Multiple Linear Regression Model – Part I

Johns Hopkins Engineering

625.461 Statistical Models and Regression

Module 3 – Lecture 3B

Johns Hopkins Whiting School of Engineering

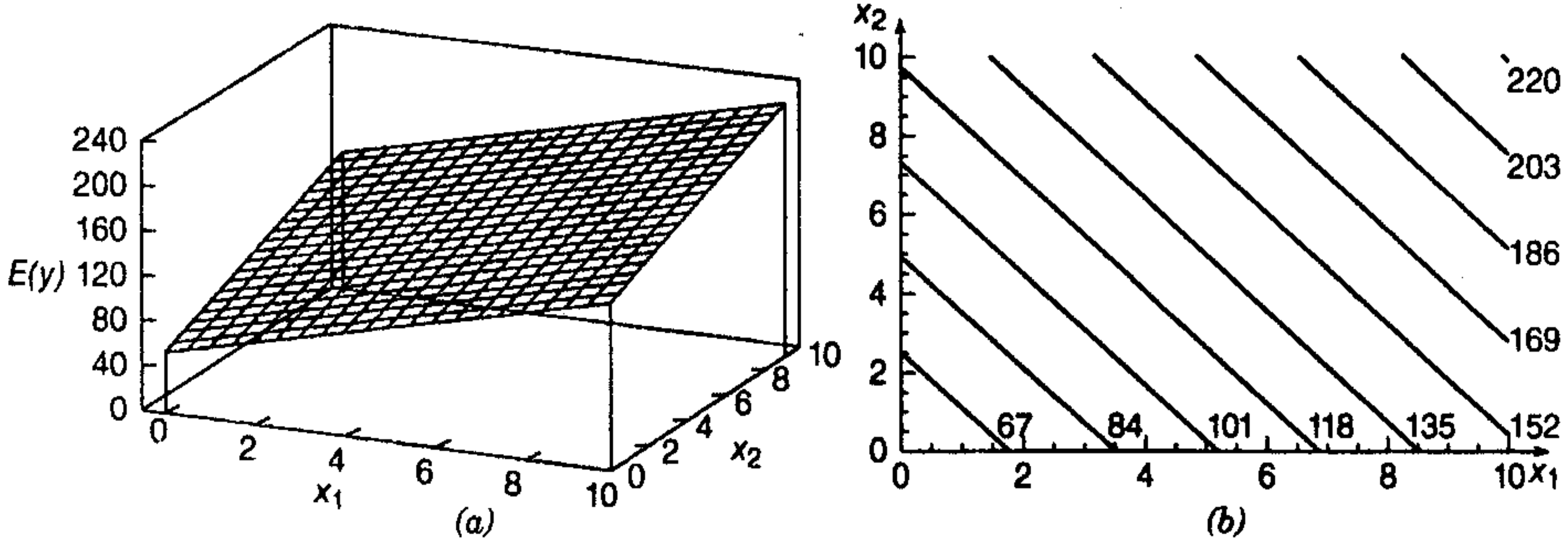
Multiple Linear Regression Model - Basics

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

Random error ε has mean zero

$$\Rightarrow E(y \mid x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

Model: $E(y \mid x_1, x_2) = 50 + 10x_1 + 7x_2$



Meaning of Regression Coefficient

k regressors

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \varepsilon$$

The parameter β_j represents the expected change of y per unit change of x_j <u>when all the remaining variables x_i ($i \neq j$) are held constant</u>.

Multiple linear regression models are empirical models or approximating functions

More General Multiple Linear Regression Models

Models that are more complex in structure may still be analyzed by multiple linear regression techniques, e.g., polynomial regression model:

