Appendix: Detailed Calculations in " Refinement on the maximum spectral radius of C_6^+ -free graphs with given size."[*]

Zenghui Shen, Jianping Li[†]

College of Sciences, National University of Defense Technology,
Changsha, China, 410073.

April 6, 2025

1 Introduction

This document contains the detailed computational steps involved in the proofs of certain lemmas presented in the manuscript " Refinement on the maximum spectral radius of C_6^+ -free graphs with given size. " The proofs of these lemmas involve the comparison of complex parametric polynomials. Specifically, the proofs require calculating the difference of squares between two parametric polynomials and systematically substituting lower bounds for certain parameters. The aim is to eliminate variables from the resulting polynomials and ultimately determine their signs.

In each step of parameter substitution and elimination, it is essential to analyse the monotonicity of the polynomial with respect to the substituted parameter. This typically involves proving that the polynomial is monotonically increasing in that parameter. Moreover, these parametric polynomials involve multiple variables and higher degrees, making the computational process lengthy and intricate, though not technically complex. Due to the extensive nature of these calculations, they are not included in the main text.

Verifying the correctness of these calculations can be tedious. To facilitate this, we have provided Python scripts that automate the verification process. These scripts not only validate the final results but also check for monotonicity during the elimination process.

This appendix, therefore, serves both as a detailed account of the necessary computations and as a guide for using the provided code to verify them.

2 Detailed calculations in proofs of Lemmas 3.10, 3.11, 3.13 .

- Specific calculations in Lemma 3.10

$$\psi_3^2(m,l,r,a) - \psi_4^2(m,l,r,a)$$
$$= ((48a + 64l + 64r + 104)m^2 + (160a^3 + 640a^2l + 640a^2r + 1040a^2 + 768al^2 + 1792alr$$

$$
\begin{aligned}
&+ 2560al + 768ar^2 + 2688ar + 1296a + 256l^3 + 1152l^2r + 1312l^2 + 1152lr^2 + 3520lr \\
&+ 1312l + 256r^3 + 1504r^2 + 1248r - 832)m + 48a^5 + 320a^4l + 320a^4r + 520a^4 + 768a^3l^2 \\
&+ 1792a^3lr + 2560a^3l + 768a^3r^2 + 2688a^3r + 1184a^3 + 768a^2l^3 + 3456a^2l^2r + 3936a^2l^2 \\
&+ 3456a^2lr^2 + 10560a^2lr + 3488a^2l + 768a^2r^3 + 4512a^2r^2 + 3296a^2r - 3224a^2 + 256al^4 \\
&+ 2560al^3r + 1664al^3 + 4864al^2r^2 + 12032al^2r + 1376al^2 + 2560alr^3 + 13056alr^2 \\
&+ 6976alr - 9056al + 256ar^4 + 2432ar^3 + 608ar^2 - 14880ar - 12864a + 512l^4r - 256l^4 \\
&+ 2048l^3r^2 + 3328l^3r - 1728l^3 + 2048l^2r^3 + 8576l^2r^2 + 1728l^2r - 6816l^2 + 512lr^4 \\
&+ 4352lr^3 + 1216lr^2 - 18880lr - 14080l - 2496r^3 - 16096r^2 - 24320r - 8800)\sqrt{4m-7} \\
&+ 16m^3 + (240a^2 + 640al + 640ar + 1040a + 384l^2 + 896lr + 1280l + 384r^2 + 1344r \\
&+ 648)m^2 + (240a^4 + 1280a^3l + 1280a^3r + 2080a^3 + 2304a^2l^2 + 5376a^2lr + 7680a^2l \\
&+ 2304a^2r^2 + 8064a^2r + 3552a^2 + 1536al^3 + 6912al^2r + 7872al^2 + 6912alr^2 + 21120alr \\
&+ 6976al + 1536ar^3 + 9024ar^2 + 6592ar - 6448a + 256l^4 + 2560l^3r + 1664l^3 + 4864l^2r^2 \\
&+ 12032l^2r + 1376l^2 + 2560lr^3 + 13056lr^2 + 6976lr - 9056l + 256r^4 + 2432r^3 + 608r^2 \\
&- 14880r - 12864)m + 16a^6 + 128a^5l + 128a^5r + 208a^5 + 384a^4l^2 + 896a^4lr + 1280a^4l \\
&+ 384a^4r^2 + 1344a^4r + 312a^4 + 512a^3l^3 + 2304a^3l^2r + 2624a^3l^2 + 2304a^3lr^2 + 7040a^3lr \\
&+ 832a^3l + 512a^3r^3 + 3008a^3r^2 + 704a^3r - 4576a^3 + 256a^2l^4 + 2560a^2l^3r + 1664a^2l^3 \\
&+ 4864a^2l^2r^2 + 12032a^2l^2r - 1312a^2l^2 + 2560a^2lr^3 + 13056a^2lr^2 + 704a^2lr - 18016a^2l \\
&+ 256a^2r^4 + 2432a^2r^3 - 2080a^2r^2 - 24288a^2r - 17400a^2 + 1024al^4r - 512al^4 + 4096al^3r^2 \\
&+ 6656al^3r - 5248al^3 + 4096al^2r^3 + 17152al^2r^2 - 4608al^2r - 22816al^2 + 1024alr^4 + 8704alr^3 \\
&- 5632alr^2 - 62400alr - 37344al - 6784ar^3 - 42720ar^2 - 57376ar - 11776a - 512l^5 \\
&+ 1024l^4r^2 - 3584l^4 + 2048l^3r^3 + 6144l^3r^2 - 6400l^3r - 10176l^3 + 1024l^2r^4 + 7168l^2r^3 \\
&- 4480l^2r^2 - 38208l^2r - 16160l^2 + 1024lr^4 - 7424lr^3 - 53696lr^2 - 61888lr - 8320l - 512r^5 \\
&- 4608r^4 - 24896r^3 - 46944r^2 - 17280r + 8736
\end{aligned}
$$

