Matrix Algebra, Random Vectors

Vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad \mathbf{x}' = [x_1, \ldots, x_n]$$

$$c\mathbf{x} = \begin{bmatrix} cx_1 \\ \cdot \\ \cdot \\ \cdot \\ cx_n \end{bmatrix}$$

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n + y_n \end{bmatrix}$$

A set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are *linearly dependent* if there exists constants $c_1, \ldots, c_k$, not all zero, such that

$$\sum_{i=1}^{k} c_i \mathbf{x}_i = \mathbf{0}$$

That is, at least one vector in the set can be written as a linear combination of the other vectors.

Matrices

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 2 & 4 \end{bmatrix} \qquad 2\mathbf{A} = \begin{bmatrix} 6 & -2 & 4 \\ 2 & 10 & 8 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & 2 \\ -3 & 5 \\ 1 & 6 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3 \cdot 4 + (-1)(-3) + 2 \cdot 1 & 13 \\ -7 & 51 \end{bmatrix}$$

Symmetric ($\mathbf{S} = \mathbf{S}'$)

$$\mathbf{S} = \begin{bmatrix} 34 & -1.5 \\ -1.5 & 0.5 \end{bmatrix}$$

Identity matrix: all diagonal elements are one, all off-diagonal elements are zero

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of matrix $\mathbf{A}$

$\mathbf{BA} = \mathbf{AB} = \mathbf{I}$ Label $\mathbf{B} = \mathbf{A}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} -.2 & .4 \\ .8 & -.6 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/5 \end{bmatrix}$$

Orthogonal matrix $\mathbf{Q}$

$$\mathbf{Q}\mathbf{Q}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I} \qquad \text{or} \qquad \mathbf{Q}' = \mathbf{Q}^{-1}$$

$$\mathbf{Q} = \begin{bmatrix} .6 & .8 \\ -.8 & .6 \end{bmatrix} \qquad \mathbf{Q}^{-1} = \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalue λ, Eigenvector $\mathbf{x} \neq \mathbf{0}$ of $\mathbf{A}$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

If $\mathbf{x}'\mathbf{x} = 1$ (i.e., x has length unity) then we often denote it by $\mathbf{e}$

A square symmetric matrix $\mathbf{A}$ of dimension k×k has k pairs of eigenvalues and eigenvectors:

$$(\lambda_1, \mathbf{e}_1) , \ldots., (\lambda_k, \mathbf{e}_k); \quad \lambda_1 \geq \cdots \geq \lambda_k$$

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$$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$$

$$\lambda_1 = 6 \qquad \mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\lambda_2 = -4 \qquad \mathbf{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Spectral decomposition of a k×k square symmetric matrix $\mathbf{A}$

$$\mathbf{A} = \lambda_1 \mathbf{e}_1 \mathbf{e}_1' + \cdots + \lambda_k \mathbf{e}_k \mathbf{e}_k'$$

Nonnegative definite matrix $\mathbf{A}$

quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$

Positive definite matrix $\mathbf{A}$

quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$

Derive eigenvalues of a k×k symmetric matrix A :

Solve for λ the characteristic equation

$$| \mathbf{A} - \lambda \mathbf{I}_k | = 0$$

$$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$$

$$| \mathbf{A} - \lambda \mathbf{I}_2 | = \begin{vmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 25 = 0$$

$\lambda_1 = 6 \qquad \lambda_2 = -4$

Derive eigenvectors of a k×k symmetric matrix A :

Solve $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$

$$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \qquad \lambda_1 = 6$$

$$\mathbf{Ae}_1 = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = \begin{bmatrix} e_{11} - 5e_{12} \\ -5e_{11} + e_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 6e_{11} \\ 6e_{12} \end{bmatrix} \qquad \Rightarrow e_{11} = -e_{12} = 1/\sqrt{2}$$

Symmetric matrix **A** is positive definite iff every eigenvalue of **A** is positive

Symmetric matrix **A** is nonnegative definite iff all of its eigenvalues are greater than or equal to zero

$$\mathbf{A} = \sum_{i=1}^{k} \lambda_i \mathbf{e}_i \mathbf{e}_i'$$

$$\mathbf{P} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k]$$

$$\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}' \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}$$

$$\mathbf{A}^{-1} = \sum_{i=1}^{k} \frac{1}{\lambda_i} \mathbf{e}_i \mathbf{e}_i'$$

