CS215 Homework Assignment-2

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Contents

1 Q1

Ans-1 – $X_1, X_2, X_3, X_4 - - - X_n$ be $n \geq 0$

$Y_1 = \max(X_1, X_2, X_3, - - - X_n)$

$F_{Y_1}(x) = P(X_{max} \leq x)$

so in this every $X_i \leq x$

$F_{Y_1}(x) = P(X_1 \leq x, X_2 \leq x, - - - X_n \leq x)$

since all random variables are independent –

$= P(X_1 \leq x) \cdot P(X_2 \leq x) \cdot - - - P(X_n \leq x)$

$F_{Y_1}(x) = F_{X_1}(x) \cdot F_{X_2}(x) - - - - F_{X_n}(x)$

similarly,

$Y_2 = \min(X_1, X_2, X_3 - - - X_n)$

$X_{min} \leq x$

$F_{Y_1}(x) = P(X_{min} \leq x)$

$= 1 - P(X_{min} > x)$

$= 1 - P(X_1 > x, X_2 > x, X_3 > x, - - - X_n > x)$

$= 1 - P(X_1 > x) \cdot P(X_2 > x) - - - P(X_n > x)$

$= 1 - (1 - F_{X_1}(x))(1 - F_{X_2}(x)) - - - (1 - F_{X_n}(x))$

$f_{Y_1}(x) = \frac{d}{dx}(F_{X_1}(x) \cdot F_{X_2}(x) - - - F_{X_n}(x))$

$= f_{X_1}(x) F_{X_2}(x) - - F_{X_n}(x) + F_{X_1}(x) f_{X_2}(x) - - - F_{X_n}(x)$

$+ F_{X_1}(x) F_{X_2}(x) f_{X_3}(x) - - F_{X_n}(x)$

$+ F_{X_1}(x) F_{X_2}(x) - - - f_{X_n}(x).$

Figure 1: Caption

Similarly,

$$f_{Y_2}(x) = f_{X_1}(x)[1-F_{X_2}(x)][1-F_{X_3}(x)] \cdots [1-F_{X_k}(x)] +$$
$$[1-F_{X_1}(x)] f_{X_2}(x) \cdots [1-F_{X_k}(x)] +$$
$$\vdots$$
$$[1-F_{X_1}(x)][1-F_{X_2}(x)] \cdots f_{X_k}(x).$$

If all $F_{X_i}(x)$ are same $\forall i \in \{1,2,3 \ldots k\}$

then $F_{Y_1}(x) = [F_X(x)]^n$

$$f_{Y_1}(x) = n\,[F_X(x)]^{n-1} f_X(x)$$

$$F_{Y_2}(x) = 1 - (1-F_X(x))^k$$

$$f_{Y_2}(x) = (n).(1-F_X(x))^{n-1} f_X(x)$$

Figure 2: Caption

2 Q2

Ans-2 - Given — X belongs to Gaussian Mixture Model (GMM)

$$X \sim \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)$$

$$\sum_{i=1}^{K} p_i = 1 \quad \forall \; 0 \le p_i \le 1$$

(p_i is mixing probability)

Now,

$X \sim \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)$ is the probability distribution function

$$\text{MGF of } X = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)\, dx$$

$$= \sum_{i=1}^{K} p_i \int_{-\infty}^{\infty} e^{tx} \mathcal{N}(\mu_i, \sigma_i^2)\, dx$$

We know that

$$\boxed{\phi_X(t) = \int_{-\infty}^{\infty} e^{tx} \mathcal{N}(\mu, \sigma^2)\, dx = e^{\mu t + \frac{\sigma^2 t^2}{2}}} \quad \text{— (1)}$$

MGF of gaussian

From (1)

$$\text{MGF}(x) = \boxed{\sum_{i=1}^{K} p_i e^{\mu_i t + \frac{\sigma_i^2 t^2}{2}}}$$

