The Power of Sparse and Low-Rank Optimization Paradigms for Network Densification

Yuanming Shi

School of Information Science and Technology

ShanghaiTech University

Outline

- **Introduction**
- **Two Vignettes:**
  - **Group Sparse Beamforming** for Green Cloud-RAN
  - **Low-Rank Matrix Completion** for Topological Interference Management
- **Summary**

Part I: Introduction

Ultra Mobile Broadband

- **Era of mobile data traffic deluge**

10x

Data growth by 2019

497M

Mobile devices added in 2014

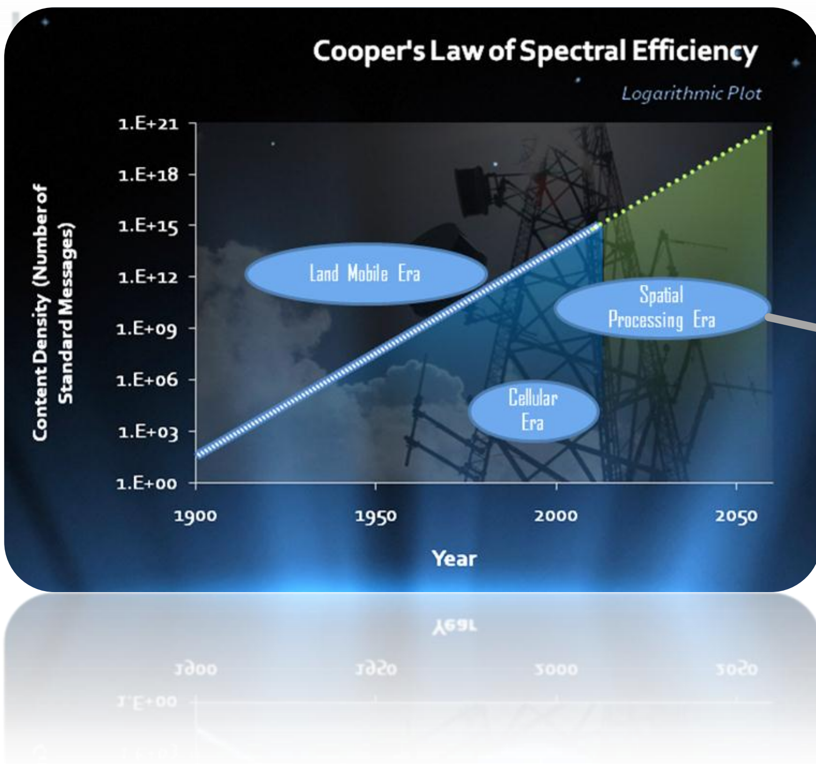
72%

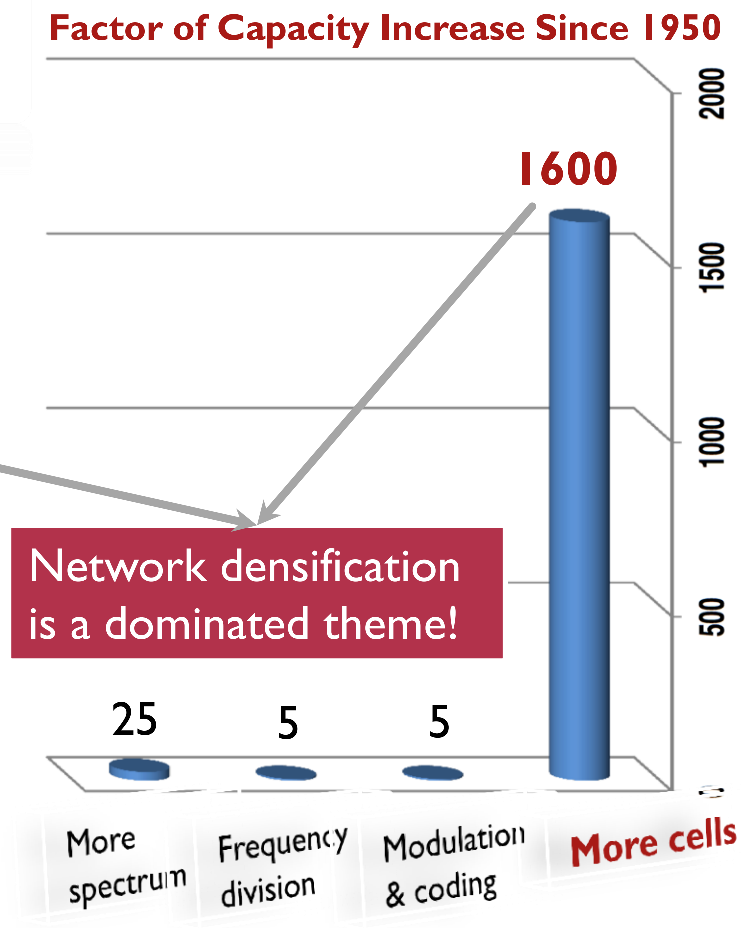
Video traffic by 2019

Source: Cisco VNI Mobile, 2015

Solution?

