Rules

You may not use any materials other than what is on this exam.
You may not discuss this exam in any manner with anyone other than the instructors.

Computability

Note: Throughout we use the angle brackets to mean "encoding of." Thus, $< M, w >$ should be read as "the encoding of a (machine,string) pair" in problem 2a, and $< M_1, M_2 >$ should be read as "the encoding of a (machine, machine) pair" in problem 4.

1. [10 points] Give the transition function (either as a diagram or a table) for a Turing machine that decides the language $L = \{w \mid \text{the last (i.e. rightmost) symbol of } w \text{ is a } 1\}$. Assume $\Sigma = \{0, 1\}$.

2. Indicate (no proof is necessary) whether the following languages are recognizable and whether their complement is recognizable.
   a. [4 points] $L_a = \{< M, w > \mid M \text{ loops on input } w\}$.
   b. [4 points] $L_b = \{< M > \mid M \text{ doesn't moves past the 10th tape square on input } 100101\}$.
   c. [4 points] A useless state in a Turing machine is one that is not reached on any input string. Consider $L_c = \{< M > \mid M \text{ has no useless states}\}$.

3. Consider the language $I_{TM} = \{< M_1, M_2 > \mid L(M_1) \cap L(M_2) \neq \emptyset\}$, i.e. the set of descriptions of pairs of Turing machines whose languages overlap.
   a. [10 points] Is I_{TM} recognizable? If it is, then explain how to construct a recognizer. If it is not, then prove that it is unrecognizable.
   b. [10 points] Is $\overline{I_{TM}}$ recognizable? If it is, then explain how to construct a recognizer. If it is not, then prove that it is unrecognizable.

   Note: Note that you cannot apply Rice's theorem directly, as I_{TM} encodes a *pair* of machines.

4. [10 points] Strings are defined to be finite. Suppose, however, that we allowed them to have infinite length. Would the set of infinite-length strings over {0,1} still be countable? Argue why or why not.

Complexity

1. [8 points] The *traveling salesman problem* (TSP) can be defined as follows.
   Input: A complete graph $G = (V, E)$, a cost function $C : E \to Z^+$ (from edges to positive integers), and an integer k.
   Decision: Is there a Hamiltonian cycle (one that visits each vertex exactly once before returning to its start) such that the sum of the cost of the edges is at most k?

   Show that the traveling salesman problem is *in NP*. (Don't misread. That says "in NP," not "is NP-complete.")

2. [8 points] Recall that the *3CNF satisfiability problem* can be defined as follows.
   Input: A boolean formula f consisting of a conjunction (logical AND) of clauses, each having *exactly* 3 *distinct* literals, e.g. $(a \vee \overline{b} \vee c) \wedge (b \vee c \vee \overline{d}) \wedge \ldots$
   Decision: Is there a satisfying assignment for the formula f?

   Let 4CNF satisfiability be defined in the analogous way where each clause must have *exactly* 4 *distinct* literals.

   Give a polynomial reduction from 3CNF to 4CNF and prove its correctness.. Do not merely cite results from the lessons.

3. Consider the traveling salesman problem as defined in problem 1. Let the $V = \{1, \ldots n\}$ and let the cost function c be represented by a $n \times n$ matrix. Suppose that you had access to an algorithm $A(c, k)$ that could solve the decision version of the problem in polynomial time.
   a. [8 points] Argue that the minimum cost of a Hamiltonian cycle could also be found in polynomial time.
   b. [12 points] Argue that a minimum cost Hamiltonian cycle could be found in polynomial time.

   Note: An algorithm for part a) might say "the minimum cost is 5403," while an algorithm for part b) might say "the minimum cost cycle is represented by the sequence of vertices a,c,e,f,b,d,a."

4. [12 points] The *subset sum problem* can be defined as follows.
   Input: A multiset of positive integers $\{a_1, \ldots, a_m\} \subset Z^+$ and an integer k.

   Decision: Is there a subset $\{a_1, \ldots, a_m\} \subseteq A$ such that $\sum_{a \in S} a = k$?

The *partition problem* can be defined like so.

Input: A multiset of positive integers $\{a_1, \ldots, a_m\} \subset Z^+$.

Decision: Let $T = \sum_{i=1}^{m} a_i$. Is there a subset $S \subseteq \{a_1, \ldots, a_m\}$ such that $\sum_{a \in S} a = T/2$?

Give a polynomial reduction from subset sum to partition. Hint: Try including two new giant numbers.