Rotating Consensus for Double-Integrator Multi-Agent Systems with Communication Delay

Xiongtao Shi[a], Yonggang Li[a,*], Yanhua Yang[a], Bei Sun[a]

[a]*School of Automation, Central South University, Changsha, 410083, China*

Abstract

This paper deals with the rotating consensus problem for a class of double-integrator multi-agent systems with and without communication delay. Compared with some existing results, the multi-agent systems are assumed to have a directed graph rather than an undirected graph. For the case without communication delay, we design a distributed control protocol by using the local relative information and obtain the sufficient and necessary condition on the lower bound of control parameter for rotating consensus problem with directed graph. For the case with uniform communication delay, we further obtain the sufficient and necessary condition on the upper bound of communication delay by the frequency domain analysis method. Finally, some numerical simulations are provided to illustrate the correctness of our results.

Keywords: Multi-agent systems, rotating consensus, communication delay, directed graph.

1. Introduction

Owing to its flexibility and scalability, the distributed control of multi-agent systems has been widely investigated in recent years [1]. To mention a few, it has been applied in cooperative robotics [2], unmanned aerial vehicles (UAV) [3], formation control of spacecraft [4], sensor networks [5] and many other scenarios. As a fundamental problem, the so-called consensus problem [6] requires the states of all agents to converge to a common value of interest, which has been used widely in various fields including robotics formations [7], flocking [8, 9] and multi-agent coordination [10]. Many researchers have got many theoretical results on this topic [11]. For example, in [12], the author deduced the necessary condition for the consensus of the multi-agent systems with the help of geometry method for the directed graph. In [14], the authors further considered event-triggered output consensus problem for heterogeneous multi-agent systems with nonuniform communication delay. Readers are referred to the recent review articles [15, 16] for more details.

Besides, when there exists information exchange between agents, the communication delay is unavoidable due to the unexpected disturbances such as congestion of the communication channels [17], the motion of wireless sensors [18] and the finite process time of interactions among agents [19], which may affect the control performance and even unstabilize the controlled systems severely. In [20], the authors considered the consensus problem with the constant communication delay and got the tolerable upper bound. In [21], the authors deduced the sufficient condition for the consensus problem with considering time-varying communication delay by using Lyapunov method. In [22], the authors obtained the conservative upper bound of the nonuniform communication delay by frequency domain analysis method for the undirected graph.

Apart from the consensus problem, more and more researchers have paid a lot of attention to rotating consensus problem because of its great potential in engineering, which could be applied in satellite formation [23], autonomous spacecraft rendezvous [24], unmanned robots [25] and so on. For the rotating consensus problem, all agents will not only reach consensus but also move around a common point. In [26], the authors firstly considered the rotating consensus and rotating formation problem for the undirected graph. In [27], the authors studied the mean-square composite-rotating consensus problem for the undirected graph and obtained the sufficient condition with considering the communication noises.

In the aspect of rotating consensus, the communication delay is also considered in some works. In [28], the rotating consensus is achieved with the proposed distributed control protocol under some nonuniform communication delay for an undirected connected graph with the help of frequency domain analysis approach. In [29], the author considered the H_∞ rotating consensus with time-delay by using the Lyapunov–Krasovskii function, which could not be used to derive the tolerable upper bound for the communication delay. We recommend the readers to refer to [30, 31] for more related results.

However, as far as the authors know, owing to the difficulty of analysis, for the delay-free rotating consensus problem with directed graph, the sufficient and necessary condition has not been obtained. Threfore, how to derive the sufficient and necessary condition on this case is worthy of investigating. Furthermore, with considering the unavoidable communication delay between agents, it is necessary to consider the influence of communication delay.

*Corresponding author
Email address: liyonggang@csu.edu.cn (Yonggang Li)

Compared with the results for the undirected graph [28] or obtained sufficient condition by using Lyapunov–Krasovskii method in [29], how to obtain the sufficient and necessary condition for the corresponding tolerable upper bound of communication delay for the directed graph is still worthy of investigating. Therefore, in this paper, for the double-integrator multi-agent systems which has a directed graph, we further investigate its rotating consensus problem with taking communication delay into consideration.

As an extension of existing study on rotating consensus problem [26], we firstly propose a new way to derive the sufficient and necessary condition to achieve the rotating consensus for a directed graph which has a directed spanning tree. Furthermore, for the case that the communication delay is taken into consideration, the accurate tolerable upper bound is deduced by analyzing the relationship between the communication delay and the roots of the characteristic equations.

Therefore, the main contributions of this paper are listed as follows:

1. We obtain the sufficient and necessary conditions for the rotating consensus with the directed graph which has a spanning tree. Compared with the rotating consensus for undirected graph in [26, 28], the case considering the directed graph is more general. By using the algebraic analysis method, we obtain the accurate lower bound of the control parameter, which includes the results in [26] as a special case of our result.

2. Compared with the obtained sufficient condition on upper bound of the communication delay for the rotating consensus problem with undirected graph in [26], we obtain the sufficient and necessary condition when considering the uniform communication delay for the rotating consensus problem with directed graph. With the help of frequency domain analysis method, the accurate upper bound of communication delay is deduced.

3. In this paper, the representation of the rotating consensus is different from previous results in [26, 28]. By using the exponential representation with respect to time t, we get the corresponding dynamic change of rotating position and velocity for every agent more specifically.

This paper is organized as follows: basic concepts and definitions are introduced in Section 2; the case without communication delay is analyzed in Section 3; the case with communication delay is considered in Section 4; some numerical simulation results are shown in Section 5; and the conclusion is given in section 6 finally.

2. Problem formulation

Notations: $\mathbb{R}$ indicates the set of real; $\mathbb{C}$ denotes the set of complex; $\mathbb{R}^+$ denotes the set of positive real; $M_{m\times n}$ represents the matrix with m rows and n columns; $\mathbf{1}_m = [1, \cdots, 1]^T$ represents a column vector with m elements; I_m indicates the m dimensional identity matrix; $\mathbf{0}_m = [0, \cdots, 0]^T$ represents a column vector with m elements; $\mathbf{0}$ represents a zero vector or a zero matrix with appropriate dimensions; x^T denotes the transposition of x; j denotes the imaginary unit; $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real and imaginary part of a complex, respectively; $\det(\cdot)$ denotes the determinant of a matrix; $\mathrm{diag}(d_1, d_2, \cdots, d_n)$ is a diagonal matrix whose diagonal entries are $d_1, d_2, ..., d_n$ and all the other entries are zero.

Define $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ as a directed graph with n agents, where $\mathcal{V} = \{1, 2, ..., n\}$ is the agents set, $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$ is the edges set and $\mathcal{A} = [a_{ik}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix, where $a_{ik} = 1 (i \neq k)$ if $(k, i) \in \mathcal{E}$ and $a_{ik} = 0$ otherwise. The agent i's neighbor set is defined as $\mathcal{N}_i = \{k \in \mathcal{V} : (k, i) \in \mathcal{E}\}$, which means that the agent i can obtain information from agents k, $k \in \mathcal{N}_i$. A spanning tree is a subgraph of $\mathcal{G}$ in which there exists at least one access including all agents. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}\{d_1, d_2, ..., d_N\}$, $d_i = \sum_{k\in\mathcal{N}_i} a_{ik}$.

