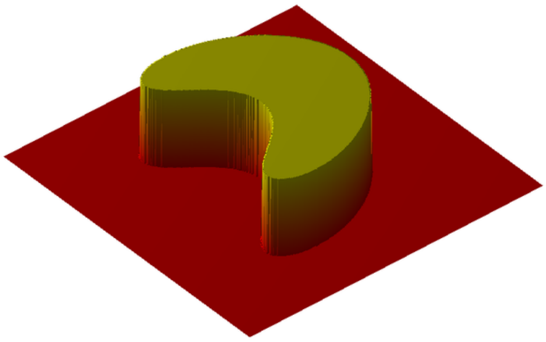
Indicator function



The graph of the indicator function of a two-dimensional subset of a square.

In mathematics, an **indicator function** or a **characteristic function** is a function defined on a set X that indicates membership of an element in a subset A of X, having the value 1 for all elements of A and the value 0 for all elements of X not in A. It is usually denoted by a bold or blackboard bold 1 symbol with a subscript describing the event of inclusion.

1 Definition

The indicator function of a subset A of a set X is a function

$$\mathbf{1}_A\colon X \to \{0,1\}$$

defined as

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if} x \in A, \\ 0 & \text{if} x \notin A. \end{cases}$$

The Iverson bracket allows the equivalent notation, $[x \in A]$, to be used instead of $\mathbf{1}_A(x)$.

The function $\mathbf{1}_A$ is sometimes denoted I_A, χ_A or even just A. (The Greek letter χ appears because it is the initial letter of the Greek word *characteristic*.)

2 Remark on notation and terminology

- The notation 1_A is also used to denote the identity function of A.
- The notation χ_A is also used to denote the characteristic function in convex analysis.

A related concept in statistics is that of a dummy variable (this must not be confused with "dummy variables" as that term is usually used in mathematics, also called a bound variable).

The term "characteristic function" has an unrelated meaning in probability theory. For this reason, probabilists use the term **indicator function** for the function defined here almost exclusively, while mathematicians in other fields are more likely to use the term *characteristic function* to describe the function which indicates membership in a set.

3 Basic properties

The *indicator* or *characteristic* function of a subset A of some set X, maps elements of X to the range $\{0,1\}$.

This mapping is surjective only when A is a non-empty proper subset of X. If $A \equiv X$, then $\mathbf{1}A = 1$. By a similar argument, if $A \equiv \emptyset$ then $\mathbf{1}A = 0$.

In the following, the dot represents multiplication, $1\cdot 1 = 1$, $1\cdot 0 = 0$ etc. "+" and "−" represent addition and subtraction. " $\cap$ " and " $\cup$ " is intersection and union, respectively.

If A and B are two subsets of X, then

$$\mathbf{1}_{A\cap B} = \min\{\mathbf{1}_A, \mathbf{1}_B\} = \mathbf{1}_A \cdot \mathbf{1}_B,$$

$$\mathbf{1}_{A\cup B} = \max\{\mathbf{1}_A, \mathbf{1}_B\} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_A \cdot \mathbf{1}_B,$$

and the indicator function of the complement of A i.e. A^C is:

$$\mathbf{1}_{A^\complement} = 1 - \mathbf{1}_A$$

More generally, suppose $A_1, \dots, A_n$ is a collection of subsets of X. For any $x \in X$:

$$\prod_{k \in I} (1 - \mathbf{1}_{A_k}(x))$$

is clearly a product of 0s and 1s. This product has the value 1 at precisely those $x \in X$ which belong to none of the sets Ak and is 0 otherwise. That is

$$\prod_{k\in I}(1-\mathbf{1}_{A_k}) = \mathbf{1}_{X-\bigcup_k A_k} = 1-\mathbf{1}_{\bigcup_k A_k}.$$

Expanding the product on the left hand side,

$$\mathbf{1}_{\bigcup_k A_k} = 1 - \sum_{F\subseteq\{1,2,\dots,n\}} (-1)^{|F|}\mathbf{1}_{\bigcap_F A_k} = \sum_{\emptyset\neq F\subseteq\{1,2,\dots,n\}} (-1)^{|F|+1}\mathbf{1}_{\bigcap_F A_k}$$

where $|F|$ is the cardinality of F. This is one form of the principle of inclusion-exclusion.

As suggested by the previous example, the indicator function is a useful notational device in combinatorics. The notation is used in other places as well, for instance in probability theory: if X is a probability space with probability measure $\mathbb{P}$ and A is a measurable set, then $\mathbf{1}_A$ becomes a random variable whose expected value is equal to the probability of A :

$$\operatorname{E}(\mathbf{1}_A) = \int_X \mathbf{1}_A(x)\,d\mathbb{P} = \int_A d\mathbb{P} = \operatorname{P}(A)$$

This identity is used in a simple proof of Markov's inequality.

In many cases, such as order theory, the inverse of the indicator function may be defined. This is commonly called the generalized Möbius function, as a generalization of the inverse of the indicator function in elementary number theory, the Möbius function. (See paragraph below about the use of the inverse in classical recursion theory.)

