Answer the questions below. Submit answers to 4 and 5 to Udacity, the others to T-square.

1a) Show that the following linear program is infeasible.

$$
\begin{array}{ll}
\text{min} & 3x_1 - 2x_2 \\
\text{s.t.} & x_1 + x_2 \le 2 \\
 & -x_1 + -2x_2 \le -6 \\
 & x_1, x_2 \ge 0.
\end{array}
$$

Solution:

Suppose that x_1, x_2 did satisfy the constraints of the program. Then adding the two inequalities together gives $-x_2 \le -4$ or $x_2 \ge 4$. This would imply, however, that $4 \le x_2 \le x_1 + x_2 \le 2$, so we have a contradiction.

1b) Show that the following linear program is unbounded.

$$
\begin{array}{ll}
\text{max} & 2x_1 + 5x_2 \\
\text{s.t.} & -2x_1 + x_2 \le -1 \\
 & -x_1 - 2x_2 \le 2 \\
 & x_1, x_2 \ge 0.
\end{array}
$$

Solution:

Consider some arbitrary value $V \ge 1$. Then setting $x_1 = V/2$ and $x_2 = 0$ gives a feasible solution that with this value.

2a) Give an example of a linear program for which the feasible region is not bounded but the optimal value is finite.

Solution:

An easy to visualize 2D example is

$$
\begin{array}{ll}
\text{min} & x_1 + x_2 \\
\text{s.t.} & 2x_1 + x_2 \ge 6 \\
 & x_1 + 2x_2 \ge 6 \\
 & x_1, x_2 \ge 0.
\end{array}
$$

More trivially,

$$
\begin{array}{ll}
\text{min} & x \\
\text{s.t.} & x \ge 0
\end{array}
$$

2b)

Solution: Construct an example of a primal problem that has no feasible solutions and whose dual problem also has no feasible solutions.

Consider the primal problem

$$
\begin{array}{ll}
\text{max} & 1 \cdot x
\end{array}
$$

s.t. $\quad 0 \cdot x \leq -1$

$x \geq 0$

whose dual is

min $\quad -1 \cdot y$

s.t. $\quad 0 \cdot y \geq 1$

$y \geq 0$.

3. Consider the following optimization problem

min $\quad |u| + |v| + |w|$

s.t. $\quad u + v \leq 1$

$2u + w = 3$.

and convert it into a linear program of the form

min $\quad c^T x$

s.t. $\quad Ax = b$

$x \geq 0$.

(Hint: you will want to introduce two new non-negative variables for each of u, v, w.)

Solution:

First, we introduce a non-negative slack variable s so that the inequality $x + y \leq 1$ becomes $x + y + s = 1$. Then we add new non-negative variables and the associated constraints $x_p - x_n = x$, $y_p - y_n = y$, and $z_p - z_n = z$. Thus, the problem becomes

min $\quad |x| + |y| + |z|$

s.t. $\quad x + y + s = 1$

$2x + z = 3$.

$x_p - x_n = x$

$y_p - y_n = y$

$z_p - z_n = z$

$x_p, x_n, y_p, y_n, z_p, z_n, s \geq 0$

Of course, $|x| + |y| + |z| = |x_p - x_n| + |y_p - y_n| + |z_p - z_n| \leq x_p + x_n + y_p + y_n + z_p + z_n$, This inequality is an equality, however, when $x_p = \max\{0, x\}$ and $x_n = \max\{0, -x\}$, etc ,values that are always feasible as long as the choice of x, y, z are feasible for the original problem. Hence, we may redefine the objective function as $x_p + x_n + y_p + y_n + z_p + z_n.$ Eliminating the other occurrences of x, y, z by substitution yields the problem

min $\quad x_p + x_n + y_p + y_n + z_p + z_n.$

s.t. $\quad x_p - x_n + y_p - y_n + s = 1$

$2x_p - 2x_n + z_p - z_n = 3$.

$x_p, x_n, y_p, y_n, z_p, z_n, s \geq 0$

which is of the desired form.

4. Express the following problem as a linear program. Given an $m \times n$ matrix A and a vector b of length m, find a vector x such that $||Ax - b||_1$ is minimized. In other words, find $x_1 \ldots x_n$ such that $\sum_{i=1}^{m} |b_i - \sum_{j=1}^{n} a_{ij} x_j|$ is minimized. Implement your solution here.