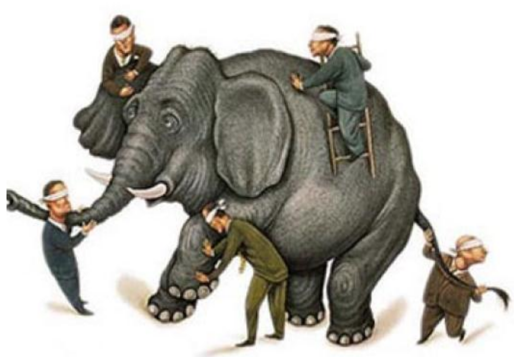
Marty Pens Cooper's Law: Data Over Usable Spectrum Doubles Every 30 Months – 1997

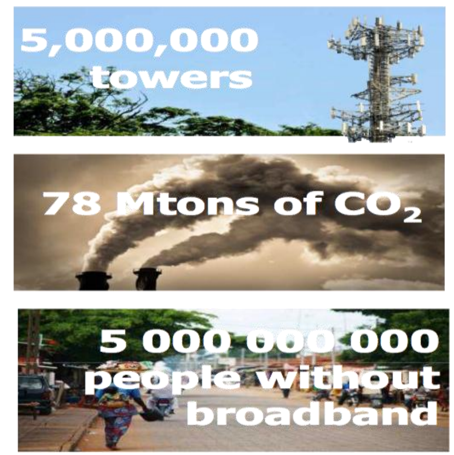




Challenges: Green, Flexibility, Scalability

- **Networking issues:**
  - Huge network power consumption
  - Massive channel state information acquisition

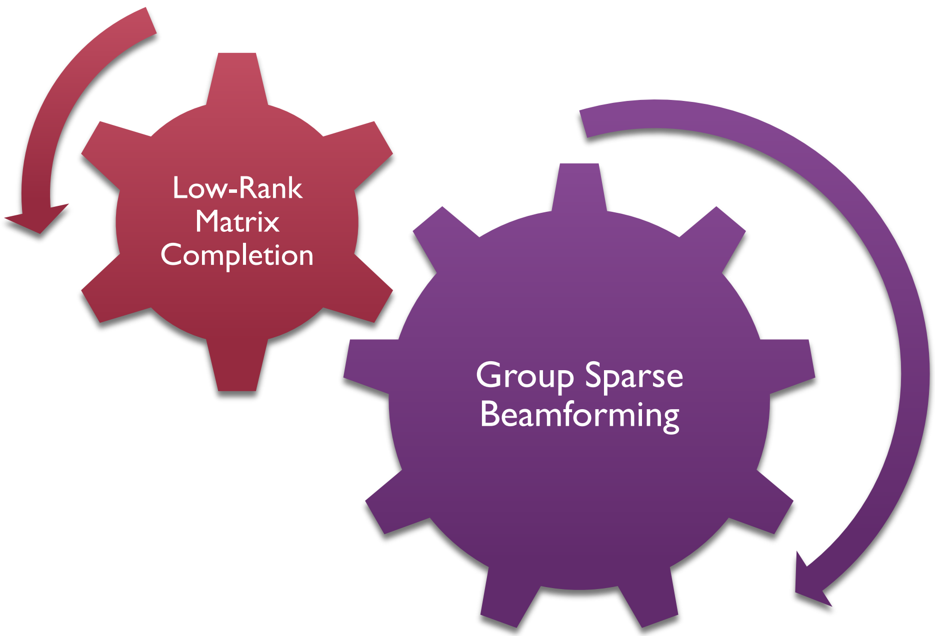




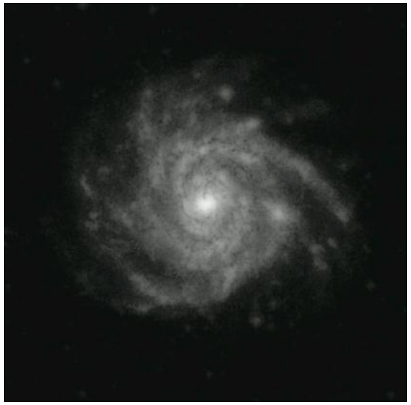
Credit: Alcatel-Lucent, 2013

- **Computing issues:**
  - Large-scale performance optimizations
  - Critical for latency