$$
\begin{aligned}
\geq\ &((256a + 512l + 512)r^4 + (768a^2 + 2560al + 3968a + 2048l^2 + 7168l + 2048)r^3 + (768a^3 \\
&+ 3456a^2l + 5792a^2 + 4864al^2 + 18176al + 10016a + 2048l^3 + 13184l^2 + 17792l - 4384)r^2 \\
&+ (320a^4 + 1792a^3l + 3008a^3 + 3456a^2l^2 + 13120a^2l + 8576a^2 + 2560al^3 + 17152al^2 + 26816al \\
&+ 2112a + 512l^4 + 6144l^3 + 17920l^2 + 7232l - 16064)r + 48a^5 + 320a^4l + 520a^4 + 768a^3l^2 \\
&+ 2880a^3l + 1984a^3 + 768a^2l^3 + 5216a^2l^2 + 8768a^2l + 1976a^2 + 256al^4 + 3200al^3 + 10528al^2 \\
&+ 7296al - 5184a + 256l^4 + 2432l^3 + 4064l^2 - 5504l - 10360)\sqrt{8l + 8r + 13} + (256a^2 + 1024al \\
&+ 3072a + 1024l^2 + 6656l + 3072)r^4 + (512a^3 + 2560a^2l + 7040a^2 + 4096al^2 + 25600al \\
&+ 21504a + 2048l^3 + 22016l^2 + 43008l + 1664)r^3 + (384a^4 + 2304a^3l + 5568a^3 + 4864a^2l^2 \\
&+ 28416a^2l + 26528a^2 + 4096al^3 + 44800al^2 + 96896al + 32544a + 1024l^4 + 20992l^3 \\
&+ 80256l^2 + 68608l - 33632)r^2 + (128a^5 + 896a^4l + 1824a^4 + 2304a^3l^2 + 12160a^3l \\
&+ 11264a^3 + 2560a^2l^3 + 27392a^2l^2 + 60992a^2l + 27936a^2 + 1024al^4 + 23552al^3 + 95616al^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 104256al - 512a + 5632l^4 + 40448l^3 + 80256l^2 + 7104l - 68448)r + 16a^6 + 128a^5l + 208a^5 \\
&+ 384a^4l^2 + 1760a^4l + 1512a^4 + 512a^3l^3 + 5184a^3l^2 + 11392a^3l + 5824a^3 + 256a^2l^4 \\
&+ 6272a^2l^3 + 26528a^2l^2 + 32288a^2l + 6360a^2 + 2560al^4 + 20736al^3 + 47456al^2 + 21440al \\
&- 18016a + 2560l^4 + 13824l^3 + 11360l^2 - 31968l - 37384 \\
\geq\ &(((256a + 1792)l^4 + (768a^2 + 10880a + 39296)l^3 + (768a^3 + 15584a^2 + 105760a + 231776)l^2 \\
&+ (320a^4 + 8256a^3 + 79232a^2 + 320448a + 411328)l + 48a^5 + 1480a^4 + 17920a^3 + 100568a^2 \\
&+ 219168a - 1240))\sqrt{8l + 37} + (256a^2 + 5632a + 28672)l^4 + (512a^3 + 13952a^2 + 128256a \\
&+ 379392)l^3 + (384a^4 + 12096a^3 + 152480a^2 + 848096a + 1651808)l^2 + (128a^5 + 4448a^4 \\
&+ 68608a^3 + 540128a^2 + 1980416a + 2307168)l + 16a^6 + 592a^5 + 10440a^4 + 103552a^3 \\
&+ 539736a^2 + 1102784a - 251656 \\
\geq\ &16a^6 + (144\sqrt{5} + 720)a^5 + (5400\sqrt{5} + 15272)a^4 + (80832\sqrt{5} + 184768)a^3 + (588456\sqrt{5} \\
&+ 1246552)a^2 + (1969536\sqrt{5} + 4065184)a + 2048856\sqrt{5} + 4115384 > 0, \\
&(m \geq 2l + 2r + 5, r \geq 3, l \geq 1 \text{ and } a \geq 0).
\end{aligned}
$$

