Nonlinear Regression – Part I

Johns Hopkins Engineering

625.461 Statistical Models and Regression

Module 13 – Lecture 13B

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Nonlinear Regression Model

$$\mathbf{y} = f(\mathbf{x}, \boldsymbol{\beta}) + \varepsilon$$

where f is a <u>nonlinear function in $\boldsymbol{\beta}$</u> .

Ex. $y = \theta_1 e^{\theta_2 x} + \varepsilon$

Assume: $\mathrm{E}(\varepsilon) = 0$, $\mathrm{Var}(\varepsilon) = \sigma^2$

Nonlinear Least Squares Method

$$y_i, x_{i1}, x_{i2}, \ldots, x_{ik}, \text{ for } i = 1, \ldots, n$$

To minimize with respect to $\boldsymbol{\theta}$ the following function:

$$S(\boldsymbol{\theta}) = \sum_{i=1}^{n} [y_i - f(\mathbf{x}_i, \boldsymbol{\theta})]^2$$

Nonlinear Least Squares Method

To find the least-squares estimators, we must differentiate $S(\boldsymbol{\theta})$ with respect to each element of $\boldsymbol{\theta}$. Let $p = k + 1$. The p normal equations are:

$$\sum_{i=1}^{n} [y_i - f(\mathbf{x}_i, \boldsymbol{\theta})] \left[\frac{\partial f(\mathbf{x}_i, \boldsymbol{\theta})}{\partial \theta_j} \right]_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}} = 0 \qquad j = 1, \ldots, p$$

An Example

Ex. $y = \theta_1 e^{\theta_2 x} + \varepsilon$

The LS normal equations are:

$$\sum_{i=1}^{n} \left[y_i - \hat{\theta}_1 e^{\hat{\theta}_2 x_i} \right] e^{\hat{\theta}_2 x_i} = 0$$

$$\sum_{i=1}^{n} \left[y_i - \hat{\theta}_1 e^{\hat{\theta}_2 x_i} \right] \hat{\theta}_1 x_i e^{\hat{\theta}_2 x_i} = 0$$

No simple closed-form solution exists. Iterative methods must be used to find the values of the LS estimators.

Geometric Representation for Nonlinear Least Squares

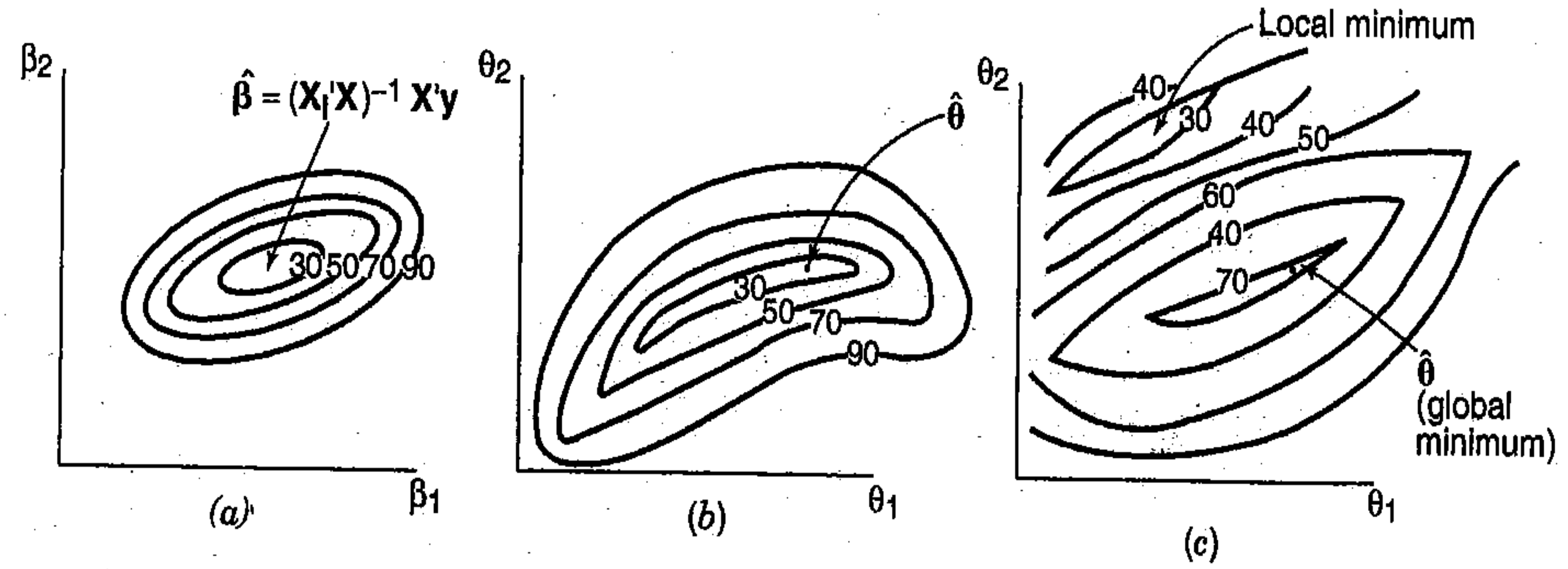

Figure 12.1 Contours of the residual-sum-of-squares function: (*a*) linear model; (*b*) nonlinear model; (*c*) nonlinear model with local and global minima.

Maximum Likelihood Estimation

If the error terms in the model are normally and independently distributed with constant variance. We can also use MLE by maximizing the likelihood function:

$$L(\boldsymbol{\theta},\sigma^2)=\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}[y_i-f(\mathbf{x}_i,\boldsymbol{\theta})]^2\right]$$

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