CS 215

Homework 3

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1. Question 1 :

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Answer 1

a) X_1 = no. of books to be picked so to get 1 distinct colour.

$\Rightarrow X_1 = 1$ (Success in 1st trial)

When already $(i-1)$ coloured books are detected, so,

To pick ~~at~~ i^{th} colour book $\Rightarrow$ Prob. $= 1 - P$(picking one of $(i-1)$ colour book)

$$P_i = 1 - \left(\frac{i-1}{n}\right) = \frac{n-i+1}{n} = P(\text{success})$$

b) $$P(X_i = k) = P\begin{pmatrix}\text{failure to pick a new} \\ \text{colour book in } (k-1) \\ \text{attempt}\end{pmatrix} \times P\begin{pmatrix}i^{th} \text{ colour book} \\ \text{picked in} \\ k^{th} \text{ attempt}\end{pmatrix}$$

$$= \underbrace{(1-P_i)(1-P_i)\ldots\ldots(1-P_i)}_{k-1 \text{ times}} \times \underset{\text{success}}{P}$$

$$= (1-P_i)^{k-1} P_i$$

So, X_i is indeed a Geometric R.V. with parameter

$$P_i = \frac{n-i+1}{n} = 1 - \left(\frac{i-1}{n}\right)$$

c) $$E(X_i) = \sum_{k=1}^{\infty} k P(X_i = k) = \sum_{k=1}^{\infty} k P_i (1-P_i)^{k-1}$$

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$$E(X_i) = P_i \sum_{k=1}^{\infty} k(1-P_i)^{k-1} = P \sum_{k=1}^{\infty} k\left(\frac{i-1}{n}\right)^{k-1}$$

~~E(Xi)=~~

$$\Rightarrow \sum_{k=1}^{\infty} (1-P_i)^k = (1-P_i) + (1-P_i)^2 + \dots \infty = \text{Infite GM}$$

$$\sum_{k=1}^{\infty} (1-P_i)^k = \frac{(1-P_i)}{1-(1-P_i)} = \frac{1}{P_i} - 1$$

differentiating both sides,

$$\sum_{k=1}^{\infty} k(1-P_i)^{k-1}(-1) = \frac{-1}{P_i^2} \rightarrow \sum_{k=1}^{\infty} k(1-P_i)^{k-1} = 1/P_i^2$$

So, finally

$$E(X_i) = P_i \sum_{k=1}^{\infty} k(1-P_i)^{k-1} = P_i(1/P_i^2)$$

$$\Rightarrow E(X_i) = 1/P \text{ or } 1/P_i \quad \therefore \text{Proved}$$

d) Now.

$$E(X^{(n)}) = E(X_1 + X_2 + \dots + X_n)$$

$$= E[X_1] + E[X_2] + E[X_3] + \dots + E[X_n]$$

$$= \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} + \dots + \frac{1}{P_n}$$

$$= \sum_{i=1}^{n} \frac{n}{n-i+1} = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \dots + \frac{n}{1}$$

$$= n\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

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e) $$var(X^{(n)}) = var(X_1 + X_2 + \dots + X_n)$$

$$= E\left[\left(X_1+X_2+X_3\dots+X_n - E(X_1) - E(X_2)\dots E(X_n)\right)^2\right]$$

$$= E\left[\left[(X_1-E(X_1)) + (X_2-E(X_2)) + \dots + (X_n-E(X_n))\right]^2\right]$$

$$= E[(X_1-E(X_1))]^2 + E[(X_2-E(X_2))]^2 + \dots + E[(X_n-E(X_n))^2]$$

$$+ \sum_{\substack{i=1, j=1 \\ i \neq j}}^{i=n, j=n} \underbrace{E[[X_i - E(X_i)][X_j - E(X_j)]]}_{\text{due to Independence of } X_i \,\&\, X_j}$$

$$= E(X_i - E(X_i))\, E(X_j - E(X_j))$$

$$= 0 \times 0 = 0$$

So, finally,

$$var(X^{(n)}) = var(X_1) + var(X_2) \dots + var(X_n)$$

$$= \left(E(X_1^2) - (E(X_1))^2\right) + \left(E(X_2^2) - (E(X_2))^2\right) + \dots + \left(E(X_n^2) - (E(X_n))^2\right)$$