In this paper, we consider a group of n agents, and the ith agent has the following dynamics:

$$\begin{aligned} &\dot{x}_i(t) = v_i(t) \\ &\dot{v}_i(t) = u_i(t), i = 1, 2, \cdots, n \end{aligned} \tag{1}$$

where $x_i(t)$, $v_i(t)$, $u_i(t) \in \mathbb{C}$ represent the position, velocity and the control input of the ith agent at time t, respectively.

The main objective is to design the distributed control protocol such that the rotating consensus problem with and without communication delay can be solved. We mainly consider the case with directed graph, therefore it is assumed that the directed graph at least has a spanning tree.

Assumption 1. *The graph is directed graph which has a spanning tree.*

For the directed graph satisfying Assumption 1, its laplacian matrix $\mathcal{L}$ has the following property.

Lemma 1 ([32]). *Denote all eigenvalues of $\mathcal{L}$ as $\mu_i = a_i + jb_i, (i = 1, 2, ..., n)$, where a_i and b_i represent the real and imaginary part of eigenvalue μ_i, respectively. Then, all eigenvalues have negative real part except $\mu_1 = 0$, and all eigenvalues μ_i can be arranged as $0 = \mu_1 < \mathrm{Re}(\mu_2) \leq ... \leq \mathrm{Re}(\mu_n)$.*

Furthermore, we give the definition of rotating consensus for double-integrator multi-agent systems as follows.

Definition 1 (Rotating consensus [26])**.** *Consider the double-integrator multi-agent systems (1) with the directed graph satisfying Assumption 1. Develop a distributed control protocol, such that for all agents $i, k \in \mathcal{V}$, the following conditions are satisfied,*

$$\begin{aligned} &\lim_{t\to\infty} [x_i(t) - x_k(t)] = 0 \\ &\lim_{t\to\infty} [v_i(t) - v_k(t)] = 0 \\ &\lim_{t\to\infty} [\dot{v}_i(t) - j\omega v_i(t)] = 0, \end{aligned} \tag{2}$$

where ω is the angular velocity of agent i, then, it is said that the rotating consensus is achieved. For simplicity of the following analysis, it is assumed that $\omega = 1$ and the general case with $\omega \neq 1$ can be discussed in a similar way.

3. Consensus analysis without communication delay

In this section, we firstly consider the case without communication delay. The distributed control protocol is designed as follows

$$\begin{aligned} u_i(t) = jv_i(t) - \sum_{k \in N_i} a_{ik}(x_i(t) - x_k(t)) \\ -\lambda \sum_{k \in N_i} a_{ik}(v_i(t) - v_k(t)), \end{aligned} \tag{3}$$

where $i \in 1, 2, \cdots, n$, $\lambda \in \mathbb{R}^+$ is the control parameter to be designed. Let $z(t) = [x^T(t), v^T(t)]^T$, where $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ and $v(t) = [v_1(t), v_2(t), ..., v_n(t)]^T$. The multi-agent system (1) with control protocol (3) can be rewritten as

$$\dot{z}(t) = \phi z(t), \tag{4}$$

where $\phi = \begin{bmatrix} 0 & I_n \\ -\mathcal{L} & jI_n - \lambda\mathcal{L} \end{bmatrix}$. It follows from (4) that

$$z(t) = e^{\phi t} z(0). \tag{5}$$

Thus, the spectral property of matrix ϕ plays an important role in convergence of the multi-agent systems. Then, to get the eigenvalues of matrix ϕ, we need to calculate the solutions of $\det(sI_{2n} - \phi) = 0$, which is the characteristic polynomial of matrix ϕ. Therefore, the eigenvalues of matrix ϕ are equal to the solutions of polynomials

$$s^2 + (\lambda\mu_i - j)s + \mu_i = 0, i \in \{1, 2, ..., n\}, \tag{6}$$

where μ_i are the eigenvalues of matrix $\mathcal{L}$. Then, one has the solutions of (6) as follows

$$\begin{aligned} s_{i1} &= \tfrac{-(\lambda\mu_i - j) - \sqrt{(\lambda\mu_i - j)^2 - 4\mu_i}}{2} \\ s_{i2} &= \tfrac{-(\lambda\mu_i - j) + \sqrt{(\lambda\mu_i - j)^2 - 4\mu_i}}{2}, \end{aligned} \tag{7}$$

where s_{i1}, s_{i2} are roots of matrix ϕ. Thus, for the special eigenvalue $\mu_1 = 0$ of matrix $\mathcal{L}$, one gets the corresponding two special eigenvalues $s_{11} = 0$ and $s_{12} = j$.

To reach the convergence of rotating consensus, one needs that the eigenvalues $s_{i1}, s_{i2}, i \in \{2, ..., n\}$ have negative real part, namely $\mathrm{Re}(s_{i1}), \mathrm{Re}(s_{i2}) < 0, i \in \{2, ..., n\}$. Therefore, we give the following lemma and its proof in advance.

Lemma 2. *Let* $\rho_\pm = \frac{j - \eta\mu \pm \sqrt{(j - \eta\mu)^2 - 4\mu}}{2}$, *where* $\mu = a + jb$, $a \in R^+, b \in \mathbb{R}$, $\eta \in \mathbb{R}^+$. *Then,* $\mathrm{Re}(\rho_\pm) < 0$ *if and only if*

$$\eta > \tfrac{ab + \sqrt{(ab)^2 + 4b^2(a^3 + ab^2)}}{2(a^3 + ab^2)}. \tag{8}$$

Proof: The real part $\mathrm{Re}(j - \eta\mu) = -\eta a$ is negative with $\eta > 0, a > 0$. Then the property that $\mathrm{Re}(\rho_\pm)$ is negative implies that $\mathrm{Re}^2(j - \eta\mu) > \mathrm{Re}^2(\sqrt{(j - \eta\mu)^2 - 4\mu})$. Suppose

$$(\sqrt{(j - \eta\mu)^2 - 4\mu})^2 = (k_1 + jk_2)^2. \tag{9}$$

For the case $k_1 = 0$, $\mathrm{Re}(\rho_\pm) = \frac{-\eta a}{2} < 0$. For the case that $k_1 \neq 0$, by separating the real and imaginary parts of (9), one has

$$\begin{aligned} \eta^2 a^2 - (1 - \eta b)^2 - 4a = k_1^2 - k_2^2 \\ -\eta a(1 - \eta b) - 2b = k_1 k_2. \end{aligned} \tag{10}$$

Thus, from the below one of (10), one has

$$k_2 = \tfrac{-\eta a(1 - \eta b) - 2b}{k_1}. \tag{11}$$

Substituting (11) into upper one of (10), we get

$$\begin{aligned} k_1^4 - (\eta^2 a^2 - (1 - \eta b)^2 - 4a)k_1^2 - \\ (\eta a(1 - \eta b) + 2b)^2 = 0. \end{aligned} \tag{12}$$

By solving (12), we get

$$\begin{aligned} 2k_1^2 = (\eta^2 a^2 - (1 - \eta b)^2 - 4a) + \\ \sqrt{(\eta^2 a^2 - (1 - \eta b)^2 - 4a)^2 + 4(\eta a(1 - \eta b) + 2b)^2}. \end{aligned} \tag{13}$$

