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15. Study of the Dynamics of Laser System Using Lorenz Model

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ABSTRACT

A semi-classical model of a laser system is proposed by coupling a classical optical field in a cavity described by Maxwell's equations with a system of two-level atoms described by optical Bloch equations. The resulting set of equations called the Maxwell-Bloch equations are found to be isomorphic to the equations of the Lorenz model (1963) of weather prediction. This isomorphism, first pointed out by H. Haken in 1975 is made use in studying the dynamics of this simple laser model both numerically and analytically. It is found that the pumping rate act as the control parameter that determines the dynamics of the system. The laser system exhibits a threshold behavior governed by the rate of pumping. Also lasing action leads to a spontaneous symmetry breaking.

Key words: Optical Bloch equations, Maxwell-Bloch equations, Two-level atom, Lorenz model, Chaos..

Introduction

Several simple models have been proposed to study the interaction between atomic states and radiation. [1, 6, 7, 9]. An exact treatment requires treating both atom and field as quantum systems as in the Janes-Cumming's model [7]. However, several interesting features emerge even in a semi-classical model [6].We consider a classical radiation field interacting with a two-level atom [3]. The radiation field is assumed to have a plane wave form

$$\vec{E}(t) = \hat{\varepsilon} E_0 \cos(\omega t) \tag{1.1}$$

Here, $\hat{\varepsilon}$ is the unit polarization vector of the field. Note that we are ignoring the spatial dependence of the field, only writing down the field at the location of the atom. This is appropriate in the dipole approximation or long-wavelength approximation, where we assume that the wavelength of the field is much longer than the size of the atom, so that we can neglect any variations of the field over the extent of the atom. This is generally appropriate for optical transitions, since atomic dimensions are very small compared to optical wavelengths. It is convenient to decompose the field into its positive- and negative-rotating components $\mathbf{E}(+)$ and $\mathbf{E}(-)$:

$$\vec{E}(t) = E^{+}(t) + E^{-}(t) \tag{1.2}$$

with $E^{\pm} \sim e^{-i(\pm\omega)t}$.

Our assumption of the atom as a two-state system is justified because we are considering near-resonant interactions, so that the transitions to other levels are negligible. We will label the ground and excited levels as |g> and |e>, respectively, and we will denote the resonant frequency by ω_0 (that is, the energy splitting of the pair of states is $\hbar\omega_0$). We will define $\Delta := \omega - \omega_0$ to be the detuning of the laser field from the atomic resonance.

We can write the total Hamiltonian for the atom and field as a sum of the free atomic Hamiltonian H_A and the atom–field interaction Hamiltonian H_{AF}:

$$H = H_A + H_{AF} \tag{1.3}$$

The atomic free-evolution Hamiltonian is given by

$$H_A = \hbar\omega_0 |e\rangle\langle e| \tag{1.4}$$

if we take the ground-state energy to be zero. The atom-field interaction Hamiltonian is

$$H_{AF} = -\mathrm{d} \cdot \mathrm{E} \tag{1.5}$$

where **d** is the atomic dipole operator .

The atomic Hamiltonian commutes with the parity operator whereas the dipole operator anti-commutes with it. This leads to the vanishing of the diagonal elements of the dipole moment operator in the energy basis.

$$\langle g|\mathbf{d}|g\rangle = \langle e|\mathbf{d}|e\rangle = 0 \tag{1.6}$$

The off-diagonal matrix elements $\langle g\,|\mathbf{d}|\,e\rangle = \langle e\,|\mathbf{d}|\,g\rangle^*$ are non-vanishing if the states have opposite parity. Thus the dipole operator couples the ground and excited states, but does not produce any first-order shift of either state. If we choose the phase of the matrix element $\langle g\,|\mathbf{d}|\,e\rangle$ as real, the dipole operator can be written as

$$\mathbf{d} = \langle g\,|\mathbf{d}|\,e\rangle\,(\sigma + \sigma^\dagger) \tag{1.7}$$

where $\sigma = |g\rangle\langle e|$ is the atomic lowering operator. We can write the total atom–field Hamiltonian as

$$H = H_A + H_{AF} = \hbar\omega_0\sigma^\dagger\sigma - \langle g|\mathbf{d}|e\rangle \cdot \mathbf{E}\,(\sigma + \sigma^\dagger) \tag{1.8}$$

where $\sigma^\dagger\sigma = |e\rangle\langle e|$ is the excited-state projection operator. Now, let us decompose the dipole operator as

$$\mathbf{d} = \langle g|\mathbf{d}|e\rangle(\sigma + \sigma^\dagger) = \mathbf{d}^{(+)} + \mathbf{d}^{(-)} \tag{1.9}$$

where $\mathbf{d}^{(+)} \sim \sigma$ and $\mathbf{d}^{(-)} \sim \sigma^\dagger$.