$$= \sum_{i=1}^{n} E(X_i^2) - \sum_{i=1}^{n} (E(X_i))^2$$

$$\Rightarrow E(X_i) = \sum_{k=1}^{\infty} k^2 P(X_i = k) = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1}$$

$$\sum_{k=1}^{\infty} (1-p)^k = \frac{1-p}{1-(1-p)} = \frac{1}{p} - 1$$

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{+1}{p^2} \rightarrow \sum_{k=1}^{\infty} k(1-p)^k = \frac{1}{p^2} - \frac{1}{p}$$

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$$(-1)\sum_{k=1}^{\infty} k^2(1-p)^{k-1} = \frac{-2}{p^3} + \frac{1}{p^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} k^2(1-p)^{k-1} = \frac{2}{p^3} - \frac{1}{p^2} \Rightarrow E(x_i^2) = p\left(\frac{2}{p^3} - \frac{1}{p^2}\right)$$

$$= \frac{2}{p^2} - 1/p$$

So,

$$\mathrm{var}(x^{(n)}) = \sum_{i=1}^{n} \frac{2}{p_i^2} - \frac{1}{p_i} - \sum_{i=1}^{n}\left(\frac{1}{p_i}\right)^2$$

$$= \sum_{i=1}^{n} \frac{1}{p_i^2} - \frac{1}{p_i} < \sum_{i=1}^{n} \frac{1}{p_i^2}$$

So,

$$\mathrm{var}(x^{(n)}) < \sum_{i=1}^{n} \frac{1}{p_i^2} = \sum_{i=1}^{n}\left(\frac{n}{n-i+1}\right)^2$$

$$= n^2\left(\frac{1}{n^2} + \frac{1}{(n-1)^2} + \dots + \frac{1}{1^2}\right)$$

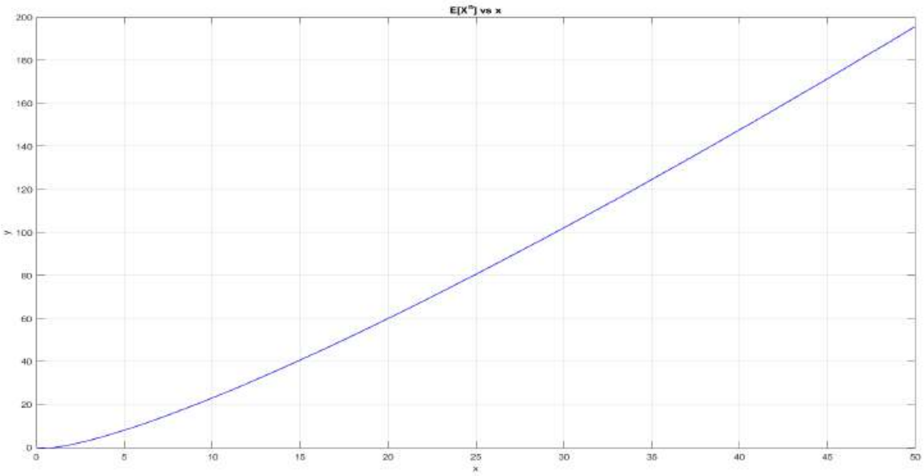
$$< n^2\left(\frac{1}{1^2} + \frac{1}{2^2} + \dots\dots + \infty\right)$$

So,

$$\mathrm{var}(x^{(n)}) < n^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = n^2\left(\frac{\pi^2}{6}\right)$$

1. f:

- The plot of $E[X^n]$ vs x



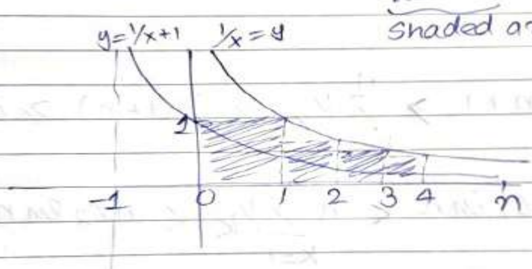
(f) as we know already,

$$E(X^{(n)}) = n \sum_{i=1}^{n} \frac{1}{i}$$

As we know that for $x > 0$,

$$\frac{1}{x+1} < \frac{1}{x}$$

So, $$\int_0^n \frac{1}{x+1}\,dx < \underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{\text{shaded area}} < 1 + \sum_{k=1}^{n} \frac{1}{k} < 1 + \int_1^n \frac{dx}{x}$$



