Johns Hopkins Engineering

625.464 Computational Statistics

Rejection Sampling

Module 4 Lecture 4C

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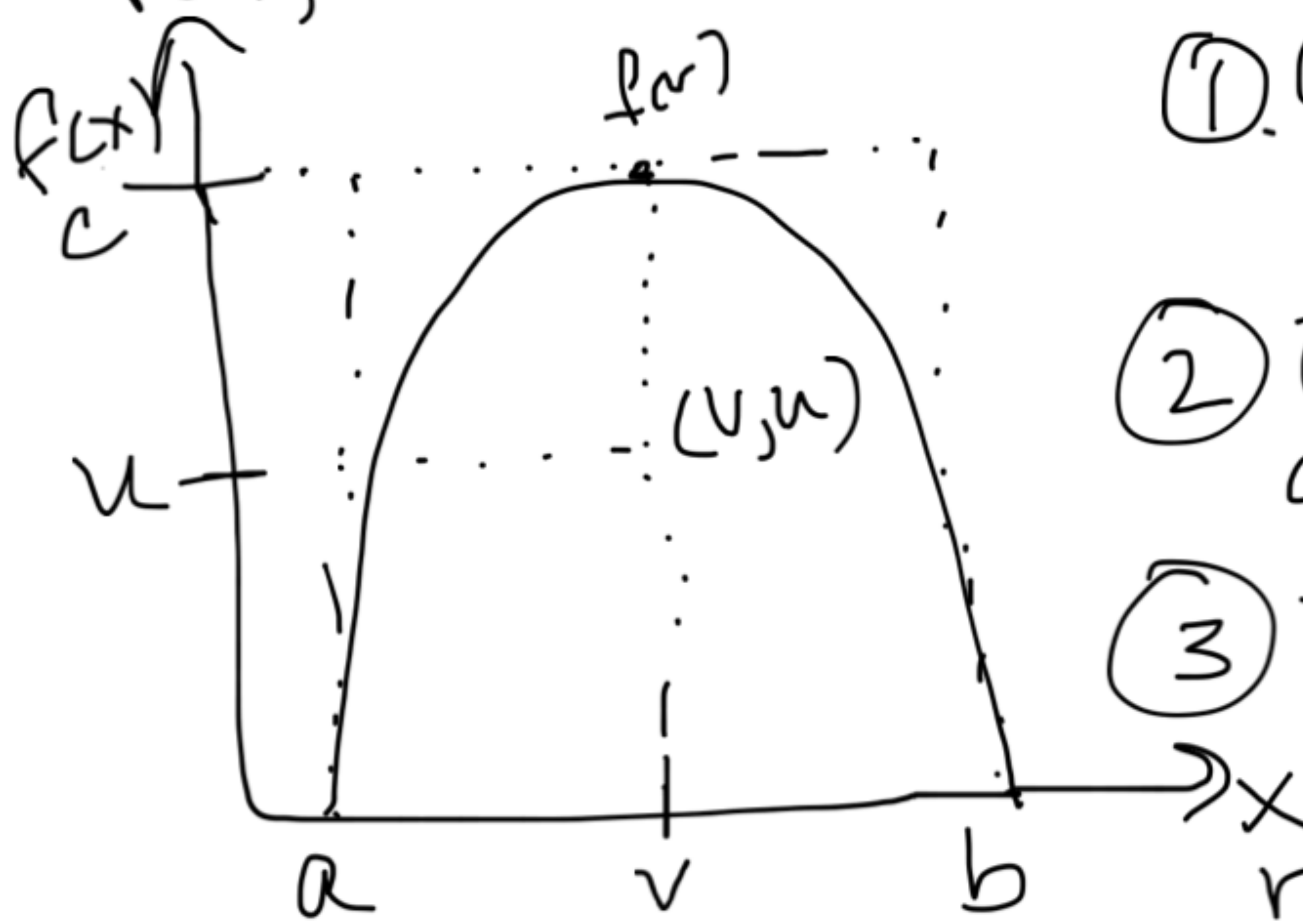
Rejection Sampling
The Basic Idea

$f(x)$ – can be calculated up to a proportionality constant.