- Specific calculations in Lemma 3.11:

$$
\begin{aligned}
&\psi_5^2(m, r, t, a) - \psi_6^2(m, r, t, a) \\
=\ &(32m^2 + (384r^2 + (768a + 448t - 128)r + 320a^2 + 384at - 64a + 96t^2 - 64t - 640)m \\
&+ (512a + 256t - 768)r^3 + (1152a^2 + 1280at - 896a + 288t^2 - 768t - 4736)r^2 \\
&+ (768a^3 + 1344a^2t - 384a^2 + 640at^2 - 704at - 6528a + 64t^3 - 256t^2 - 3840t - 2816)r \\
&+ 160a^4 + 384a^3t - 64a^3 + 288a^2t^2 - 192a^2t - 2144a^2 + 64at^3 - 160at^2 - 2496at - 1408a \\
&- 32t^3 - 544t^2 - 384t + 1152)\sqrt{4m - 7} + (320a + 384r + 192t - 32)m^2 + (640a^3 \\
&+ 2304a^2r + 1152a^2t - 192a^2 + 2304ar^2 + 2688art - 768ar + 576at^2 - 384at - 4288a \\
&+ 512r^3 + 1280r^2t - 896r^2 + 640rt^2 - 704rt - 6528r + 64t^3 - 160t^2 - 2496t - 1408)m \\
&+ 64a^5 + 384a^4r + 192a^4t - 32a^4 + 768a^3r^2 + 896a^3rt - 256a^3r + 192a^3t^2 - 128a^3t \\
&- 2176a^3 + 512a^2r^3 + 1280a^2r^2t - 896a^2r^2 + 640a^2rt^2 - 704a^2rt - 9216a^2r + 64a^2t^3 \\
&- 160a^2t^2 - 3840a^2t - 1184a^2 + 512ar^3t - 1536ar^3 + 576ar^2t^2 - 1536ar^2t - 12160ar^2 \\
&+ 128art^3 - 512art^2 - 10816art - 4736ar - 64at^3 - 1760at^2 - 320at + 6784a - 1024r^4 \\
&+ 128r^3t^2 - 1280r^3t - 4864r^3 + 64r^2t^3 - 544r^2t^2 - 7168r^2t - 4480r^2 - 64rt^3 - 2304rt^2 \\
&- 1280rt + 10496r - 96t^3 + 416t^2 + 4736t + 2944 \\
\geq\ &((512a + 256t)r^3 + (1152a^2 + 1280at + 640a + 288t^2 + 512t - 1792)r^2 + (768a^3 + 1344a^2t \\
&+ 256a^2 + 640at^2 + 832at - 512a + 64t^3 + 384t^2 - 384t - 4096)r + 160a^4 + 384a^3t \\
&- 64a^3 + 288a^2t^2 + 128a^2t + 416a^2 + 64at^3 + 224at^2 + 512at - 1920a + 64t^3 + 192t^2 \\
&- 1024t - 1920)\sqrt{8r + 4t + 25} + (512a^2 + 512at + 3072a + 128t^2 + 1792t - 1024)r^3 \\
&+ (768a^3 + 1280a^2t + 3712a^2 + 576at^2 + 6144at + 6016a + 64t^3 + 2016t^2 + 3072t - 12544)r^2
\end{aligned}
$$