So, finally,

$$\int_0^n \frac{1}{x+1}\,dx < \sum_{k=1}^{n} \frac{1}{k} < 1 + \int_1^n \frac{dx}{x}$$

$$\ln(1+n) < \sum_{k=1}^{n} 1/k < 1 + \ln n$$

$\frac{\ln(1+n)}{n}$ and $\frac{1+\ln n}{n}$ will have discrete finite value and tends to a finite value as $n \to \infty$.

So, we can say that

$\exists c_1$ such that $c_1 \leq \frac{\ln(1+n)}{\ln n}$

$c_1 \ln n \leq \ln(1+n)$

AND

$\exists c_2$ such that $c_2 \geq \frac{\ln(n)+1}{\ln n}$

$c_2 \ln n \geq \ln(n)+1$.

So,

$c_2 \ln n \geq \ln n + 1 \geq \sum_{k=1}^{n} 1/k \geq \ln(1+n) \geq c_1 \ln n$

So, $n c_1 \ln n \leq n \sum_{k=1}^{n} 1/k \leq n c_2 \ln n$

$\Rightarrow c_1(n \ln n) \leq E(X^{(n)}) \leq c_2(n \ln n)$

So, Thus, $E(X^{(n)}) = \Theta f(n) = \Theta n \ln(n)$

So, $\underline{f(n) = n \ln(n)}$

2. Question 2 :

Question 2 :-

(a) Given F is an invertible CDF. Hence, it has to be strictly increasing. Since, for any two values
$F(a) \neq F(b) \quad \forall\, a \neq b$ for any ~~CDF~~ invertible fn

Now,
To prove, $v_i = F^{-1}(u_i)$ follows distribution F.
where u_i is generated from a random fn

Given, if we prove $P[v_i \leq z] = F(z)$ for any given z. And since the distribution fn is strictly increasing, which implies,

$\Rightarrow v_i \leq z$ and $F(v_i) \leq F(z)$ are same

Hence, $P[v_i \leq z] = P[F(v_i) \leq F(z)]$
$= P[u_i \leq F(z)]$

and since,
$P[u_i \leq F(z)] = F(z)$
u_i being randomly gen var.

$\therefore P[v_i \leq z] = F(z)$

Hence, proved.

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(b) Here, $F_e(x) = \sum_{i=1}^{n} \frac{1(Y_i \leq x)}{n}$ and

$D = \max_x |F_e(x) - F(x)|$. Meanwhile, E quantifies the maximum absolute difference b/w true distribution $F(x)$ and CDFs of Y_i

$$\therefore \quad E = \max |F(x) - F(Y_i)|$$

Since, F is increasing f^n, replacing $F(x) = y$, then,

$$D = \max |y - F(Y_i)|$$

Now,
lets consider that $Y_i = F^{-1}(U_i)$ as proven ~~ist~~ in (a). Hence, the distribution of Y_i and U_i are ~~samtt~~ same.

$$\therefore \quad P(D \leq d) = P(E \leq d) \quad \text{also}$$
$$P(D \geq d) = P(E \geq d)$$

This allows us to understand the error in generating random samples using inv. method in terms of the error in obtaining a uniform distribution. ~~(E4)~~

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3. Question 3 :

Answer 3

3 (a)

z is corrupted with values from Gauss. Dist.

X & Y known correctly.

