2. Consider the following mixture of two normal densities:

$$\mathbf{p}(x;\theta) = \pi\phi(x;\ \mu_1,\sigma_1^2) + (1-\pi)\phi(x;\ \mu_2,\sigma_2^2)$$

where $\theta = (\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$

a. Show the complete-data log-likelihood function:

$$L(\theta; x, z) = \prod_{i=1}^{n} [\pi\phi(x_i;\mu_1,\sigma_1^2)]^{z_i}\,[(1-\pi)\phi(x_i;\mu_2,\sigma_2^2)]^{(1-z_i)}$$

$$= \prod_{i=1}^{n} [\pi\phi(x_i;\mu_1,\sigma_1^2)]^{z_i} \prod_{i=1}^{n} [(1-\pi)\phi(x_i;\mu_2,\sigma_2^2)]^{(1-z_i)}$$

$$l(\theta; x, z) = \log L(\theta; x, z) = \sum_{i=1}^{n} \log[[\pi\phi(x_i;\mu_1,\sigma_1^2)]^{z_i}] + \sum_{i=1}^{n} \log[(1-\pi)\phi(x_i;\mu_2,\sigma_2^2)]^{(1-z_i)}$$

$$= \sum_{i=1}^{n} z_i[\log\pi + \log\phi(x_i;\mu_1,\sigma_1^2)] + \sum_{i=1}^{n} (1-z_i)[\log(1-\pi) + \log\phi(x_i;\mu_2,\sigma_2^2)]$$

$$\boxed{= \log\pi \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i \log\phi(x_i;\mu_1,\sigma_1^2) + \log(1-\pi)\left(n - \sum_{i=1}^{n} z_i\right) + \sum_{i=1}^{n} (1-z_i)\log\phi(x_i;\mu_2,\sigma_2^2)}$$

b. Find $Q\left(\theta|\theta^{(k)}\right)$ in terms of $E\left[Z_i|x_i,\theta^{(k)}\right]$

$$Q\left(\theta|\theta^{(k)}\right) = E\left[l(\theta; y)|x,\theta^{(k)}\right]E\left[l(\theta; x, z)|x,\theta^{(k)}\right]$$

$$= E\left\{\log\pi^{(k)} \sum_{i=1}^{n} Z_i + \sum_{i=1}^{n} Z_i \log\phi\left(x_i;\mu_1^{(k)},(\sigma_1^2)^{(k)}\right)\right.$$
$$\left. + \log\left(1-\pi^{(k)}\right)\left(n - \sum_{i=1}^{n} Z_i\right) + \sum_{i=1}^{n} (1-Z_i)\log\phi\left(x_i;\mu_2^{(k)},(\sigma_2^2)^{(k)}\right)\right\}$$

$$\boxed{\begin{aligned} &= \log\pi^{(k)} \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right] + \log\phi\left(x_i;\mu_1^{(k)},(\sigma_1^2)^{(k)}\right) \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right] + \\ &\log\left(1-\pi^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right]\right) + \log\phi\left(x_i;\mu_2^{(k)},(\sigma_2^2)^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right]\right)\end{aligned}}$$

c. Show $E\left[Z_i|x_i,\theta^{(k)}\right]$

$$E\left[Z_i|x_i,\theta^{(k)}\right] = P\left[Z_i = 1|x_i,\theta^{(k)}\right]\cdot 1 + P\left[Z_i = 0|x_i,\theta^{(k)}\right]\cdot 0$$

$$= P\left[Z_i = 1 | x_i, \theta^{(k)}\right] = \frac{P\left[x_i, Z_i = 1 | \theta^{(k)}\right]}{P[x_i; \theta^{(k)}]}$$

$$= \frac{\pi^{(k)}\phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right)^{(1)} \left[(1-\pi^{(k)})\phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)\right]^{(1-(1))}}{\pi^{(k)}\phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right) + (1-\pi^{(k)})\phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)}$$

$$\boxed{= \frac{\pi^{(k)}\phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right)}{\pi^{(k)}\phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right) + (1-\pi^{(k)})\phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)}}$$

d. Show $\pi^{(k+1)}$

Let $\eta_i^{(k)} = E\left[Z_i | x_i, \theta^{(k)}\right]$ and $\eta^{(k)} = \sum_{i=1}^n \eta_i^{(k)}$.

