Jared Yu
Module 11 Discussion

1. In a logistic regression model with only one regressor x and its corresponding regression coefficient β. What does β mean? Suppose that we obtain the maximum likelihood estimator of β. Discuss how to construct a confidence interval of the odds ratio.

<u>Ans:</u>

The question seems slightly unclear, so for the sake of the discussion I will be assuming that it is referring to

$$\hat{y}_i = \frac{1}{1 + \exp\left[-\left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right)\right]}.$$

To understand the parameter β (which in this case is β_1), we can look at the input into the sigmoid function,

$$\hat{\eta}(x_i) = \hat{\beta}_0 + \hat{\beta}_1 x_i,$$

which is the fitted value of the linear predictor at a particular value of x (i.e., x_i). We can look also at the fitted value for $x_i + 1$, which is as follows,

$$\hat{\eta}(x_i + 1) = \hat{\beta}_0 + \hat{\beta}_1 (x_i + 1).$$

Then taking the difference between these two we get,

$$\hat{\eta}(x_i + 1) - \hat{\eta}(x_i) = \hat{\beta}_1.$$

The $\hat{\eta}(\cdot)$ functions are also known as the *log-odds*, so we can rewrite the difference as follows,

$$\hat{\eta}(x_i + 1) - \hat{\eta}(x_i) = \ln \text{odds}_{x_i+1} - \ln \text{odds}_{x_i} = \ln\left(\frac{\text{odds}_{x_i+1}}{\text{odds}_{x_i}}\right) = \hat{\beta}_1.$$

Taking the antilog of the above difference yields the *odds ratio*,

$$\hat{O}_R = \frac{\text{odds}_{x_i+1}}{\text{odds}_{x_i}} = e^{\hat{\beta}_1}.$$

In the textbook, it is said to be, "the estimated increase in the probability of success associated with a one-unit change in the value of the predictor variable." This can be generalized to a d-unit change as a change of $e^{d\hat{\beta}_1}$. So, for example, if for example β_1 is associated with height in centimeters, then a 1-centimeter increase in height is associated with a $e^{\hat{\beta}_1}$ change in probability of success for the response y.

To construct a confidence interval for the odds ratio, we first need to look at the test for the individual model coefficient. The hypothesis test is as follows,

$$H_0: \beta_j = 0, H_1: \beta_j \neq 0.$$

The test statistic is as follows,

$$Z_0 = \frac{\hat{\beta}_j}{se\left(\hat{\beta}_j\right)},$$

which follows a standard normal distribution. The standard error in the denominator can be found by finding the covariance matrix of $\hat{\boldsymbol{\beta}}$,

$$Var\left(\hat{\boldsymbol{\beta}}\right) = -\mathbf{G}\left(\hat{\boldsymbol{\beta}}\right)^{-1} = (\mathbf{X}'\mathbf{V}\mathbf{X})^{-1},$$

where $\mathbf{G}$ is the Hessian matrix found from the log-likelihood function. From here, it can be seen that the $(1-\alpha)\%$ confidence interval for β_j would be as follows,

$$\hat{\beta}_j - Z_{\frac{\alpha}{2}}\, se\left(\hat{\beta}_j\right) \leq \beta_j \leq \hat{\beta}_j + Z_{\frac{\alpha}{2}}\, se\left(\hat{\beta}_j\right).$$

In this confidence interval, we have that $Z_{\frac{\alpha}{2}}$ is the critical value at the $\frac{\alpha}{2}$ confidence level for the standard normal distribution. We have from before that $\hat{O}_R = e^{\hat{\beta}_j}$, so the above confidence interval can be changed to the following,

$$\boxed{\exp\left[\hat{\beta}_j - Z_{\frac{\alpha}{2}}\, se(\hat{\beta}_j)\right] \leq O_R \leq \exp\left[\hat{\beta}_j + Z_{\frac{\alpha}{2}}\, se(\hat{\beta}_j)\right]}.$$

The above gives a confidence interval of the odds ratio.

2. Consider a logistic regression model with a linear predictor that includes an interaction term, say $\mathbf{x}'\boldsymbol{\beta} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$. Does the odds ratio associated with the regressor x_2 have the same interpretation as in the case where the linear predictor does not have the interaction term? Provide mathematical development to support your answer.

Ans:

We have now that the (fitted) logistic regression model is as follows,

$$\hat{y} = \frac{1}{1 + \exp\left[-\left(\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_{12} x_1 x_2\right)\right]}.$$

Using a similar log-odds function as before from problem 1, we have the following for the odds ratio of x_2,

$$\hat{\eta}(x_2 + 1) - \hat{\eta}(x_2) = \ln \text{odds}_{x_2+1} - \ln \text{odds}_{x_2}$$
$$= \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 (x_2 + 1) + \hat{\beta}_{12} x_1 (x_2 + 1) - \left(\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_{12} x_1 x_2\right)$$
$$= \hat{\beta}_2 + \hat{\beta}_{12} x_1$$

Then again taking the antilog, we obtain

$$\hat{O}_R = \frac{\ln \text{odds}_{x_2+1}}{\ln \text{odds}_{x_2}} = e^{\hat{\beta}_2 + \hat{\beta}_{12} x_1}.$$

It can be seen quite evidently that when excluding the interaction term, we'd only be left with $e^{\hat{\beta}_2}$, which is the same as before with only β_1 in the model. Therefore, the interpretation is different, and we'd have to consider the value of $e^{\hat{\beta}_{12} x_1}$.