Let, $z_i = ax_i + by_i + c + \epsilon_i$

where ϵ is a R.V. (Gaussian) $\sim N(0, \sigma^2)$

$$P(z_i | x_i, y_i, a, b, c) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{(z_i - (ax_i + by_i + c))^2}{2\sigma^2}\right)}$$

as $z_i \sim N(ax_i + by_i + c, \sigma^2)$

$$\sum_{i=1}^{n} \log_e P(z_i | x_i, y_i, a, b, c) = -\sum_{i=1}^{n} \frac{(z_i - (ax_i + by_i + c))^2}{2\sigma^2} - n \log(\sqrt{2\pi}\,\sigma)$$

$$L = -\sum_{i=1}^{n} \frac{(z_i - (ax_i + by_i + c))^2}{2\sigma^2} - n \log(\sqrt{2\pi}\,\sigma)$$

To maximise L,

$$\frac{\partial L}{\partial a} = -\sum_{i=1}^{n} \frac{2(z_i - ax_i - by_i - c)(-x_i)}{2\sigma^2} - 0 = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i z_i = a \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i y_i + c \sum_{i=1}^{n} x_i$$

$$\frac{\partial L}{\partial b} = -\sum_{i=1}^{n} \frac{2(z_i - ax_i - by_i - c)(-y_i)}{2\sigma^2} = 0$$

$$\Rightarrow \sum_{i=1}^{n} z_i y_i = a\sum_{i=1}^{n} x_i y_i + b\sum_{i=1}^{n} y_i^2 + c\sum_{i=1}^{n} y_i$$

$$\frac{\partial L}{\partial c} = -\sum_{i=1}^{n} \frac{2(z_i - ax_i - by_i - c)(-1)}{2\sigma^2} = 0$$

$$\Rightarrow \sum_{i=1}^{n} z_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i + c(n)$$

So, To sum up all 3 above eq

$$\begin{bmatrix} \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \\ \Sigma x_i y_i & \Sigma y_i^2 & \Sigma y_i \\ \Sigma x_i & \Sigma y_i & n \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \Sigma z_i x_i \\ \Sigma z_i y_i \\ \Sigma z_i \end{bmatrix}$$

(matrix form)

and.

$$a\begin{bmatrix} \Sigma x_i^2 \\ \Sigma x_i y_i \\ \Sigma x_i \end{bmatrix} + b\begin{bmatrix} \Sigma x_i y_i \\ \Sigma y_i^2 \\ \Sigma y_i \end{bmatrix} + c\begin{bmatrix} \Sigma x_i \\ \Sigma y_i \\ n \end{bmatrix} = \begin{bmatrix} \Sigma z_i x_i \\ \Sigma z_i y_i \\ \Sigma z_i \end{bmatrix}$$

(Vector form)

(b) $z = a_1 x^2 + a_2 y^2 + a_3 xy + a_4 x + a_5 y + a_6$

→ where z is corrupted with values from Gaussian dist.

→ x, y are exact.

Let, RV. $z_i = a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6 + \varepsilon_i$

where $\varepsilon_i \sim N(0, \sigma^2)$

So, $z_i \sim N(a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6, \sigma^2)$

$P(z_i | a_1, a_2, a_3, a_4, a_5, a_6, x_i, y_i)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(z_i - (a_1 x_i^2 + a_2 x_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6))^2}{2\sigma^2}}$$

$$\sum_{i=1}^{n} \ln P(z_i | a_1, a_2, a_3 \dots, y_i) = L$$

$$L = -n \log_e(\sqrt{2\pi\sigma^2}) - \sum_{i=1}^{n} \frac{(z_i - (a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6))^2}{2\sigma^2}$$

To maximize L,

$$\frac{\partial L}{\partial a_1} = -\sum_{i=1}^{n} \frac{2(-x_i^2)(z_i - (a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6))}{2\sigma^2} = 0$$

$$\Rightarrow \sum z_i x_i^2 = a_1 \sum x_i^4 + a_2 \sum y_i^2 x_i^2 + a_3 \sum x_i^3 y_i + a_4 \sum x_i^3 + a_5 \sum x_i^2 y_i + a_6 \sum x_i^2$$

