date : 08 | 08 | 2022

ch: Differential Equations (DE)

DE

Ordinary DE

$y = F(x)$

$\frac{dy}{dx} = F(x, y)$

Partial DE

$y = F(x, t)$

$\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}$

- First order initial value Problems (FOIVP)

$$\frac{dy}{dx} = f(x, y) \;;\; y(x_0) = y_0 \quad - \text{(I)}$$

Boundary value problem (BVP)

x_0 ——— x_n x

FOIVP

1. Taylor's series Method

$y(x)$ at $x = x_0$ then Taylor's series of $y(x)$ at $x = x_0$ is given by

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \dots$$

→ If $x - x_0 = h \Rightarrow x = x_0 + h$

$$\Rightarrow y(x_0 + h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \dots$$

Ex. solve: $\frac{dy}{dx} = x^2+y^2$; $y(0)=1$ by finding 1^{st} 5-non-zero terms & hence find (i) $y(0.1)$ (ii) $y(0.2)$ (iii) $y(0.4)$

→ $\frac{dy}{dx} = x^2+y^2$ $\qquad y(0)=1$

by $y(x_0)=y_0$

$x_0 = 0$ and $y_0 = 1$

& $y' = x^2+y^2$

⇒

$y'(0) = x^2+y^2 = 0^2+1^2 = 1$

$y''(0) = 2x + 2yy' = 0+2(1)(1) = 2$

$y'''(0) = 2 + 2[yy'' + (y')^2]$

$= 2[1 + yy'' + (y')^2] = 2[1+2+(1)^2]$

$= 2[4] = 8$

$y^{iv}(0) = 2[y'y'' + yy''' + 2y'y'']$

$= 2[(1\times 2) + (1\times 8) + 2(1)(2)]$

$= 2[2+8+4] = 28$

$y^{v}(0) = 2[y'y''' + y''y'' + y'y''' + yy^{iv} + 2(y'')^2 + 2y'y''']$

$= 2[(1\times 8) + (2)^2 + (1\times 8) + (1\times 28) + 2(2)^2 + 2(1)(8)]$

$= 2[8+4+8+28+8+16]$

$= 2(72) = 144$

Now,

$y(0) = 1$; $y'(0)=1$; $y''(0)=2$; $y'''(0)=8$;

$y^{iv}(0) = 28$; $y^{v}(0) = 144$

Then,

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{iv}(x_0) + \cdots$$

$$y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{\cancel{2}}(\cancel{2}) + \frac{(0.1)^3}{\cancel{6}\,3}(\cancel{8}\,4) + \cdots$$

$$\left[y(x) \cancel{= 1 + (x - x_0)} \\ = 1 + x + \frac{2x^2}{2} + \frac{x^3}{6} \times 8 + \frac{28x^4}{24} + \frac{x^5 \times 144}{120} \right]$$

$y(0.1) = 1.1115 + \ldots$
$y(0.2) = 1.2525$
$y(0.4) = 1.6752$

solve IVP

Q. $\frac{dy}{dx} = x^2 + y^2$; $y(0) = 1$ & find $y(0.4)$

by using $h = 0.2$ $\quad y(0.6) = ?$

→ $\frac{dy}{dx} = x^2 + y^2$

*3 terms min. (if not mentioned)

& by $y(x_0) = y_0 \Rightarrow x_0 = 0$ & $y_0 = 1$

$x - x_0 = h \Rightarrow x = x_0 + h$

$y'(0) = x^2 + y^2 = 0 + 1 = 1$

$y''(0) = 2x + 2yy' = 2$

$y'''(0) = 2 + 2[yy'' + (y')^2] = 8$

$y^{iv}(0) = 2[y'y'' + yy''' + 2y'y''] = 28$

$y^{v}(0) = 2[y'y''' + y''y'' + y'y''' + yy^{iv} + 2(y'')^2 + 2y'y''']$
$= 144$

Ex.