Rejection Sampling
The Basic Idea

Suppose we want random #'s from $f(x)$, where $a < x \le b$ and $f(x) \le c$.



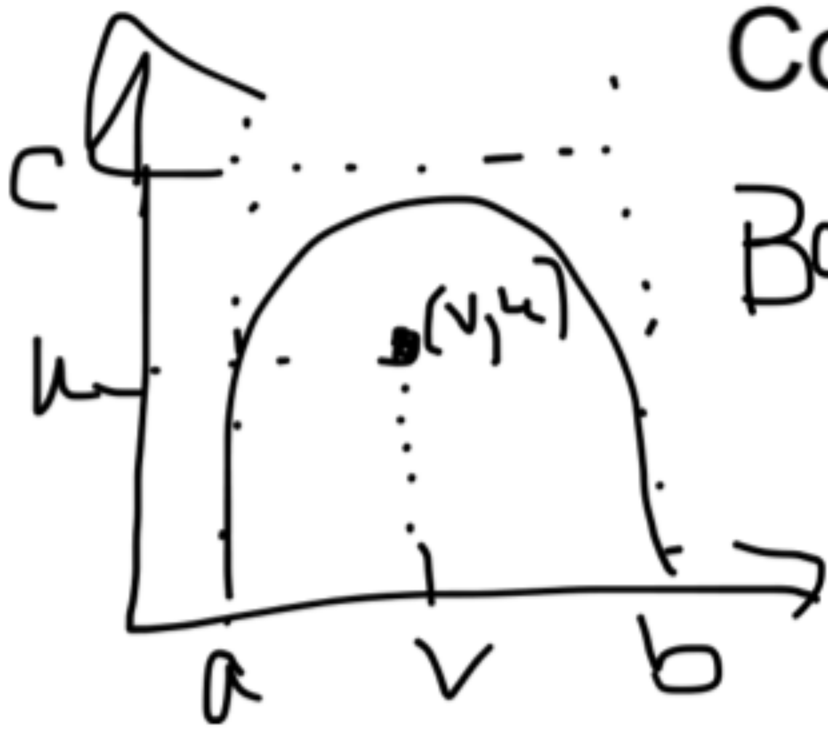
① Create a "box" around f
$[a,b] \times [0,c]$

② Generate $v \sim U[a,b]$
and $u \sim U[0,c]$
ie (v,u)

③ if $u \le f(v)$ (point under curve)
accept v as your random variable from density f

④ o.w. reject & go back to 1

Comments On Rejection Sampling



Basic idea: if $u \le f(v)$

(1) If v is accepted, (v,u) lies below the graph of $f(x)$ $\therefore$ $P(v \le d)$ is proportional to the area under the graph and left of d

So CDF of $v = F(X)$

(2) The probability of rejection

$$1 - P(\text{acceptance}) = 1 - \frac{\text{area und } f(x)}{\text{area of box}}$$

$$= 1 - \frac{1}{(b-a)\cdot(c)}$$

(3) can improve…

Rejection Sampling
A bit more formally

$$g \qquad g(x)$$

$$e(x) = g(x)/\alpha$$

Let g denote another density from which we know how to sample and for which $g(x)$ can be easily calculated.