https://www.udacity.com/course/viewer#!/c-ud557/l-1209378918/m-2871868559

Solution:

```
def labsdev(A,b):
    """Input: A numpy array A with shape = (m,n) and a numpy array b.
      Output: A numpy array x that minimizes |b - Ax|_1.
    """
    #Not given
    (m,n) = A.shape

    xid = range(n)
    yid = range(m)
    zid = range(m)

    #problem definition
    beginModel('basic')
    verbose(False)

    #create variables
    x = var(xid, 'X') #The original X
    y = var(yid, 'Y') #The surplus variables
    z = var(zid, 'Y') #The slack variables

    #set objective
    minimize(sum(y[i] for i in yid) + sum(z[i] for i in zid))

    #set constraints
    ry = st(sum(x[j]*float(A[i][j]) for j in xid) - y[i] <= float(b[i]) for i in yid)
    rz = st(sum(x[j]*float(A[i][j]) for j in xid) + z[i] >= float(b[i]) for i in zid)
    None <= x

    solve() #solve and report

    endModel() #Good habit: do away with the problem

    return np.array([x[i].primal for i in xid])
```

5. Let $A=(a_{ij})$ for $1 \le i \le m$ and $1 \le j \le n$ be a matrix with m rows and n columns. Such a matrix defines a two-person game as follows. Two players, Row and Column play a game where Row selects a row i and Column selects a column j. If $a_{ij} > 0$ Row receives a payoff amount of a_{ij} . If $a_{ij} < 0$, Row pays an amount of $-a_{ij}$ to Column. The payoff matrix A is known to both players.

   Suppose Row picks the i-th row with probability p_i and announces this vector p. Knowing this vector, Column will choose column j that minimizes Row's expected payout. Thus, the expected payout is $z = \min_j \sum_{i=1}^{m} p_i a_{ij}$. Naturally, Row then will want to choose the vector $(p_1, \ldots, p_m)$ so as to maximize this quantity. Express Row's problem as a linear program. (Of course $p_1, \ldots p_n \ge 0$ and $\sum_{i=1}^{n} p_i = 1$.)

   Implement your procedure here

   https://www.udacity.com/course/viewer#!/c-ud557/l-1209378918/m-3379798710

   Solution:

```
def rowStrategy(A):
   (m,n) = A.shape

   xid = range(m)
   zid = [0]

   #problem definition
   pymprog.beginModel('basic')
   pymprog.verbose(False)

   #create variables
   x = pymprog.var(xid, 'X')
   z = pymprog.var(zid, 'Z')

   #set objective
   pymprog.maximize(z[0])

   #set constraints
   pymprog.st(sum(x[i] for i in xid) == 1.0)
   for j in range(n):
      pymprog.st(sum(x[i]*float(A[i][j]) for i in xid) >= z[0])
   None <= z[0]
```

```
pymprog.solve() #solve and report

pymprog.endModel() #Good habit: do away with the problem

return np.array([x[i].primal for i in xid])
```

6. Prove that for any $m \times n$ matrix A and vector b of length m, exactly one of the following holds.
    a. There is a vector $x \geq 0$ such that $Ax = b$.
    b. There is a vector y such that $y^T A \geq 0$ and $y^T b < 0$.

    Hint: Use substitution to show both statements cannot be true for the same matrix A. To show that at least one must be true, consider the following linear program.

    $$\begin{array}{ll} \text{min} & b^T y \\ \text{s.t.} & A^T y \geq 0 \end{array}$$

    and find its dual. Use the Duality Theorem to complete the result.

Solution:

Suppose that there exist x and y satisfying $x \geq 0$, $Ax = b$, $y^T b < 0$, and $y^T A \geq 0$. A dot product of two nonnegative vectors must be non-negative so, $0 \leq (y^T A)x = y^T(Ax) = y^T b < 0$, which is a contradiction. Thus, both statements cannot be true.

On the other hand, suppose that statement b) is false. Then for every vector y such that $y^T A \geq 0$, we have that $y^T b \geq 0$. Under these assumptions, the program

$$\begin{array}{ll} \text{min} & b^T y \\ \text{s.t.} & A^T y \geq 0 \end{array}$$

has a finite optimum solution at $y = 0$. By the Duality Theorem this means that the dual program must also have a finite optimum. In this case, the dual program is

$$\begin{array}{ll} \text{max} & 0^T x \\ \text{s.t.} & A x = b \\ & x \geq 0. \end{array}$$

The finite optimum is also a feasible solution, so statement a) must be true.

Alternative Solution

Both cannot be true as before.

To show that both cannot be false, suppose that a is false. Thus, the point b is not in the set $\{Ax \mid x \geq 0\}$. This set is convex, so there must be a separating hyperplane which can represented by the vector y. That is to say, $y^T b < y^T Ax$ for all $x \geq 0$. In particular, this applies to the vector $x = 0$ and we conclude that $y^T b < 0$.

To show that $y^T A \geq 0$, suppose not. That is, $y^T A e_j < 0$ for some unit vector e_j. Then for some positive α, $y^T A(\alpha e_j) < y^T b$. But $\alpha e_j \geq 0$, so this violates properties of y.