CS 215
Homework 2

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1. Question 1 :

Date

Answer 1

Given, n Independent identically distributed RV $x_1, x_2, x_3 \dots x_n$.

and $Y_1 = \max(x_1, x_2, \dots x_n)$
$Y_2 = \min(x_1, x_2, \dots x_n)$

Also,
cdF of $X = F(x)$ and PdF of $X = F'(x)$

(cdF) $F_{Y_1}(x) = P(\text{max among the } x < x)$
$= P(x_1 < x, x_2 < x, x_3 < x \dots x_n < x)$

as all x_is are Independant.
$= P(x_1 < x) P(x_2 < x) \dots P(x_n < x)$

$= F_{x_1}(x) F_{x_2}(x) \dots F_{x_n}(x)$

(Pdf) $F'_{Y_1}(x) = \frac{d}{dx} F_{Y_1}(x) = F_{x_1}(x) F_{x_2}(x) \dots F_{x_n}(x) \times \left[\frac{F'_{x_1}(x)}{F_{x_1}(x)} \dots + \frac{F'_{x_2}(x)}{F_{x_2}(x)} \right]$

If all $F_{x_i}(x)$ are Equal then,

cdF $\Rightarrow (F_x(x))^n = F_{Y_1}(x)$

pdF $\Rightarrow F'_{Y_1}(x) = n F'_x(x) (F_x(x))^{n-1}$

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For Y_2,

cdf $F_{Y_2}(x) = P(\text{min. among } X_i \leq x)$

$= 1 - P(\text{min. among } X_i \geq x)$

$= 1 - P(\text{all } X_i\text{'s} \geq x)$

$= 1 - P(X_1 \geq x, X_2 \geq x, \ldots X_n \geq x)$

as all X_i are Independant,

$= 1 - P(X_1 \geq x) P(X_1 \geq x) \ldots P(X_n \geq x)$

$= 1 - [1 - P(X_1 \leq x)][1 - P(X_2 \leq x)] \ldots [1 - P(X_n \leq x)]$

$= 1 - (1 - F_{X_1}(x))(1 - F_{X_2}(x)) \ldots (1 - F_{X_n}(x))$

Pdf $F'_{Y_2}(x) = +(1 - F_{X_1}(x)) \ldots (1 - F_{X_n}(x)) \times \left[\frac{F'_{X_1}(x)}{1 - F'_{X_1}(x)} + \frac{F'_{X_2}(x)}{1 - F'_{X_2}(x)} \ldots + \frac{F'_{X_n}(x)}{1 - F'_{X_n}(x)}\right]$

If all $F_{X_i}(x)$ are Equal & $= F_X(x)$

Then,

cdf $= 1 - (1 - F_X(x))^n = F_{Y_2}(x)$

Pdf $= F'_{Y_2}(x) = (1 - F_X(x))^{n-1} F'_X(x)$

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2. Question 2 :

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Answer 2

Given - X belongs to a Gaussian Mixt. model (GMM)

$$X \sim \sum_{i=1}^{k} P_i \, N(\mu_i, \sigma_i^2)$$

$$\sum_{i=1}^{k} P_i = 1$$

[$N(\mu_i, \sigma_i^2)$ represents a Gaussian dist. with mean μ_i & var. σ_i^2]

So,

$$\text{MGF of } X = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \sum_{i=1}^{k} P_i N(\mu_i, \sigma_i^2)\, dx$$

$$= \sum_{i=1}^{k} \cancel{e^{tx}} P_i \underbrace{\int_{-\infty}^{\infty} e^{tx} N(\mu_i, \sigma_i^2)\, dx}$$

MGF of a Gaussian $\Rightarrow \phi_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$\Rightarrow M_X(t) = \sum_{i=1}^{k} P_i \, e^{\mu_i t + \frac{\sigma_i^2 t^2}{2}}$$

$$\text{So, } E(X) \Rightarrow \frac{d}{dt} M_X(t)\Big|_{t=0} = \sum_{i=1}^{k} P_i \, e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu_i + \sigma^2 t)\Big|_{t=0}$$

$$\Rightarrow E(X) = \sum_{i=1}^{k} P_i \mu_i$$

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$$\operatorname{var}(X) = \frac{d^2}{dt^2} MgF_X(t)\Big|_{t=0} - \left(\frac{d}{dt} M_X(t)\Big|_{t=0}\right)^2$$

$$\Rightarrow \operatorname{var}(X) = \sum_{i=1}^{k} p_i(\mu_i^2 + \sigma_i^2) - \left(\sum_{i=1}^{k} p_i \mu_i\right)^2$$

Given, $Z = \sum_{i=1}^{k} p_i X_i$ & $X_i \sim N(\mu_i, \sigma_i^2)$

Gaussian with $\underbrace{\text{mean}}_{E(X_i)}$ μ_i & $\underbrace{\text{var}}_{E[(X_i-\mu_i)^2]}$ σ_i^2.