Thus, $\mathrm{Re}(\rho_\pm) < 0$ if and only if $2\mathrm{Re}^2(j - \eta\mu) - 2\mathrm{Re}^2(\sqrt{(j - \eta\mu)^2 - 4\mu}) = 2\eta^2 a^2 - 2k_1^2 > 0$, one has

$$\begin{aligned} (\eta^2 a^2 + (1 - \eta b)^2 + 4a) - \\ \sqrt{(\eta^2 a^2 - (1 - \eta b)^2 - 4a)^2 - 4(\eta a(1 - \eta b) + 2b)^2} > 0 \end{aligned}$$

Namely,

$$\begin{aligned} (\eta^2 a^2 + (1 - \eta b)^2 + 4a)^2 - \\ (\eta^2 a^2 - (1 - \eta b)^2 - 4a)^2 - 4(\eta a(1 - \eta b) + 2b)^2 > 0. \end{aligned}$$

With a series of calculations, one gets that $\eta > \frac{ab + \sqrt{a^2 b^2 + 4b^2(a^3 + ab^2)}}{2(a^3 + ab^2)}$. Then, we could conclude that $\rho_\pm$ have negative real part if and only if the parameter $\eta > \frac{ab + \sqrt{a^2 b^2 + 4b^2(a^3 + ab^2)}}{2(a^3 + ab^2)}$. ■

From the above analysis, we could get the following theorem.

Theorem 1. *Consider the double-integrator multi-agent systems* (1) *with a directed graph. Under Assumption 1, the rotating consensus problem is solved by control protocol* (3) *if and only if* $\lambda > \max_{i \in \{2, \cdots, n\}} \{\frac{a_i b_i + \sqrt{a_i^2 b_i^2 + 4b_i^2(a_i^3 + a_i b_i^2)}}{2(a_i^3 + a_i b_i^2)}\}$, *where* $\mu_i = a_i + jb_i$ *is the ith eigenvalue of matrix* $\mathcal{L}$. *Specially,* $x_i(t) = q^T x(0) + j(1 - e^{jt})q^T v(0)$, $v_i(t) = e^{jt} q^T v(0)$, *where* $q^T \mathcal{L} = 0, q^T \mathbf{1}_n = 1$.

Proof: By solving the dynamic equation $\dot{z}(t) = \phi z(t)$, one gets $z(t) = e^{\phi t} z(0)$. With the fact that the matrix ϕ has two eigenvalues 0 and j, and other eigenvalues are denoted as $s_{i1}, s_{i2}, i \in \{2, ..., n\}$, the matrix ϕ can be rewritten as following Jordan canonical form

$$\begin{aligned} \phi &= PJP^{-1} \\ &= \begin{bmatrix} p_1 & p_2 & \cdots & p_{2n} \end{bmatrix} \begin{bmatrix} 0 & 0 & \mathbf{0} \\ 0 & j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J' \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{2n} \end{bmatrix} \end{aligned}$$

where $J' \in \mathbb{C}^{(2n-2)\times(2n-2)}$ is the Jordan upper diagonal block matrix corresponding to eigenvalues $s_{i1}, s_{i2}, i \in$

$\{2,...,n\}$ in (7). $p_i \in \mathbb{C}^{2n\times 1}$ is the right eigenvector or generalized right eigenvector of matrix ϕ, and $q_i \in \mathbb{C}^{1\times 2n}$ is the left eigenvector or generalized left eigenvector of matrix ϕ. Without loss of generality, we choose $p_1 = \begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix}$ and $p_2 = \begin{bmatrix} \mathbf{1}_n \\ j\mathbf{1}_n \end{bmatrix}$, which can be verified that p_1, p_2 are the right eigenvectors of ϕ associated with eigenvalues 0 and j, respectively. We choose $q_1 = \begin{bmatrix} q^T & jq^T \end{bmatrix}$ and $q_2 = \begin{bmatrix} \mathbf{0}_n^T & -jq^T \end{bmatrix}$, where $q \in \mathbb{R}^{n\times 1}, q^T\mathcal{L} = 0, q^T\mathbf{1}_n = 1$. Then, it can be verified that q_1, q_2 are the left eigenvectors of ϕ associated with eigenvalues 0 and j, respectively. Besides, $q_1p_1 = 1, q_2p_2 = 1, q_1p_2 = 0, q_2p_1 = 0$ are also hold. Then we can get

$$
\begin{aligned}
e^{\phi t} &= Pe^{Jt}P^{-1} \\
&= \begin{bmatrix} p_1 & p_2 & \cdots & p_{2n} \end{bmatrix} \begin{bmatrix} 1 & 0 & \mathbf{0} \\ 0 & e^{jt} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & e^{J't} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{2n} \end{bmatrix}.
\end{aligned}
$$

By using Lemma 2, for the eigenvalues s_{i1}, s_{i2}, $i \in \{2...,n\}$ of matrix ϕ, one can get that all eigenvalues s_{i1}, s_{i2} have negative real parts except two eigenvalues 0 and j if and only if $\lambda > \max\limits_{i\in\{2,\cdots,n\}}\{\frac{a_ib_i+\sqrt{a_i^2b_i^2+4b_i^2(a_i^3+a_ib_i^2)}}{2(a_i^3+a_ib_i^2)}\}$, which means that all diagonal elements of matrix J' have negative real part. Thus, as time tend to infinity, one has $\lim\limits_{t\to\infty} e^{J't} \to 0_{(2n-2)\times(2n-2)}$. Then, we could get that

$$
\begin{aligned}
\lim_{t\to\infty} z(t) &= e^{\phi t}z(0) \\
&= Pe^{Jt}P^{-1}z(0) \\
&\to \begin{bmatrix} \mathbf{1}_np^T & j(1-e^{jt})\mathbf{1}_np^T \\ 0_{n\times n} & e^{jt}\mathbf{1}_np^T \end{bmatrix} z(0).
\end{aligned} \tag{14}
$$

By using (14), we can get the position and velocity of agent i as follows:

$$
\begin{aligned}
&\lim_{t\to\infty} x_i(t) \to p^Tx(0) + j(1-e^{jt})p^Tv(0) \\
&\lim_{t\to\infty} v_i(t) \to e^{jt}p^Tv(0).
\end{aligned} \tag{15}
$$

Furthermore, the condition $\dot{v}_i(t) - jv_i(t) = je^{jt}p^Tv(0) - je^{jt}p^Tv(0) = 0$ is satisfied. Therefore, with control protocol (3), the rotating consensus is achieved as defined in Definition 1. ■

If the graph $\mathcal{G}$ is an undirected graph, one can get that the corresponding Laplacian matrix $\mathcal{L}$ is a symmetric matrix and its eigenvalues are real. For the undirected graph, it is assumed that this graph is connected. Namely, every agent could get information from other agents directly or indirectly. It is straightforward to verify that the eigenvalues s_{i1}, s_{i2} of matrix ϕ have negative real parts if and only if $\lambda > 0$, which means that the conditions of Theorem 1 are satisfied. As a result, we have the following corollary.

Corollary 1. *Consider multi-agent system* (1) *with connected undirected graph, the rotating consensus is achieved with control protocol* (3) *if and only if control parameter* $\lambda > 0$. *Specially,* $\lim\limits_{t\to\infty} x_i(t) = \frac{1}{n}\sum_{i=1}^n x_i(0) + \frac{1}{n}j(1 - e^{jt})\sum_{i=1}^n v_i(0)$, $\lim\limits_{t\to\infty} v_i(t) = \frac{1}{n}e^{jt}\sum_{i=1}^n v_i(0)$, $\dot{v}_i(t) - jv_i(t) = 0$.