In the **rotating-wave approximation** (RWA), in which terms rotating at optical frequencies are replaced by their zero average value, and which is at the same level of accuracy as that of two-level approximation, the atom–field interaction Hamiltonian is

$$H_{AF} = \frac{\hbar\Omega}{2}(\sigma\, e^{\omega t i} + \sigma^\dagger\, e^{-i\omega t}) \tag{1.10}$$

where we have assumed $E^{(+)}{}_0$ to be real.

$$\Omega := \frac{2\langle g|\hat{\varepsilon}.\,\mathbf{d}|e\rangle E_0^{(+)}}{\hbar} \tag{1.13}$$

is called the Rabi frequency. The Rabi frequency characterizes the strength of the atom–field coupling. Also, if ρ is the density matrix of the atom, we can prove that, in the rotating frame,

$$<\sigma> = \tilde{\rho}_{eg}$$
$$<\sigma^\dagger> = \tilde{\rho}_{ge}$$
$$\sigma^\dagger\sigma = \rho_{ee}$$
$$\sigma\sigma^\dagger = \rho_{gg} \tag{1.14}$$

If we write the atomic state as

$$|\psi> = c_g|g>+ c_e|e>, \tag{1.15}$$

The Schrodinger equation for the expansion coefficients will lead to the following equations of motion:

$$\partial_t \rho_{ee} = i\frac{\Omega}{2}(\tilde{\rho}_{eg} - \tilde{\rho}_{ge}),$$
$$\partial_t \rho_{gg} = -i\frac{\Omega}{2}(\tilde{\rho}_{eg} - \tilde{\rho}_{ge})$$
$$\partial_t \tilde{\rho}_{ge} = i\Delta\tilde{\rho}_{eg} + i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg})$$
$$\partial_t \tilde{\rho}_{eg} = -i\Delta\tilde{\rho}_{ge} - i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg})$$

To incorporate spontaneous emission, we modify the above equations by adding some extra terms:

$$\partial_t \rho_{ee} = i\frac{\Omega}{2}(\tilde{\rho}_{eg} - \tilde{\rho}_{ge}) - \Gamma\rho_{ee}$$
$$\partial_t \rho_{gg} = -i\frac{\Omega}{2}(\tilde{\rho}_{eg} - \tilde{\rho}_{ge}) + \Gamma\rho_{ee}$$
$$\partial_t \tilde{\rho}_{ge} = -(\gamma_\perp + i\Delta)\tilde{\rho}_{ge} - i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg})$$
$$\partial_t \tilde{\rho}_{eg} = -(\gamma_\perp - i\Delta)\tilde{\rho}_{eg} + i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg}) \tag{1.16}$$

These equations are extension of Bloch's equations for nuclear magnetic resonance [2] to optical regime and hence known as the Optical Bloch equations [3, 4]. Here, Γ represents the decay of the excited state as well as the growth of the ground state population. It is also called the longitudinal relaxation rate. Damping of coherence is represented by the coefficient $\gamma_\perp$ which is called the transverse decay rate. Reciprocal of relaxation rates are called relaxation times.

2. Construction of a simple Laser Model

H. Haken has constructed a simple laser model comprised of an optical cavity (resonator), which traps light and confines it in some region of space, a gain medium, consisting of a vapor of two-level atoms uniformly filling the cavity; and some pump source that promotes the atoms to excited state.