$$
\begin{aligned}
&+ (384a^4 + 896a^3t + 1024a^3 + 640a^2t^2 + 3904a^2t + 8832a^2 + 128at^3 + 3328at^2 + 10432at \\
&\quad - 9216a + 704t^3 + 2944t^2 - 6272t - 20992)r + 64a^5 + 192a^4t - 32a^4 + 192a^3t^2 + 512a^3t \\
&\quad + 2944a^3 + 64a^2t^3 + 992a^2t^2 + 5184a^2t - 2720a^2 + 512at^3 + 2784at^2 - 2560at - 7040a \\
&\quad + 64t^4 + 448t^3 - 320t^2 - 4864t - 10368 \\
&\geq ((64a + 256)t^3 + (288a^2 + 2144a + 3936)t^2 + (384a^3 + 4160a^2 + 14528a + 9344)t + 160a^4 \\
&\quad + 2240a^3 + 11552a^2 + 16128a - 30336)\sqrt{4t + 49} + 64t^4 + (64a^2 + 896a + 3136)t^3 \\
&\quad + (192a^3 + 2912a^2 + 17952a + 30112)t^2 + (192a^4 + 3200a^3 + 28416a^2 + 97856a + 52352)t \\
&\quad + 64a^5 + 1120a^4 + 12928a^3 + 71008a^2 + 102400a - 213888 \\
&\geq 581728a + \sqrt{61}(160a^4 + 3392a^3 + 26624a^2 + 80736a + 40032) + 184192a^2 + 24256a^3 \\
&\quad + 1696a^4 + 64a^5 + 304032 > 0. \quad (m \geq 2r + t + 8, r \geq 3, t \geq 3 \text{ and } a \geq 0).
\end{aligned}
$$

- Specific calculations in Lemma 3.13:

$$
\begin{aligned}
&\psi_7^2(m, t, r, a) - \psi_8^2(m, t, r, a) \\
&= ((2a + 2r + t - 2)m + 2a^3 + 6a^2r + 3a^2t - 6a^2 + 4ar^2 + 6art - 14ar + at^2 - 7at + 2a + 2r^2t \\
&\quad - 8r^2 + rt^2 - 8rt + 4r - \frac{3}{2}t^2 + 2t + 2)\sqrt{4m - 7} + m^2 + (6a^2 + 12ar + 6at - 12a + 4r^2 + 6rt \\
&\quad - 14r + t^2 - 7t + 2)m + a^4 + 4a^3r + 2a^3t - 4a^3 + 4a^2r^2 + 6a^2rt - 14a^2r + a^2t^2 - 7a^2t - 5a^2 \\
&\quad + 4ar^2t - 16ar^2 + 2art^2 - 16art - 6ar - 3at^2 - 3at + 18a - 8r^3 + r^2t^2 - 10r^2t + 4r^2 - 3rt^2 \\
&\quad + 20r + \frac{1}{2}t^2 + 10t - 6 \\
&\geq ((a + r - \frac{1}{2})t^2 + (3a^2 + 6ar - 5a + 2r^2 - 4r + 7)t + 2a^3 + 6a^2r - 6a^2 + 4ar^2 - 10ar + 16a - 4r^2 \\
&\quad + 14r - 12)\sqrt{8r + 4t + 21} + t^3 + (a^2 + 2ar + 3a + r^2 + 5r + \frac{3}{2})t^2 + (2a^3 + 6a^2r - a^2 + 4ar^2 \\
&\quad + 8ar + 27a + 6r^2 + 18r - 23)t + a^4 + 4a^3r - 4a^3 + 4a^2r^2 - 2a^2r + 37a^2 + 8ar^2 + 54ar - 66a \\
&\quad + 8r^2 - 46r + 57 \\
&\geq (4ar^2 + (6a^2 + 2a + 10)r + 2a^3 + 10a)\sqrt{8r + 29} + (4a^2 + 16a + 24)r^2 + (4a^3 + 10a^2 + 78a + 10)r \\
&\quad + a^4 + 39a^2 + 25 \\
&\geq 378a + \sqrt{53}(2a^3 + 18a^2 + 52a + 30) + 105a^2 + 12a^3 + a^4 + 271 > 0. \\
&\quad (m \geq 2r + t + 7, t \geq 2, r \geq 3 \text{ and } a \geq 0).
\end{aligned}
$$

