Answer the questions below. For the questions involving both programming and analysis, please submit your code through Udacity and your analysis through T-square.

1. Let $G(V, E, c, s, t)$ be a network with integer capacities. The problem is to find an s-t cut in G of minimum capacity that has the smallest number of edges among all minimum capacity cuts of G. Show how to modify the capacities of G to create a new network $G' = (V, E, c', s, t)$ also with integer capacities such that any minimum capacity s-t cut in G' is a minimum capacity s-t cut in G with a minimum number of edges.

   Implement your algorithm here:

   https://www.udacity.com/course/viewer#!/c-ud557/l-1209378918/m-3410948546

2. Let $r = (r_1, \ldots, r_m)$ and $c = (c_1 \ldots c_n)$ be two vectors of positive integers such that for some K, $r_1 + \ldots + r_m = c_1 + \ldots + c_n = K$.
   a. Implement an $O(mnK)$ algorithm to find a matrix A, if it exists, with 0/1 entries, m rows and n columns such that it has exactly r_i ones in the ith row and exactly c_j ones in the jth column. Return None if no such A exists. Submit your answer here.

   https://www.udacity.com/course/viewer#!/c-ud557/l-1209378918/m-3373348543

   Hint: Construct an appropriate flow network and find a maximum flow in it.

   b. Prove the correctness of your algorithm.
   c. Prove that its running time is the $O(mnK)$. Why does it run in the claimed time bound? Hint: What will the value of the augmenting flows be? How many augmenting flows will be required?

3. A graph is said to be k-regular if the degree of each vertex is exactly k. Let $G = (L, R, E)$ be a k-regular bipartite graph.
   a. Prove that $|L| = |R|$.
   b. Prove that any such G has a perfect matching. Hint: Use Frobenius-Hall.
   c. Prove that the edges of a k-regular bipartite graph can be partitioned into k perfect matchings.

4. Let $G = (V, E)$ be an undirected graph with edge costs $w : E \to Z^+$. Let P denote the set of s – t paths for two vertices s, t in G. Let C denote the set of cuts that separate s and t (each cut is a set of edges). Prove that $\max_{p \in P} \min_{e \in p} w(e) = \min_{c \in C} \max_{e \in c} w(e)$.