For this problem set, you may find it useful to consult Ken Rosen's textbook *Discrete Math and Its Applications*.

1. Give the contrapositive of the following statement. "If every bird flies, then there is a hungry cat."

Answer:

"If there is not a hungry cat, then not every bird flies."

or

"If every cat is not hungry, then some bird does not fly."

or some other equivalent. Common mistakes include

"If there is a not hungry cat, then …"
"If …., then every bird does not fly."

2. A proposition is a statement that can be true or false but not both. Let A, B, and C be propositions. Let $\wedge$ denote logical AND, let $\vee$ denote logical OR, and let $\neg$ denote logical NOT. Argue that if $(A \vee B) \wedge (\neg B \vee C)$ is true, then $(A \vee C)$ must be true as well.

Answer

B is either true or false. If B is true then C must be true. If B is false then A must be true. So (A OR C) must be true.

A truth table with explanation is acceptable as well. Note that in the truth table

A	B	C	$(A \vee B) \wedge (\neg B \vee C)$	$(A \vee C)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

shown, we have that for every assignment where $(A \vee B) \wedge (\neg B \vee C)$ is true, $(A \vee C)$ is true as well.

3. We use the notation $A \Rightarrow B$ to indicate that A implies B. This new proposition $A \Rightarrow B$ is true except when A is true and B is false. We write $A \Leftrightarrow B$ when either both A and B are true or both are false. Argue that $A \Leftrightarrow B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$.

Answer

There is no escaping trying all cases here. For convenience we use the fact that $A \Rightarrow B$ is equivalent to $\neg A \vee B$ and $B \Rightarrow A$ is equivalent to $\neg B \vee A$ to write “if $A \Rightarrow B$ and $B \Rightarrow A$” as $(\neg A \vee B) \wedge (\neg B \vee A)$.

A	B	$(\neg A \vee B) \wedge (\neg B \vee A)$	$A \Leftrightarrow B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

The last two columns are identical, meaning that the two logical expressions are equivalent.

4. We will use the notation $|\cdot|$ to indicate the number of elements in the set or its *cardinality*, e.g. $|A|$ is the number of elements in the set A. Consider four sets A,B,C,D such that the intersection of any three is empty. Use the inclusion-exclusion to give an expression for $|A \cup B \cup C \cup D|$ without using any union $(\cup)$ symbols.

Answer

$|A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$

5. State the formal definition of $O(n)$, and show that the function $f(n) = (n^4 + n^2 - 9)/(n^3 + 1)$ is $O(n)$.

Answer

A function $f(n)$ is $O(n)$ if there are constants c and k such that for all $n > k$, it holds that $f(n) \leq cn$.

For example,

$$f(n) < \frac{(n^4+n^2)}{n^3} = \frac{n+1}{n} < 2n$$

for $n > 1$.

Another acceptable definition is that $\limsup_{n\to\infty}\frac{f(n)}{n}$ is a constant. In this case, the constant is 1.

6. Let A be a set. We use the notation $P(A)$ to indicate the power set of A, which consist of all subsets of A. For example, if $A = \{0,1\}$, then $P(A) = \{\{\},\{0\},\{1\},\{0,1\}\}$. Consider $Q(n) = P(\{1,\ldots n\}) - \{\{\}\}$ and use an inductive argument to show that the sum

$\sum_{\{a_1,\ldots a_k\} \in Q(n)} \frac{1}{a_1 \ldots a_k} = n$.

(For example, the expansion for $n = 3$ is $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3}+\frac{1}{1\cdot 2\cdot 3} = 3.$)

Answer

We first consider the base case where $n = 1$. Here $Q(1) = \{\{1\}\}$ and the sum is just 1. For the inductive step, we note that

$$Q(n+1) = Q(n) \cup \{\{n+1\}\} \cup \{\{n+1\} \cup S | S \in Q(n)\}.$$

Therefore, we can break the sum up into three terms.

$$\sum_{\{a_1,\ldots a_k\} \in Q(n+1)} \frac{1}{a_1 \ldots a_k} = \left(\sum_{\{a_1,\ldots a_k\} \in Q(n)} \frac{1}{a_1 \ldots a_k}\right) + \frac{1}{n+1} + \left(\sum_{\{a_1,\ldots a_k\} \in Q(n)} \frac{1}{a_1 \ldots a_k}\frac{1}{n+1}\right).$$

Applying the inductive hypothesis gives

$$\sum_{\{a_1,\ldots a_k\} \in Q(n+1)} \frac{1}{a_1 \ldots a_k} = n + \frac{1}{n+1} + \frac{n}{n+1} = n+1,$$

which is the desired result.

7 . Prove that the set of all languages over $\{0,1\}$ that have a bounded maximum string length is countable.

Answer

Following the argument given in the Udacity quiz gives are argument like this one. Consider some arbitrary positive integer k. Let S_k be the set of languages where the maximum string length is k. Because there are only a finite number of strings of length at most k (there are $2^{k+1} - 1$ to be exact) , there can only be a finite number of subsets of these strings (there are $2^{2^{k+1}-1}$ to be exact), and therefore only a finite number of languages with maximum string length k. Therefore, S_k is a finite set of languages. The set of languages that have bounded maximum string length is the union of all the S_k, i.e. $\cup_{k\in Z^+} S_k$. Since a countable union of finite sets is countable, we have that the set of languages with bounded maximum string length is countable.

Alternative Answers (not essential to understand).

Another class of arguments begin with the recognition that the set of languages with bounded maximum string length is just the set of finite languages (i.e. languages with a finite number of strings in them). This follows from the observation that any finite language must have a longest string, and for any given maximum length there are only a finite number that are of the same length or less.

From here, there are again multiple possible proofs.

Alternative 1

The most straightforward is that any finite language can be uniquely represented as a string using standard set notion. Therefore, we have an injective map from finite languages to strings. Since the set of all strings is countable, the set of finite languages must be as well.

Alternative 2

Alternatively, one can let C_k be the set of languages containing exactly k strings.

For every k, C_k is countable. Proving this takes some work. The same argument of counting diagonal-by-diagonal will show that the cartesian product of any two countable sets is countable. It follows that the set of sequences of k strings (i.e. $(\{0,1\}^*)^k$) is countable. The straightforward mapping from sequences to sets is surjective, showing that is C_k is countable.

The set of all finite languages is $\cup_{k \in Z^+} C_k$. Since this is a countable union of countable sets, it is countable.