Remark 1. *It is noted that the result of Corollary 1 is consistent with the result in paper [26] for the case with undirected graph. Thus, the results for the directed graph in this paper are the extension for the paper [26]. Besides, from (14), we could get the corresponding position and velocity of all agents by using the Euler formula e^{jt} in (15). Different from the representation in [26, 28], where the disagreement systems are mainly used to prove the convergence of the rotating consensus, we use the specific equations to obtain the dynamic change of position and speed with changing on time, which is clearer and more specific.*

4. Consensus analysis with communication delay

In this section, we further consider the case with communication delay. In this part, it is assumed that all delays are equal, namely $\tau_{ik} = \tau, (k,i) \in \mathcal{E}$. To reach rotating consensus, we design the following control protocol

$$
\begin{aligned}
u_i(t) = jv_i(t) &- \sum_{k\in N_i} a_{ik}(x_i(t-\tau) - x_k(t-\tau)) \\
&- \lambda \sum_{k\in N_i} a_{ik}(v_i(t-\tau) - v_k(t-\tau)),
\end{aligned} \tag{16}
$$

where $i \in 1, 2, \cdots, n$, $\lambda \in \mathbb{R}^+$, $\tau \in \mathbb{R}^+$ are the control parameter and uniform communication delay, respectively.

Define $\hat{x}_i(t) = x_i(t) - x_1(t), \hat{v}_i(t) = v_i(t) - v_1(t), i = 2, 3, \cdots, n$ and the state error vector $\hat{z}(t) = [\hat{x}^T(t), \hat{v}^T]^T$, where $\hat{x}(t) = [\hat{x}_2(t), ..., x_n(t)]^T, \hat{v}(t) = [\hat{v}_2(t), ..., v_n(t)]^T$. With control protocol (16), one has the disagreement systems as $\dot{\hat{z}}(t) = \varphi_1\hat{z}(t) + \varphi_2\hat{z}(t-\tau)$, where $\varphi_1 = \begin{bmatrix} 0 & I_{n-1} \\ 0 & jI_{n-1} \end{bmatrix}$, $\varphi_2 = \begin{bmatrix} 0 & 0 \\ -\hat{\mathcal{L}} & -\lambda\hat{\mathcal{L}} \end{bmatrix}$ and $\hat{\mathcal{L}} = \mathcal{L}_{22} + \mathbf{1}_{(n-1)}\alpha^T$, where

$$
\mathcal{L}_{22} = \begin{bmatrix} d_2 & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & d_3 & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n2} & -a_{n3} & \cdots & d_n \end{bmatrix}, \alpha = \begin{bmatrix} a_{12} \\ a_{13} \\ \vdots \\ a_{1n} \end{bmatrix}.
$$

From [20], the eigenvalues of $\hat{\mathcal{L}}$ are $\mu_2, \mu_3, \ldots, \mu_n$. Thus, for the characteristic equation $\det(sI_{2n-2} - \varphi_1 - \varphi_2e^{-\tau s}) = 0$, one has

$$
\begin{aligned}
&\det(sI_{2n-2} - \varphi_1 - \varphi_2e^{-\tau s}) \\
&= \det\left(sI_{2n-2} - \begin{bmatrix} 0 & I_{n-1} \\ 0 & jI_{n-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -\hat{\mathcal{L}} & -\lambda\hat{\mathcal{L}} \end{bmatrix} e^{-\tau s}\right) \\
&= \det\left(\begin{bmatrix} sI_{n-1} & -I_{n-1} \\ e^{-\tau s}\hat{\mathcal{L}} & sI_{n-1} - jI_{n-1} + \lambda e^{-\tau s}\hat{\mathcal{L}} \end{bmatrix}\right) \\
&= \det(sI_{n-1}(sI_{n-1} - jI_{n-1} + \lambda e^{-\tau s}\hat{\mathcal{L}}) + e^{-\tau s}\hat{\mathcal{L}}) \\
&= \det((s^2 - js)I_{n-1} + e^{-\tau s}(\lambda s + 1)\hat{\mathcal{L}}) \\
&= \textstyle\prod_{i=2}^n((s^2 - js) + e^{-\tau s}(\lambda s + 1)\mu_i) \\
&= \textstyle\prod_{i=2}^n g_i(s) \\
&= 0,
\end{aligned}
$$

where

$$g_i(s) = (s^2 - js) + e^{-\tau s}(\lambda s + 1)\mu_i. \tag{17}$$

To further obtain the upper bound of communication delay, we need the following lemmas.

Lemma 3 ([33])**.** *From Corollary 2.4 of [33], We can know that for an exponential polynomial like $h(s, e^{-\tau s}) = q(s)e^{-\tau s} + p(s)$, where $q(s) = b_1 s^{n-1} + b_2 s^{n-2} + ... + b_n, p(s) = s^n + a_1 s^{n-1} + ... + a_n$, if $h(s, e^{-\tau s})$ is stable when $\tau = 0$ and $h(s, e^{-\tau s})$ is unstable if there exists some $\tau > 0$, then there must exit some τ^* such that $h(s, e^{-\tau s})$ is stable when $\tau < \tau^*$ and $h(s, e^{-\tau s})$ is unstable when $\tau > \tau^*$, where $h(s, e^{-\tau^* s}) = 0, s = jw$.*

Lemma 4. *There are only two real roots of equation*

$$\omega^4 - 2\omega^3 + (1 - |\mu|^2\lambda^2)\omega^2 - |\mu|^2 = 0, \tag{18}$$

where $\lambda \in \mathbb{R}^+$, $\mathrm{Re}(\mu) < 0$. And the solutions of (18) *are*

$$\begin{aligned} 2\omega_1 &= -(-1 + \sqrt{(|\mu|^2\lambda^2 + y)}) \\ &\quad + \sqrt{(-1 + \sqrt{(|\mu|^2\lambda^2 + y)})^2 - 4(\tfrac{1}{2}y + \sqrt{(\tfrac{1}{4}y^2 + |\mu|^2)})}, \\ 2\omega_2 &= -(-1 + \sqrt{(|\mu|^2\lambda^2 + y)}) \\ &\quad - \sqrt{(-1 + \sqrt{(|\mu|^2\lambda^2 + y)})^2 - 4(\tfrac{1}{2}y + \sqrt{(\tfrac{1}{4}y^2 + |\mu|^2)})}, \end{aligned}$$

where

$$\begin{aligned} & y = z + \tfrac{1}{3}(1 - |\mu|^2\lambda^2) \\ & z = \begin{cases} z_1 = u + v & (\text{if} \quad \mathrm{Im}(z_1) = 0) \\ z_2 = \eta u + \eta^2 v & (\text{if} \quad \mathrm{Im}(z_2) = 0) \\ z_3 = \eta^2 u + \eta v & (\text{if} \quad \mathrm{Im}(z_3) = 0) \end{cases} \\ & \begin{cases} \eta = \frac{-1 + j\sqrt{3}}{2} \\ \delta = (\frac{q}{2})^2 + (\frac{p}{3})^3 \\ u = (-\frac{q}{2} + \sqrt{\delta})^{\frac{1}{3}} \\ v = -\frac{p}{3u} \end{cases} \\ & \begin{cases} p = 4|\mu|^2 - \frac{1}{3}(1 - |\mu|^2\lambda^2)^2 \\ q = 4|\mu|^4\lambda^2 \end{cases} \end{aligned}$$

Proof: The proof is given in Appendix. ■

With above analysis, we get the following theorem.