$E(x) = \text{MGF}_x'(0) \rightarrow$ derivative of MGF of x at $t=0$

$$E(x) = \left[\sum_{i=1}^{K} p_i \left(e^{\cancel{\mu_i t + \frac{\sigma_i^2 t^2}{2}}}\right)\left(\mu_i + \cancel{\sigma_i^2 t}\right)\right]_{t=0}$$

$$\boxed{E(x) = \sum_{i=1}^{K} p_i \mu_i} \quad \text{— (2)}$$

$$Var(x) = \text{MGF}_x''(0) - (\text{MGF}_x'(0))^2$$

$$= \sum_{i=1}^{K} (p_i) \left[\left(e^{\cancel{\mu_i t + \frac{\sigma_i^2 t^2}{2}}}\right)\left(\mu_i + \cancel{\sigma_i^2 t}\right)^2 + \left(e^{\cancel{\mu_i t + \frac{\sigma_i^2 t^2}{2}}}\right)\left(\sigma_i^2\right) \right]_{t=0} - \left(\sum_{i=1}^{K} p_i \mu_i\right)^2$$

Figure 3: Caption

$$\boxed{Var(X) = \sum_{i=1}^{k} p_i(\mu_i^2 + \sigma_i^2) - \left(\sum_{i=1}^{k} p_i\mu_i\right)^2}$$

Given $Z = \sum_{i=1}^{k} p_i X_i$, $X_i \sim N(\mu_i, \sigma_i^2)$ ③

$$E(Z) = \sum_{i=1}^{k} p_i E(X_i)$$

from ③

$E(X_i) = \mu_i \quad \forall i \in [1,2,3 \ldots k]$

$$= \boxed{\sum_{i=1}^{k} p_i\mu_i}$$

$$Var(Z) = E\left(\left(Z - E(Z)\right)^2\right)$$

$$= E\left(\left(\sum_{i=1}^{k} p_i X_i - \sum_{i=1}^{k} p_i\mu_i\right)^2\right)$$

$$= E\left(\sum_{i=1}^{k} p_i^2 (X_i-\mu_i)^2\right) + \sum_{\substack{i=1 \\ i\neq j}}^{k}\sum_{j=1}^{k} E\left(p_i(X_i-\mu_i)\, p_j(X_j-\mu_j)\right)$$

$$= \sum_{i=1}^{k} p_i^2 E(X_i-\mu_i)^2 + \sum_{\substack{i=1 \\ i\neq j}}^{k}\sum_{j=1}^{k} p_i p_j \cancel{E(X_i-\mu_i)}^{0} \times E(X_j-\mu_j)$$

Given $\{X_i\}_{i=1}^{k}$ are independent random variables.

So, $E\left[(X_i-\mu_i)(X_j-\mu_j)\right]$

$= E(X_i-\mu_i)E(X_j-\mu_j)$ ④

Figure 4: Caption

we know that

$$E(x_i-\mu_i)=0 \quad \text{and} \quad E(x_i-\mu_i)^2=\sigma_i^2$$

$$\therefore \boxed{Var(Z)=\sum_{i=1}^{K} p_i^2\sigma_i^2}$$

$$\text{MGF of } Z \Rightarrow \phi_X(t)=E(e^{tZ})$$

$$\Rightarrow E\left(e^{t\sum p_i x_i}\right)$$

$$\Rightarrow E\left(e^{tp_1x_1+tp_2x_2+\cdots tp_kx_k}\right)$$

$$\Rightarrow E\left(e^{tp_1x_1}\cdot e^{tp_2x_2} e^{tp_3x_3} \ldots e^{tx_kp_k}\right)$$

$$= E(e^{tp_1x_1})\cdot E(e^{tp_2x_2}) \ldots E(e^{tp_kx_k})$$

$$= \prod_{i=1}^{K} \phi_{x_i}(tp_i)$$

$$= \prod_{i=1}^{K} e^{\mu_i tp_i + \frac{\sigma_i^2(tp_i)^2}{2}}$$

$$\boxed{\text{using } \phi_X(t)=e^{\mu t+\frac{\sigma^2t^2}{2}}}$$

for gaussian

$$= e^{t\sum p_i\mu_i + \frac{t^2}{2}\sum p_i^2\sigma_i^2}$$

$$= e^{tE(Z)+\frac{t^2}{2}Var(Z)}$$

Figure 5: Caption

PDF of Z

We know that sum of independent Gaussian random variables is also a Guassian variable (By taking MGFs of each random variable and multiplying them).