$\star$ $$E(Z) = \sum_{i=1}^{k} p_i E(X_i)$$

$$\Rightarrow E(Z) = \sum_{i=1}^{k} p_i \mu_i$$

$\star$ $$\operatorname{var}(Z) = E\left[(Z - E(Z))^2\right]$$

$$= E\left[\left(\sum_{i=1}^{k} p_i X_i - \sum_{i=1}^{k} p_i \mu_i\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{k} X_i p_i - \sum_{i=1}^{k} \mu_i p_i\right)^2\right]$$

$$= E\left(\left(\sum_{i=1}^{k} p_i (X_i - \mu_i)\right)^2\right)$$

$$= \left[E\left(\sum_{i=1}^{k} (X_i - \mu_i)^2 p_i^2\right) + E\left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (X_i - \mu_i) p_i \, p_j (X_j - \mu_j)\right)\right]$$

[as all X_i's are independant RV, $(i = 1, \dots, k)$

$$E[(X_i - \mu_i)(X_j - \mu_j)] = \underbrace{E(X_i - \mu_i)}_{0} \, \underbrace{E(X_j - \mu_j)}_{0}$$

$$= 0$$

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So,

$$var(z) = \cancel{E(\Sigma}\ \sum_{i=1}^{k} p_i^2 \underbrace{E(x_i-\mu_i)^2}_{variance} + 0$$

$$\Rightarrow \underline{var(z) = \sum_{i=1}^{k} p_i^2 \sigma_i^2}$$

MgF of $z \Rightarrow$

$$\phi_z(t) = E(e^{tz})$$

$$= E(e^{t\sum_{i=1}^{k} p_i x_i})$$

$$= E(e^{tp_1x_1} \cdot e^{tp_2x_2} \cdots e^{tp_kx_k})$$

$$= E(e^{tp_1x_1})E(e^{tp_2x_2}) \cdots E(e^{tp_kx_k})$$

$$= \prod_{i=1}^{k} \phi_{x_i}(tp_i) = \prod_{i=1}^{k} e^{\mu_i t p_i + \sigma_i^2 \frac{t^2 p_i^2}{2}}$$

$$= e^{t\underbrace{\sum_{i=1}^{k} \mu_i p_i}_{E(z)} + \frac{t^2}{2} \underbrace{\sum_{i=1}^{k} \sigma_i^2 p_i^2}_{var(z)}}$$ already Proved

$$= e^{tE(z) + \frac{t^2}{2} var(z)}$$

PdF of $z \Rightarrow$

$$\frac{1}{\sqrt{var(z)}\sqrt{2\pi}} e^{-\frac{(z-E(z))^2}{2(var(z))}}$$

or $\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$

$$= \frac{1}{\sqrt{2\pi \sum_{i=1}^{k} p_i^2 \sigma_i^2}} \exp\left(\frac{-(z-\sum_{i=1}^{k} p_i \mu_i)^2}{2\sum_{i=1}^{k} p_i^2 \sigma_i^2}\right)$$

$\left[z = \sum_{i=1}^{k} p_i x_i\right]$

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3. Question 3 :

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Answer 3

Given, a Random variable x,

& Let Y be a RV $= X - E(X)$

$Y = X - \mu$

So, using RV $(Y+b)^2$ and a constant $(\tau+b)^2$ $\{>0\}$.