Similarly,

$$\frac{\partial L}{\partial a_2} = 0 \rightarrow \sum y_i^2 z_i = a_1 \sum x_i^2 y_i^2 + a_2 \sum y_i^4 + a_3 \sum x_i y_i^3 + a_4 \sum x_i y_i^2 + a_5 \sum y_i^3 + a_6 \sum y_i^2$$

$$\frac{\partial L}{\partial a_3} = 0 \rightarrow \sum x_i y_i z_i = a_1 \sum x_i^3 y_i + a_2 \sum x_i y_i^3 + a_3 \sum x_i^2 y_i^2 + a_4 \sum x_i^2 y_i + a_5 \sum x_i y_i^2 + a_6 \sum x_i y_i$$

$$\frac{\partial L}{\partial a_4} = 0 \rightarrow \sum x_i z_i = a_1 \sum x_i^3 + a_2 \sum x_i y_i^2 + a_3 \sum x_i^2 y_i + a_4 \sum x_i^2 + a_5 \sum x_i y_i + a_6 \sum x_i$$

$$\frac{\partial L}{\partial a_5} = 0 \rightarrow \sum y_i z_i = a_1 \sum x_i^2 y_i + a_2 \sum y_i^3 + a_3 \sum x_i y_i^2 + a_4 \sum x_i y_i + a_5 \sum y_i^2 + a_6 \sum y_i$$

$$\frac{\partial L}{\partial a_6} = 0 \rightarrow \sum z_i = a \sum x_i^2 + a_2 \sum y_i^2 + a_3 \sum x_i y_i + a_4 \sum x_i + a_5 \sum y_i + a_6 (n)$$

So, summing up above 6 eq. in matrix form,

$$\begin{bmatrix} \sum z_i x_i^2 \\ \sum z_i y_i^2 \\ \sum x_i y_i z_i \\ \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix} = \begin{bmatrix} \sum x_i^4 & \sum y_i^2 x_i^2 & \sum x_i^3 y_i & \sum x_i^3 & \sum x_i^2 y_i & \sum x_i^2 \\ \sum x_i^2 y_i^2 & \sum y_i^4 & \sum x_i y_i^3 & \sum x_i y_i^2 & \sum y_i^3 & \sum y_i^2 \\ \sum x_i^3 y_i & \sum x_i y_i^3 & \sum x_i^2 y_i^2 & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i y_i \\ \sum x_i 3 & \sum x_i y_i^2 & \sum x_i^2 y_i & \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i^2 y_i & \sum y_i^3 & \sum x_i y_i^2 & \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i^2 & \sum y_i^2 & \sum x_i y_i & \sum x_i & \sum y_i & n \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

& in vector form,

$$a_1\begin{bmatrix}\Sigma x_i^4\\ \Sigma x_i^2y_i^2\\ \Sigma x_i^3y_i\\ \Sigma x_i^3\\ \Sigma x_i^2y_i\\ \Sigma x_i^2\end{bmatrix}+a_2\begin{bmatrix}\Sigma x_i^2y_i^2\\ \Sigma y_i^4\\ \Sigma x_iy_i^3\\ \Sigma x_iy_i^2\\ \Sigma y_i^3\\ \Sigma y_i^2\end{bmatrix}+a_3\begin{bmatrix}\Sigma x_i^3y_i\\ \Sigma x_iy_i^3\\ \Sigma x_i^2y_i^2\\ \Sigma x_i^2y_i\\ \Sigma x_iy_i^2\\ \Sigma x_iy_i\end{bmatrix}+a_4\begin{bmatrix}\Sigma x_i^3\\ \Sigma x_iy_i^2\\ \Sigma x_i^2y_i\\ \Sigma x_i^2\\ \Sigma x_iy_i\\ \Sigma x_i\end{bmatrix}+a_5\begin{bmatrix}\Sigma x_i^2y_i\\ \Sigma y_i^3\\ \Sigma x_iy_i^2\\ \Sigma x_iy_i\\ \Sigma y_i^2\\ \Sigma y_i\end{bmatrix}$$

$$+a_6\begin{bmatrix}\Sigma x_i^2\\ \Sigma y_i^2\\ \Sigma x_iy_i\\ \Sigma x_i\\ \Sigma y_i\\ n\end{bmatrix}=\begin{bmatrix}\Sigma z_ix_i^2\\ \Sigma z_iy_i^2\\ \Sigma z_ix_iy_i\\ \Sigma x_iz_i\\ \Sigma y_iz_i\\ \Sigma z_i\end{bmatrix}$$

3. c :

