Johns Hopkins Engineering

625.464 Computational Statistics

Introduction to Function Estimation
Orthogonal Functions

Module 10 Lecture 10C

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

A Few Definitions

Def: If each function in a linear space H can be expressed as a linear combination of functions in a set G, then G is a **basis** (generating set, spanning set) of H.

Def: A set of functions $\{g_i(x)\}$ is said to be **orthoganal** if

$$\langle g_i, g_j \rangle = \int_D g_i(x) g_j(x) w(x)\, dx = \begin{cases} 0 & i \neq j \\ \lambda_i > 0 & i = j \end{cases}$$

Additionally, if $\langle g_i, g_i \rangle = \int_D g_i^2(x) w(x)\, dx = 1 \ \forall i$ then the functions are **orthonormal**.

A Few More Definitions

- If a set $\{g_i(x)\}$ is orthoganal with some $\lambda_i \neq 1$ then $\left\{\frac{g_i(x)}{\sqrt{\lambda_i}}\right\}$ is orthonormal.
- If $\{g_i(x)\}$ is a set of orthoganal functions, then it is a linearly independent set.

~~Ex~~ $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$

$0 \leq x \leq 2\pi$ is an orthoganal family known as the Fourier Trig Family.

Gram-Schmidt Otrhonomalization

From any linearly ind. set, $\{q_i(x)\}$, we can always construct an orthoganal set. $\{\tilde{q}_i(x)\}$

~~pf~~ Gram-Schmidt Ortho.

$$\tilde{q}_1 = q_1$$

$$\tilde{q}_2 = q_2 - \frac{\langle \tilde{q}_1, q_2 \rangle}{\langle \tilde{q}_1, \tilde{q}_1 \rangle} \tilde{q}_1$$

$$\tilde{q}_3 = q_3 - \frac{\langle \tilde{q}_1, q_3 \rangle}{\langle \tilde{q}_1, \tilde{q}_1 \rangle} \tilde{q}_1 - \frac{\langle \tilde{q}_2, q_3 \rangle}{\langle \tilde{q}_2, \tilde{q}_2 \rangle} \tilde{q}_2$$

$$\vdots$$

$$\tilde{q}_k = q_k - \sum_{i=1}^{k-1} \frac{\langle \tilde{q}_i, q_k \rangle}{\langle \tilde{q}_i, \tilde{q}_i \rangle} \tilde{q}_i$$

$$\boxed{\left\{ \frac{\tilde{q}_i}{\|\tilde{q}_i\|} \right\} \text{ orthonormal set}}$$

Why Otrhonormal Functions?

Often useful to rep. function of interest $f(x)$ as a linear combo of "simpler" functions $g_0(x), g_1(x), \ldots$

$$f(x) = \sum_{k=0}^{\infty} c_k g_k(x)$$

- good idea to use $\{g_i(x)\}$ that is an orthonormal basis.

Given $\{g_k(x)\}$ how do I find c_k?

Finding the Linear Representation

$$f(x) = \sum_{k=0}^{\infty} c_k g_k(x)$$

Q: Given $\{g_k(x)\}$ how to find c_k?

A: If f is con't and integrable over D, then

$$c_k = \langle f, g_k \rangle$$

$\{c_k\}$ are called the Fourier coeff of f w.r.t $\{g_k\}$.

Finding the Linear Representation

In practice we approx f with

$$\tilde{f} = \sum_{k=0}^{j} c_k g_k(x)$$

has error

$$f - \sum_{k=0}^{j} c_k g_k(x) = f - \tilde{f}$$

$$\text{mSE} \quad \|f-\tilde{f}\|^2 = \frac{1}{|D|} \| f - \sum_{k=0}^{j} c_k g_k \|^2$$

(1) The Fourier coef $\{c_k\}$ minimize the mSE when $\{g_k\}$ are orthonormal

(2) We will approx $\tilde{f}$ then est the coef

(3) What basis should we use?

- Fourier trig
- ortho poly
- spline