Using equation (1.14), expectation values of the Pauli spin operators can be written as

$$\langle\sigma_x\rangle = \langle\sigma\rangle + \langle\sigma^\dagger\rangle = (\tilde{\rho}_{eg} + \tilde{\rho}_{ge})$$

$$\langle\sigma_y\rangle = i\langle\sigma\rangle - i\langle\sigma^\dagger\rangle = i(\tilde{\rho}_{eg} - \tilde{\rho}_{ge})$$

$$\langle\sigma_z\rangle = \rho_{ee} - \rho_{gg}$$

Using these relations, the optical Bloch equations can be written in the Bloch vector form as:

$$\begin{aligned} \partial_t\langle\sigma_x\rangle &= \Delta\langle\sigma_y\rangle - \gamma_\perp\langle\sigma_x\rangle \\ \partial_t\langle\sigma_y\rangle &= -\Delta\langle\sigma_x\rangle - \Omega\langle\sigma_z\rangle - \gamma_\perp\langle\sigma_y\rangle \\ \partial_t\langle\sigma_z\rangle &= \Omega\langle\sigma_y\rangle - \Gamma(\langle\sigma_z\rangle + 1) \qquad (2.1) \end{aligned}$$

Bloch vectors can be given the following interpretation: $\langle\sigma_z\rangle$ gives the degree, of atomic excitation or it is a measure of population inversion. $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ are respectively the real and imaginary parts of the dipole operator $\mathbf{d}^{(+)}$. Consider the interaction of a single cavity field mode with an ensemble of quantum-mechanical two-level atoms of number density N that fill the cavity. We assume that initially all atoms are in the ground state and the cavity field is a slowly varying field and the amplitude of the cavity field as well as the polarization has no space variation. Under these assumptions, $\langle\sigma_x\rangle$ becomes irrelevant in this model and instead the variation of the field amplitude gains significance. We incorporate the decay of the field also. Similarly, in the equation for variation of $\langle\sigma_z\rangle$, we add the extra term $-R(\langle\sigma_z\rangle - 1)$ to account for the increase of population inversion due to the pumping. The general case with detuning has been studied by Zeghlache and Mandel [10]. We consider only the situation at resonance so that detuning $\Delta = \mathbf{0}$. It follows that

$$\partial t\langle E_0\rangle = \frac{N d_{ge}\omega_0}{\varepsilon_0}\langle\sigma_y\rangle - \frac{k}{2}E_0$$

$$\partial t\langle\sigma_y\rangle = \frac{d_{ge}E_0}{\hbar}\langle\sigma_z\rangle - \gamma_\perp\langle\sigma_y\rangle$$

$$\partial t\langle\sigma_z\rangle = \frac{d_{ge}E_0}{\hbar}\langle\sigma_y\rangle - \Gamma_R[\langle\sigma_z\rangle - \langle\sigma_{z0}\rangle] \qquad (2.2)$$

These equations are called the Maxwell-Bloch equations [3, 5]. Here, E_0 is the slowly varying cavity-field amplitude , the field is exactly resonant with the atoms; $\Gamma_R := \Gamma + R$; R is a pumping rate for the atoms to the excited state (by some auxiliary pump mechanism, not the cavity field); $\langle\sigma_z\rangle := -(\Gamma - R)/(\Gamma + R)$ is the steady-state atomic inversion in the absence of coupling to the cavity field; κ is the energy decay rate of the cavity; and $d_{ge} := \langle g|d_z|e\rangle$ is the matrix element that appears in the Rabi frequency. Introducing the parameters

$$r = \frac{\langle\sigma_z\rangle_0}{\langle\sigma_z\rangle_{ss}}; \quad \sigma = \frac{k}{2\gamma_\perp}; \qquad \mathrm{b} = \frac{\Gamma_R}{\gamma_\perp}$$

where E_{0ss}, $\langle\sigma_y\rangle_{SS}$ and $\langle\sigma_z\rangle_{SS}$ denoting the steady-state values of E_0, $\langle\sigma_y\rangle$, and $\langle\sigma_z\rangle$ respectively, and making a scaling of the coordinates as

$$\tau = \gamma_\perp t\,; \; x = \sqrt{b(r-1)}\;\frac{E_0}{(E_0)_{ss}} \quad ; \; y = \sqrt{b(r-1)}\frac{\langle\sigma_y\rangle}{\langle\sigma_y\rangle_{ss}} \; ; \; z = \frac{\langle\sigma_z\rangle_0 - \langle\sigma_z\rangle}{\langle\sigma_z\rangle_{ss}}.$$

the equations are transformed to [6, 3]

$$\partial_\tau(x) = -\sigma(x - y)$$

$$\partial_\tau(y) = -y - xz + rx$$

$$\partial_\tau(z) = xy - bz \tag{2.3}$$

These equations are the well-known **Lorenz equations,** first appeared in the weather model proposed by Lorenz [5, 6, 8]. Thus, the dynamics of our laser model is reduced to that of the Lorenz system. In this laser model, the parameter σ can be interpreted as a measure of the decay of the amplitude of the cavity field, r can be interpreted as scaled pumping rate, and b can be interpreted as the rate with which the population of the upper level exceeds the lower level. The coordinate x is related to the cavity field amplitude; y is related to the maximum value of the transverse component of the dipole moment of the atoms; z is the offset value of population inversion from its steady state value in the absence of coupling.