Square-root matrix of p.d. $\mathbf{A}$

$$\mathbf{A}^{1/2} = \sum_{i=1}^{k} \sqrt{\lambda_i}\, \mathbf{e}_i \mathbf{e}_i'$$

$$(\mathbf{A}^{1/2})' = \mathbf{A}^{1/2}$$

$$\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$$

Random vectors and matrices **X**

The elements of **X** are random variables

E(**X**): expectation of **X** (i.e., applying expectation to each element of **X**)

Ex 2.12

$$E(\mathbf{X}) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(.3) + (0)(.3) + (1)(.4) \\ (0)(.8) + (1)(.2) \end{bmatrix} = \begin{bmatrix} .1 \\ .2 \end{bmatrix}$$

$$\mathbf{X} = [X_1, \ldots, X_p]'$$

Mean vector

$$\mathrm{E}(\mathbf{X}) = [\mu_1, \ldots, \mu_p]' \equiv \boldsymbol{\mu}' \qquad \mu_i = \mathrm{E}(X_i)$$

Covariance matrix $\mathrm{Cov}(\mathbf{X}) \equiv \Sigma$

$$\boldsymbol{\Sigma} = [\sigma_{ij}]_{p \times p} = E(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'$$

$$\sigma_{ij} = \mathrm{E}(X_i - \mu_i)(X_j - \mu_j)$$

Correlation matrix Corr($\mathbf{X}$) $\equiv$ ρ

$$\boldsymbol{\rho} = [\rho_{ij}]_{p \times p}$$

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$$

$$\mathbf{V} = diag(\sigma_{11}, \ldots, \sigma_{pp})$$

$$\boldsymbol{\rho} = (\mathbf{V}^{1/2})^{-1} \boldsymbol{\Sigma} (\mathbf{V}^{1/2})^{-1}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix} \qquad (\mathbf{V}^{1/2})^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$\boldsymbol{\rho} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/6 & 1/5 \\ 1/6 & 1 & -1/5 \\ 1/5 & -1/5 & 1 \end{bmatrix}$$

Partitioning covariance matrix

$$\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_q \\ \cdots\cdots \\ X_{q+1} \\ \vdots \\ X_p \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{bmatrix} \qquad E(\mathbf{X}) = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \cdots\cdots \\ \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix}$$

$$Cov(\mathbf{X}) = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}$$

$$\mathbf{\Sigma}_{ij} = E(\mathbf{X}^{(i)} - \boldsymbol{\mu}^{(i)})(\mathbf{X}^{(j)} - \boldsymbol{\mu}^{(j)})'$$

$$\mathbf{\Sigma}_{ij} = \mathbf{\Sigma}_{ji}'$$

$$Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) = \mathbf{\Sigma}_{12}$$

Mean and covariance for linear combination of random variables

$$\mathbf{X} = [X_1, \ldots, X_p]' \qquad \mathbf{C} = [c_1, \ldots, c_p]'$$

$$\boldsymbol{\mu}_{\mathbf{X}} = E(\mathbf{X}) \qquad \boldsymbol{\Sigma}_{\mathbf{X}} = Cov(\mathbf{X})$$

$$E(\mathbf{C}'\mathbf{X}) = \mathbf{C}'\boldsymbol{\mu}_{\mathbf{X}}$$

$$Cov(\mathbf{C}'\mathbf{X}) = \mathbf{C}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{C}$$

Partitioning sample mean vector and sample covariance matrix in the same way as partitioning population mean vector and covariance matrix

Row (column) rank of a matrix is the maximum number of linearly independent rows (columns).

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

Row rank = 2
Column rank = 2

$$-2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The row rank and the column rank of a matrix are equal.
Rank of a matrix is either row rank or column rank.

A square matrix **A** is nonsingular if

$\mathbf{Ax} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.

$\mathbf{A}=[\mathbf{a}_1, \ldots, \mathbf{a}_k] \quad \mathbf{x} = [x_1, \ldots, x_k]'$

$\mathbf{Ax} = x_1\mathbf{a}_1 + \cdots + x_k\mathbf{a}_k$

Nonsingularity $\Leftrightarrow$ Columns of A are linearly independent