Theorem 2. *Consider the multi-agent system* (1) *with directed graph. Under Assumption 1, the rotating consensus is achieved with control protocol* (16) *if and only if the control parameter $\lambda > \max_{i\in\{2,\cdots,n\}}\{\frac{a_i b_i + \sqrt{a_i^2 b_i^2 + 4b_i^2(a_i^3 + a_i b_i^2)}}{2(a_i^3 + a_i b_i^2)}\}$ and the communication delay $\tau < \min_{i\in\{2,\cdots,n\}}\{\frac{arctan(\frac{a_i\lambda\omega_i + b_i}{a_i - b_i\lambda\omega_i})}{\omega_i}\}$, where ω_i is the solution of $\omega_i^4 - 2\omega_i^3 + (1 - |\mu_i|^2\lambda^2)\omega_i^2 - |\mu_i|^2 = 0$ and can be obtained by using Lemma 4.*

Proof: Firstly, for the case with $\tau = 0$, we get that the characteristic polynomial in (17) is degenerated to $g_i(s) = (s^2 - js) + (\lambda s + 1)\mu_i$ for the non-zero $\mu_i \neq 0$, which is same with the results of (6) with μ_i, $s^2 + (\lambda\mu_i - j)s + \mu_i = 0$ for the non-zero μ_i. It is straightforward to get that the multi-agent system (1) could achieve the rotating consensus for the case that $\tau = 0$. Moreover, for the case that communication delay is infinite $\tau = \infty$, the multi-agent system can be seen as n independent systems. Namely, any agent can not obtain information from its neighbors. Thus, in this case, every agent move independently, which makes the consensus could not be achieved as we designed. Therefore, base on Lemma 3, it is known that there exists a maximum time delay τ_{max} such that the system (1) achieve rotating consensus if $\tau < \tau_{max}$ while the system (1) is unstable if $\tau > \tau_{max}$. Thus, to obtain the maximum communication delay τ_{max}, based on Lemma 3, we need to calculate the corresponding delay τ_i with $g_i(s) = 0, i = 2, \cdots, n$ when $s = j\omega_i$. Then, by separating the real and imaginary parts of $g_i(j\omega_i) = 0$, one has

$$\begin{aligned} & -\omega_i^2 + \omega_i + \\ & \cos(\tau_i\omega_i)(a_i - b_i\lambda\omega_i) + \sin(\tau_i\omega_i)(a_i\lambda\omega_i + b_i) = 0 \\ & \cos(\tau_i\omega_i)(a_i\lambda\omega_i + b_i) - \sin(\tau_i\omega_i)(a_i - b_i\lambda\omega_i) = 0. \end{aligned} \tag{19}$$

Then, one has

$$\begin{aligned} \cos(\tau_i\omega_i) &= \tfrac{(\omega_i^2 - \omega_i)(a_i - b_i\lambda\omega_i)}{(a_i\lambda\omega_i + b_i)^2 + (a_i - b_i\lambda\omega_i)^2} \\ \sin(\tau_i\omega_i) &= \tfrac{(\omega_i^2 - \omega_i)(a_i\lambda\omega_i + b_i)}{(a_i\lambda\omega_i + b_i)^2 + (a_i - b_i\lambda\omega_i)^2}. \end{aligned} \tag{20}$$

With $\sin^2(\tau_i\omega_i) + \cos^2(\tau_i\omega_i) = 1$, one has

$$\omega_i^4 - 2\omega_i^3 + (1 - |\mu_i|^2\lambda^2)\omega_i^2 - |\mu_i|^2 = 0. \tag{21}$$

where $|\mu_i| = a_i^2 + b_i^2$. With $\tan(\tau_i\omega_i) = \frac{\sin(\tau_i\omega_i)}{\cos(\tau_i\omega_i)}$, we can obtain that communication delay $\tau_i = \frac{\arctan(\frac{a_i\lambda\omega_i + b_i}{a_i - b_i\lambda\omega_i})}{\omega_i}$ for any corresponding eigenvalue μ_i, where the solution ω_i of (21) can be solved by using Lemma 4. Then, it is concluded that the system is stable with communication delay $\tau < \tau_{max} = \min_{i\in\{2,\cdots,n\}}\{\frac{\arctan(\frac{a_i\lambda\omega_i + b_i}{a_i - b_i\lambda\omega_i})}{\omega_i}\}$. ■

Similarly, for the case that the communication graph is undirected, the corresponding non-zero eigenvalues μ_i of matrix $\mathcal{L}$ is real, namely $b_i = 0$. Thus, when there exists uniform communication delay τ, the corresponding maximum delay is simplified as $\tau_{max} = \min_{i\in\{2,\cdots,n\}}\{\frac{\arctan(\lambda\omega_i)}{\omega_i}\}$. Based on Theorem 2, we can obtain the following corollary.

Corollary 2. *Consider the multi-agent system* (1) *with a connected undirected graph, the rotating consensus is achieved with control protocol* (16) *if and only if control parameter $\lambda > 0$ and $\tau < \min_{i\in\{2,\cdots,n\}}\{\frac{\arctan(\lambda\omega_i)}{\omega_i}\}$, where ω_i is the solution of $\omega_i^4 - 2\omega_i^3 + (1 - |\mu_i|^2\lambda^2)\omega_i^2 - |\mu_i|^2 = 0$, which can be solved by using Lemma 4.*

Remark 2. *In this paper, the rotating consensus problem with uniform communication delay is solved by the frequency domain analysis method. However, the nonuniform communication delay problem is more common in the real scenarios. For the rotating consensus problem that the communication delay is nonuniform, it could not be directly solved by current method. It is worthy of investigating the rotating consensus problem with nonuniform communication delay to adapt to more general case in the future.*

5. Simulation examples

In this section, we will give several numerical simulation examples to illustrate the correctness of the obtained theorem results.

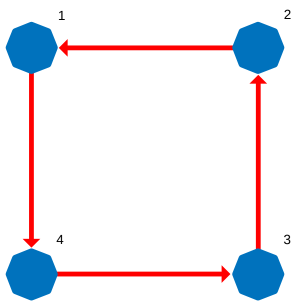

Fig. 1 Directed graph $\mathcal{G}$.

In this part, the given directed communication graph is shown in Fig. 1, which is consisted of 4 agents represented with nodes $1, 2, 3, 4$. The red line represent the information exchange between agents. With above connection shown in Fig. 1, the corresponding adjacency matrix and Laplacian matrix are

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

respectively. Thus, the eigenvalues of matrix $\mathcal{L}$ are $\mu_1 = 0, \mu_2 = 1 + j$, $\mu_3 = 1 - j$, $\mu_4 = 2$. Firstly, we consider the case without communication delay. By using Theorem 1, one gets that if $\lambda > 1$, the rotating consensus could be achieved with proposed control protocol. Thus, the parameters are chosen as $\lambda = 1.5, \lambda = 0.5$. For the case that $\lambda = 1.5 > 1$, the simulation results are given in Fig. 2, respectively.