Now,

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \dots$$

$$y(0+0.2) = 1 + (0.2)(1) + \frac{(0.2)^2}{2}(2)$$

$$= 1 + 0.2 + 0.04 \approx \cancel{1.44}\ 1.2525$$

$$y(0.2+0.2) = y(0.2) + hy'(0.2) + \frac{h^2}{2} y''(0.2) + \dots$$

$$= 1.2525 + (0.2)\, y'(0.2) + \frac{(0.2)^2}{2} y''(0.2) + \dots$$

Now,

$$y'(0.2) = x^2 + y^2 = (0.2)^2 + (1.2525)^2 \approx 1.6088$$

$$y''(0.2) = 2x + 2yy' = 2(0.2) + 2(1.2525)(1.6088) = 4.4299$$

$$y'''(0.2) = 2(1 + yy'' + (y')^2) = 18.2734$$

$$\therefore y(0.4) = 1.2525 + (0.2)(1.6088) + \frac{(0.04)}{2}(4.4299)$$

$$= 1.6629$$

2. Euler's Method

$$\frac{dy}{dx} = f(x, y) \; ; \; y(x_0) = y_0$$

1^{st} approximation : $y_1 = y_0 + h f(x_0, y_0)$

~~2nd~~

n^{th} approximation : $y_{n+1} = y_n + h f(x_n, y_n)$

when $y_n = y(x_n)$; $x_n = x_0 + nh$

$x_{n+1} = x_n + h$.

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Ex. Find $y(0.04)$ for $y' = y$; $y(0) = 1$ with
(a) $h = 0.01$ & (b) $h = 0.005$

→ $\frac{dy}{dx} = f(x, y)$

and here, $y' = y$

$\therefore f(x, y) = y$

and, $y(0) = 1 \Rightarrow x_0 = 0$ and $y_0 = 1$

(a) $h = 0.01$

$n = 0$; $y_1 = y(0.01) = y_0 + h y_0$
$= 1 + (0.01) = 1.01$

$n = 1$; $y_2 = y(0.02) = y_1 + h y_1$
$= 1.01 + (0.01)(1.01)$
$= 1.0201$

Then,

n	x_n	y_n	$y_{n+1} = y_n + h y_n$
0	0	1	1.01
1	0.01	1.01	1.0201
2	0.02	1.0201	1.0303
3	0.03	1.0303	1.0406

$\therefore y(0.04) = 1.0406$ → Ans.

(b) $h = 0.005$

$n = 0$; $y_1 = y_0 + h y_0 = 1 + (0.005)(1)$
$= 1.005$

n	x_n	y_n	$y_{n+1} = y_n + h y_n$
0	0	1	1.005
1	0.005	1.005	1.0100
2	0.010	1.0100	1.0151
3	0.015	1.0151	1.0202

4	0.020	1.0202	1.0253
5	0.025	1.0253	1.0304
6	0.030	1.0304	1.0355
7	0.035	1.0355	1.0407

$y(0.04) = 1.0408$

For exact sol^n

$y' = y$

$\int \frac{dy}{y} = \int dx \Rightarrow \ln y = x + \ln c$

$\Rightarrow y = ce^x$

$\Rightarrow y = e^x$

$\therefore \cancel{y = e^x}$

	$y = e^x$	$y_{n+1} = y_n + h y_n$	$E = y_E - y_n$
0.01	1.0100		
0.02			
0.03			
0.04			

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continued...

3. modified Euler's method.

$$y_{n+1} = y_n = \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

where,

$$y_{n+1} = y_n + h f(x_n, x_n)$$

Ex. Find $\frac{dy}{dx} = \frac{2y}{x}$; $y(1) = 2$; $h = 0.25$.
$y(2) = ?$

$\rightarrow$ $\frac{dy}{dx} = \frac{2y}{x} = f(x, y)$ $\quad$ $y(1) = 2$
$h = 0.25$

$\Rightarrow$ $x_0 = 1$ & $y_0 = 2$, $h = 0.25$

x	1	1.25	1.5	1.75	x_2
y	x_0	x_1	x_2	x_3	x_4

$n=0$ $\quad y_1 = y(x_1) = y(1.25) = y_0 + \frac{0.25}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$

~~$y_1 = y_0 + 25 (f y_1, y_0)$~~

for $y_1^* = y_0 + 0.25 (x_0, y_0)$
$= 2 + 0.25 (1R)$
$= 2 + 0.25 (2 \times 2/1)$
$= 3$

y_2

$\Rightarrow y_1 = y(1.25) = 2 + \frac{0.25}{2}\left[f(1,2) + f(1.25, 3)\right]$

$= 2 + \frac{0.25}{2}\left[\frac{2\times 2}{1} + \frac{2\times 3}{1.25}\right]$

$y(1.5) = 3.1 = y_1$

$n=1$

$y_2 = y(x_2) = y(1.5) = y_1 + \frac{0.25}{2}\left[f(x,y) + f(x_2, y_2^*)\right]$

where, $y_2^* = y_1 + hf(x_1, y_1)$

$= 3.1 + 0.25 f(1.25, 3.1)$

$= 3.1 + 0.25\left[2 \times \frac{3.1}{1.25}\right] = 4.34$

$\Rightarrow y_2 = y(1.5) = 3.1 + \frac{0.25}{2}\left[f(1.25, 3.1) + f(1.5, 4.34)\right]$