3. Results and Discussion

In figure 1 and figure 2, the parameter x is plotted against the re-scaled time parameter τ for two slightly different sets of initial values $(x = 0, y = 1, z = 0)$ and $(x = 0.1, y = 1, z = 0)$. In both cases, x is found to be oscillating with increasing amplitude and with kinks. This is indicative of the growing amplitude of the cavity field with and the laser spikes. However, the trajectories are very much different even though their qualitative behavior do not vary considerably. Oscillations and spiking is more evident for large values of r as in figure 3 where (r=28, x=2, y=5, z=6) . Here the system shows a chaotic behavior. More or less the same behavior is observed in the dynamics of the parameters y and z also. (Figure 4 and figure 5). The parameter r is the one which determine the dynamics of the system. Hence it is interesting to study the variation of x with the controlling parameter r, which is given in figure 6. It shows all features of a chaotic system including periodic windows. The phase space trajectories in the x-z and y-z planes shows the typical chaotic attractor (figure 7 and figure 8):

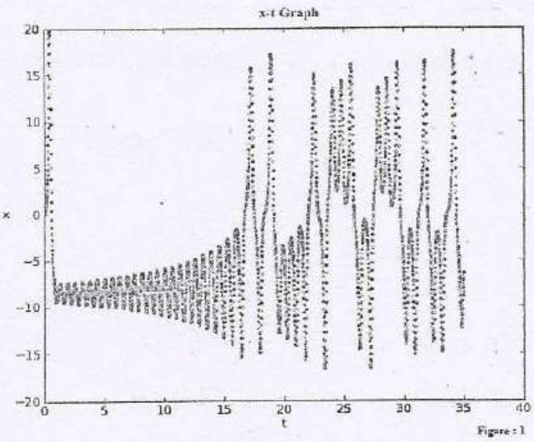

Figure : 1

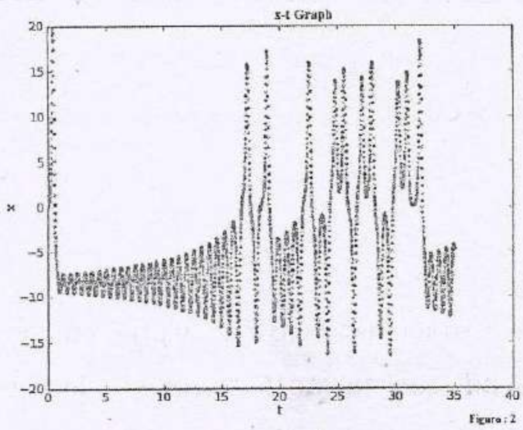

Figure : 2

In chaotic systems exhibiting some symmetry, it is customary to introduce invariant coordinates. The dynamical equations of the Lorenz model are invariant under rotation around the z-axis by π radians. This helps to construct the invariant coordinates $u = x^2 - y^2; v = 2xy; w = z$. Equations involving these coordinates are called image equations and phase portraits are called invariant images. We have constructed an invariant image of the Lorenz system which is shown in figure 9.

4. Conclusion

In this project, we have made use of a simple model of a laser system by considering the interaction of a classical radiation field with a two level atomic system. The Maxwell-Bloch equations describing such a system represents a three dimensional dynamical system and is isomorphic to the equations of the model introduced by Lorenz to predict weather. We studied the dynamics of the system and the stability of