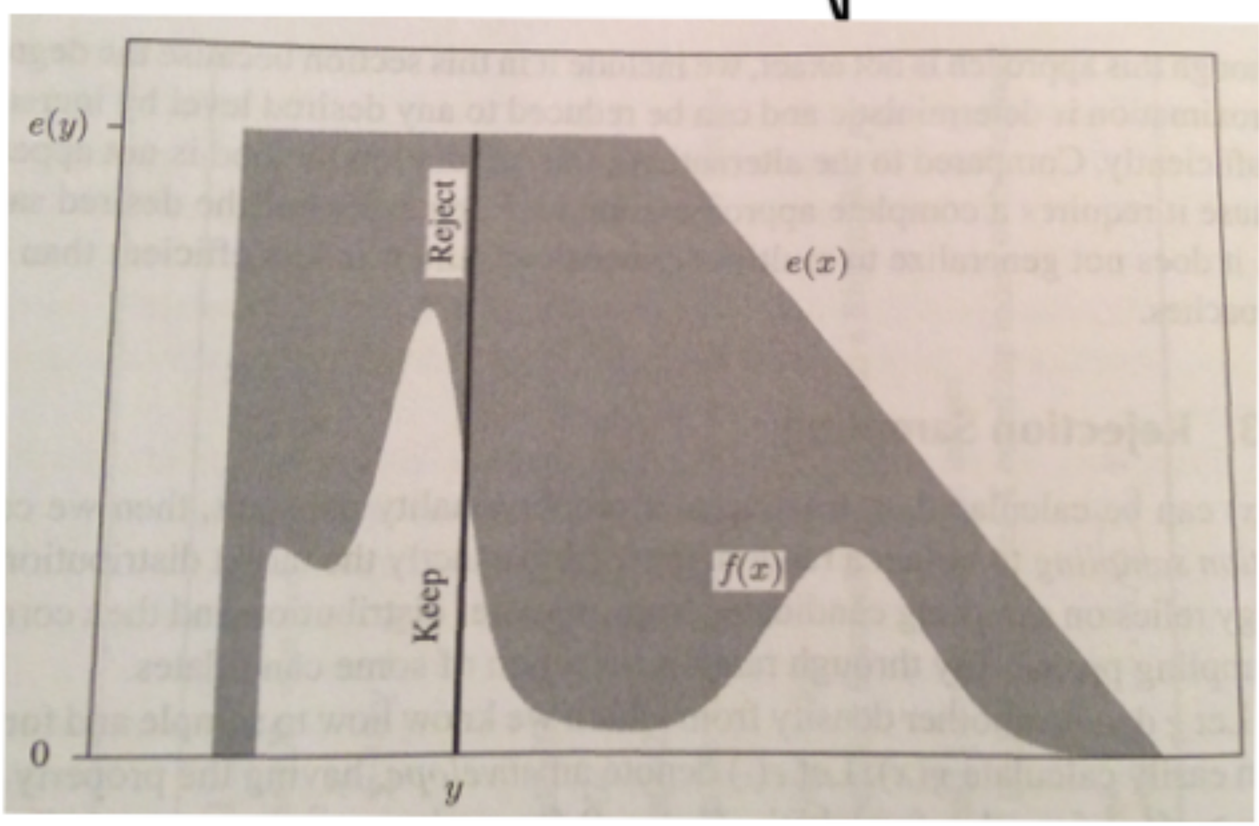
Let $e(\cdot)$ denote an envelope with the prop

$e(x) = g(x)/\alpha \geq f(x) \quad \forall x$ s.t. $f(x) > 0$; some positive $\alpha \leq 1$

Rejection Sampling Algorithm

Given: f, g, e

1. Sample $y \sim g$
2. Sample $u \sim U(0,1)$
3. If $u > f(y)/e(y) = f(y)/g(y)/\alpha$ reject
4. O.W we keep $x = y$



Proof that Rejection Sampling Works

Proposition:

The variable X in the R-S method is drawn from f.

Pf

$$P(X\le y) = P\left(Y\le y \,\middle|\, u \le \frac{f(y)}{e(y)}\right) = \frac{P\left(Y\le y \,\&\, u \le \frac{f(y)}{e(y)}\right)}{P\left(u \le \frac{f(y)}{e(y)}\right)}$$

$$= \frac{\int_{-\infty}^{y}\int_{0}^{f(z)/e(z)} du\, g(z)\, dz}{\int_{-\infty}^{\infty}\int_{0}^{f(z)/e(z)} du\, g(z)\, dz} = \frac{\int_{-\infty}^{y} \frac{f(z)}{e(z)}\, g(z)\, dz}{\int_{-\infty}^{\infty} \frac{f(z)}{e(z)}\, g(z)\, dz}$$

$$= \frac{\alpha \int_{-\infty}^{y} f(z)\, dz}{\alpha \int_{-\infty}^{\infty} f(z)\, dz} = \int_{-\infty}^{y} f(z)\, dz \quad \square$$