$= 3.1 + \frac{0.25}{2}\left[f(1.25, 3.1) + f(1.5, 4.34)\right]$

$= y(1.5) = 4.44$

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$n=2$:

$y_3 = y(x_3) = y(1.75)$

$= y_2 + \frac{h}{2}\left[f(x_2, 2) + f(x_3, y_3^*)\right]$

Let $y_3^* = y_2 + h f(x_2, y_2) = 5.92$

$\Rightarrow y_3 = y(1.75) = 6.03$

$n=3$

...

~~Eg: y'~~

Ex. $y' + 2xy^2$

Sol^n ~~Some~~: $\frac{dy}{dx} = -2xy^2$

$\frac{dy}{dx} = 2xy^2$

$\Rightarrow \int \frac{dy}{y^2} = \int x \, dx$

$-1/y^3 = 1$

$x_0 = 0$ & $y_0 = 1$

~~The guass Tip Tip~~

Ex. $y' + 2xy^2 = 0$ $y(0) = 1$, $y(1) = 2$

Sol^n:

$\frac{dy}{dx} = -2xy^2$

$f(x, y) = (-2xy^2)$ $x_0 = 0$ & $y = 1$

$h = 0.5$

x	0	0.5	1
	x_0	x_1	x_2

$n=0$

$y_1 = y(x_1) = y(0.5) = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$

...

4. R-K Method

RK1 → Euler's

RK2 → modified Euler's $y_{n+1} = y_{n+1}$ $(K + K_2)$

Let $K_1 = hf(x_1, x_0)$

$K_2 = hf(g, cut)$

$x = h$

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* RK-2

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

M. $K_1 = hf(x_0, y_0)$

$K_2 = hf(x_0 + h, y_0 + K_1)$

* RK-4

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

M $K_1 = hf(x_0, y_0)$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$K_4 = hf(x_0 + h, y_0 + K_3)$

Ex: $y' = x+y \quad y(0)=1 \; ; \; h = 0.1 \; ; \; y(0.2) = 2$

→ Given:

$f(x,y) = x+y$,

$y(0) = 1 \Rightarrow x_0 = 0$ and $y_0 = 1$

$n=0 \quad y_1 = y(x_1) = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1 f(0,1) = 0.1$

$k_2 = hf(x_0 + h/2, y_0 + \frac{k_1}{2})$

$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$

$= 0.1 f(0.0500, 1.0500)$

$= 0.11$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right)$

$= 0.1 f(0.0500, 1.0550)$

$= 0.1105$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1 f(0 + 0.1, 1 + 0.1105)$

$= 0.1 f(0.1, 1.1105)$

$= 0.1211$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0.1 + 2(0.11) + 2(0.1105) + 0.1211)$

$y_{1_{new}} = 1.1104 \Rightarrow y(0.1) = 1.1104$

so, $x_0 = 0.1$ & $y_0 = 1.1104$

$n = 1$

$y_2 = y(x_2) = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0)$
$= 0.1 f(0.1, 1.1104)$
$= 0.1210$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$
$= 0.1 f\left(0.1 + \frac{0.1}{2}, 1.1104 + \frac{0.1210}{2}\right)$

$= 0.1 f(0.1500, 1.1709)$
$= 0.1321$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$= 0.1 f(0.1 + 0.1/2, 1.1104 + 0.1321/2)$
$= 0.1 f(0.1500, 1.1765)$
$= 0.1327$

$k_4 = hf(x_0 + h, y_0 + k_3)$
$= 0.1(0.1 + 0.1, 1.1104 + 0.1327)$
$= 0.1(0.2, 1.2431)$
$= 0.1443$

$y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.1104 + \frac{1}{6}(0.1210 + 2(0.1321) + 2(0.1327) + 0.1443)$

$= 1.2429$

• Adams-Bashforth-Moulton predictor-corrector method of fourth order.

correct upto 5 decimal places.

