Computational Electromagnetics

FDTD (Finite Difference Time Domain)

Maxwell's Equations :-

Effect = Cause

① $\nabla \cdot \vec{D}(t) = \rho_v(t)$ — Gauss' Law

② $\nabla \cdot \vec{B}(t) = 0$ — No Magnetic Charge

③ $\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$ — Faraday's Law

④ $\nabla \times \vec{H}(t) = \vec{J}(t) + \frac{\partial \vec{D}(t)}{\partial t}$ — Ampere's Circuital Law

Current Continuity :-

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

Constitutive Relations :-

$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$ (Electric Response)

$\vec{B}(t) = \underbrace{[\mu(t)]}_{\text{tensor}} * \vec{H}(t)$ (Magnetic Response)

D : Electric flux density
B : Magnetic flux density
E : Electric field intensity
H : Magnetic field intensity

Lorentz Force Law :-

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$

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① **Linear, isotropic, non-dispersive materials :-**

$D(t) = \varepsilon \vec{E}(t)$ (linear relation)

② **Dispersive materials :-**

$\vec{D}(t) = \varepsilon(t) * \vec{E}(t)$ (convolution)

③ **Anisotropic materials :-**

$D(t) = [\varepsilon] \vec{E}(t)$ (tensor)

④ **Non-linear materials :-**

$$D(t) = \varepsilon_0 \chi_e^{(1)} E(t) + \varepsilon_0 \chi_e^{(2)} E^2(t) + \cdots$$

Material refractive index :

$$n = \sqrt{\mu_r \varepsilon_r}$$

Material impedance :

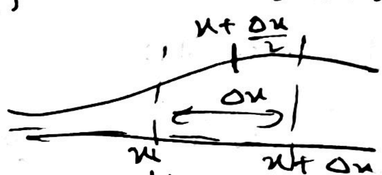
$$\eta \simeq \frac{|\vec{E}|}{|\vec{H}|}$$

$$\eta = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Finite Difference Approximations :-

$$\boxed{\frac{df_{1.5}}{dx} \approx \frac{f_2 - f_1}{\Delta x}}$$

(first order derivative)



* Each term in a finite-difference equation must exist at the same point in time and space.

$$\frac{\partial f(x)}{\partial x} + f(x) = 0$$

$$\rightarrow \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{exists at } x + \frac{\Delta x}{2}} + \underbrace{f(x)}_{\text{exists at } x} = 0$$

↳ Simulation Unstable

Solution :-

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} + f\left(x + \frac{\Delta x}{2}\right) = 0$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{f(x+\Delta x) + f(x)}{2} = 0$$

(Averaging)
(Interpolation)

Approximation of Time Derivatives in Maxwell's eqns:-

$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t}$$

$$\Rightarrow \nabla \times \vec{E}(t) = -\mu \frac{\vec{H}(t+\Delta t) - \vec{H}(t)}{\Delta t}$$

↳ Unstable

$$\nabla \times \vec{H}(t) = \varepsilon \frac{\vec{E}(t+\Delta t) - \vec{E}(t)}{\Delta t}$$

↳ Unstable

Solution:- (Staggering $\vec{E}$ and $\vec{H}$ in time)

$$\nabla \times \vec{E}(t) = -\mu \frac{\vec{H}(t+\Delta t/2) - \vec{H}(t-\Delta t/2)}{\Delta t}$$

$$\cancel{\nabla \times \vec{H}(t) \quad \varepsilon \vec{E}}$$

$$\nabla \times \vec{H}(t+\Delta t/2) = \varepsilon \frac{\vec{E}(t+\Delta t) - \vec{E}(t)}{\Delta t}$$