$$\frac{\delta Q\left(\theta|\theta^{(k)}\right)}{\delta \pi^{(k)}} = \frac{\delta \begin{Bmatrix} \log \pi^{(k)} \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right] + \log \phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right) \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right] + \\ \log\left(1-\pi^{(k)}\right)\left(n - \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right]\right) + \log \phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)\left(n - \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right]\right) \end{Bmatrix}}{\delta \pi^{(k)}}$$

$$= \frac{\eta^{(k)}}{\pi^{(k)}} - \frac{n - \eta^{(k)}}{1 - \pi^{(k)}} = \overset{set\ to}{\triangleq} 0$$

$$\Rightarrow \frac{\eta^{(k)}}{\pi^{(k)}} = \frac{n-\eta^{(k)}}{1-\pi^{(k)}} \Rightarrow \frac{1-\pi^{(k)}}{\pi^{(k)}} = \frac{n-\eta^{(k)}}{\eta^{(k)}} \Rightarrow \frac{1}{\pi^{(k)}} = \frac{n}{\eta^{(k)}} \Rightarrow \pi^{(k)} = \frac{\eta^{(k)}}{n}$$

$$\boxed{\Rightarrow \pi^{(k+1)} = \frac{\eta^{(k)}}{n} = \frac{1}{n}\sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right]}$$

e. Show $\mu_1^{(k+1)}$, $(\sigma_1^2)^{(k+1)}$, $\mu_2^{(k+1)}$, and $(\sigma_2^2)^{(k+1)}$

$$\frac{\delta Q\left(\theta|\theta^{(k)}\right)}{\delta \mu_1^{(k)}} = \frac{\delta \begin{Bmatrix} \log \pi^{(k)} \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right] + \log \phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right) \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right] + \\ \log\left(1-\pi^{(k)}\right)\left(n - \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right]\right) + \log \phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)\left(n - \sum_{i=1}^n E\left[Z_i|x_i, \theta^{(k)}\right]\right) \end{Bmatrix}}{\delta \mu_1^{(k)}}$$

$$= \eta^{(k)}\left[-\frac{1}{2}\left((\sigma_1^2)^{(k)}\right)^{-1}(2)\left(x_i - \mu_1^{(k)}\right)(-1)\right] = \eta^{(k)}\frac{\left(x_i - \mu_1^{(k)}\right)}{(\sigma_1^2)^{(k)}} \overset{set\ to}{\triangleq} 0$$

$$\Rightarrow \eta^{(k)} x_i = \eta^{(k)} \mu_1^{(k)} \Rightarrow \mu_1^{(k)} = \frac{1}{\eta^{(k)}} \eta^{(k)} x_i$$

$$\boxed{\Rightarrow \mu_1^{(k+1)} = \frac{1}{\eta^{(k)}} \sum_{i=1}^{n} \eta_i^{(k)} x_i}$$

$$\frac{\delta Q\left(\theta|\theta^{(k)}\right)}{\delta(\sigma_1^2)^{(k)}} = \frac{\delta \left\{ \begin{matrix} \log \pi^{(k)} \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right] + \log \phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right) \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right] + \\ \log\left(1-\pi^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right]\right) + \log \phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right]\right) \end{matrix} \right\}}{\delta(\sigma_1^2)^{(k)}}$$

$$= \eta^{(k)} \left\{ -\frac{1}{2(\sigma_1^2)^{(k)}} + \frac{\left(x_i - \mu_1^{(k)}\right)^2}{2((\sigma_1^2)^{(k)})^2} \right\} \overset{set\ to}{\triangleq} 0$$

