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Module 3 Discussion

A set of n subjects give data on a response variable y and two regressors, x_1 and x_2. The data are fitted to two different models

Model 1: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$
Model 2: $y = \alpha_0 + \alpha_1(x_1 - \bar{x}_1) + \alpha_2(x_2 - \bar{x}_2) + \eta$

where $\bar{x}_k$ is the simple average of x_k values from this set of data, $k = 1,2$; ε and η are random errors with mean zero.

Discuss with mathematical arguments whether the ordinary least-squares estimator of β_i under Model 1 is equal to the ordinary least-squares estimator of α_i under Model 2, for $i = 0,1,2$. State the assumptions in your discussion. Discuss whether or not the regressors are random or non-random change your discussion.

Ans:

Let **X** be the design matrix for $\boldsymbol{X}$, the data matrix with dimensions $(n \times k)$. It has dimensions $n \times p$, where $p = k + 1$, since it includes the column of 1's in the first position. Let **y** be the $n \times 1$ vector of the observations. An assumption for OLS is that $n < k$, $E(\varepsilon) = 0$, $Var(\varepsilon) = \sigma^2$, and that the errors are uncorrelated. The same applies to η in place of ε.

Starting with Model 1, the sample regression model is as follows,

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i \tag{1}$$

$$= \beta_0 + \sum_{j=1}^{k} \beta_j x_{ij} + \varepsilon_i, \qquad i = 1,2,\cdots,n. \tag{2}$$

The least-squares function to minimize then is,

$$S(\beta_0, \beta_1, \beta_2) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{k} \beta_j x_{ij} \right)^2. \tag{3}$$

For β_0, the following is the case:

$$\left.\frac{\partial S}{\partial \beta_0}\right|_{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2} = -2 \sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \sum_{j=1}^{k} \hat{\beta}_j x_{ij} \right) = 0 \tag{4}$$

$$n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_{i1} + \hat{\beta}_2 \sum_{i=1}^{n} x_{i2} = \sum_{i=1}^{n} y_i \tag{5}$$

It also follows then that,

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}_1 - \hat{\beta}_2 \bar{x}_2, \tag{6}$$

by dividing both sides of Equation (5) by n. Using the least-squares normal equations, the OLS estimates of β_i for $i = 0,1,2$ in Model 1, are as follows,

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}. \tag{7}$$

An assumption for this is that $(\mathbf{X}'\mathbf{X})^{-1}$ exists, which is possible if the regressors (i.e., x_1 and x_2) are linearly independent. The next step is to find $(\mathbf{X}'\mathbf{X})^{-1}$. Let

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} \end{bmatrix},$$

then

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n & \sum_{i=1}^{n} x_{i1} & \sum_{i=1}^{n} x_{i2} \\ \sum_{i=1}^{n} x_{i1} & \sum_{i=1}^{n} x_{i1}^2 & \sum_{i=1}^{n} x_{i1}x_{i2} \\ \sum_{i=1}^{n} x_{i2} & \sum_{i=1}^{n} x_{i2}x_{i1} & \sum_{i=1}^{n} x_{i2}^2 \end{bmatrix}.$$

Note: To simplify notation, allow $\sum_{i=1}^{n}(\cdot)$ to be shortened to $\sum(\cdot)$.

Furthermore, let $SSX_1 = \sum x_{i1}^2 - \frac{(\sum x_{i1})^2}{n}, SSX_2 = \sum x_{i2}^2 - \frac{(\sum x_{i2})^2}{n}, SSX_{12} = \sum x_{i1}x_{i2} - \frac{\sum x_{i1} \sum x_{i2}}{n}$.