Update Equations:-

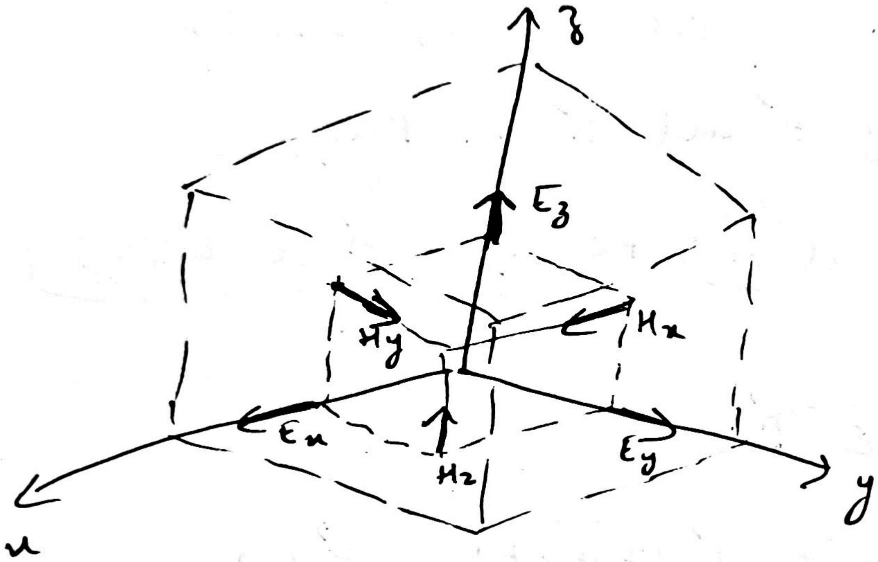
$$\vec{E}\Big|_{t+\Delta t} = \vec{E}\Big|_{t} + \frac{\Delta t}{\varepsilon}\left(\nabla \times \vec{H}\Big|_{t+\frac{\Delta t}{2}}\right)$$

$$\vec{H}\Big|_{t+\frac{\Delta t}{2}} = \vec{H}\Big|_{t-\frac{\Delta t}{2}} - \frac{\Delta t}{\mu}\left(\nabla \times \vec{E}\Big|_{t}\right)$$

Yee Grid Scheme

Assigning a function value to a specific point within the grid.

↳ Stagger the position of each field component within the grid cells



↳ results in divergence free grid

$$\nabla \cdot (\varepsilon \vec{E}) = 0$$

$$\nabla \cdot (\mu \vec{H}) = 0$$

↳ Maxwell's curl equations are approximated.

Normalization:

$$\vec{\tilde{H}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \, H$$

$$\Rightarrow \quad \nabla \times \vec{E} = -\frac{[\mu_r]}{c_0} \frac{\partial \vec{\tilde{H}}}{\partial t}$$

$$\nabla \times \vec{\tilde{H}} = \frac{[\varepsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t}$$

Curl Eqns :-

(assume anisotropic materials and ignore off-diagonal tensor elements)

$$\nabla \times \vec{E} = -\frac{[\mu_r]}{c_0}\frac{\partial \vec{H}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$

Finite Difference Eqns :-

$$\frac{E_z^{i,j+1,k}\big|_t - E_z^{i,j,k}\big|_t}{\Delta y} - \frac{E_y^{i,j,k+1}\big|_t - E_y^{i,j,k}\big|_t}{\Delta z}$$

$$= -\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\varepsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

$$\frac{\tilde{H}_z^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k}\big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$= \frac{\varepsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\big|_{t+\Delta t} - E_x^{i,j,k}\big|_t}{\Delta t}$$

Reduction to 1-dimension :-

* Cancel out $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ terms.
* Longitudinal terms $\vec{E_z}$ and $\vec{H_z}$ are 0.
* ignore i and j indices (positions in x & y directions)

$$\frac{E_y^{k+1}\big|_t - E_y^{k}\big|_t}{\Delta z} = -\frac{\mu_{xx}^{k}}{c_0}\,\frac{\tilde{H}_x^{k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k}\big|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