By Markov's Inea.

$$P\left((Y+b)^2 \geq (b+\tau)^2\right) \leq \frac{E\left((Y+b)^2\right)}{(b+\tau)^2}$$

$$= \frac{E(Y^2) + E(2bY) + E(b^2)}{(b+\tau)^2}$$

$$\left[\begin{array}{l} E(2bY) = 2bE(Y) = 2bE(X-\mu) = 2b(0) = 0 \\ \& \quad E(b^2) = b^2 \ \{\text{const.}\} \end{array}\right]$$

$$= \frac{E[(X-\mu)^2] + 0 + b^2}{(b+\tau)^2}$$

$$= \frac{\sigma^2 + b^2}{(b+\tau)^2}, \quad \sigma = \text{var. of } X.$$

Let, $\frac{\sigma^2+b^2}{(b+\tau)^2} = f \rightarrow \frac{df}{db} = \frac{b}{(b+\tau)^2} - \frac{2(\sigma^2+b^2)}{(b+\tau)^3} = 0$

$$\Rightarrow b = \frac{\sigma^2}{\tau}$$

f takes max. value at $b = \frac{\sigma^2}{\tau}$

(min. at $b \rightarrow \infty$)

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So, $P((Y+b)^2 \geq (b+\tau)^2) \leq \frac{\sigma^2+b^2}{(b+\tau)^2} \leq f_{max}$

$$= \frac{\sigma^2 + \left(\frac{\sigma^2}{\tau}\right)^2}{\left(\frac{\sigma^2}{\tau} + \tau\right)^2}$$

$$= \frac{\frac{\sigma^2}{\tau^2}(\sigma^2+\tau^2)}{\frac{1}{\tau^2}(\sigma^2+\tau^2)^2}$$

$$= \frac{\sigma^2}{\tau^2+\sigma^2}$$

So,

$$P((Y+b)^2 \geq (b+\tau)^2) \leq \frac{\sigma^2}{\tau^2+\sigma^2} \quad \text{for all } b.$$

So, $b=0$,

$$P(Y^2 \geq \tau^2) \leq \frac{\sigma^2}{\tau^2+\sigma^2}$$

OR

$$P((X-\mu)^2 \geq \tau^2) \leq \frac{\sigma^2}{\tau^2+\sigma^2}$$

For $t \geq 0$,

$$P(Y^2 \geq \tau^2) = P(Y \geq t) + P(Y \leq -t) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

Finally, $P(Y \geq t), P(Y \leq -t) \geq 0$, So,

$$P(Y \geq t) \leq P(Y^2 \geq \tau^2) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

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So,

$$\Rightarrow P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2} \quad \text{.....} \star$$

we can also write that,

$$P(Y \leq -\tau) \leq P(Y^2 \geq \tau^2) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$P(X-\mu \leq -\tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2} \rightarrow \text{here } \tau > 0$$

Substituting τ by $-\tau$,

$$P(X-\mu \leq -(-\tau)) \leq \frac{\sigma^2}{\sigma^2+(-\tau)^2} \quad ; (-\tau) > 0 \;=\; \tau < 0$$

$$P(X-\mu \leq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$1 - P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\Rightarrow P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\text{.....} \star$$

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4. Question 4 :

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Answer- 4

For $t > 0$, and Random variable X,

$$P(X \geq x) = P(tX \geq tx)$$
$$= P(e^{tX} \geq e^{tx})$$

As the new random variable $e^{tX} > 0$ and Parameter $e^{tx} > 0$,

So, By Markov's Inequality,

$$P(X \geq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}}$$

$$\Rightarrow P(X \geq x) \leq e^{-tx}\, \phi_X(t) \qquad \left\{\phi_X(t) = E(e^{tX})\right\} \quad \text{.......(1)}$$

For $t < 0$,

$$P(\underbrace{X \leq x}_{tX \geq tx}) = P(e^{tX} \geq e^{tx})$$

And By Markov's Ineq as Earlier,

$$P(X \leq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}}$$

$$\Rightarrow P(X \leq x) \leq e^{-tx}\, \phi_X(t) \quad \text{.......(2)}$$

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Now,

Given X = sum of n Independent Bernoulli RV

$$X = X_1 + X_2 + \dots + X_n$$

& $E(X_i) = p_i$ & $\mu = \sum_{i=1}^{n} p_i$

In eq. 1, as $t > 0$ and replacing x with $(1+\delta)\mu$,

we get,

$$P(X \geq (1+\delta)\mu) \leq \frac{E(e^{tX})}{e^{t(1+\delta)\mu}} = \frac{\Phi_X(t)}{e^{t(1+\delta)\mu}} \quad \dots (3)$$

as $X = X_1 + X_2 + \dots + X_n$

where $X_1, X_2, \dots X_n$ are all Independent.