So pdf Z is

$$f(Z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ = mean of Z

σ = standard deviation of Z

$$\mu = E(X) = \sum_{i=1}^{K} p_i \mu_i$$

$$\sigma^2 = Var(X) = \sum_{i=1}^{K} p_i \sigma_i^2$$

$$\therefore f(Z) = \frac{1}{\sqrt{2\pi \sum_{i=1}^{k} p_i \sigma_i^2}} e^{-\frac{\left(x - \sum_{i=1}^{k} p_i \mu_i\right)^2}{2\sum_{i=1}^{k} p_i \sigma_i^2}}$$

Figure 6: Caption

3 Q3

we can check that $f''(b)$ is $-ve$ at this point. So, this is maxima.

$$\Rightarrow \quad P\left((X-\mu+b)^2 \geq (a+b)^2\right) \leq \frac{\sigma^2+\frac{\sigma^4}{a^2}}{\left(a+\frac{\sigma^2}{a}\right)^2}$$

$$P\left((X-\mu+b)^2 \geq (a+b)^2\right) \leq \frac{\sigma^2}{a^2+\sigma^2}$$

Since this is true $\forall b$, take $b=0$.

$$\Rightarrow \quad P\left((X-\mu)^2 \geq a^2\right) \leq \frac{\sigma^2}{a^2+\sigma^2}$$

Replace a by τ.

$$\Rightarrow \quad P\left((X-\mu)^2 \geq \tau^2\right) \leq \frac{\sigma^2}{\tau^2+\sigma^2}$$

Now, $P\left((X-\mu)^2 \geq \tau^2\right) = P(X-\mu \geq \tau) + P(X-\mu \leq -\tau)$

where τ is a (+ve) number.

$$\Rightarrow \quad P(X-\mu \geq \tau) \leq P\left((X-\mu)^2 \geq \tau^2\right) \leq \frac{\sigma^2}{\tau^2+\sigma^2}$$

$\boxed{\text{First part proved}}$

Now, if τ is -ve,

$$P\left((X-\mu)^2 \geq \tau^2\right) = P(X-\mu \leq \tau) + P(X-\mu \geq -\tau)$$

$$\Rightarrow \quad P(X-\mu \leq \tau) \leq P\left((X-\mu)^2 \geq \tau^2\right) \leq \frac{\sigma^2}{\tau^2+\sigma^2}$$

Figure 7: Caption

3) From Markov's identity,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Now, since X is a random variable, $X-\mu-b$ is also a random variable and sq is $(X-\mu-b)^2$

so, taking this and applying Markov's identity,

$$P\left((X-\mu+b)^2 \geq (a+b)^2\right) \leq \frac{E\left((X-\mu+b)^2\right)}{(a+b)^2}$$

$$E\left((X-\mu+b)^2\right) = E\left((X-\mu)^2+b^2+2b(X-\mu)\right)$$

$$= E\left((X-\mu)^2\right) + E\left(b^2\right) + E\left(2b(X-\mu)\right)$$

Now $E\left(2b(X-\mu)\right) = 0$ since $E(X-\mu) = E(X) - \mu = 0$

so,

$$P\left((X-\mu+b)^2 \geq (a+b)^2\right) \leq \frac{E\left((X-\mu)^2\right) + E\left(b^2\right)}{(a+b)^2}$$

$$P\left((X-\mu+b)^2 \geq (a+b)^2\right) \leq \frac{\sigma^2+b^2}{(a+b)^2},$$

where σ^2 is the variance of X.