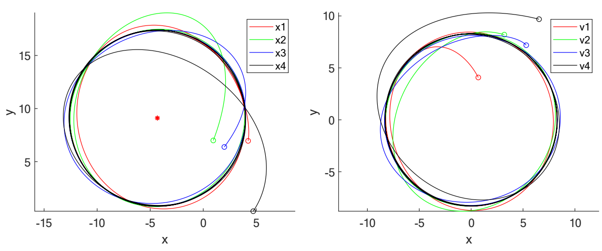

Fig. 2 All agents' positions (left) and velocities (right) when $\lambda = 1.5, \tau = 0.0$.

It is shown that as time changes, the position and velocity of all agents tend to reach consensus. And finally, all agents reach consensus on position and velocity, and they all move around a common center point shown by red point in above left figure, in which the rotating consensus is achieved as expected.

Furthermore, for the case that $\lambda = 0.5 < 1$, the corresponding simulation results are given in Fig. 3.

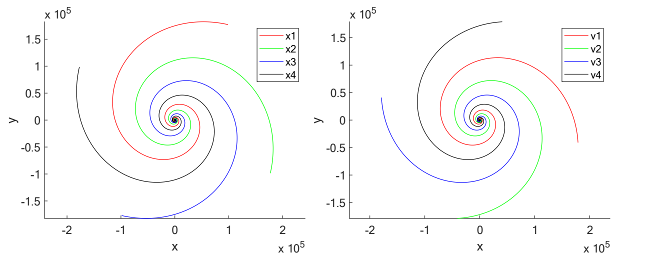

Fig. 3 All agents' positions (left) and velocities (right) when $\lambda = 0.5, \tau = 0.0$.

It is shown that the position and velocity of all agents will diverge to the infinity as time tend to infinity. The given simulation result shows that the rotating consensus could not be obtained, which is consistent with our theoretical analysis. Thus, we could get that the system reaches rotating consensus when $\lambda > 1$ and tends to divergence when $\lambda \leq 1$, which is consistent with Theorem 1.

When we further consider the case with communication delay, we choose control parameter as $\lambda = 2.0$. By a series of calculations, we get that $\omega_{21} = -1.9225$, $\omega_{22} = 3.8522$; $\omega_{31} = -1.9225$, $\omega_{32} = 3.8522$; $\omega_{41} = -3.0533$, $\omega_{42} = 5.0198$ corresponding the eigenvalues μ_2, μ_3 and μ_4. Therefore, all corresponding time delay are $\tau_{21} = 0.2762$, $\tau_{22} = 0.5781$; $\tau_{31} = 1.0932$, $\tau_{32} = 0.1704$; $\tau_{41} = 0.4613$, $\tau_{42} = 0.2931$. Thus, by Theorem 2, the upper bound is $\tau_{max} = 0.1704$. Thus, we choose the communication delay as $\tau = 0.19$ and $\tau = 0.15$. For the case that communication delay $\tau = 0.15 < \tau_{max}$, the simulation results are shown in Fig. 4.

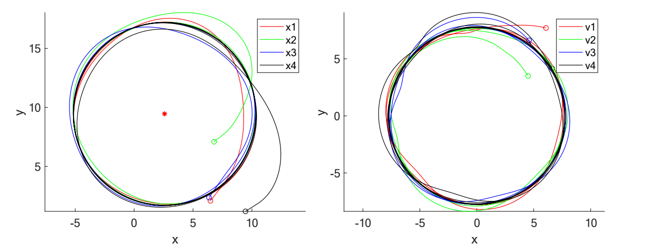

Fig. 4 All agents' positions (left) and velocities (right) when $\lambda = 2.0, \tau = 0.15$.

As shown in the above simulation results, it is obvious that the position and velocity of all agents reach consensus, which is consistent with the theoretical analysis.

For the case that communication delay $\tau = 0.19 > \tau_{max}$, the simulation results are shown in Fig. 5.

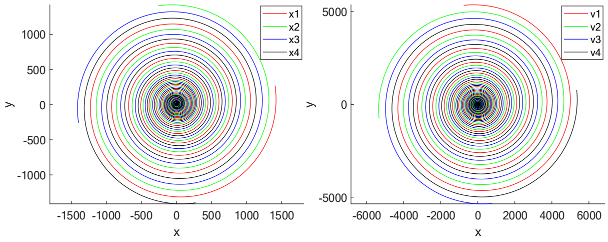

Fig. 5 All agents' positions (left) and velocities (right) when $\lambda = 2.0, \tau = 0.19$.

As shown in above figures, the system tends to divergence when $\tau \geq \tau_{max} = 0.1704$, which is consistent with Theorem 2.

Thus, we could get that the system reaches rotating consensus when $\tau \leq \tau_{max}$ and tends to divergence when $\tau \geq \tau_{max}$, which is consistent with Theorem 2.

6. Conclusion

In this paper, the rotating consensus problem for the double-integrator multi-agent systems is studied with taking communication delay into consideration for the directed graph. With the distributed rotating consensus control protocol, the accurate lower bound of control parameter is obtained. Furthermore, with considering uniform communication delay, the upper bound of the communication delay is derived out with the help of frequency domain analysis method. In the future, we would further consider the more general case that the communication delay is nonuniform and time-varying.

Appendix

The proof of Lemma 4:

Proof: For convenience, we define $b_1 = -2$, $c_1 = 1 - |\mu|^2\lambda^2$, $e_1 = -|\mu|^2$. Then, equation (18) can be rewritten as

$$\omega^4 + b_1\omega^3 + c_1\omega^2 + e_1 = 0. \tag{22}$$

Define a function $\alpha(\omega) = \omega^4 + b_1\omega^3 + c_1\omega^2 + e_1$, the derivative of $\alpha(\omega)$ is

$$\begin{aligned}\dot{\alpha}(\omega) &= 4\omega^3 + 3b_1\omega^2 + 2c_1\omega \\ &= \omega(4\omega^2 + 3b_1\omega + 2c_1).\end{aligned} \tag{23}$$

Define $\beta(\omega) = \dot{\alpha}(\omega)$. Due to the fact that $(3b_1)^2 - 32c_1 = 4 + 32|\mu|^2\lambda^2 > 0$, the curve example of (23) is similar with Fig. 6.

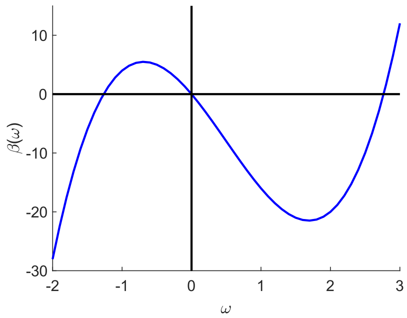

Fig. 6 Curve example of $\beta(\omega) = \omega(4\omega^2 - 6\omega - 14)$.

Thus, with the fact that $\alpha(0) = e_1 = -|\mu|^2 < 0$, the curve example of (22) is similar with Fig. 7.

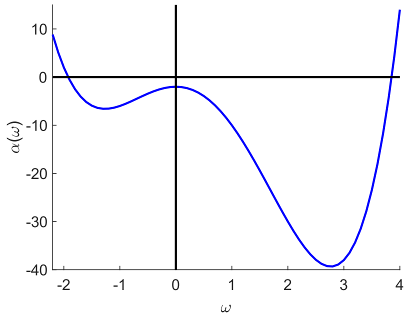

Fig. 7 Curve example of $\alpha(\omega) = \omega^4 - 2\omega^3 - 7\omega^2 - 2$.