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \varepsilon$$

simply by letting

$$x_1 = x,\ x_2 = x^2, \text{ and } x_3 = x^3$$

More General Multiple Linear Regression Models

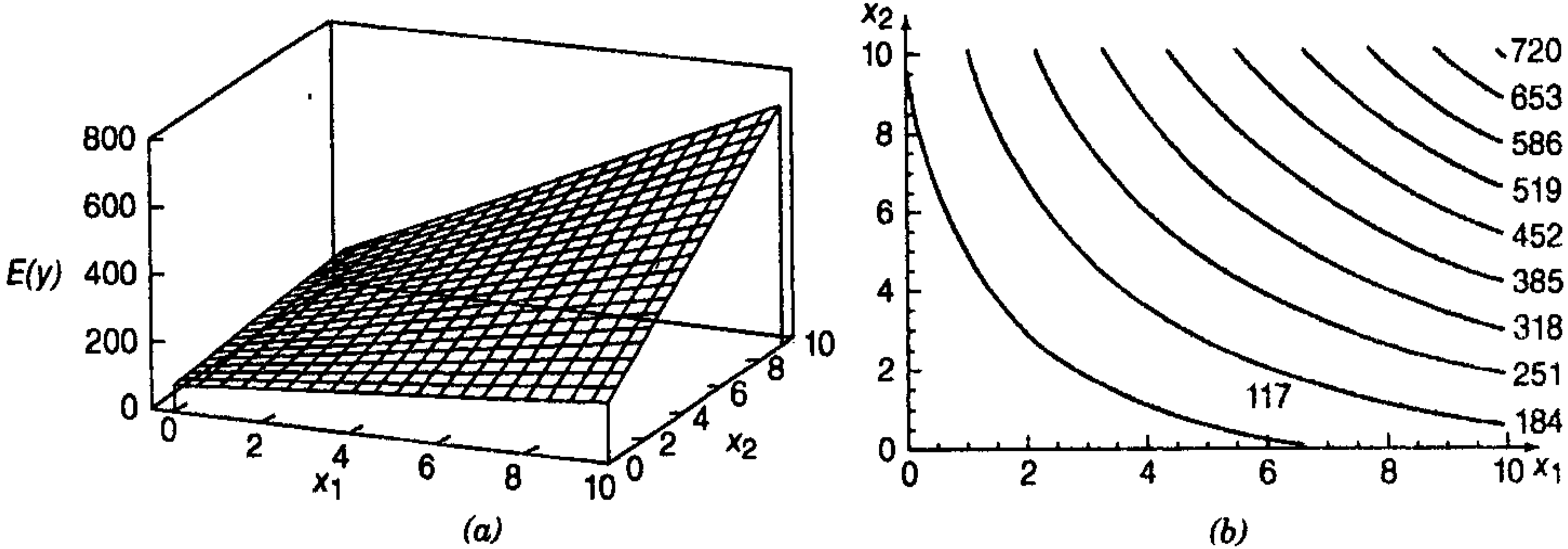
Models that include "interaction effects" may still be analyzed by multiple linear regression techniques, e.g.,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \varepsilon$$

simply by letting

$$x_3 = x_1 x_2 \text{ and } \beta_3 = \beta_{12}$$

Model: $E(y \mid x_1, x_2) = 50 + 10x_1 + 7x_2 + 5x_1x_2$



Data and Model Structures

n independent data: $(y_i\,, x_{1i}\,, \ldots, x_{ki})$

$$y_i = \beta_0 + \beta_1 x_{1i} + \ldots + \beta_k x_{ki} + \varepsilon_i \;, \qquad i = 1, \ldots, n$$

$\varepsilon_1, \ldots, \varepsilon_n$ are statistically independent or uncorrelated with mean zero and constant variance σ^2

Data Layout

TABLE 3.1 Data for Multiple Linear Regression

		Regressors			
Observation, i	Response, y	x_1	x_2	$\dots$	x_k
1	y_1	x_{11}	x_{12}	$\dots$	x_{1k}
2	y_2	x_{21}	x_{22}	$\dots$	x_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	y_n	x_{n1}	x_{n2}	$\dots$	x_{nk}

Least-Squares Estimation:

$$S(\beta_0, \beta_1, \ldots, \beta_k) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{k} \beta_j x_{ij} \right)^2$$

Solve

$$\left.\frac{\partial S}{\partial \beta_0}\right|_{\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k} = -2 \sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \sum_{j=1}^{k} \hat{\beta}_j x_{ij} \right) = 0$$

$$\left.\frac{\partial S}{\partial \beta_j}\right|_{\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k} = -2 \sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \sum_{j=1}^{k} \hat{\beta}_j x_{ij} \right) x_{ij} = 0, \quad j = 1, 2, \ldots, k$$

Simple to Use Matrix Form

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Least-Squares Estimation in Matrix Form:

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{n} \varepsilon_i^2 = \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$$

Solve

$$\left.\frac{\partial S}{\partial \boldsymbol{\beta}}\right|_{\hat{\boldsymbol{\beta}}} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{0}$$

$$\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$$

L-S estimator of $\boldsymbol{\beta}$: $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$

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