$$
\begin{aligned}
&\psi_9^2(m, k_1, t_1, a) - \psi_{10}^2(m, k_1, t_1, a) \\
&= ((8a + 8k_1 + 8t_1 + 2)m + 8a^3 + 24a^2k_1 + 24a^2t_1 + 6a^2 + 16ak_1^2 + 64ak_1t_1 - 16ak_1 \\
&\quad + 16at_1^2 - 32at_1 - 34a + 32k_1^2t_1 - 32k_1^2 + 32k_1t_1^2 - 48k_1t_1 - 60k_1 - 40t_1^2 - 52t_1 \\
&\quad - 20)\sqrt{4m - 7} + 4m^2 + (24a^2 + 48ak_1 + 48at_1 + 12a + 16k_1^2 + 64k_1t_1 - 16k_1 + 16t_1^2 \\
&\quad - 32t_1 - 34)m + 4a^4 + 16a^3k_1 + 16a^3t_1 + 4a^3 + 16a^2k_1^2 + 64a^2k_1t_1 - 16a^2k_1 + 16a^2t_1^2 \\
&\quad - 32a^2t_1 - 62a^2 + 64ak_1^2t_1 - 64ak_1^2 + 64ak_1t_1^2 - 96ak_1t_1 - 176ak_1 - 80at_1^2 - 160at_1
\end{aligned}
$$

$$- 54a - 32k_1^3 + 64k_1^2 t_1^2 - 64k_1^2 t_1 - 96k_1^2 - 160k_1 t_1^2 - 144k_1 t_1 - 20k_1 + 72t_1^2 + 68t_1 + 36$$

$$\geq ((16a + 32k_1 - 24)t_1^2 + (24a^2 + 64ak_1 - 16a + 32k_1^2 - 16k_1 - 24)t_1 + 8a^3 + 24a^2 k_1 + 6a^2$$

$$+ 16ak_1^2 - 10a - 16k_1^2 - 32k_1 - 14)\sqrt{8k_1 + 8t_1 + 5} + 32t_1^3 + (16a^2 + 64ak_1 + 16a + 64k_1^2$$

$$+ 72)t_1^2 + (16a^3 + 64a^2 k_1 + 16a^2 + 64ak_1^2 + 96ak_1 + 8a + 96k_1^2 - 16k_1 - 48)t_1 + 4a^4$$

$$+ 16a^3 k_1 + 4a^3 + 16a^2 k_1^2 + 32a^2 k_1 + 10a^2 + 32ak_1^2 - 8ak_1 - 18a - 64k_1^2 - 88k_1 - 30$$

$$\geq ((16a + 16)k_1^2 + (24a^2 + 64a - 16)k_1 + 8a^3 + 30a^2 - 10a - 62)\sqrt{8k_1 + 13} + (16a^2$$

$$+ 96a + 96)k_1^2 + (16a^3 + 96a^2 + 152a - 104)k_1 + 4a^4 + 20a^3 + 42a^2 + 6a + 26$$

$$\geq 694a + 182\sqrt{29}a - 30\sqrt{29} + 78\sqrt{29}a^2 + 8\sqrt{29}a^3 + 298a^2 + 52a^3 + 4a^4 + 202 > 0.$$

$$(m \geq 2k_1 + 2t_1 + 3, t_1 \geq 1, k_1 \geq 2 \text{ and } a \geq 0).$$