The following steps will attempt to find the inverse of $\mathbf{X}'\mathbf{X}$. The row operations will be abbreviated with R1, R2, and R3.

$$\left\langle \begin{array}{ccc|ccc} n & \sum x_{i1} & \sum x_{i2} & 1 & 0 & 0 \\ \sum x_{i1} & \sum x_{i1}^2 & \sum x_{i1}x_{i2} & 0 & 1 & 0 \\ \sum x_{i2} & \sum x_{i2}x_{i1} & \sum x_{i2}^2 & 0 & 0 & 1 \end{array} \right\rangle$$

Divide R1 by n:

$$\left\langle \begin{array}{ccc|ccc} 1 & \frac{\sum x_{i1}}{n} & \frac{\sum x_{i2}}{n} & \frac{1}{n} & 0 & 0 \\ \sum x_{i1} & \sum x_{i1}^2 & \sum x_{i1}x_{i2} & 0 & 1 & 0 \\ \sum x_{i2} & \sum x_{i2}x_{i1} & \sum x_{i2}^2 & 0 & 0 & 1 \end{array} \right\rangle$$

$$\left\langle \begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ \sum x_{i1} & \sum x_{i1}^2 & \sum x_{i1}x_{i2} & 0 & 1 & 0 \\ \sum x_{i2} & \sum x_{i2}x_{i1} & \sum x_{i2}^2 & 0 & 0 & 1 \end{array} \right\rangle$$

R2 – R1*$\sum x_{i1}$:

$$\left\langle \begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & \sum x_{i1}^2 - \frac{(\sum x_{i1})^2}{n} & \sum x_{i1}x_{i2} - \frac{\sum x_{i1} \sum x_{i2}}{n} & -\bar{x}_1 & 1 & 0 \\ \sum x_{i2} & \sum x_{i2}x_{i1} & \sum x_{i2}^2 & 0 & 0 & 1 \end{array} \right\rangle$$

$$\left\langle \begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & SSX_1 & SSX_{12} & -\bar{x}_1 & 1 & 0 \\ \sum x_{i2} & \sum x_{i2}x_{i1} & \sum x_{i2}^2 & 0 & 0 & 1 \end{array} \right\rangle$$

R3 – R1*$\sum x_{i2}$:

$$\left\langle\begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & SSX_1 & SSX_{12} & -\bar{x}_1 & 1 & 0 \\ 0 & \sum x_{i2}x_{i1} - \frac{\sum x_{i1}\sum x_{I2}}{n} & \sum x_{i2}^2 - \frac{(\sum x_{i2})^2}{n} & -\frac{\sum x_{i2}}{n} & 0 & 1 \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & SSX_1 & SSX_{12} & -\bar{x}_1 & 1 & 0 \\ 0 & SSX_{12} & SSX_2 & -\bar{x}_2 & 0 & 1 \end{array}\right\rangle$$

Divide R2 by SSX_1:

$$\left\langle\begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & 1 & \frac{SSX_{12}}{SSX_1} & \frac{-\bar{x}_1}{SSX_1} & \frac{1}{SSX_1} & 0 \\ 0 & SSX_{12} & SSX_2 & -\bar{x}_2 & 0 & 1 \end{array}\right\rangle$$

R3 – R2 * SSX_{12}:

$$\left\langle\begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & 1 & \frac{SSX_{12}}{SSX_1} & \frac{-\bar{x}_1}{SSX_1} & \frac{1}{SSX_1} & 0 \\ 0 & 0 & SSX_2 - \frac{SSX_{12}^2}{SSX_1} & -\bar{x}_2 + \frac{\bar{x}_1 SSX_{12}}{SSX_1} & -\frac{SSX_{12}}{SSX_1} & 1 \end{array}\right\rangle$$

Divide R3 by $SSX_2 - \frac{SSX_{12}^2}{SSX_1}$:

$$\left\langle\begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & 1 & \frac{SSX_{12}}{SSX_1} & \frac{-\bar{x}_1}{SSX_1} & \frac{1}{SSX_1} & 0 \\ 0 & 0 & 1 & c & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{array}\right\rangle$$

$$c = \frac{-\bar{x}_2 + \frac{\bar{x}_1 SSX_{12}}{SSX_1}}{SSX_2 - \frac{SSX_{12}^2}{SSX_1}} = \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2}$$

$$\left\langle\begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & 1 & \frac{SSX_{12}}{SSX_1} & \frac{-\bar{x}_1}{SSX_1} & \frac{1}{SSX_1} & 0 \\ 0 & 0 & 1 & \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{array}\right\rangle$$