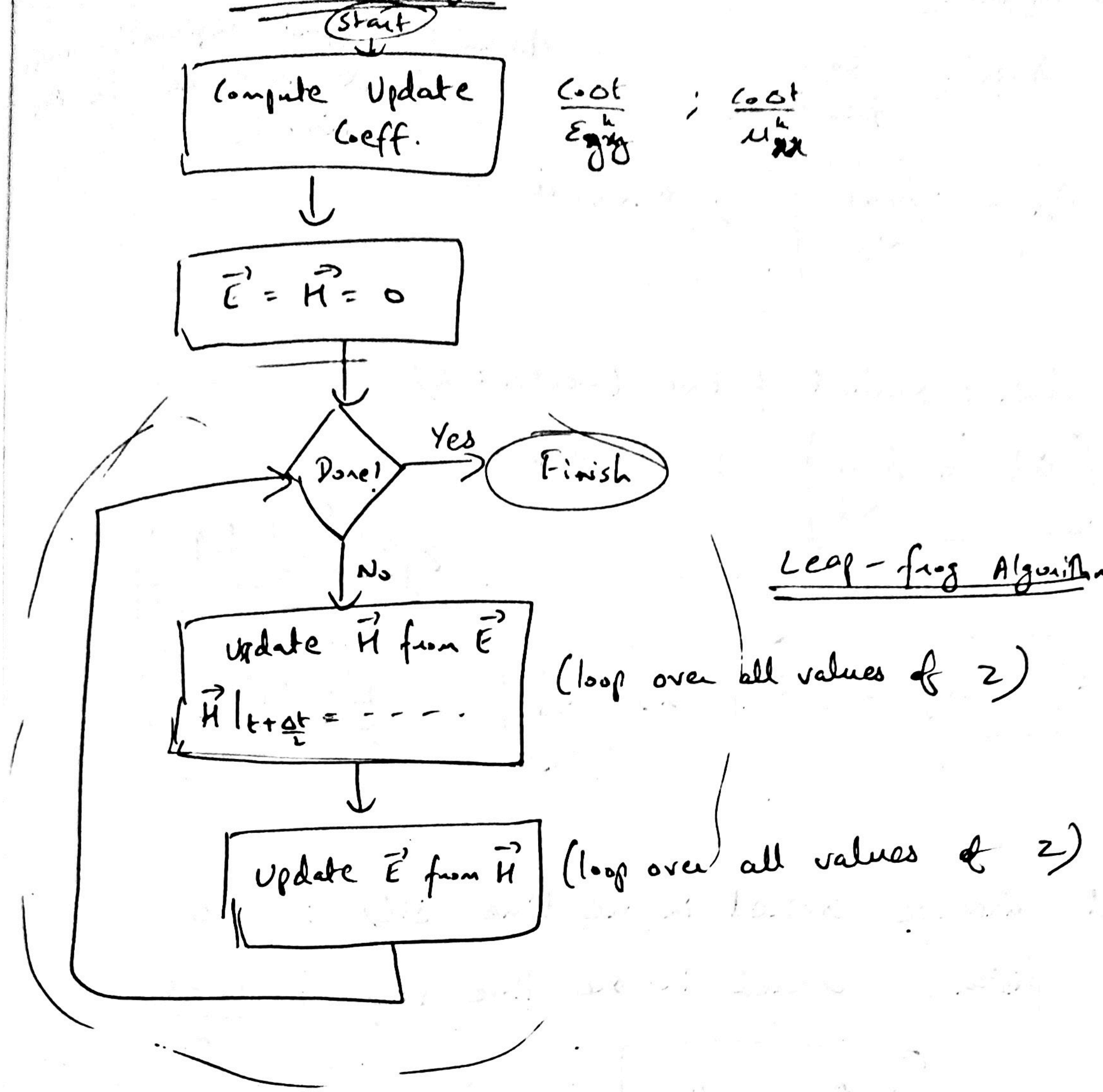
$$\frac{\tilde{H}_x^{k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\varepsilon_{yy}^{k}}{c_0}\,\frac{E_y^{k}\big|_{t+\Delta t} - E_y^{k}\big|_t}{\Delta t}$$