So, $\Phi_X(t) = \prod_{i=1}^{n} \Phi_{X_i}(t) = \prod_{i=1}^{n} E(e^{tX_i})$

For any Bernoulli RV, X_i,

$$MGF = \Phi_{X_i}(t) = (1 - P_i + P_i e^t) = 1 + P_i(e^t - 1)$$

$$\left[\begin{array}{c} \text{as } 1 + x \leq e^x, \text{ replacing } x \Rightarrow P_i(e^t-1), \\ 1 + P_i(e^t - 1) \leq e^{P_i(e^t-1)} \end{array} \right]$$

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Cont. from eq. 3,

$$P(X \geq (1+\delta)\mu) \leq \frac{\phi_X(t)}{e^{t(1+\delta)\mu}} = \frac{\phi_{X_1}(t)\phi_{X_2}(t)\ldots\phi_{X_n}(t)}{e^{t(1+\delta)\mu}}$$

$$\leq \frac{e^{P_1(e^t-1)}\, e^{P_2(e^t-1)}\, e^{P_3(e^t-1)} \ldots e^{P_n(e^t-1)}}{e^{t(1+\delta)\mu}}$$

$$= \frac{e^{(e^t-1)(P_1+P_2 \ldots + P_n)}}{e^{t(1+\delta)\mu}}$$

$$= \frac{e^{(e^t-1)\mu}}{e^{t(1+\delta)\mu}} \qquad \left\{ \text{as } \sum_{i=1}^{n} P_i = \mu \right\}$$

Thus Proved,

$$P(X \geq (1+\delta)\mu) \leq \frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}} \quad \forall\, t.$$

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as above is true for all t,

So, $$P(X \geq (1+\delta)\mu) \leq \left(\frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}}\right)_{min}$$

To find min., $$\frac{d}{dt}\left(e^{\mu(e^t-1) - t(1+\delta)\mu}\right) = 0$$

$$e^{\mu(e^t-1) - t(1+\delta)\mu}\left(\mu e^t - (1+\delta)\mu\right) = 0$$

$$e^t = 1+\delta$$

$$\Rightarrow t = \ln(1+\delta)$$

So, The final Bound is,

$$P(X \geq (1+\delta)\mu) \leq \frac{e^{\mu(\delta+1-1)}}{e^{\mu(1+\delta)\ln(1+\delta)}} = \frac{e^{\mu\delta}}{(1+\delta)^{\mu(1+\delta)}}$$