$$\Rightarrow \frac{\eta^{(k)}}{2(\sigma_1^2)^{(k)}} = \eta^{(k)} \frac{\left(x_i - \mu_1^{(k)}\right)^2}{2((\sigma_1^2)^{(k)})^2} \Rightarrow (\sigma_1^2)^{(k)} = \frac{1}{\eta^{(k)}} \sum_{i=1}^{n} \eta_i^{(k)} \left(x_i - \mu_1^{(k)}\right)^2$$

$$\boxed{\Rightarrow (\sigma_1^2)^{(k+1)} = \frac{1}{\eta^{(k)}} \sum_{i=1}^{n} \eta_i^{(k)} \left(x_i - \mu_1^{(k+1)}\right)^2}$$

$$\frac{\delta Q\left(\theta|\theta^{(k)}\right)}{\delta \mu_2^{(k)}} = \frac{\delta \left\{ \begin{matrix} \log \pi^{(k)} \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right] + \log \phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right) \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right] + \\ \log\left(1-\pi^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right]\right) + \log \phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i,\theta^{(k)}\right]\right) \end{matrix} \right\}}{\delta \mu_2^{(k)}}$$

$$= \left(n - \eta^{(k)}\right) \left\{ -\frac{1}{2(\sigma_2^2)^{(k)}} (2)\left(x_i - \mu_2^{(k)}\right)(-1) \right\} = \frac{\left(n - \eta^{(k)}\right)\left(x_i - \mu_2^{(k)}\right)}{(\sigma_2^2)^{(k)}} \overset{set\ to}{\triangleq} 0$$

$$\Rightarrow \frac{\left(n - \eta^{(k)}\right) x_i}{(\sigma_2^2)^{(k)}} = \frac{\left(n - \eta^{(k)}\right) \mu_2^{(k)}}{(\sigma_2^2)^{(k)}} \Rightarrow \mu_2^{(k)} = \frac{1}{n - \eta^{(k)}} \left(n - \eta^{(k)}\right) x_i$$

$$\boxed{\Rightarrow \mu_2^{(k+1)} = \frac{1}{n - \eta^{(k)}} \sum_{i=1}^{n} \left(1 - \eta_i^{(k)}\right) x_i}$$

$$\frac{\delta Q\left(\theta|\theta^{(k)}\right)}{\delta(\sigma_2^2)^{(k)}}$$

$$= \frac{\delta\left\{\begin{matrix}\log \pi^{(k)} \sum_{i=1}^{n} E\left[Z_i|x_i, \theta^{(k)}\right] + \log \phi\left(x_i; \mu_1^{(k)}, (\sigma_1^2)^{(k)}\right) \sum_{i=1}^{n} E\left[Z_i|x_i, \theta^{(k)}\right] + \\ \log\left(1-\pi^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i, \theta^{(k)}\right]\right) + \log \phi\left(x_i; \mu_2^{(k)}, (\sigma_2^2)^{(k)}\right)\left(n - \sum_{i=1}^{n} E\left[Z_i|x_i, \theta^{(k)}\right]\right)\end{matrix}\right\}}{\delta(\sigma_2^2)^{(k)}}$$

$$= \left(n - \eta^{(k)}\right)\left\{-\frac{1}{2(\sigma_2^2)^{(k)}} + \frac{\left(x_i - \mu_2^{(k)}\right)^2}{2((\sigma_2^2)^{(k)})^2}\right\} \overset{set\ to}{\triangleq} 0$$

$$\Rightarrow \frac{\left(n - \eta^{(k)}\right)}{2(\sigma_2^2)^{(k)}} = \frac{\left(n - \eta^{(k)}\right)\left(x_i - \mu_2^{(k)}\right)^2}{2((\sigma_2^2)^{(k)})^2} \Rightarrow (\sigma_2^2)^{(k)} = \frac{1}{n - \eta^{(k)}}\left(n - \eta^{(k)}\right)\left(x_i - \mu_2^{(k)}\right)^2$$

$$\boxed{\Rightarrow (\sigma_2^2)^{(k+1)} = \frac{1}{n - \eta^{(k)}} \sum_{i=1}^{n} \left(1 - \eta_i^{(k)}\right)\left(x_i - \mu_2^{(k+1)}\right)^2}$$