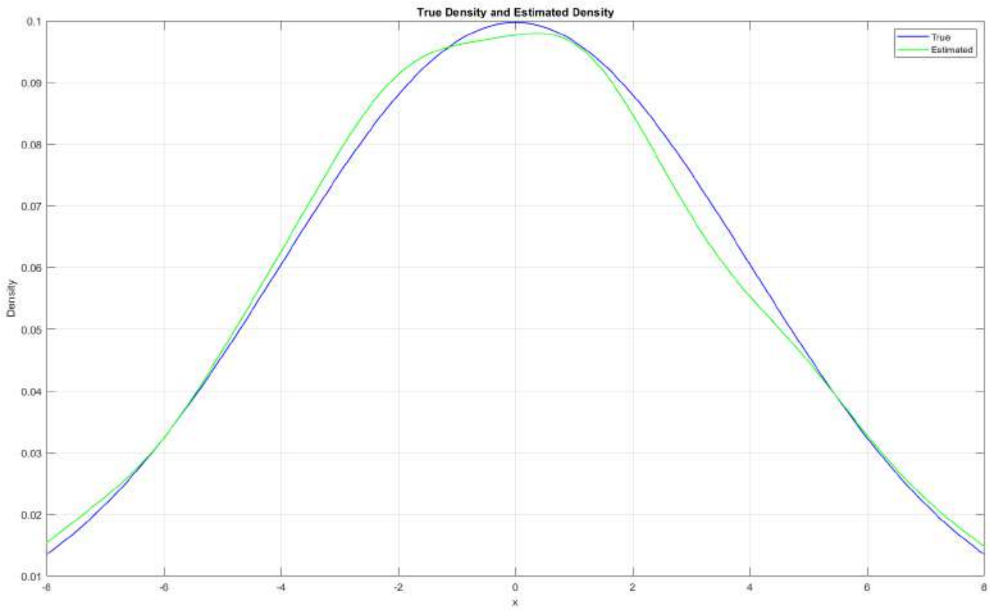
- To run the code, just run the file named as *q3.m* in MatLab
- Specify the path of *XYZ.txt* (*if changed*)

