1. Give the contrapositive of the following statement. “If every bird flies, then there is a hungry cat.”

   If there is not a hungry cat, then every bird does not fly.

   Or equivalently, “if a cat is not hungry, then every bird does not fly”

2. A proposition is a statement that can be true or false but not both. Let A, B, and C be propositions. Let $\wedge$ denote logical AND, let $\vee$ denote logical OR, and let $\neg$ denote logical NOT. Argue that if $(A \vee B) \wedge (\neg B \vee C)$ is true, then $(A \vee C)$ must be true as well.

   Here is the truth table for this scenario:

A	B	C	(A OR B) AND (NOT B OR C)	A OR C
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

   We want to show that if $(A \vee B) \wedge (\neg B \vee C)$ is true, then $(A \vee C)$ is also true. From the definition of a conditional statement, this means that if $(A \vee B) \wedge (\neg B \vee C)$ is true, then $(A \vee C)$ must also be true. Otherwise, $(A \vee C)$ can be either true or false. From the truth table above, we see that this condition is satisfied. When $(A \vee B) \wedge (\neg B \vee C)$ is true, $(A \vee C)$ is also true. When $(A \vee B) \wedge (\neg B \vee C)$ is not true, then $(A \vee C)$ does not have any condition on it.

   A different way of thinking about this is to look at the statement itself. In order for $(A \vee B) \wedge (\neg B \vee C)$ to be true, $(A \vee B)$ must be true and $(\neg B \vee C)$ must also be true. Observe that if A is true and C is true, then $(A \vee B) \wedge (\neg B \vee C)$ is also true. Also observe that the only case where A is not true and $(A \vee B) \wedge (\neg B \vee C)$ is true is when B and C are both true. In all these cases, $(A \vee C)$ will also be true because at least A or C will be true in each of these cases.

3. We use the notation $A \Rightarrow B$ to indicate that A implies B. This new proposition$A \Rightarrow B$ is true except when A is true and B is false. We write $A \Leftrightarrow\ B$ when either both A and B are true or both are false. Argue that $A \Leftrightarrow B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$.

   Using the truth table below, we see that the two statements are equivalent. From the definition of a biconditional statement, this proves that $A \Leftrightarrow B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$

A	B	A implies B	B implies A	(A implies B) AND (B implies A)	$A \Leftrightarrow B$
F	F	T	T	T	T
F	T	T	F	F	F
T	F	F	T	F	F
T	T	T	T	T	T

4. We will use the notation $|\cdot|$ to indicate the number of elements in the set or its *cardinality*, e.g. $|A|$ is the number of elements in the set A. Consider four sets A,B,C,D such that the intersection of any three is empty. Use the inclusion-exclusion principle to give an expression for $|A\ \cup B \cup C \cup D|$ without using any union ($\cup$) symbols.

   From the definition, of the inclusion-exclusion principle, we know that the cardinality of a union of sets is the sum of the cardinality of each individual set minus the cardinality of where those sets intersect. Since the intersection of any three sets is empty, we have the following simplified expression:

$$|A\ \cup B \cup C \cup D| = |A| + |B| + |C| + |D| -$$
$$|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| - |A \cap B \cap C \cap D|$$

5. State the formal definition of $O(n)$, and use the definition to show that the function $f(n)\ =\ (n^4\ +\ n^2 - 9)/(n^3 + 1)$ is $O(n)$.

   Big O notation is used to describe the performance of an algorithm in the worst case scenario as the input size grows. Formally, assuming real valued functions f and g and real variable x, "if, for sufficiently large values of *x*, the values of $|f|$ are less than those of a multiple of $|g|$, then f is of order at most g, or f(x) is O(g(x))" (*Discrete Mathematics with Applications*, Epp). So in this case, as n tends to infinity, we want to show that

$$|f(n)| \leq M|n|$$

   As n goes to infinity, f(n) will be dominated by the largest terms. So

$$\lim_{n\to\infty} f(n) = \frac{n^4}{n^3} = n$$

   So, in this case, M = 1 and we have shown that f(n) is O(n).

6. For every positive integer n, let $[n]$ denote the set $\{1, \dots n\}$. Use an inductive argument to show that $\sum_{S \subseteq [n]: S \neq \emptyset} \quad \prod_{a \in S} \quad a^{-1} = n$.

An alternative way to phrase the question is as follows. Let A be a set. We use the notation $P(A)$ to indicate the power set of A, which consist of all subsets of A. For example, if $A = \{0,1\}$, then $P(A) = \{\{\}, \{0\}, \{1\}, \{0,1\}\}$. Consider $Q(n) = P(\{1, \dots n\}) - \{\{\}\}$ and use an inductive argument to show that the sum $\sum_{\{a_1, \dots a_k\} \in Q(n)} \quad \frac{1}{a_1 \dots a_k} = n$.

(For example, the expansion for $n = 3$ is $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} = 3$.)

Step 1: Proof for n=1
Q(n) = {1}
So

$$\sum_{\{a_1, \dots, a_k\} \in Q(n)} \frac{1}{a_1 \dots a_k} = \frac{1}{1} = 1$$

So we have proven the sum is correct for n=1

Step 2: Show that if the sum is true for n=k is true, then the sum is also true for n=k+1
So we assume that

$$\sum_{\{a_1, \dots, a_k\} \in Q(n)} \frac{1}{a_1 \dots a_k} = n$$

is true. Now we need to show that

$$\sum_{\{a_1, \dots, a_{k+1}\} \in Q(n+1)} \frac{1}{a_1 \dots a_{k+1}} = n + 1$$

is also true.
We have

$$\sum_{\{a_1, \dots, a_{k+1}\} \in Q(n+1)} \frac{1}{a_1 \dots a_{k+1}}$$

Writing in terms of the summation from step 1, we have

$$\left(\sum_{\{a_1, \dots, a_k\} \in Q(n)} \frac{1}{a_1 \dots a_k} \right) + \frac{1}{n+1} \left(\sum_{\{a_1, \dots, a_k\} \in Q(n)} \frac{1}{a_1 \dots a_k} \right) + \frac{1}{n+1}$$

Substituting in our assumptions, we have

$$n + \frac{n}{n+1} + \frac{1}{n+1}$$

$$= \frac{n(n+1)}{n+1} + \frac{n}{n+1} + \frac{1}{n+1}$$

$$\frac{(n+1)^2}{n+1} = n+1$$

So we have shown that

$$\sum_{\{a_1,\dots,a_{k+1}\} \in Q(n+1)} \frac{1}{a_1 \dots a_{k+1}} = n+1$$

7. Prove that the set of all languages over $\{0,1\}$that have a bounded maximum string length is countable.

   I adapted the proof from the quiz "A False Proof"

   1. Let S_i be the set of languages containing strings of length at most i where i ≤ M, the maximum string length
   2. Each S_i is finite.
   3. The set of all languages is $\bigcup_{i=0}^{M} S_i$
   4. A countable union of finite sets is countable.

The reason the proof is acceptable now is because the maximum string length is bounded, where before it was not.