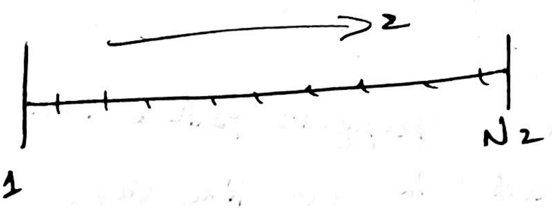
Update Equation :-

$$E_y^{k}\big|_{t+\Delta t} = E_y^{k}\big|_t + \left(\frac{c_0\,\Delta t}{\varepsilon_{yy}^{k}}\right)\left[\frac{\tilde{H}_x^{k}\big|_{t+\frac{\Delta t}{2}} - \tilde{H}^{k-1}\big|_{t+\frac{\Delta t}{2}}}{\Delta z}\right]$$

$$\tilde{H}_x^{k}\big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^{k}\big|_{t-\frac{\Delta t}{2}} + \left(\frac{c_0\,\Delta t}{\mu_{xx}^{k}}\right)\left[\frac{E_y^{k+1}\big|_t - E_y^{k}\big|_t}{\Delta z}\right]$$

FDTD Algo :





Dirichlet Boundary Condition :-

Assume fields outside the grid (boundary) are 0.

$$\tilde{H}_x^k\Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k\Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k \left(\frac{E_y^{k+1}\big|_t - E_y^k\big|_t}{\Delta z} \right) \quad ; \quad k < N_z$$

$$\tilde{H}_x\Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k\Big|_{t-\frac{\Delta t}{2}} + m_{Hx}^k \left(\frac{0 - E_y^{N_z}\big|_t}{\Delta z} \right) \quad ; \quad k = N_z$$

$$E_y^k\Big|_{t+\Delta t} = E_y^k\Big|_t + m_{Ey}^k \left(\frac{\tilde{H}^1\big|_{t+\frac{\Delta t}{2}} - 0}{\Delta z} \right) \quad ; \quad k = 1$$

Grid resolution :-

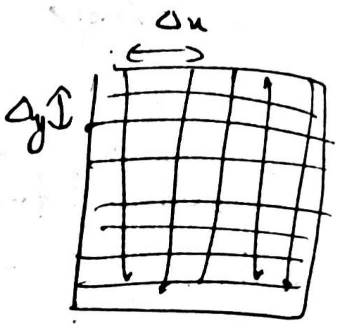
$$\lambda_{min} = \frac{c_0}{f_{max}\, n_{max}}$$

n_{max} : largest refractive index found anywhere in grid.

$$\boxed{\Delta_\lambda = \frac{\lambda_{min}}{N_\lambda}} \quad ; \; N_\lambda \geq 10$$

d_{min} : smallest feature (mechanical)

$$\boxed{\Delta d \approx \frac{d_{min}}{N_d}} \quad ; \; N_d \geq 1$$



$$\boxed{\Delta x = \Delta y = \min[\Delta_\lambda, \Delta d]}$$

Numerical distance covered in one time step : Δz

Physical distance covered in one time step : $\frac{c_0 \Delta t}{n}$

$$\boxed{\frac{c_0 \Delta t}{n} < \Delta z}$$

↳ wave cannot propagate farther than a single unit cell in one time step.

$$\Rightarrow \boxed{\Delta t < \frac{n \Delta z}{c_0}} \qquad n \approx 1 \text{ (smallest refractive index)}$$

$$\rightarrow \boxed{t < \frac{\Delta z}{c_0}} \rightarrow \text{for 1D}$$

Courant stability condition :-

$$\boxed{\Delta t < \frac{1}{c_0 \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \left(\frac{1}{\Delta z}\right)^2}}} \rightarrow \text{for 2D/3D}$$