Now, we find the max. of $\frac{\sigma^2+b^2}{(a+b)^2}$ with respect to b.

$$f = \frac{\sigma^2+b^2}{(a+b)^2}$$

$$f' = 0 \Rightarrow \frac{2b}{(a+b)^2} - \frac{2(\sigma^2+b^2)}{(a+b)^3} = 0 \Rightarrow b = \frac{\sigma^2}{a}$$

Figure 8: Caption

$$P(X-\mu \leq \tau) = 1 - P(X-\mu > \tau)$$

$$\Rightarrow \quad 1 - P(X-\mu > \tau) \leq \frac{\sigma^2}{\tau^2+\sigma^2}$$

$$\Rightarrow \quad P(X-\mu > \tau) \geq 1 - \frac{\sigma^2}{\tau^2+\sigma^2}$$

Now $P(X-\mu \geq \tau) = P(X-\mu = \tau) + P(X-\mu > \tau)$

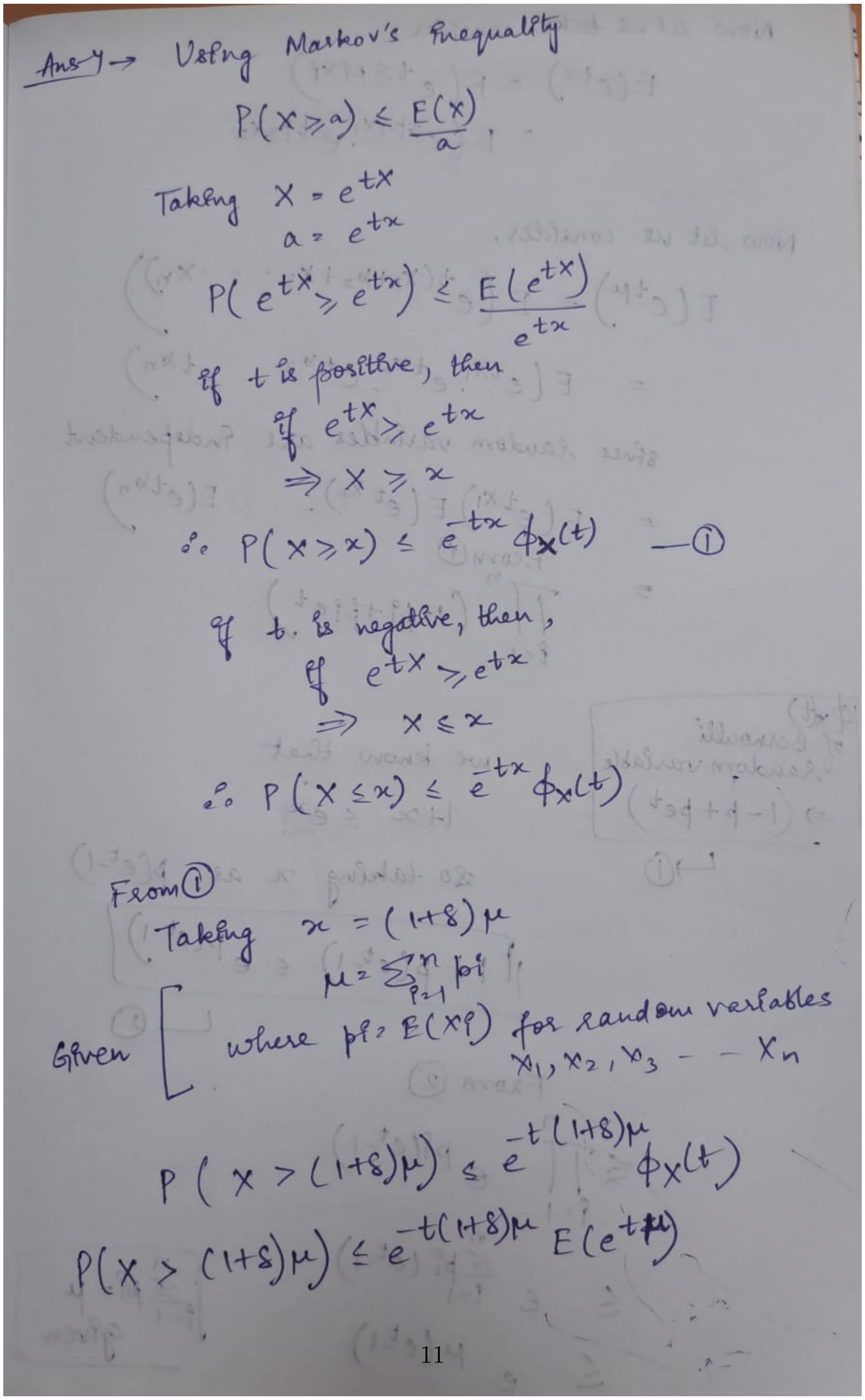
$$\Rightarrow \quad P(X-\mu \geq \tau) \geq P(X-\mu > \tau)$$

$$\Rightarrow \quad P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\tau^2+\sigma^2}$$