Then, it is concluded that the function $\alpha(\omega)$ only has two real roots, where one is positive and another is negative. Rewrite equation (22) as follows

$$\omega^4 + b_1\omega^3 = -c_1\omega^2 - e_1. \tag{24}$$

Adding $(\frac{1}{2}b_1\omega)^2$ to both sides of (24), we have

$$(\omega^2 + \tfrac{1}{2}b_1\omega)^2 = (\tfrac{1}{4}b_1^2 - c_1)\omega^2 - e_1. \tag{25}$$

Adding $(\omega^2 + \frac{1}{2}b_1\omega)y + \frac{1}{4}y^2$ to both sides of (25), one has

$$\begin{aligned}&[(\omega^2 + \tfrac{1}{2}b_1\omega) + \tfrac{1}{2}y]^2 \\ &\quad = (\tfrac{1}{4}b_1^2 - c_1 + y)\omega^2 + \tfrac{1}{2}b_1\omega y + \tfrac{1}{4}y^2 - e_1.\end{aligned} \tag{26}$$

If $(\frac{1}{2}b_1 y)^2 - 4(\frac{1}{4}b_1^2 - c_1 + y)(\frac{1}{4}y^2 - e_1) = 0$, the following equation is held

$$\begin{aligned}&(\tfrac{1}{4}b_1^2 - c_{1i} + y)\omega^2 + \tfrac{1}{2}b_1\omega y + \tfrac{1}{4}y^2 - e_1 \\ &\quad = \left[\sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}\omega_i + \sqrt{(\tfrac{1}{4}y^2 - e_1)}\right]^2.\end{aligned} \tag{27}$$

Then, we solve the requirement of (27) as follows

$$\begin{aligned}&(\tfrac{1}{2}b_1 y)^2 - 4(\tfrac{1}{4}b_1^2 - c_1 + y)(\tfrac{1}{4}y^2 - e_1) \\ &= \tfrac{1}{4}b_1^2 y^2 - (\tfrac{1}{4}b_1^2 y^2 - c_1 y^2 + y^3) + (b_1^2 e_1 - 4c_1 e_1 + 4e_1 y) \\ &= -(y^3 - c_1 y^2 - 4e_1 y + 4c_1 e_1 - b_1^2 e_1) \\ &= -(y^3 + b_2 y^2 + c_2 y + d_2),\end{aligned}$$

where $b_2 = -c_1$, $c_2 = -4e_1$, $d_2 = 4c_1e_1 - b_1^2e_1$. Define $y = z - \frac{1}{3}b_2$. It is obvious that

$$y^3 + b_2y^2 + c_2y + d_2 = z^3 + pz + q,$$

where $p = c_2 - \frac{1}{3}b_2^2$, $q = d_2 + \frac{2}{27}b_2^3 - \frac{1}{3}b_2c_2$. By solving $z^3 + pz + q = 0$, we get three roots

$$\begin{aligned} z_1 &= u + v \\ z_2 &= \eta u + \eta^2 v \\ z_3 &= \eta^2 u + \eta v, \end{aligned}$$

where $\eta = \frac{-1+j\sqrt{3}}{2}$, $\delta = (\frac{q}{2})^2 + (\frac{p}{3})^3$, $u = (-\frac{q}{2} + \sqrt{\delta})^{\frac{1}{3}}$, $v = -\frac{p}{3u}$.

Due to the fact that the solutions of (22) only have two real roots, there only has one real root in z_1, z_2 and z_3. Find a real value in z_1, z_2, z_3, and define it as z. Then,

$$y = z - \tfrac{1}{3}b_2. \tag{28}$$

With (28), equation (26) can be rewritten as

$$\begin{aligned} &[(\omega^2 + \tfrac{1}{2}b_1\omega) + \tfrac{1}{2}y]^2 \\ &= \left[\sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}\omega + \sqrt{(\tfrac{1}{4}y^2 - e_1)}\right]^2. \end{aligned} \tag{29}$$

From (29), we can get following two equations

$$(\omega^2 + \tfrac{1}{2}b_1\omega) + \tfrac{1}{2}y = \sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}\omega + \sqrt{(\tfrac{1}{4}y^2 - e_1)}$$

$$(\omega^2 + \tfrac{1}{2}b_1\omega) + \tfrac{1}{2}y = -\sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}\omega - \sqrt{(\tfrac{1}{4}y^2 - e_1)}$$

Thus, one has

$$\omega^2 + b_3\omega + c_3 = 0 \tag{30}$$

$$\omega^2 + b_4\omega + c_4 = 0, \tag{31}$$

where $b_3 = \frac{1}{2}b_1 - \sqrt{(\frac{1}{4}b_1^2 - c_1 + y)}$, $c_3 = \frac{1}{2}y - \sqrt{(\frac{1}{4}y^2 - e_1)}$, $b_4 = \frac{1}{2}b_1 + \sqrt{(\frac{1}{4}b_1^2 - c_1 + y)}$, $c_4 = \frac{1}{2}y + \sqrt{(\frac{1}{4}y^2 - e_1)}$. Define $\delta_1 = b_3^2 - 4c_3$, $\delta_2 = b_4^2 - 4c_4$. Due to the fact that there only have two real roots for (23), we can get $\delta_1 \geq 0 > \delta_2$ if $\delta_1 > \delta_2$, and $\delta_2 \geq 0 > \delta_1$ if $\delta_2 > \delta_1$. Then, we get $\delta_2 - \delta_1$ as follows

$$\begin{aligned} &\delta_2 - \delta_1 = [b_3^2 - 4c_3] - [b_4^2 - 4c_4] \\ &= \left\{\left[\tfrac{1}{2}b_1 - \sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}\right]^2 - 4\left[\tfrac{1}{2}y - \sqrt{(\tfrac{1}{4}y^2 - e_1)}\right]\right\} \\ &\quad -\left\{\left[\tfrac{1}{2}b_1 + \sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}\right]^2 - 4\left[\tfrac{1}{2}y + \sqrt{(\tfrac{1}{4}y^2 - e_1)}\right]\right\} \\ &= 4\sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)} + 8\sqrt{(\tfrac{1}{4}y^2 - e_1)} > 0. \end{aligned}$$

With $\delta_2 - \delta_1 > 0$, we have $\delta_2 \geq 0 > \delta_1$. Then, the equation (31) is solvable. And the final solutions are $\omega_1 = \frac{-b_4+\sqrt{\delta_2}}{2}$, $\omega_2 = \frac{-b_4-\sqrt{\delta_2}}{2}$. Then we get one root

$$\begin{aligned} &2\omega_1 = -b_4 + \sqrt{\delta_2} \\ &= -(\tfrac{1}{2}b_1 + \sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}) \\ &\quad +\sqrt{(\tfrac{1}{2}b_1 + \sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)})^2 - 4(\tfrac{1}{2}y + \sqrt{(\tfrac{1}{4}y^2 - e_1)})} \\ &= -(-1 + \sqrt{(|\mu|^2\lambda^2 + y)}) \\ &\quad +\sqrt{(-1 + \sqrt{(|\mu|^2\lambda^2 + y)})^2 - 4(\tfrac{1}{2}y + \sqrt{(\tfrac{1}{4}y^2 + |\mu|^2)})} \end{aligned}$$

where y can be calculated by solving (28). And another is

$$\begin{aligned} &2\omega_2 = -b_4 - \sqrt{\delta_2} \\ &= -(\tfrac{1}{2}b_1 + \sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)}) \\ &\quad -\sqrt{(\tfrac{1}{2}b_1 + \sqrt{(\tfrac{1}{4}b_1^2 - c_1 + y)})^2 - 4(\tfrac{1}{2}y + \sqrt{(\tfrac{1}{4}y^2 - e_1)})} \\ &= -(-1 + \sqrt{(|\mu|^2\lambda^2 + y)}) \\ &\quad -\sqrt{(-1 + \sqrt{(|\mu|^2\lambda^2 + y)})^2 - 4(\tfrac{1}{2}y + \sqrt{(\tfrac{1}{4}y^2 + |\mu|^2)})} \end{aligned}$$

where y can be calculated by solving (28). ■

Acknowledgment

This project is financially supported by National Natural Science Foundation of China (61988101).

