# Nonlinear Regression – Part III

Johns Hopkins Engineering

## 625.461 Statistical Models and Regression

Module 13 – Lecture 13D

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

# Parameter Estimation

Linearization followed by the Gauss-Newton iteration method of parameter estimation

Linearization is accompanied by a Taylor series expansion of $f(\mathbf{x}, \boldsymbol{\theta})$ about the initial point $\boldsymbol{\theta}_0 = [\theta_{10}, \theta_{20}, \ldots, \theta_{p0}]'$ with only the linear terms retained.

# Parameter Estimation and Iterative Process

This yields

$$f(\mathbf{x}_i, \boldsymbol{\theta}) = f(\mathbf{x}_i, \boldsymbol{\theta}_0) + \sum_{j=1}^{p} \left[ \frac{\partial f(\mathbf{x}_i, \boldsymbol{\theta})}{\partial \theta_j} \right]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} (\theta_j - \theta_{j0})$$

Set

$$f_i^0 = f(\mathbf{x}_i, \boldsymbol{\theta}_0)$$

$$\beta_j^0 = \theta_j - \theta_{j0}$$

$$Z_{ij}^0 = \left[ \frac{\partial f(\mathbf{x}_i, \boldsymbol{\theta}_0)}{\partial \theta_j} \right]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$$

# Parameter Estimation and Iterative Process

We then have

$$y_i - f_i^0 = \sum_{j=1}^{p} \beta_j^0 Z_{ij}^0 + \varepsilon_i, \quad i = 1, 2, \ldots, n$$

$$\mathbf{y}_0 = \mathbf{Z}_0 \boldsymbol{\beta}_0 + \boldsymbol{\varepsilon}$$

Then the LS estimator of $\boldsymbol{\beta}_0$ :

$$\hat{\boldsymbol{\beta}}_0 = (\mathbf{Z}_0'\mathbf{Z}_0)^{-1} \mathbf{Z}_0'\mathbf{y}_0 = (\mathbf{Z}_0'\mathbf{Z}_0)^{-1} \mathbf{Z}_0'(\mathbf{y} - \mathbf{f}_0)$$

# Parameter Estimation and Iterative Process

Since $\beta_0 = \theta - \theta_0$ , the revised estimates of $\theta$ is

$$\hat{\theta}_1 = \hat{\beta}_0 + \theta_0$$

We may now place $\hat{\theta}_1$ as the initial estimates $\theta_0$ to the above equations on slides 4-5 and then produce another set of revised estimates, say $\hat{\theta}_2$ , and so forth.

# Parameter Estimation and Iterative Process

In general, we have at the $k$th iteration

$$\hat{\boldsymbol{\theta}}_{k+1} = \hat{\boldsymbol{\theta}}_k + \hat{\boldsymbol{\beta}}_k = \hat{\boldsymbol{\theta}}_k + (\mathbf{Z}_k'\mathbf{Z}_k)^{-1}\mathbf{Z}_k'(\mathbf{y} - \mathbf{f}_k)$$

where

$$\mathbf{Z}_k = [Z_{ij}^k]$$

$$\mathbf{f}_k = [f_1^k, f_2^k, \ldots, f_n^k]'$$

$$\hat{\boldsymbol{\theta}}_k = [\theta_{1k}, \theta_{2k}, \ldots, \theta_{pk}]'$$

# Parameter Estimation and Iterative Process

This iterative process continues until converges, that is, until

$$\left[\left(\hat{\theta}_{j,k+1}-\hat{\theta}_{jk}\right)/\hat{\theta}_{jk}\right]<\delta, \quad j=1,2,\ldots,p$$

where $\delta$ is a small number, say $10^{-6}$.

# Parameter Estimation Iterative Process

At each iteration, the residual sum of squares $S(\hat{\theta}_k)$ should be evaluated to ensure that a reduction in its value has been obtained.

JOHNS HOPKINS
WHITING SCHOOL
*of* ENGINEERING