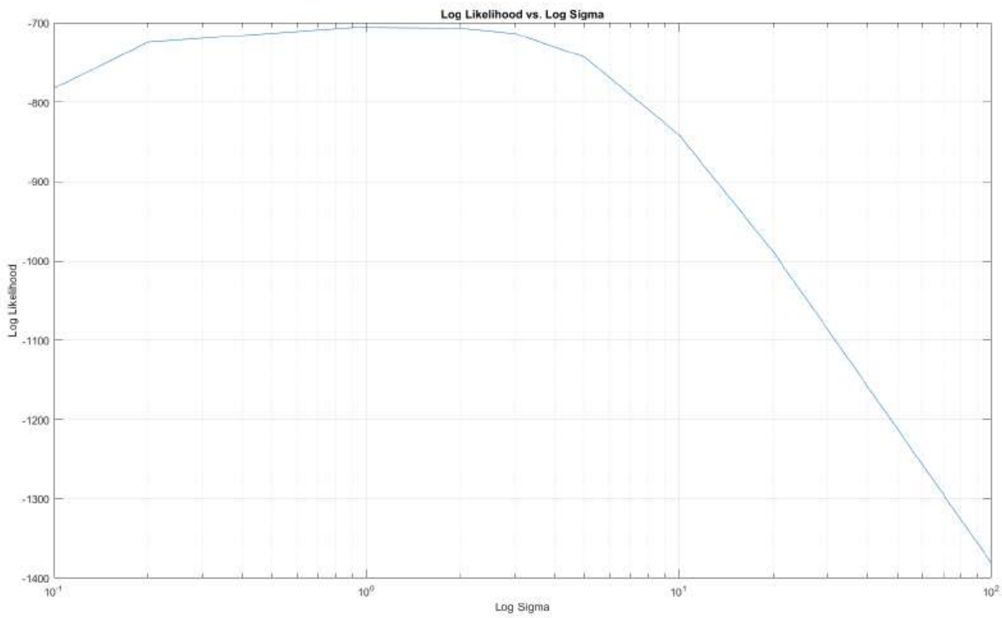
Command Window

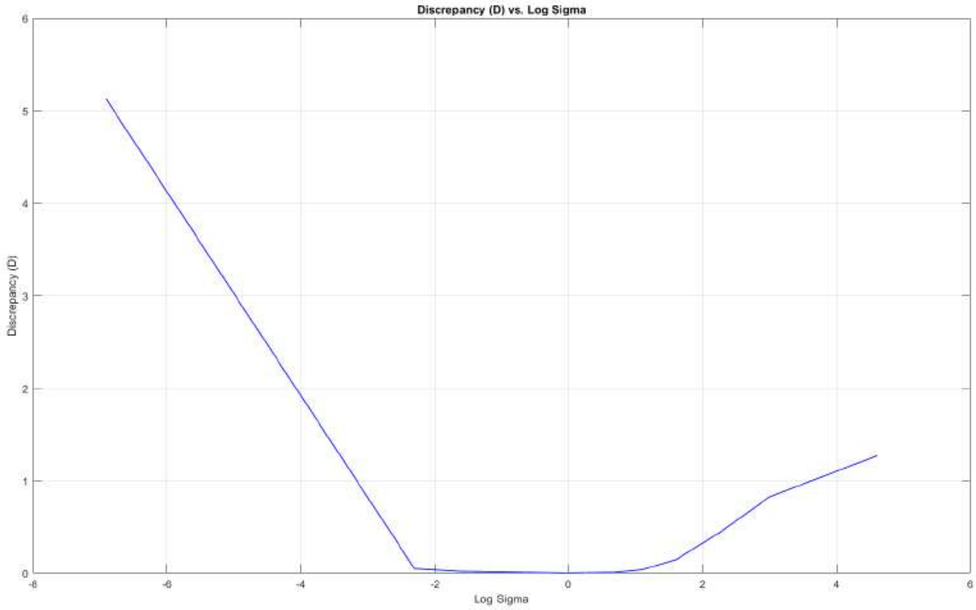
```
Predicted eq. of the plane : z = 10.0022x + 19.9980y + 29.9516
Predicted noise variance : 23.0685
Predicted std deviation : 4.8030
Note that i calculated the error based on the (Least-Squares Regression)
fx >>
```

4. Question 4 :

- To run the code, just run the file named as *q4.m* in MatLab






Discrepancy (D) vs. Log Sigma
Discrepancy (D)
Log Sigma
6
5
4
3
2
1
0
-8
-6
-4
-2
0
2
4
6

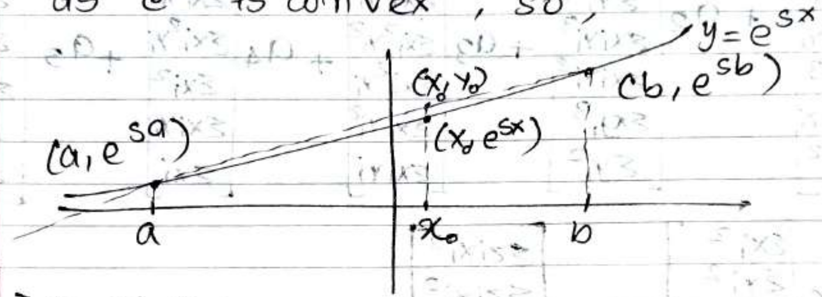
5. Question 5 :

Answer-5

5)

(a) Given, $E(X)=0$

as e^{sx} is convex, so,



↳ eq. of line.

$$\left(\frac{y - e^{sa}}{x-a}\right) = \left(\frac{e^{sb}-e^{sa}}{b-a}\right)$$

$$y = \left(\frac{e^{sb}-e^{sa}}{b-a}\right)(x-a) + e^{sa}$$

$$y = x\left(\frac{e^{sb}-e^{sa}}{b-a}\right) + \frac{be^{sa}-ae^{sb}}{b-a}$$

Now

$$y_0 = e^{sb}\left(\frac{x_0-a}{b-a}\right) + \left(\frac{b-x_0}{b-a}\right)e^{sa}$$

as, $y_0 \geqslant e^{sx_0}$

so,

$$e^{x_0 s} \leq e^{sb}\left(\frac{x_0-a}{b-a}\right) + e^{sa}\left(\frac{b-x_0}{b-a}\right)$$