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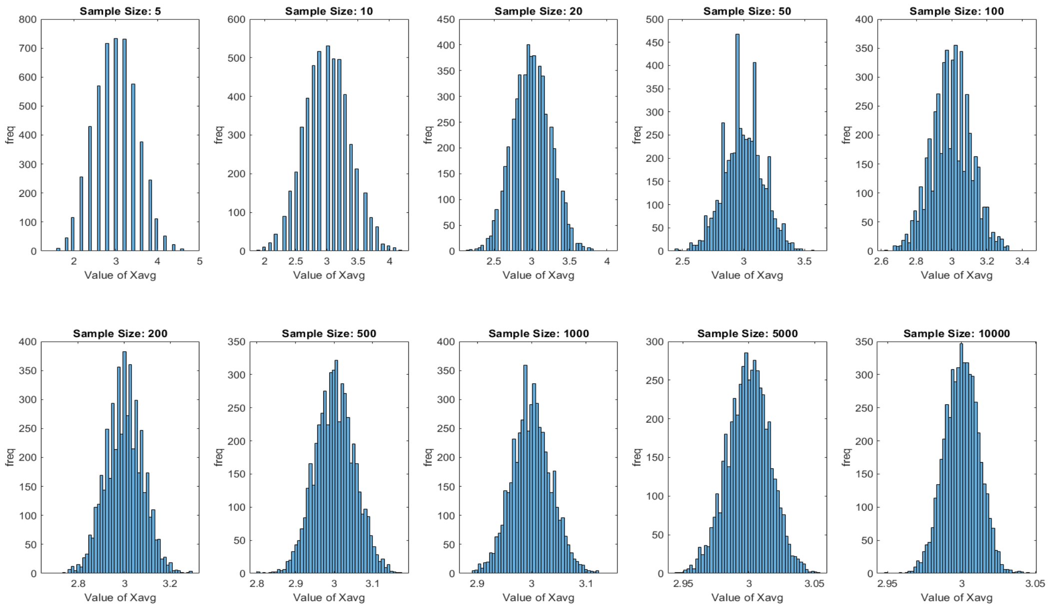
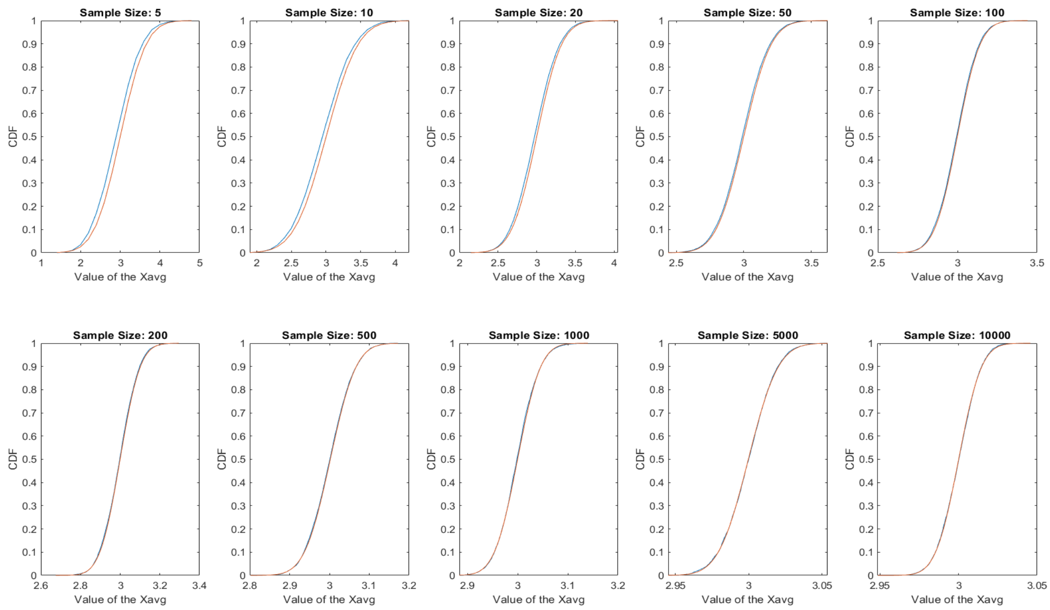
5. Question 5 :

- To run the code, just run the file named as *q5.m* in MatLab

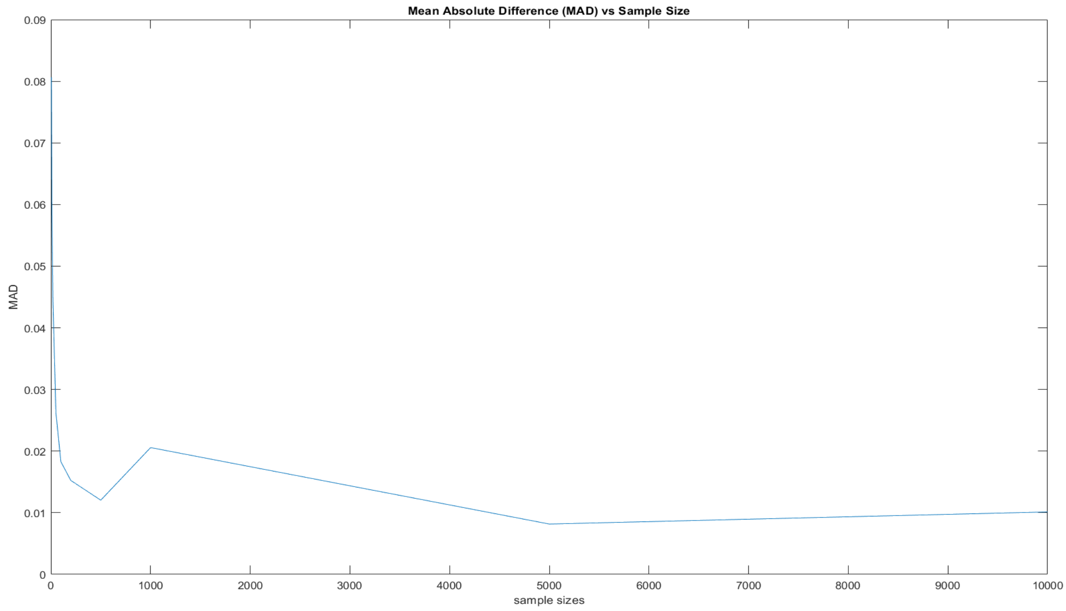
5.1 plot a :



5.2 plot b :