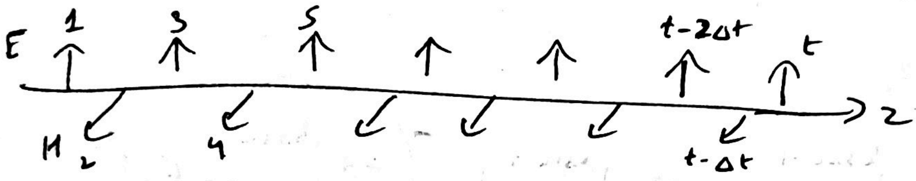
$$\boxed{\Delta t = \frac{n_{bc}\,\Delta z}{2 c_0}}$$

n_{bc} : boundary refractive index

0.5 factor for safety margin.

↳ wave travels the distance of one grid in exactly two time steps.

$$E_y^{N_z+1}\big|_t = E_y^{N_z}\big|_{t-2\Delta t}$$



Source : Short pulse $g(t) = e^{-\left(\frac{t-t_0}{\tau}\right)^2}$

↳ Gaussian pulse

(1) max freq = f_{max}

$B = f_{max} = \frac{1}{\pi\tau} \Rightarrow \tau \lesssim \frac{1}{\pi f_{max}}$

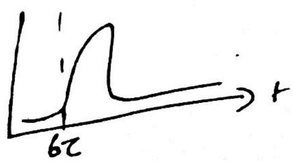
$$\boxed{\tau \simeq \frac{0.5}{f_{max}}}$$

$$\boxed{\Delta t = \frac{\tau}{N_t}}$$; $N_t \geq 10$

(3) minimum of Δt obtained by Courant condition and max. freq. condition.

Delay for introducing Source :- (No step function)

$$\boxed{t_0 = 6\tau}$$

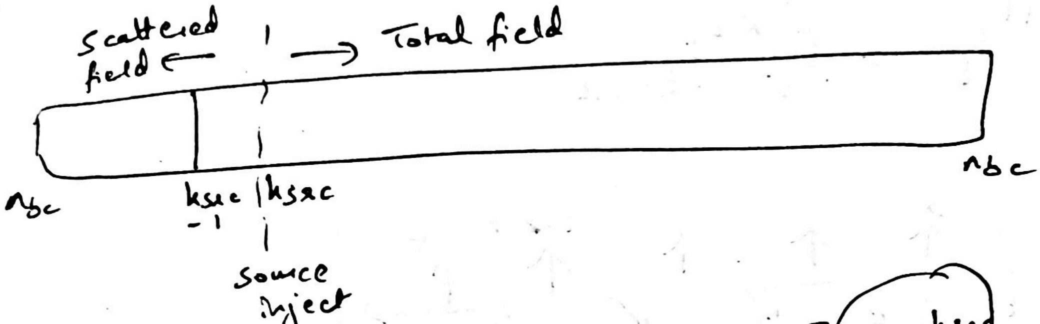


Soft Source:

$$\tilde{H}_x^k\Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k\Big|_{t+\frac{\Delta t}{2}} + g_H\big|_k$$

$$\tilde{E}_y^k\Big|_{t+\Delta t} = \tilde{E}_y^k\Big|_{t+\Delta t} + g_E\big|_k$$

Total Field / Scattered Field



$$\tilde{H}_x^{ksrc-1}\Big|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^{ksrc-1}\Big|_{t-\frac{\Delta t}{2}} + \left(m_{Hx}\right)^{ksrc-1}\left[\frac{E_y\big|_t^{ksrc} - E_y\big|_t^{ksrc-1}}{\Delta z}\right]$$

total field term in scattered field region

Correction:-

Subtract source E_y^{ksrc} from E_y^{ksrc} to make it look like a scattered field.