References

[1] R. Abdulghafor, S. Shah Abdullah, S. Turaev, M. Othman, An overview of the consensus problem in the control of multi-agent systems, Automatika: časopis za automatiku, mjerenje, elektroniku, računarstvo i komunikacije 59 (2) (2018) 143–157.

[2] Z. Peng, G. Wen, A. Rahmani, Y. Yu, Distributed consensus-based formation control for multiple nonholonomic mobile robots with a specified reference trajectory, International Journal of Systems Science 46 (8) (2015) 1447–1457.

[3] A. A. Najm, I. K. Ibraheem, A. T. Azar, A. J. Humaidi, Genetic optimization-based consensus control of multi-agent 6-DoF UAV system, Sensors 20 (12) (2020) 3576.

[4] H. Cai, J. Huang, The leader-following attitude control of multiple rigid spacecraft systems, Automatica 50 (4) (2014) 1109–1115.

[5] S. Bosse, A. Lechleiter, A hybrid approach for structural monitoring with self-organizing multi-agent systems and inverse numerical methods in material-embedded sensor networks, Mechatronics 34 (2016) 12–37.

[6] W. Ren, R. W. Beard, E. M. Atkins, A survey of consensus problems in multi-agent coordination, in: Proceedings of the 2005, American Control Conference, 2005., IEEE, 2005, pp. 1859–1864.

[7] X. Dong, Y. Zhou, Z. Ren, Y. Zhong, Time-varying formation tracking for second-order multi-agent systems subjected to switching topologies with application to quadrotor formation flying, IEEE Transactions on Industrial Electronics 64 (6) (2016) 5014–5024.

[8] R. Olfati-Saber, Flocking for multi-agent dynamic systems: Algorithms and theory, IEEE Transactions on Automatic Control 51 (3) (2006) 401–420.

[9] Q. Zhang, Z. Gong, Z. Yang, Z. Chen, Distributed convex optimization for flocking of nonlinear multi-agent systems, International Journal of Control, Automation and Systems 17 (5) (2019) 1177–1183.

[10] D. Panagou, D. M. Stipanović, P. G. Voulgaris, Distributed coordination control for multi-robot networks using lyapunov-like barrier functions, IEEE Transactions on Automatic Control 61 (3) (2015) 617–632.

[11] Y. Cao, W. Yu, W. Ren, G. Chen, An overview of recent progress in the study of distributed multi-agent coordination, IEEE Transactions on Industrial informatics 9 (1) (2012) 427–438.

[12] W. Ren, E. Atkins, Distributed multi-vehicle coordinated control via local information exchange, International Journal of Robust and Nonlinear Control 17 (10-11) (2007) 1002–1033.

[13] H. Zhang, F. L. Lewis, Z. Qu, Lyapunov, adaptive, and optimal design techniques for cooperative systems on directed communication graphs, IEEE Transactions on Industrial Electronics 59 (7) (2011) 3026–3041.

[14] C. Deng, W. Che, Z. Wu, A dynamic periodic event-triggered approach to consensus of heterogeneous linear multiagent systems with time-varying communication delays, IEEE Transactions on Cybernetics (2020).

[15] Q. Wang, H. Gao, F. Alsaadi, T. Hayat, An overview of consensus problems in constrained multi-agent coordination, Systems Science & Control Engineering: An Open Access Journal 2 (1) (2014) 275–284.

[16] Z. H. Ismail, N. Sariff, A survey and analysis of cooperative multi-agent robot systems: Challenges and directions, in: Applications of Mobile Robots, IntechOpen, 2018, pp. 8–14.

[17] J. Ni, L. Liu, C. Liu, J. Liu, Fixed-time leader-following consensus for second-order multiagent systems with input delay, IEEE Transactions on Industrial Electronics 64 (11) (2017) 8635–8646.

[18] Y. Li, H. Li, S. Wang, Finite-time consensus of finite field networks with stochastic time delays, IEEE Transactions on Circuits and Systems II: Express Briefs (2020).

[19] Q. Ma, S. Xu, Consensus switching of second-order multiagent systems with time delay, IEEE Transactions on Cybernetics (2020).

[20] W. Hou, M. Fu, H. Zhang, Z. Wu, Consensus conditions for general second-order multi-agent systems with communication delay, Automatica 75 (2017) 293–298.

[21] A. Petrillo, A. Salvi, S. Santini, A. S. Valente, Adaptive synchronization of linear multi-agent systems with time-varying multiple delays, Journal of the Franklin Institute 354 (18) (2017) 8586–8605.

[22] P. Lin, M. Dai, Y. Song, Consensus stability of a class of second-order multi-agent systems with nonuniform time-delays, Journal of the Franklin Institute 351 (3) (2014) 1571–1576.

[23] V. Kapila, A. G. Sparks, J. M. Buffington, Q. Yan, Spacecraft formation flying: Dynamics and control, Journal of Guidance, Control, and Dynamics 23 (3) (2000) 561–564.

[24] L. S. Breger, J. P. How, Safe trajectories for autonomous rendezvous of spacecraft, Journal of Guidance, Control, and Dynamics 31 (5) (2008) 1478–1489.

[25] J. G. Manathara, D. Ghose, Rendezvous of multiple uavs with collision avoidance using consensus, Journal of Aerospace Engineering 25 (4) (2011) 480–489.

[26] P. Lin, Y. Jia, Distributed rotating formation control of multi-agent systems, Systems & Control Letters 59 (10) (2010) 587–595.

[27] L.-p. Mo, S. Guo, Y. Yu, Mean-square composite-rotating consensus of second-order systems with communication noises, Chinese Physics B 27 (7) (2018) 070504.

[28] Y. Li, Y. Huang, P. Lin, W. Ren, Distributed rotating consensus of second-order multi-agent systems with nonuniform delays, Systems & Control Letters 117 (2018) 18–22.

[29] P. Li, K. Qin, M. Shi, Distributed robust H_∞ rotating consensus control for directed networks of second-order agents with mixed uncertainties and time-delay, Neurocomputing 148 (2015) 332–339.

[30] Y. Huang, Y. Li, W. Hu, Distributed rotating formation control of second-order leader-following multi-agent systems with nonuniform delays, Journal of the Franklin Institute 356 (5) (2019) 3090–3101.

[31] R. Ding, W. Hu, Y. Yang, Rotating consensus control of double-integrator multi-agent systems with event-based communication, Science China Information Sciences 63 (2020) 1–10.

[32] R. Olfati-Saber, R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Transactions on Automatic Control 49 (9) (2004) 1520–1533.

[33] S. Ruan, J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, Dynamics of Continuous Discrete and Impulsive Systems Series A 10 (2003) 863–874.