R2 – R3 * $\frac{SSX_{12}}{SSX_1}$:

$$\left\langle\begin{array}{ccc|ccc} 1 & \bar{x}_1 & \bar{x}_2 & \frac{1}{n} & 0 & 0 \\ 0 & 1 & 0 & e & f & -\frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ 0 & 0 & 1 & \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{array}\right\rangle$$

$$e = \frac{-\bar{x}_1}{SSX_1} - \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \times \frac{SSX_{12}}{SSX_1} = \frac{-1}{SSX_1}\left[\bar{x}_1 + \frac{-\bar{x}_2 SSX_1 SSX_{12} + \bar{x}_1 SSX_{12}^2}{SSX_1 \times SSX_2 - SSX_{12}^2}\right]$$

$$= \frac{-1}{SSX_1}\left[\frac{\bar{x}_1 SSX_1 SSX_2 - \bar{x}_2 SSX_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2}\right] = \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}$$

$$f = \frac{1}{SSX_1} - \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \times \frac{SSX_{12}}{SSX_1} = \frac{1}{SSX_1}\left[1 + \frac{SSX_{12}^2}{SSX_1 \times SSX_2 - SSX_{12}^2}\right]$$

$$= \frac{1}{SSX_1}\left[\frac{SSX_1 \times SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}\right] = \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}$$

$$\left\langle \begin{matrix} 1 & \bar{x}_1 & \bar{x}_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} \frac{1}{n} & 0 & 0 \\ \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & -\frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{matrix} \right. \right\rangle$$

R1 – R3*$\bar{x}_2$:

$$\left\langle \begin{matrix} 1 & \bar{x}_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} \frac{1}{n} - \frac{-\bar{x}_2^2 SSX_1 + \bar{x}_1 \bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{\bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & -\frac{\bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & -\frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{matrix} \right. \right\rangle$$

R1 – R2*$\bar{x}_1$:

$$\left\langle \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} g & h & i \\ \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & -\frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{matrix} \right. \right\rangle$$

$$g = \frac{1}{n} - \frac{-\bar{x}_2^2 SSX_1 + \bar{x}_1 \bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} - \frac{\bar{x}_1 \bar{x}_2 SSX_{12} - \bar{x}_1^2 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}$$

$$= \frac{1}{n} - \frac{-\bar{x}_2^2 SSX_1 - \bar{x}_1^2 SSX_2 + 2\bar{x}_1 \bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2}$$

$$h = \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}$$

$$i = \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2}$$

$$\left\langle \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} \frac{1}{n} - \frac{-\bar{x}_2^2 SSX_1 - \bar{x}_1^2 SSX_2 + 2\bar{x}_1 \bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & -\frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{matrix} \right. \right\rangle$$

The following has been shown:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} \frac{1}{n} - \frac{-\bar{x}_2^2 SSX_1 - \bar{x}_1^2 SSX_2 + 2\bar{x}_1\bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} & -\frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \\ \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{-SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} & \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \end{bmatrix}$$

Where

$$SSX_1 = \sum x_{i1}^2 - \frac{(\sum x_{i1})^2}{n}, SSX_2 = \sum x_{i2}^2 - \frac{(\sum x_{i2})^2}{n}, SSX_{12} = \sum x_{i1}x_{i2} - \frac{\sum x_{i1} \sum x_{i2}}{n}.$$

Then, from equation (7):

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & \cdots & 1 \\ x_{11} & \cdots & x_{n1} \\ x_{12} & \cdots & x_{n2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_{i1} y_i \\ \sum x_{i2} y_i \end{bmatrix}$$

From this result it follows that for $\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$**:**

$$\hat{\beta}_0 = \frac{\sum y_i}{n} - \frac{-\bar{x}_2^2 SSX_1 - \bar{x}_1^2 SSX_2 + 2\bar{x}_1\bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum y_i + \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i + \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i$$

$$\hat{\beta}_1 = \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum y_i + \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i$$

$$\hat{\beta}_2 = \frac{-\bar{x}_2 SSX_1 + \bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i + \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i$$

