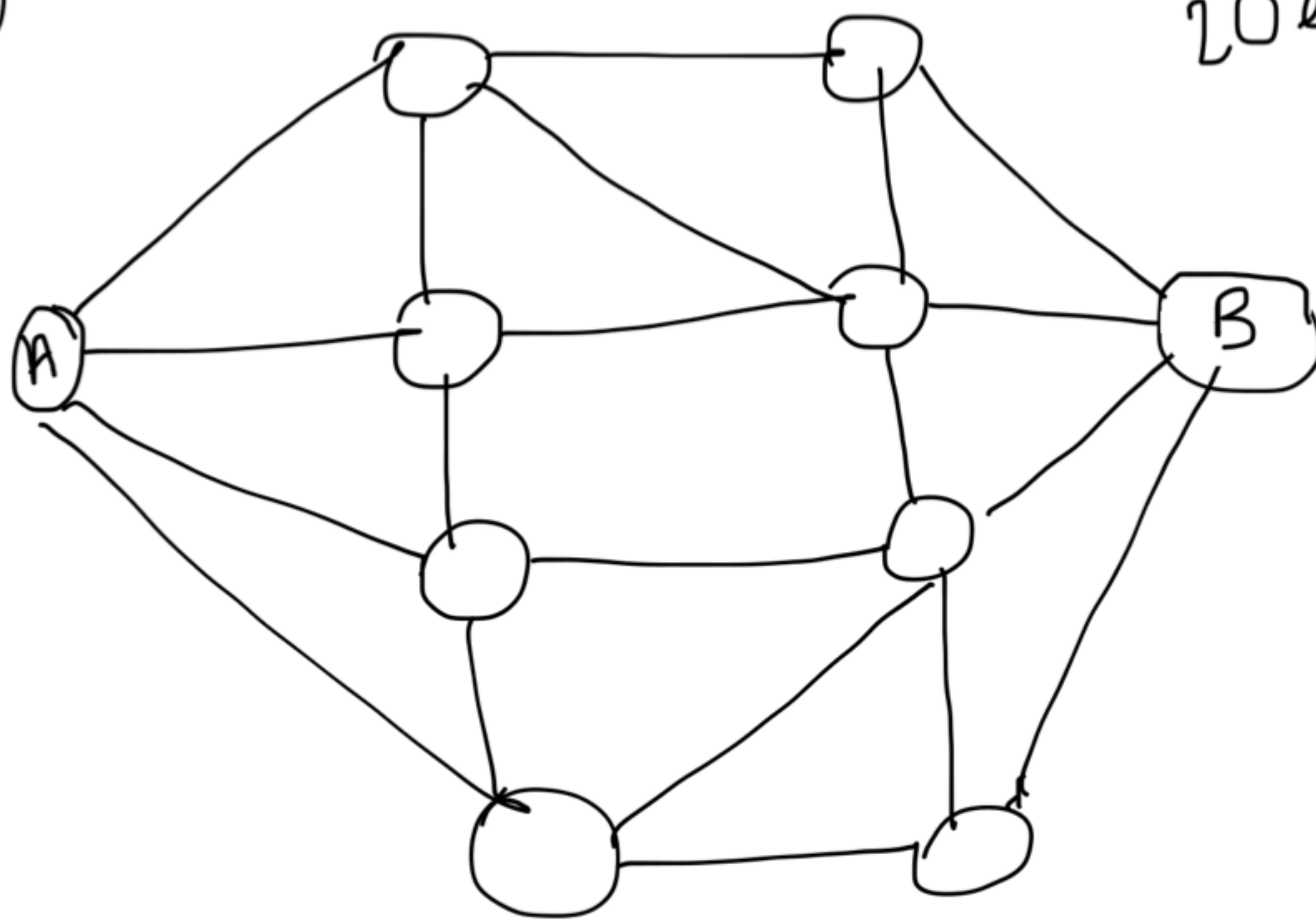
Module 5 Lecture 5B

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Network Failure Probability Example

