Transformation and Weighting – Part II

Johns Hopkins Engineering

625.461 Statistical Models and Regression

Module 8 – Lecture 8D

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Transformation to Linearize the Model

Suppose that

$$y = \beta_0 e^{\beta_1 x} \varepsilon$$

Applying log transformation will yield

$$\ln y = \ln \beta_0 + \beta_1 x + \ln \varepsilon$$

$$y' = \beta_0' + \beta_1 x + \varepsilon'$$

Figure 5.4 and Table 5.4 (page 177-178)

Analytical Methods for Selecting a Transformation

Transformation on y: Box-Cox Method

Goal: transform y to correct nonnormality and/or nonconstant variance

A useful class of transformation is the power transformation y^λ, where λ is a parameter to be determined

Generalized and Weighted Least Squares

Nonconstant error variance can also be managed by the method of weighted least-squares.
In simple linear regression, minimize the weighted sum of sequares

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} w_i (y_i - \beta_0 - \beta_1 x_i)^2$$

where $w_i \propto 1/\mathrm{Var}(y_i)$

Generalized and Weighted Least Squares

Moreover, if

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$E(\boldsymbol{\varepsilon}) = \mathbf{0}, \ \mathrm{Var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{V}$$

The OLS estimator $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ is no longer appropriate. This problem can be approached by transforming the model to a new set of observations to satisfy the standard LS assumptions.

Generalized Least Squares

V must be nonsingular and positive definite, so there exists an $n \times n$ nonsingular symmetric matrix K, such that K′K = KK = V. The K is often called ***square-root*** of V. Now let

$$\mathbf{z} = \mathbf{K}^{-1}\mathbf{y}, \quad \mathbf{B} = \mathbf{K}^{-1}\mathbf{X}, \quad \mathbf{g} = \mathbf{K}^{-1}\boldsymbol{\varepsilon}$$

The regression model becomes

$$\mathbf{z} = \mathbf{B}\boldsymbol{\beta} + \mathbf{g}$$

Generalized Least Squares

$$\begin{aligned}
\mathrm{Var}(\mathbf{g}) &= \{[\mathbf{g} - E(\mathbf{g})][\mathbf{g} - E(\mathbf{g})]'\} \\
&= E(\mathbf{g}\mathbf{g}') \\
&= E\left(\mathbf{K}^{-1}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\mathbf{K}^{-1}\right) \\
&= \mathbf{K}^{-1}E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\mathbf{K}^{-1} \\
&= \sigma^2\mathbf{K}^{-1}\mathbf{V}\mathbf{K}^{-1} \\
&= \sigma^2\mathbf{K}^{-1}\mathbf{K}\mathbf{K}\mathbf{K}^{-1} \\
&= \sigma^2\mathbf{I}
\end{aligned}$$

Generalized Least Squares

$$S(\beta) = \mathbf{g}'\mathbf{g} = \boldsymbol{\varepsilon}'\mathbf{V}^{-1}\boldsymbol{\varepsilon} = (\mathbf{y} - \mathbf{X}\beta)'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\beta)$$

$$(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})\hat{\beta} = \mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$$

$$\hat{\beta} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$$

which is called **generalized LS** estimator of β.

$$\mathrm{Var}(\hat{\beta}) = \sigma^2(\mathbf{B}'\mathbf{B})^{-1} = \sigma^2(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$$

Generalized Least Squares

$\hat{\beta}$ is BLUE of β.

TABLE 5.8 Analysis of Variance for Generalized Least Squares

Source	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Regression	$SS_R = \hat{\beta}'\mathbf{B}'\mathbf{z} = \mathbf{y}'\mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$	p	SS_R/p	MS_R/MS_{Res}
Error	$SS_{Res} = \mathbf{z}'\mathbf{z} - \hat{\beta}'\mathbf{B}'\mathbf{z} = \mathbf{y}'\mathbf{V}^{-1}\mathbf{y} - \mathbf{y}'\mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$	$n-p$	$SS_{Res}/(n-p)$	
Total	$\mathbf{z}'\mathbf{z} = \mathbf{y}'\mathbf{V}^{-1}\mathbf{y}$	n		

Generalized Least Squares

Note: GLSE gives smaller variances for the regression coefficients than OLSE.

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$E(\boldsymbol{\varepsilon}) = \mathbf{0}, \ \mathrm{Var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{V}$$

$$\hat{\beta}_{OLS} = (\mathbf{X'X})^{-1}\mathbf{X'Y}$$

$$Var(\hat{\beta}_{OLS}) = \sigma^2(\mathbf{X'X})^{-1}\mathbf{X'VX}(\mathbf{X'X})^{-1}$$

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