$\boxed{\text{Second part proved}}$

Figure 9: Caption

4 Q4



Ans 4 → Using Markov's inequality

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Taking $X = e^{tX}$

$a = e^{tx}$

$$P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}}$$

if t is positive, then

if $e^{tX} \geq e^{tx}$

$\Rightarrow X \geq x$

$\therefore P(X \geq x) \leq e^{-tx} \phi_X(t)$ — (1)

if t is negative, then,

if $e^{tX} \geq e^{tx}$

$\Rightarrow X \leq x$

$\therefore P(X \leq x) \leq e^{-tx} \phi_X(t)$

From (1)

Taking $x = (1+\delta)\mu$

$\mu = \sum_{i=1}^{n} p_i$

Given [where $p_i = E(X_i)$ for random variables $X_1, X_2, X_3 - - X_n$

$$P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \phi_X(t)$$

$$P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} E(e^{tX})$$

Figure 10: Caption

Now let us consider,

$$E(e^{t\mu}) = E\left(e^{t(X_1+X_2+X_3 \cdots X_n)}\right)$$

$$= E\left(e^{tX_1} \cdot e^{tX_2} \cdot e^{tX_3} \cdots e^{tX_n}\right)$$

since random variables are independent.

$$= E(e^{tX_1})\, E(e^{tX_2}) \cdots E(e^{tX_n})$$

From ①

$$= \prod_{i=1}^{n} (1-p_i+p_i e^t)$$

$\phi_X(t)$ of Bernoulli random variable $\Rightarrow (1-p+pe^t)$ ①

we know that

$$1+x \le e^x$$

so taking x as $p(e^t-1)$

$$\boxed{1+p(e^t-1) \le e^{p(e^t-1)}} \quad ②$$

From ②

$$\le \prod_{i=1}^{n} e^{p_i(e^t-1)}$$

$$\le e^{\sum_{i=1}^{n} p_i (e^t-1)}$$

$\sum_{i=1}^{n} p_i = \mu$ given

$$\le e^{\mu(e^t-1)}$$

Figure 11: Caption

So

$$P(x > (1+\delta)\mu) \leq \frac{e^{\mu(e^t - 1)}}{e^{(1+\delta)t\mu}}$$

Hence proved.

Figure 12: Caption

5 Q5

5.1 Plots

The plots that we got for this is are:

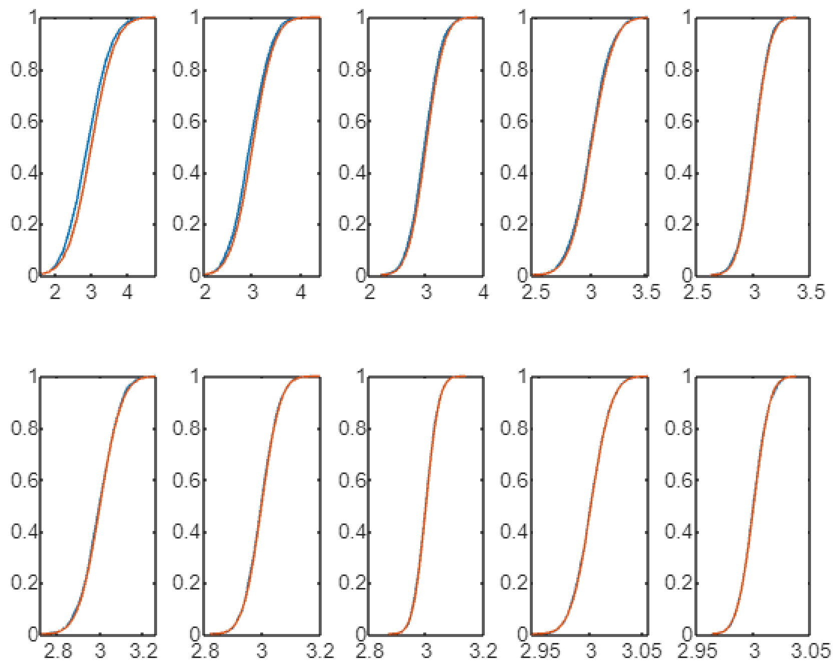


Figure 13: Part(a)