$$\tilde{H}_x\Big|_{t+\frac{\Delta t}{2}}^{ksrc-1} = \tilde{H}_x\Big|_{t-\frac{\Delta t}{2}}^{ksrc-1} + \left(m_{Hx}\right)^{ksrc-1}\left[\frac{\left(E_y\big|_t^{ksrc} - E_y^{src}\big|_t^{ksrc}\right) - E_y\big|_t^{ksrc-1}}{\Delta z}\right]$$

$$\Rightarrow \tilde{H}_x\Big|_{t+\frac{\Delta t}{2}}^{ksrc-1} = \underbrace{\tilde{H}_x\Big|_{t-\frac{\Delta t}{2}}^{ksrc-1} + \left(m_{Hx}\right)\left[\frac{E_y\big|_t^{ksrc} - E_y\big|_t^{ksrc-1}}{\Delta z}\right]}_{\text{Original update eqn.}} - \underbrace{\frac{\left(m_{Hx}\right) E_y^{src}\big|_t^{ksrc}}{\Delta z}}_{\text{Correction term}}$$

Similarly,

$$E_y\Big|_{t+\Delta t}^{ksrc} = E_y\Big|_t^{ksrc} + \left(m_y\right)\left[\frac{\tilde{H}_x\big|_{t+\frac{\Delta t}{2}}^{ksrc} - \tilde{H}_x\big|_{t+\frac{\Delta t}{2}}^{ksrc-1}}{\Delta z}\right]$$

Scattered field term in total field region.

Correction :-

$$E_y\Big|^{n_{sec}}_{t+\Delta t} = \underbrace{E_y\Big|^{n_{sec}}_{t} + (m_{E_y})\left[\frac{\tilde{H}_x\Big|^{n_{sec}}_{t+\frac{\Delta t}{2}} - \tilde{H}_x\Big|^{n_{sec}-1}_{t+\frac{\Delta t}{2}}}{\Delta z}\right]}_{\text{original update eqn.}} - \underbrace{\frac{(m_{E_y})\,\tilde{H}_x^{sec}\Big|^{n_{sec}}_{t+\frac{\Delta t}{2}}}{\Delta z}}_{\text{Correction term}}$$

$$E_y = \sin(\omega t - \beta z)$$

$$H_x = A \sin(\omega t - \beta z - \phi)$$

$$A = -\sqrt{\frac{\varepsilon_r^{sec}}{\mu_r^{sec}}}$$

$$\boxed{E_y^{sec}\Big|^{n_{sec}}_{t} = g(t)}$$

$$\tilde{H}_x^{sec}\Big|^{n_{sec}-1}_{t+\frac{\Delta t}{2}} = -\sqrt{\frac{\varepsilon_r}{\mu_r}}\, g\left(t + \frac{n_{sec}\,\Delta z}{2c_0} + \frac{\Delta t}{2}\right)$$

delay through half of grid cell (staggered)

staggered in time

Fourier Transform :-

$$F(f) \cong \Delta t \sum_{m=1}^{M} \left(e^{-j2\pi f \Delta t}\right)^{m} \cdot f(m)$$

Compute Kernel (for each freq.) :-

$$K_n = e^{-j2\pi f_n \Delta t}$$

$\downarrow$

Update Loop

$\downarrow$

$$\boxed{F_n\big|_{t+\Delta t} = F_n\big|_t + (K_n)^{m} \cdot E_y^{i,j,k}\big|_{t+\Delta t}}$$

time loop

$\downarrow$

$$\boxed{F_n = \Delta t \cdot F_n}$$

Adding Material to Grid :-

$$n_{max} = \sqrt{\mu_r \varepsilon_r}$$

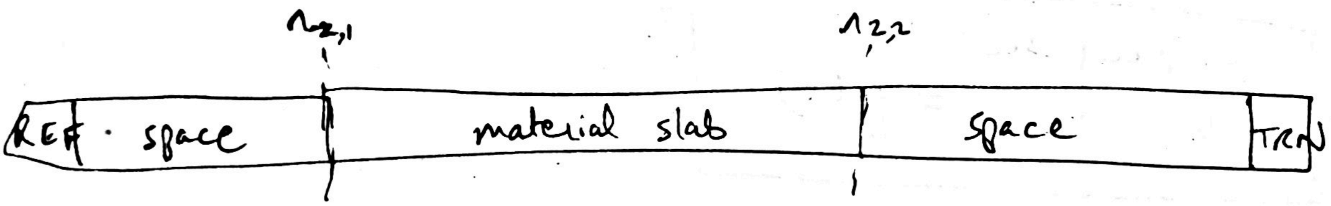
find λ_{min}.