Part II: Two Vignettes

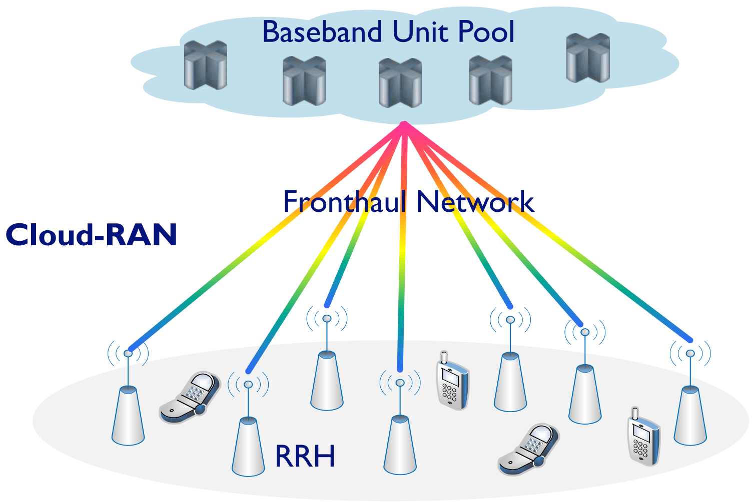


Vignette A: Group Sparse Beamforming for Green Cloud-RAN



Dense Cloud Radio Access Networks

- **Dense Cloud-RAN:** A cost-effective way for network densification and cooperation

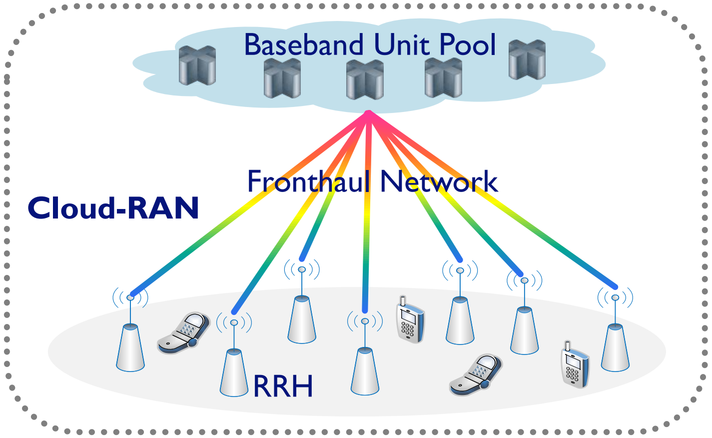


Cost-effective cooperative wireless networks to improve the network capacity and network energy efficiency

1. Centralized signal processing and resource allocation
2. Dense deployment of low-cost low-power RRHs
3. Real-time cloud infrastructure with BS virtualization