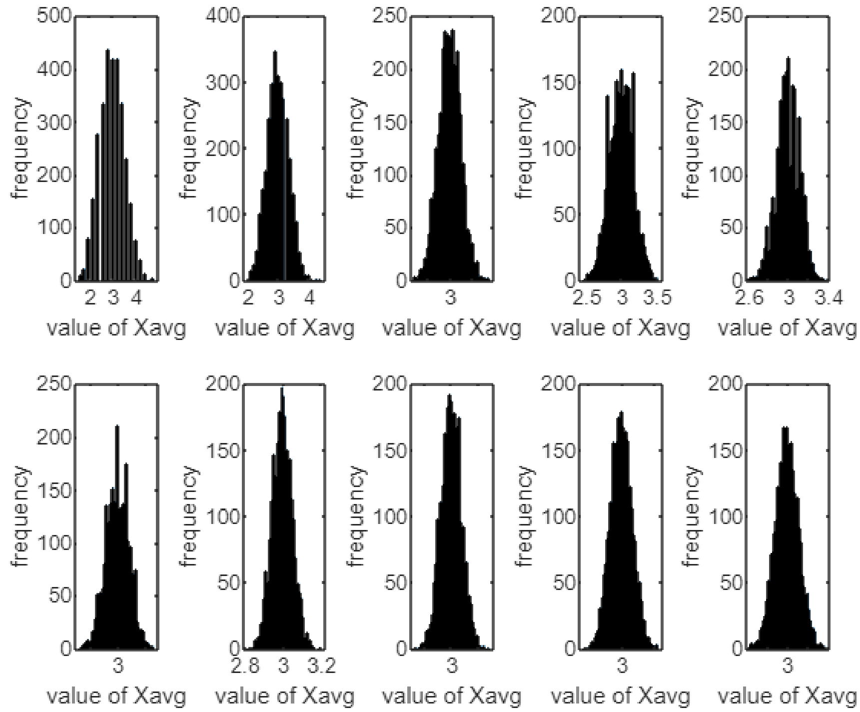


Figure 14: Part(b)

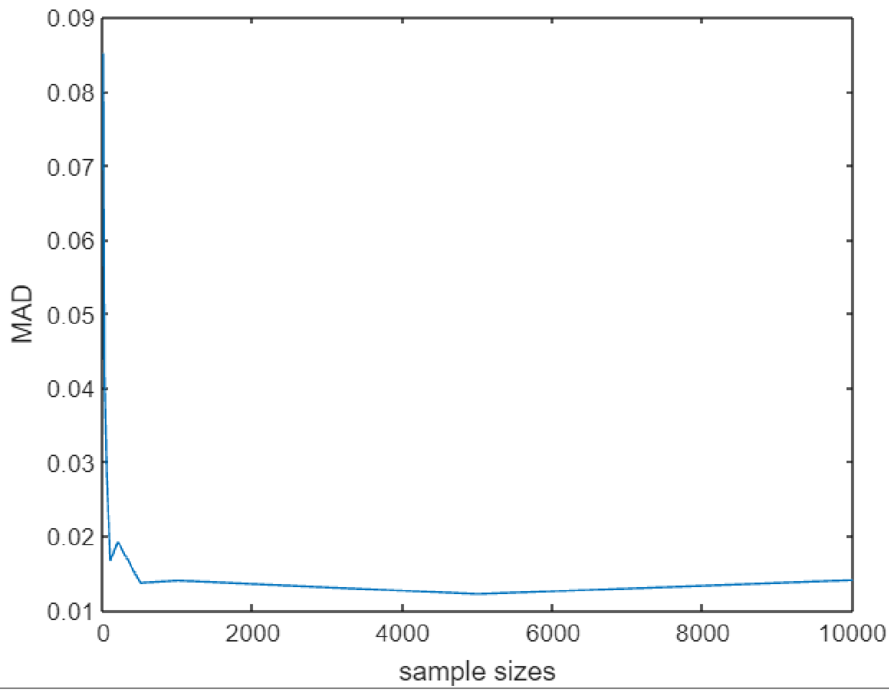


Figure 15: Part(c)

5.2 How to run the code

To run the code, open the matlab terminal where the files of the code are present. Then type "q5" to run the code.