4 Mean, variance and covariance

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $A \in \mathcal{F}$, the indicator random variable $\mathbf{1}_A\colon \Omega \rightarrow \mathbb{R}$ is defined by $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, otherwise $\mathbf{1}_A(\omega) = 0$.

Mean $\operatorname{E}(\mathbf{1}_A(\omega)) = \operatorname{P}(A)$

Variance $\operatorname{Var}(\mathbf{1}_A(\omega)) = \operatorname{P}(A)(1-\operatorname{P}(A))$

Covariance $\operatorname{Cov}(\mathbf{1}_A(\omega), \mathbf{1}_B(\omega)) = \operatorname{P}(A\cap B) - \operatorname{P}(A)\operatorname{P}(B)$

5 Characteristic function in recursion theory, Gödel's and Kleene's *representing function*

Kurt Gödel described the *representing function* in his 1934 paper "On Undecidable Propositions of Formal Mathematical Systems". (The paper appears on pp. 41–74 in Martin Davis ed. *The Undecidable*):

> "There shall correspond to each class or relation R a representing function $\varphi(x_1, \ldots, x_n) = 0$ if $R(x_1, \ldots, x_n)$ and $\varphi(x_1, \ldots, x_n) = 1$ if $\sim R(x_1, \ldots, x_n)$." (p. 42; the "~" indicates logical inversion i.e. "NOT")

Stephen Kleene (1952) (p. 227) offers up the same definition in the context of the primitive recursive functions as a function φ of a predicate P takes on values 0 if the predicate is true and 1 if the predicate is false.

For example, because the product of characteristic functions $\varphi_1 * \varphi_2 * \ldots * \varphi_n = 0$ whenever any one of the functions equals 0, it plays the role of logical OR: IF $\varphi_1 = 0$ OR $\varphi_2 = 0$ OR . . . OR $\varphi_n = 0$ THEN their product is 0. What appears to the modern reader as the representing function's logical inversion, i.e. the representing function is 0 when the function R is "true" or satisfied", plays a useful role in Kleene's definition of the logical functions OR, AND, and IMPLY (p. 228), the bounded- (p. 228) and unbounded- (p. 279ff) mu operators (Kleene (1952)) and the CASE function (p. 229).

6 Characteristic function in fuzzy set theory

In classical mathematics, characteristic functions of sets only take values 1 (members) or 0 (non-members). In fuzzy set theory, characteristic functions are generalized to take value in the real unit interval [0, 1], or more generally, in some algebra or structure (usually required to be at least a poset or lattice). Such generalized characteristic functions are more usually called membership functions, and the corresponding "sets" are called *fuzzy* sets. Fuzzy sets model the gradual change in the membership degree seen in many real-world predicates like "tall", "warm", etc.

7 Derivatives of the indicator function

A particular indicator function, which is very well known, is the Heaviside step function. The Heaviside step function is the indicator function of the one-dimensional positive half-line, i.e. the domain [0, ∞). It is well known that the distributional derivative of the Heaviside step function, indicated by $H(x)$, is equal to the Dirac delta function, i.e.

$$\delta(x) = \tfrac{dH(x)}{dx},$$

with the following property:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)dx = f(0).$$

The derivative of the Heaviside step function can be seen as the 'inward normal derivative' at the 'boundary' of the domain given by the positive half-line. In higher dimensions, the derivative naturally generalises to the inward normal derivative, while the Heaviside step function naturally generalises to the indicator function of some domain *D*. The surface of *D* will be denoted by *S*. Proceeding, it can be derived that the inward normal derivative of the indicator gives rise to a 'surface delta function', which can be indicated by δ*S*(**x**):

$$\delta_S(\mathbf{x}) = -\mathbf{n}_x \cdot \nabla_x \mathbf{1}_{\mathbf{x}\in D}$$

where *n* is the outward normal of the surface *S*. This 'surface delta function' has the following property:[1]

$$-\int_{\mathbf{R}^n} f(\mathbf{x})\,\mathbf{n}_x \cdot \nabla_x \mathbf{1}_{\mathbf{x}\in D}\; d^n\mathbf{x} = \oint_S \, f(\beta)\; d^{n-1}\beta.$$

By setting the function *f* equal to one, it follows that the inward normal derivative of the indicator integrates to the numerical value of the surface area *S*.

8 See also

- Dirac measure
- Laplacian of the indicator
- Dirac delta
- Extension (predicate logic)
- Free variables and bound variables
- Heaviside step function
- Iverson bracket
- Kronecker delta, a function that can be viewed as an indicator for the identity relation
- Multiset
- Membership function
- Simple function
- Dummy variable (statistics)
- Statistical classification
- Zero-one loss function

9 Notes

[1] Lange, Rutger-Jan (2012), "Potential theory, path integrals and the Laplacian of the indicator", *Journal of High Energy Physics* (Springer) **2012** (11): 29–30, arXiv:1302.0864, Bibcode:2012JHEP...11..032L, doi:10.1007/JHEP11(2012)032

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