find $\Delta\lambda = \dfrac{\lambda_{min}}{N_\lambda}$, $\Delta d = \dfrac{d}{N_d}$

$$dz = \min(\Delta\lambda, \Delta d)$$

$$N' = \frac{d}{dz}$$

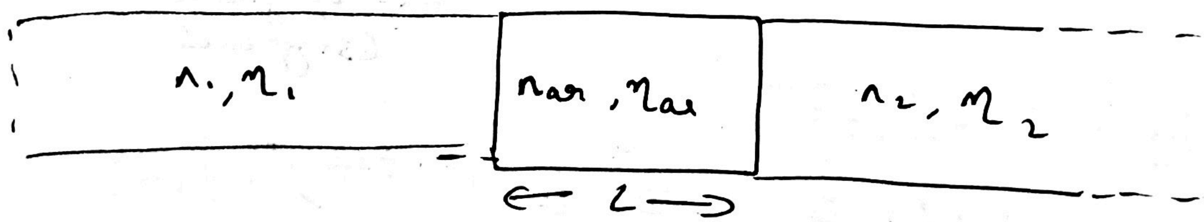
$N = \text{ceil}(N') =$ Thickness (no. of cells) of the material.



$$N_z = N_{material} + 2\,(\text{space} + \text{TRN/REF})$$

UR($n_{z,1}$: $n_{z,2}$) = ur;
ER(n_{z1} : $n_{z,2}$) = er;

Anti-Reflection layer :-



$$\cancel{\eta_{ar} = \sqrt{\eta_1 \eta_2}}$$

$$\eta_{ar} = \sqrt{\eta_1 \eta_2}$$

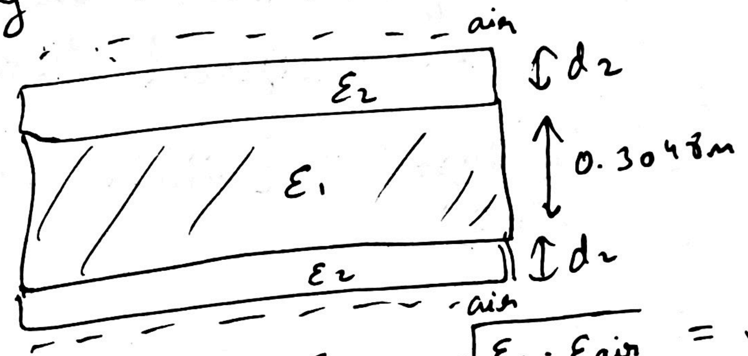
$$\boxed{L = \frac{\lambda_0}{4\, n_{ar}}}$$

* **Problem :-**

A radome is to be designed to protect an antenna operating at 2.4 GHz. For mechanical reasons, it must be constructed from 1 ft thick plastic with dielectric constant 12. Modify the design to maximize transmission through radome. Simulate the design using 1D FDTD.

1 feet = 0.3048 m

Using Anti-Reflection layer :-



$$\varepsilon_2 = \sqrt{\varepsilon_1 \cdot \varepsilon_{air}} = \sqrt{12(1)} = 3.46$$

$$n_2 = \sqrt{\varepsilon_2} = \sqrt{3.46} = 1.86$$

$$\lambda_0 \text{ (free space)} = \frac{c_0}{f_0} = \frac{3 \times 10^8}{2.4 \times 10^9}$$

$$= 12.49 \text{ cm}$$

$$= 0.1249 \text{ m}$$

$$d_2 = \frac{\lambda_0}{4 n_2} = \frac{0.1249}{4(1.86)} = 0.0167 \text{ m}$$