6 Q6

The plots for correlation coefficient and QMI for two images are:

1. For image 1-

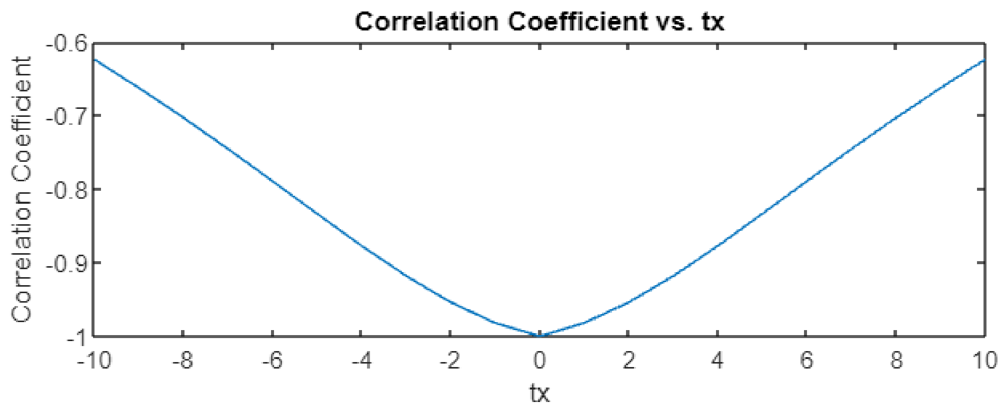


Figure 16: corr coeff v/s shift

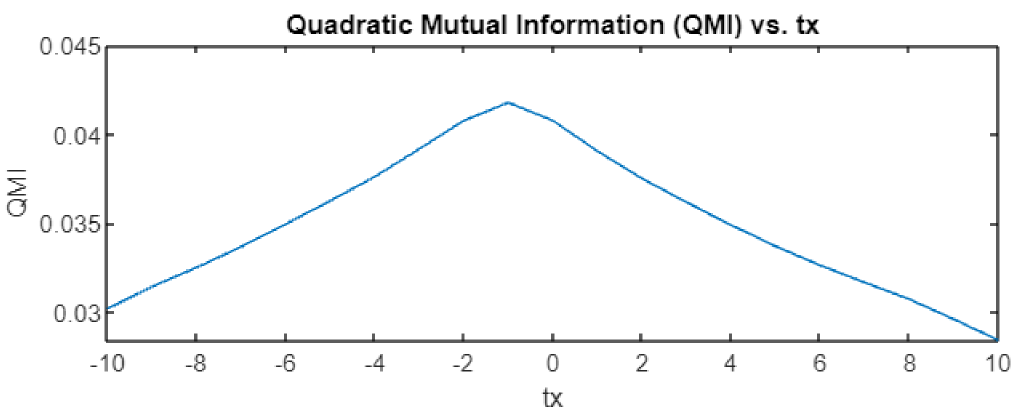


Figure 17: QMI v/s shift

2. For image 2-

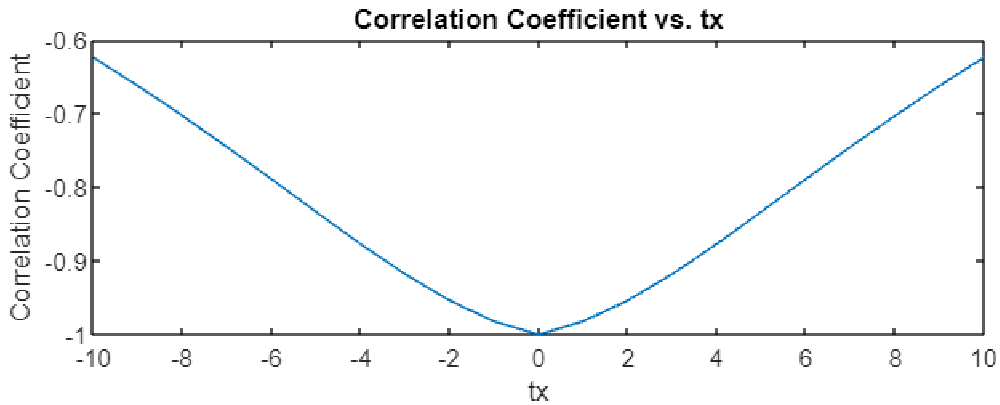


Figure 18: corr coeff v/s shift

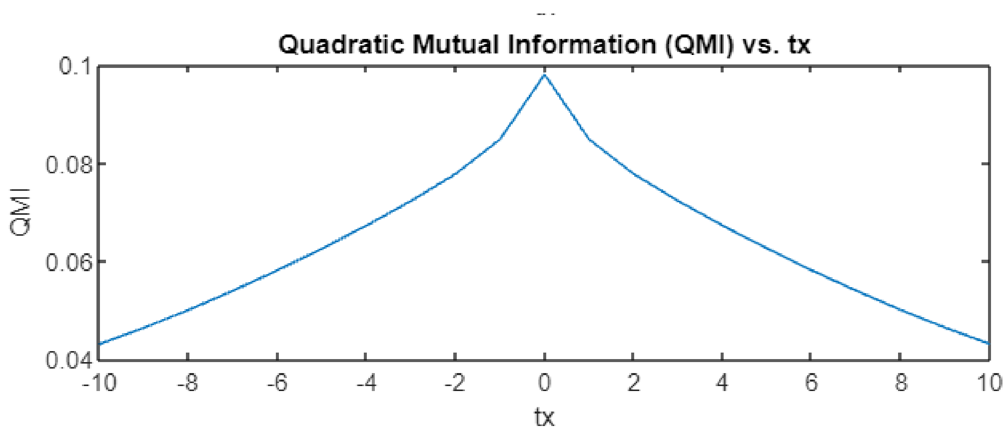


Figure 19: QMI v/s shift

6.1 Comments on the plots

In the first part, from the Correlation coefficient, we measure the linear dependence between the two images. We see that as tx approaches 0, there is not much of a linear dependence but there is some non-linear dependence as the QMI shows some significant value
In the second part, on the contrary, we see that when the two images come to an alignment, the correlation coefficient approaches 1, which means that there is a large linear dependence between the two. Also, the QMI is also increasing, which means there is some non-linear dependence between the two as well.

6.2 How to run the code

To run the code, open the matlab terminal where the files of the code are present. Then type "q6I1" to run the code for image-1(T1), and type "q6I2" to run the code for image-2(T2).