$$y_{i+1}^{(P)} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

$$f_i = f(x_i, y_i)$$

This method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}

corrector C: Adams-Moulton method of fourth order.

$$y_{i+1}^{(C)} = y_i + \frac{h}{24}[9f(x_{i+1}, y_{i+1}^{(P)}) + 19f_i - 5f_{i-1} + f_{i-2}]$$

This method requires the starting values of

y_i, f_{i-1}, y_{i-2}

Ex. evaluate $y(1.4)$ by above method if y satisfies

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

and $y(1) = 1$

$y(1.1) = 0.996$

$y(1.2) = 0.986$

$y(1.3) = 0.972$

→

$$f(x, y) = \frac{1}{x^2} - \frac{y}{x}$$

$x_0 = 1,\ x_1 = 1.1,\ x_2 = 1.2,\ x_3 = 1.3$

$y_0 = 1,\ y_1 = 0.996,\ y_2 = 0.986,\ y_3 = 0.972$

$\cancel{y_4 = y_3 + \frac{h}{24}[9f(x_4, y.}$

$f(x_0, y_0) = 0$

$f(x_1, y_1) = -0.079008$

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- Initial & Boundary value problem

Initial value problem

only one value is given

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

Boundary value problem

~~has~~ more values are given

$y(x_0) = y_0 \quad ; \quad y(x_n) = y_n$

- General second order ordinary differentiation.

$$y'' + p(x)y' + q(x)y = r(x)$$

① Boundary condition of 1^{st} kind

$y(a) = A \quad , y(b) = B.$

② Boundary condition of 2^{nd} kind.

$y'(a) = A \quad , \quad y'(b) = B$

③ 3^{rd} kind or, mixed boundary.

$a_0 y(a) + a_1 y'(a) = A$ (linear combination)

$b_0 y(b) + b_1 y'(b) = B$

① Forward diff. approximation.

$$y' = \frac{1}{h}[\Delta y_0] = \frac{1}{h}[y_1 - y_0]$$

$$y_i' = \frac{1}{h}[y_{i+1} - y_i] = \frac{1}{h}[y(x_{i+1}) - y(x_i)]$$

(2) Backward Diff. approximation.

$y' = \frac{1}{h}[\nabla y_n] = \frac{1}{h}[y_n - y_{n-1}]$

$\Rightarrow y'_i = \frac{1}{h}[y_i - y_{i-1}] = \frac{1}{h}[y(x_i) - y(x_{i-1})]$

(3) Central Diff. approximation.

$y'_i = \frac{1}{2h}[y_{i+1} - y_{i-1}]$

$= \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})]$

$y''_i = \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}]$.

Q. Derive the difference equations for the solution of the boundary value problem.

$y'' + p(x)y' + q(x)y = r(x)$

(1) $y(a) = A$. ~~(2) y~~

$y(b) = B$.

→ FD for y' & CD for y''

$\Rightarrow \frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] + p(x_i)\frac{1}{h}[y_{i+1} - y_i] + q(x_i)y_i = r(x_i)$

$\Rightarrow (1 + hP(x_i)]y_{i+1} - [2 + hP(x_i) + h^2 q(x_i)]y_i + \Delta y_{i-1} = r(x_i)$

$\Rightarrow a_{i+1}y_{i+1} + a_i y_i + a_{i-1}y_{i-1} = r(x_i)$

where

Ex. ~~xy~~

$a_{i+1} = 1 + hP(x_i)$;

$a_i = -(2 + hP(x_i) + h^2 q(x_i)]\, a_{i-1} = 1$

Ex: $xy'' + y = 0$; $y(1) = 1$ $y(2) = 2$ $h = 0.25$

Solve boundary value problem.

$\rightarrow$ $x_i y_i'' + y_i = 0$

$$\frac{x_i}{h^2}[y_{i+1} - 2y_i + y_{i-1}] + y_i = 0$$

$\Rightarrow x_i y_{i+1} - 2x_i y_i + x_i y_{i-1} + 0.0625\, y_i = 0$

$\Rightarrow x_i y_{i+1} + (0.0625 - 2x_i) y_i + x_i y_{i-1} = 0$

Taking value of i

$i = 1, 2, 3$ as.

$x = 1, 1.25, 1.50, 1.75, 2$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$.