5.3 plot c :


Mean Absolute Difference (MAD) vs Sample Size
MAD
sample sizes
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

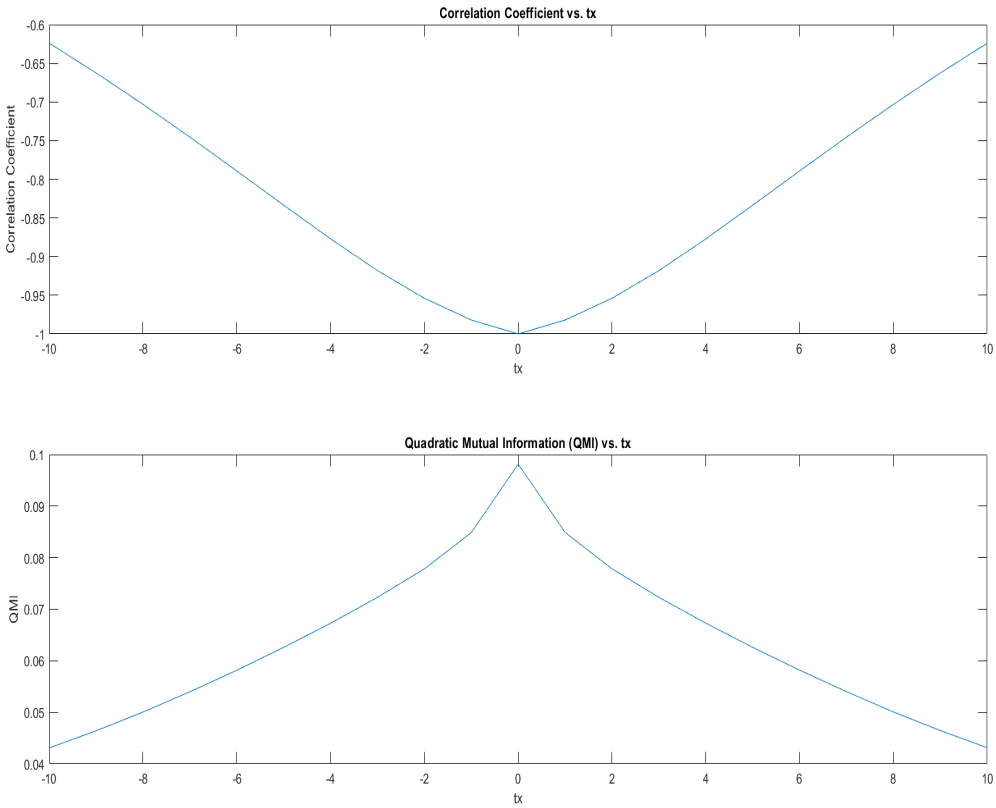
6. Question 6 :

- To run the code, run the file *q6.m*
  - to get data of *original* image, keep the *img2* as the path of *original* image.
  - to get data of the *negative* image, change the *img2* to *255-img1*.

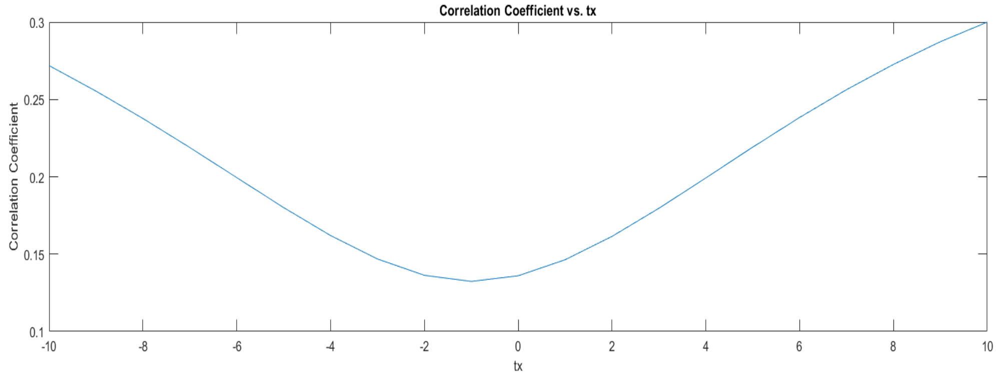
  in MatLab. *Change the path of the image according to its location*