7 Q7

The (i,j)th term of the covarience matrix is $C(X_i, X_j)$ by definition.

$$C(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j).$$

Also, MGF:

$$\phi = E\left(e^{t_1X_1+t_2X_2+\cdots}\right)$$

From ϕ, we need to get $E(X_iX_j), E(X_i), E(X_j)$

By observation, we see,

$$\frac{\partial\phi}{\partial t_i} = \frac{\partial}{\partial t_i} E\left(e^{t_1X_1+t_2X_2+\cdots}\right)$$

$$= E\left(\frac{\partial}{\partial t_i}\left(e^{t_1X_1+t_2X_2+\cdots}\right)\right)$$

$$= E\left(X_i\, e^{t_1X_1+t_2X_2+\cdots}\right)$$

Taking its value at $t_i = 0\ \forall i$,

$$\left.\frac{\partial\phi}{\partial t_i}\right|_{t_i=0\ \forall i} = E(X_i)$$

Similarly,

$$\left.\frac{\partial^2\phi}{\partial t_i\partial t_j}\right|_{t_i=0\ \forall i} = E(X_iX_j)$$

Figure 20: Caption

$$\Rightarrow C(x_i, x_j) = \frac{\partial^2 \phi}{\partial t_i \partial t_j} - \left(\frac{\partial \phi}{\partial t_i}\right)\left(\frac{\partial \phi}{\partial t_j}\right)$$

This will give the (i,j)th element of the covarience matrix $\forall\ i,j$.

Figure 21: Caption