$\therefore$ for $i = 1$

$x_1 y_2 + (0.0625 - 2x_1) y_1 + x_1 y_0 = 0$

$\Rightarrow 1.25 y_2 + (0.0625 - 2.50) y_1 + 1.25 y_0 = 0$

$\Rightarrow 1.25 y_2 - 2.437500 y_1 + 1.25 y_0 = 0$ — (1)

for $i = 2$.

$x_2 y_3 + (0.0625 - 2(x_2)) y_2 + x_2 y_1 = 0$

$\Rightarrow 1.50 y_3 - 2.9375 y_2 + 1.50 y_1 = 0$ — (2)

for $i = 3$

$x_3 y_4 + (0.0625 - 2x_3) y_3 + x_3 y_2 = 0$

$1.75 y_4 - 3.4375 y_3 + 1.75 y_2 = 0$ — (3)

we have values of $\cancel{d}y_0(1)=1$; $y_{\cancel{4}}(2)=$

$\therefore -1.25y_2 + 2.4375y_1 = 1.\cancel{2}5$

$-2.9375y_2 + 1.50y_3 \cancel{+} 1.50y_1 = 0$

$1.75y_2 - 3.\cancel{4}375y_3 = -3.5$

Now,

$AX = B$

$$A = \begin{bmatrix} -2.4375 & 1.25 & 0 \\ 1.5 & -2.9375 & 1.5 \\ 0 & 1.75 & -3.4375 \end{bmatrix}$$

$$X = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad B = \begin{bmatrix} -1.25 \\ 0 \\ -3.5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2.4375 & 1.25 & 0 & -1.25 \\ 1.5 & -2.9375 & 1.5 & 0 \\ 0 & 1.75 & -3.4375 & -3.5 \end{array}\right]$$

$R_1 \rightarrow -R_1/2.4375$

$$\begin{bmatrix} 1 & -0.512821 & 0 & 0.512821 \\ 1.5 & -2.9375 & 1.5 & 0 \\ 0 & 1.75 & -3.4375 & -3.5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 1.5R_1$

$$\begin{bmatrix} 1 & -0.512821 & 0 & 0.512821 \\ 0 & -2.168269 & 1.5 & -0.769232 \\ 0 & 1.75 & -3.4375 & -3.5 \end{bmatrix}$$

$R_2 \to -R_2/2.168269$

$$\begin{bmatrix} 1 & -0.512821 & 0 & 0.512821 \\ 0 & 1 & -0.691796 & 0.354768 \\ 0 & 1.75 & -3.4375 & -3.5 \end{bmatrix}$$

$\cancel{R_1 \to R_1 + 0.512821}$

$R_3 \to R_3 - 1.75 R_2$

$\vdots$

$y_1 = 1.35126 = y(1.25)$

$y_2 = 1.63495 = y(1.50)$

$y_3 = 1.85052 = y(1.75)$

Q. Solve the boundary value problem

$y'' + 5y' + 4y = 1$

$y(0) = 0, \quad y(1) = 0 \qquad h = 0.25$

Q. $y'' - y = x$

$y_0(0) = 0 \qquad y_4(1) = 2$

$h = 0.25$

$\rightarrow \quad y_i'' - y_i = x_i$

$\Rightarrow \dfrac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] - y_i - x_i = 0$

$= y_{i+1} - 2y_i + y_{i-1} - 0.0625 y_i - 0.0625 x_i = 0$

$= y_{i+1} - (2.0625) y_i + y_{i-1} - 0.0625 x_i = 0$

$i =$	0	1	2	3	4
$x_i =$	0	0.25	0.50	0.75	2

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for $i=1$

$y_2 - 2.0625y_1 + y_0 = (0.0625)(0.25)$

$\Rightarrow y_0 - 2.0625y_1 + y_2 = 0.0156 \quad - (1)$

for $i=2$

$y_3 - 2.0625y_2 + y_1 = (0.0625)(0.50)$

$\Rightarrow y_1 - 2.0625y_2 + y_3 = 0.0313 \quad - (2)$

for $i=3$

$y_4 - 2.0625y_3 + y_2 = (0.0625)(0.75)$

$\Rightarrow y_2 - 2.0625y_3 + y_4 = 0.0469 \quad - (3)$

$\therefore$ eq^u

$$\left.\begin{aligned} -2.0625y_1 + y_2 &= 0.0156 \\ y_1 - 2.0625y_2 + y_3 &= 0.0313 \\ y_2 - 2.0625y_3 &= -1.9531 \end{aligned}\right] \text{eq}^n\text{s.}$$