To double-check the results, it can be shown that $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}_1 - \hat{\beta}_2 \bar{x}_2$, as stated in equation (6):

$$\hat{\beta}_0 = \bar{y} - \frac{(\bar{x}_1\bar{x}_2 SSX_{12} - \bar{x}_1^2 SSX_2) + (\bar{x}_1\bar{x}_2 SSX_{12} - \bar{x}_2^2 SSX_1)}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum y_i + \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i + \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i$$

$$= \bar{y} - \frac{\bar{x}_1(\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2) + \bar{x}_2(\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1)}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum y_i + \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i + \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i$$

$$
\begin{aligned}
&= \bar{y} - \frac{\bar{x}_1(\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2) + \bar{x}_2(\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1)}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum y_i + \frac{\bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i1} y_i \\
&\quad + \frac{-\bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i1} y_i + \frac{\bar{x}_1 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i2} y_i \\
&\quad + \frac{-\bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i2} y_i \\
&= \bar{y} - \bar{x}_1 \left\{ \frac{\bar{x}_2 SSX_{12} - \bar{x}_1 SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum y_i + \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i1} y_i \right. \\
&\qquad \left. - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i2} y_i \right\} \\
&\qquad - \bar{x}_2 \left\{ \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i1} y_i \right. \\
&\qquad \left. + \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2}\sum x_{i2} y_i \right\} \\
&= \bar{y} - \bar{x}_1 \hat{\beta}_1 - \bar{x}_2 \hat{\beta}_2
\end{aligned}
$$

Now look towards Model 2:

$$y = \alpha_0 + \alpha_1(x_1 - \bar{x}_1) + \alpha_2(x_2 - \bar{x}_2) + \eta$$

Let $x_1^c = x_1 - \bar{x}_1$, $x_2^c = x_2 - \bar{x}_2$ denote the centered terms,

$$y = \alpha_0 + \alpha_1 x_1^c + \alpha_2 x_2^c + \eta.$$

Then the following also holds:

$$
\begin{aligned}
\hat{\alpha}_1 &= \frac{\bar{x}_2^c SSX_{12}^c - \bar{x}_1^c SSX_2^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}}\sum y_i + \frac{SSX_2^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}}\sum x_{i1}^c y_i \\
&\quad - \frac{SSX_{12}^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}}\sum x_{i2}^c y_i \\
\hat{\alpha}_2 &= \frac{-\bar{x}_2^c SSX_1^c + \bar{x}_1^c SSX_{12}^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}}\sum y_i - \frac{SSX_{12}^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}}\sum x_{i1}^c y_i \\
&\quad + \frac{SSX_1^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}}\sum x_{i2}^c y_i
\end{aligned}
$$

The following can also be shown:

$$
\begin{aligned}
\bar{x}_1^c &= \frac{\sum x_{i1}^c}{n} = \frac{\sum(x_{i1} - \bar{x}_1)}{n} = \frac{\sum x_{i1} - n\,\bar{x}_1}{n} = 0, \\
\bar{x}_2^c &= \frac{\sum x_{i2}^c}{n} = \frac{\sum(x_{i2} - \bar{x}_2)}{n} = \frac{\sum x_{i2} - n\,\bar{x}_2}{n} = 0,
\end{aligned}
$$

which is showing that $\sum x_{i1}^c = \sum x_{i2}^c = 0$. The following can also be shown:

$$
\begin{aligned}
SSX_1^c &= \sum x_{i1}^{c\,2} - \frac{(\sum x_{i1}^c)^2}{n} = \sum(x_{i1} - \bar{x}_1)^2 - 0 = \sum(x_{i1} - \bar{x}_1)^2 = \sum x_{i1}^2 - \frac{(\sum x_{i1})^2}{n} = SSX_1 \\
SSX_2^c &= \sum x_{i2}^{c\,2} - \frac{(\sum x_{i2}^c)^2}{n} = \sum(x_{i2} - \bar{x}_2)^2 - 0 = \sum(x_{i2} - \bar{x}_2)^2 = \sum x_{i2}^2 - \frac{(\sum x_{i2})^2}{n} = SSX_2 \\
SSX_{12}^c &= \sum x_{i1}^c x_{i2}^c - \frac{\sum x_{i1}^c \sum x_{i2}^c}{n} = \sum x_{i1}^c x_{i2}^c = \sum(x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2) = \sum x_{i1} x_{i2} - \frac{\sum x_{i1} \sum x_{i2}}{n} \\
&= SSX_{12}
\end{aligned}
$$