Or General.

$$e^{xs} \leq e^{sb}\left(\frac{x-a}{b-a}\right) + e^{sa}\left(\frac{b-x}{b-a}\right)$$

(b) So,

$$E(e^{sx}) \le E\left(e^{sb}\left(\frac{x-a}{b-a}\right) + \left(\frac{b-x}{b-a}\right)e^{sa}\right)$$

$$= E\left[e^{sa}\left[\left(\frac{b-x}{b-a}\right) + e^{s(b-a)}\left(\frac{x-a}{b-a}\right) - 1 + 1\right]\right]$$

$$= E\left[e^{sa}\left(1 + \frac{e^{s(b-a)}(x-a) + (b-x) - (b-a)}{b-a}\right)\right]$$

$$= E\left[e^{sa}\left(1 + \frac{a(1-e^{s(b-a)})}{b-a} + \frac{x(e^{s(b-a)}-1)}{b-a}\right)\right]$$

$$= E\left[e^{sa}\left(1 + \frac{a(1-e^{s(b-a)})}{b-a}\right)\right] + \underbrace{E\left[\frac{e^{sa}x(e^{s(b-a)}-1)}{b-a}\right]}$$

$$+ \frac{e^{sa}(e^{sb-sa}-1)}{b-a}\underbrace{E(X)}_{0}$$

$$= 0$$

$$= E\left[e^{sa + \log_e\left(1 + \frac{a(1-e^{sb-sa})}{b-a}\right)}\right]$$

putting $\Rightarrow s = \dfrac{h}{(b-a)}$

$$= E\left[e^{\frac{ha}{b-a} + \ln\left(1 + \frac{a(1-e^{h})}{b-a}\right)}\right]$$

So, $L(h) = \dfrac{ha}{b-a} + \ln\left(1 + \dfrac{a(1-e^{h})}{b-a}\right)$ .... (1)

hence Proved,

$$E(e^{sx}) \le E(\underbrace{e^{L(h)}}_{\text{const}}) = e^{L(h)}\Big|_{h=s(b-a)}$$

∴ Hence, Proved

(c) By eq. ①,

$$L(h) = \frac{ha}{b-a} + \log_e\left(\frac{b-a+a-ae^h}{b-a}\right)$$

$$L(h) = \frac{ha}{b-a} + \ln\left(\frac{b-ae^h}{b-a}\right)$$

$$\Rightarrow L(h) = \frac{ha}{b-a} + \ln(b-ae^h) - \ln(b-a)$$

$$\Rightarrow L'(h) = \frac{a}{b-a} + \frac{(-ae^h)}{b-ae^h}$$

$$\Rightarrow L''(h) = \frac{(-b/a)e^h}{(e^h+(-b/a))^2}$$

In $L''(h)$, substituting $(-b/a) = x$ & $e^h = y$.

Now, we know $y > 0 \ \forall h$.

$$L''(h) = \frac{xy}{(x+y)^2}$$

when $x \leq 0$ Then, $\frac{xy}{(x+y)^2} < 0$

when $x > 0$ Then, $\frac{xy}{(x+y)^2} \leq \frac{1}{4}$

$$\left[(x-y)^2 \geq 0 \rightarrow (x+y)^2 \geq 4xy \rightarrow \frac{xy}{(x+y)^2} \leq \frac{1}{4}\right]$$

Combining both we get,

$$\frac{xy}{(x+y)^2} \leq 1/4 \text{ for any value of } x, y, h$$

So, $L''(h) \leq 1/4$

$$L'(h) \leq h/4$$

$$\Rightarrow L(h) \leq \frac{h^2}{8} \leftrightarrow e^{L(h)} \leq e^{h^2/8}$$

So, finally,

(d) $\therefore E(e^{sX}) \leq e^{L(h)}\Big|_{h=s(b-a)} = e^{L(s(b-a))}$

and

$$e^{L(s(b-a))} \leq e^{\frac{s^2(b-a)^2}{8}}$$

So,

$$\therefore E(e^{sX}) \leq e^{L(s(b-a))} \leq e^{\frac{s^2(b-a)^2}{8}}$$

The IR proved.

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