$P(10^{-10} \sim 10^{-3})$

10 nodes
20 edges



Network Failure Probability Example

Let $X = (X_1, \ldots, X_{26})$

$$X_i = \begin{cases} X_i = 1 \text{ broken} \\ X_i = 0 \text{ intact} \end{cases}$$

$$b(X) = \sum_{i=1}^{26} X_i$$

$$h(X) = \begin{cases} 1 & \text{network fails} \quad - \text{ no A-B path} \\ 0 & \text{network doesn't fail} \quad - \text{ A-B exists} \end{cases}$$

$$\mu = E[h(X)]$$ ← probability of a network failure

Network Failure Probability Example

Attempt 1 : Standard Monte Carlo

$$\mu = E[h(x)]$$ ← prob network fails

$$x^1, \dots, x^n$$

$$\hat{\mu}_{mc} = \frac{1}{n} \sum_{i=1}^{n} h(x^i)$$

$$Var(\hat{\mu}_{mc}) = \frac{\mu(1-\mu)}{n}$$

$n = 100,000$

$p = .05$

$\hat{\mu} = 2.00 \times 10^{-5}$

1.41×10^{-5}

Network Failure Probability Example

Attempt 2: Importance Sampling

Let $X^{1*}, \ldots, X^{n*}$ $\quad p^* > p$

Originally $X^i \sim f$

$$\mu = \int h(x) f(x)\,dx \quad \text{where}$$

$$f(x) = p^{b(x)}(1-p)^{20-b(x)}$$

Now we are drawing $X^{i*} \sim g$

$$g(x) = p^{*\,b(x)}(1-p^*)^{20-b(x)}$$

$$w^*(X^{i*}) = \frac{f(x)}{g(x)} = \left(\frac{1-p}{1-p^*}\right)^{20}\left(\frac{p(1-p^*)}{p^*(1-p)}\right)^{b(X^{i*})}$$

Network Failure Probability Example

Our estimator will be

$$\hat{\mu}_{IS} = \frac{1}{n}\sum_{i=1}^{n} h(X^{i*})w^{*}(X^{i*})$$

$$X^{i*} \sim q$$

Network Failure Probability Example

What about variance?

$$var(\hat{\mu}_{IS}) = \frac{1}{n} var\{h(X^{i*})w^*(X^{i*})\}$$

$$= \frac{1}{n}\left\{E\left\{\left[h(x^{i*})w(X^{i*})\right]^2\right\} - \left[E\left[h(x^{i*})w(x^{i*})\right]^2\right]\right\}$$

Letting $\mathcal{C}$ be all possible networks and $\mathcal{F} \subseteq \mathcal{C}$ be the subset that fails

$$\rightarrow = \frac{1}{n}\left[\sum_{x \in \mathcal{F}} E\left[w^*(X^{i*})^2\right] - \mu^2\right]$$

$h(X)^2 = h(X) = 1$ iff network fails o.w. 0

Network Failure Probability Example

$$\mathrm{var}(\hat{\mu}_{IS}) = \frac{1}{n}\left[\sum_{x \in \mathcal{F}} \left[\underbrace{\left(\frac{1-p}{1-p^*}\right)^{20}\left(\frac{p(1-p^*)}{p^*(1-p)}\right)^{b(x)}}_{w^*(x)}\right]^2 \cdot \underbrace{p^*(x)}_{g(x)} \cdot {p^*}^{b(x)}(1-p^*)^{20-b(x)} - \mu^2\right]$$

$$= \frac{1}{n}\left[\sum_{x \in \mathcal{F}} w^*(x)\, p^{b(x)}(1-p)^{20-b(x)} - \mu^2\right]$$

$$w^*(x) \leq \left(\frac{1-p}{1-p^*}\right)^{20}\left(\frac{p(1-p^*)}{p^*(1-p)}\right)^4 \qquad b(x) \geq 4$$

$$p = .05 \rightarrow p^* = .25 \qquad \mathrm{var}(\hat{\mu}_{IS}) \leq \frac{\mu}{n}(.07 - \mu) \qquad \frac{\mathrm{var}\{MC\}}{\mathrm{var}\{IS\}} \approx 14$$

$$\hat{\mu}_{IS} = 1.01 \times 10^{-5} \qquad 1.56 \times 10^{-6}$$