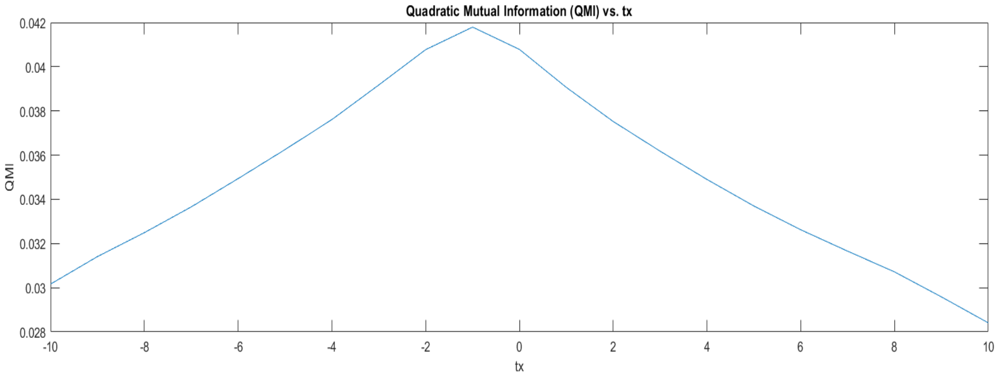
- In the below figure, the blue lines represent the data extracted from the positive i.e., *T1.jpg* and orange represents data of *T2.jpg*.
- In the first section, we calculate the linear dependence between the two images based on the correlation coefficient. As tx gets closer to 0, we observe that there is less linear dependency but more non-linear dependence since the QMI has a substantial value.
- Contrarily, in the second section, we observe that when the two images align, the correlation coefficient approaches peak, indicating a significant linear dependence between the two. Additionally, the QMI is rising as well, indicating some nonlinear connection between the two as well.

Data of the negative image



Data of the actual image





7. Question 7 :

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Answer 7

The moment generating func. of a multinomial distribution is given by,

$$\phi(t) = (p_1 e^{t_1} + p_2 e^{t_2} + p_3 e^{t_3} \dots + p_k e^{t_k})^n$$

where p_i are success probability of multinomial pmf's.

The (ij)th term of covar. matrix be C_{ij}

$$= C_{ij} = E(X_i X_j) - E(X_i)E(X_j)$$

$E(x_i)$ and $E(x_j)$ can be derived by just.

$$E(X_i) = \frac{\partial}{\partial t_i} \phi(t) \Big|_{t_k=0 \ \forall k \in \{1,\dots,n\}}$$

$$E(X_j) = \frac{\partial}{\partial t_j} \phi(t) \Big|_{t_k=0 \ | k \in \{1,\dots,n\}}$$

Similarly,

$$E(X_i X_j) = \frac{\partial}{\partial t_i}\left(\frac{\partial}{\partial t_j} \phi(t)\right)\Big|_{t_k=0 \ \forall k}$$

So, finally,

$$C(X_i;X_j) \text{ or } C_{ij} = \left(\frac{\partial^2 \phi(t)}{\partial t_i \partial t_j} - \frac{\partial \phi(t)}{\partial t_i}\frac{\partial \phi(t)}{\partial t_j}\right)\Bigg|_{t=0}$$

will yield us C_{ij}. (C_{ij} th element in covar. matrix)

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$$E(X_i) = n(p_1e^{t_1}+\dots+p_ke^{t_k})^{n-1}(p_ie^{t_i})\Big|_{t=0}$$
$$= n p_i (p_1+\dots+p_k) = \underline{n p_i}$$

Similarly,

$$E(X_j) = n p_j$$

And

$$E(X_iX_j) = \frac{\partial}{\partial t_i}\left(n(p_1e^{t_1}\dots+p_ke^{t_k})^{n-1}(p_je^{t_j})\right)\Big|_{t=0}$$
$$= np_j\left[e^{t_j}\right](n-1)(p_1e^{t_1}+\dots+p_ke^{t_k})^{n-2}p_ie^{t_i}\Big|_{t=0}$$
$$= np_j(n-1)p_i(p_1+\dots+p_k)^{n-2}$$
$$= (n^2-n)p_ip_j$$

So, $C_{ij} = E(X_iX_j) - E(X_i)E(X_j)$
$$= (n^2-n)p_ip_j - (np_i)(np_j)$$

$$\Rightarrow C_{ij} = \underline{\underline{-np_ip_j}}$$

If $i=j$, $\underline{C_{ii} = -np_i^2}$ } These are the element of our covar. matrix.

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