So, it follows that:

$$\hat{\alpha}_1 = \frac{\bar{x}_2^c SSX_{12}^c - \bar{x}_1^c SSX_2^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}} \sum y_i + \frac{SSX_2^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}} \sum x_{i1}^c y_i - \frac{SSX_{12}^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}} \sum x_{i2}^c y_i$$

$$= \frac{0 \times SSX_{12} - 0 \times SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum y_i + \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1}^c y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2}^c y_i$$

$$= \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1}^c y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2}^c y_i$$

$$= \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum (x_{i1} - \bar{x}_1) y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum (x_{i2} - \bar{x}_2) y_i$$

$$= \left\{ -\frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \bar{x}_1 \sum y_i + \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \bar{x}_2 \sum y_i \right\} + \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i$$

$$= \left\{ \frac{-\bar{x}_1 SSX_2 + \bar{x}_2 SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \right\} \sum y_i + \frac{SSX_2}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i = \hat{\beta}_1$$

$$\hat{\alpha}_2 = \frac{-\bar{x}_2^c SSX_1^c + \bar{x}_1^c SSX_{12}^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}} \sum y_i - \frac{SSX_{12}^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}} \sum x_{i1}^c y_i + \frac{SSX_1^c}{SSX_1^c \times SSX_2^c - SSX_{12}^{c\,2}} \sum x_{i2}^c y_i$$

$$= \frac{-0 \times SSX_1 + 0 \times SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum (x_{i1} - \bar{x}_1) y_i + \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum (x_{i2} - \bar{x}_2) y_i$$

$$= -\frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum (x_{i1} - \bar{x}_1) y_i + \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum (x_{i2} - \bar{x}_2) y_i$$

$$= \left\{ \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \bar{x}_1 \sum y_i - \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \bar{x}_2 \sum y_i \right\} - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i + \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i$$

$$= \left\{ \frac{\bar{x}_1 SSX_{12} - \bar{x}_2 SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \right\} \sum y_i - \frac{SSX_{12}}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i1} y_i + \frac{SSX_1}{SSX_1 \times SSX_2 - SSX_{12}^2} \sum x_{i2} y_i = \hat{\beta}_2$$

It has been shown that $\hat{\alpha}_1 = \hat{\beta}_1$ and $\hat{\alpha}_2 = \hat{\beta}_2$. However, the following can be shown:

$$\hat{\alpha}_0 = \bar{y} - \bar{x}_1^c \hat{\alpha}_1 - \bar{x}_2^c \hat{\alpha}_2 = \bar{y} - 0 \times \hat{\alpha}_1 - 0 \times \hat{\alpha}_2 = \bar{y} \neq \hat{\beta}_0$$

In conclusion it can be said that in general,

$$\boxed{\hat{\alpha}_0 \neq \hat{\beta}_0, \hat{\alpha}_1 = \hat{\beta}_1, \hat{\alpha}_2 = \hat{\beta}_2.}$$

In the process of finding the inverse of $\mathbf{X}'\mathbf{X}$, an assumption is that is required is that it is doable. That is, there is no perfect collinearity issue. In other words, $\mathbf{X}'\mathbf{X}$ exists. So,

$$SSX_1 \times SSX_2 - SSX_{12}^2 \neq 0$$
$$SSX_1 \times SSX_2 \neq SSX_{12}^2$$
$$1 \neq \frac{SSX_{12}^2}{SSX_1 \times SSX_2} = r^2$$

That means we need $r^2 \neq 1$, or $r \neq \pm 1$. If the regressors are random, then r, the sample correlation, is also a random variable. It cannot be guaranteed $r \neq \pm 1$ since it is random. This must also apply to the centered versions of the data, $x_1^c = x_1 - \bar{x}_1$, $x_2^c = x_2 - \bar{x}_2$. So, for certain sets of sample data, if the regressors are random, it is possible that the above statements do not hold. Another unlikely situation is if $\bar{x}_1 = \bar{x}_2 = 0$. In such a situation, it would hold that $\hat{\alpha}_0 = \hat{\beta}_0$, which would make $\hat{\boldsymbol{\alpha}} = \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \hat{\boldsymbol{\beta}}$.