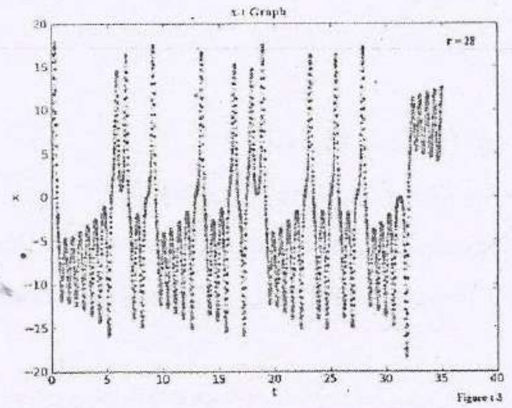

Figure : 3

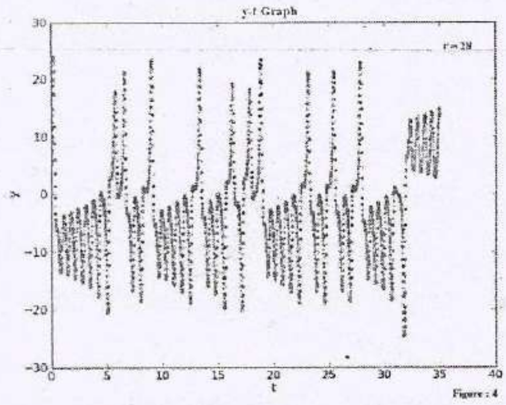

Figure : 4

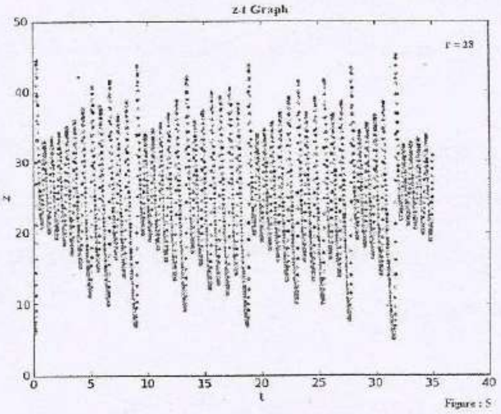

Figure : 5

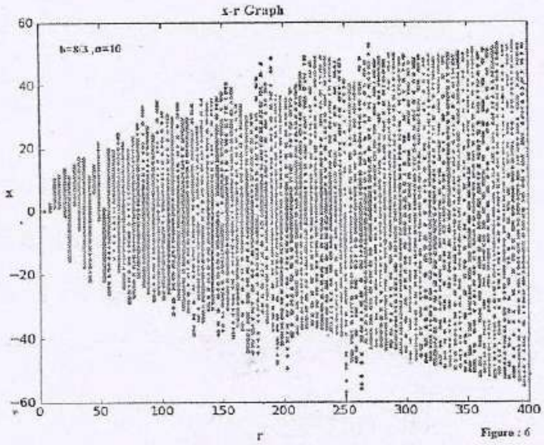

Figure : 6

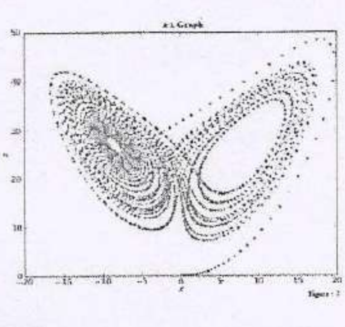

Figure : 7

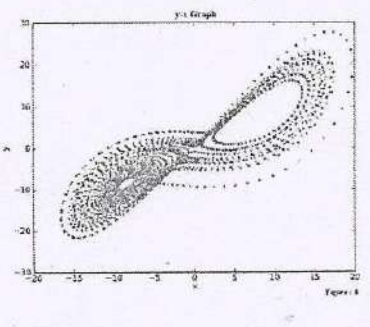

Figure : 8

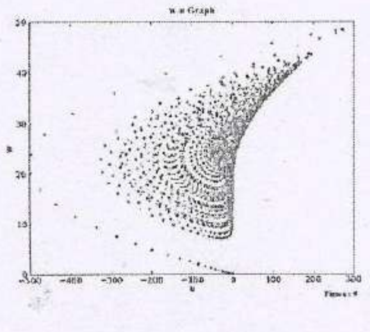
Figure : 9

the system numerically. Analogy with the Lorenz model indicated that the pumping rate was the major parameter determining the dynamics of the system. The plots of the dynamical variables against time showed oscillatory nature. However, when pumping rate is very high, the system exhibited chaotic nature. Our numerical analysis indicated the occurrence of laser spikes also. This simple model described the dynamics of a laser system almost completely. However this model neglected some important features like space variations of the cavity field amplitude and the polarization amplitude. A realistic model of a laser should incorporate such features also.

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