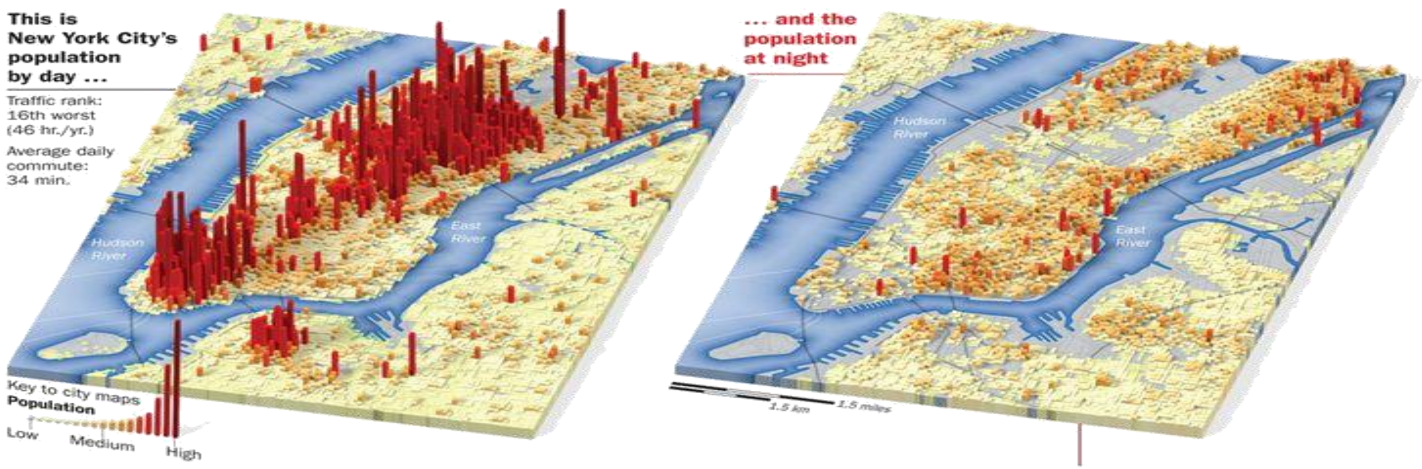
Network Power Consumption

- **Goal:** Design a green dense Cloud-RAN
- **Prior works:** Physical-layer transmit power consumption
  - Wireless power control: [Chiang, *et al.*, FT 08], [Qian, *et al.*, TWC 09], [Sorooshyari, *et al.*, TON 12], …
  - Transmit beamforming: [Sidiropoulos and Luo, TSP 2006], [Yu and Lan, TSP 07], [Gershman, et al., SPMag 10],…
- **Unique challenge:**
  - Network power consumption:
    - RRHs, fronthaul links, etc.



Network Adaptation

- **Question:** Can we provide a holistic approach for network power minimization?
- **Key observation:** Spatial and temporal mobile data traffic variation



- **Approach:** Network adaptation
  - Adaptively switch off network entities to save power

Problem Formulation

- **Goal:** Minimize network power consumption in Cloud-RAN

$$\underset{\mathbf{v}\in\mathcal{C}}{\text{minimize}} \quad f_1(\mathbf{v}) + f_2(\mathbf{v}) \qquad \text{combinatorial composite function}$$

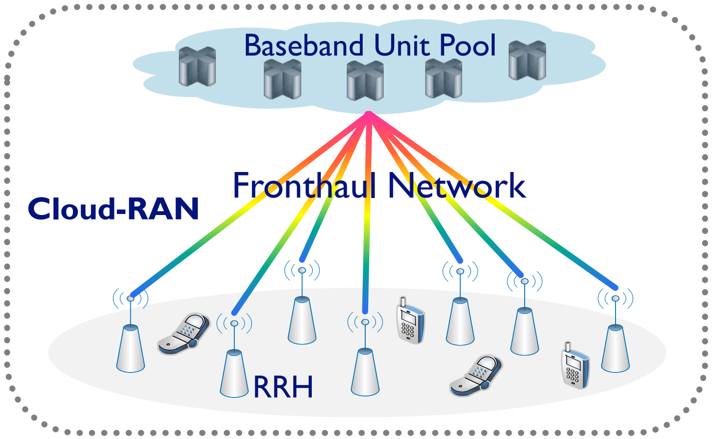
$$\text{subject to} \quad \frac{|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_k|^2}{\sum_{i\neq k}|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_i|^2 + \sigma_k^2} \geq \gamma_k, \forall k.$$

  - Fronthaul power: $f_1(\mathbf{v}) = \sum_{l=1}^{L} P_l^c I(\mathcal{T}(\mathbf{v}) \cap \mathcal{V}_l \neq \emptyset)$
  - Transmit power: $f_2(\mathbf{v}) = \sum_{l=1}^{L}\sum_{k=1}^{K}\frac{1}{\zeta_l}\|\mathbf{v}_{lk}\|_2^2$

- Prior algorithms: heuristic or computationally expensive: [Philipp, et. al, TSP 13], [Luo, et. al, JSAC 13], [Quek, et. al, TWC 13],…

Finding Structured Solutions

- **Proposal:** Group sparse beamforming framework [1]



$$\mathbf{v} = [\underbrace{\mathbf{v}_{11}^T, \ldots, \mathbf{v}_{1K}^T}_{\tilde{\mathbf{v}}_1^T}, \ldots, \underbrace{\mathbf{v}_{L1}^T, \ldots, \mathbf{v}_{LK}^T}_{\tilde{\mathbf{v}}_L^T}]^T$$

Beamforming coefficients of the first RRH, forming a **group**

- Switch off the l-th RRH $\rightarrow$ $\tilde{\mathbf{v}}_l = \mathbf{0}$, i.e., group sparsity structure in $\mathbf{v}$

[1] Y. Shi, J. Zhang, and K. B. Letaief, "Group sparse beamforming for green Cloud-RAN," *IEEE Trans. Wireless Commun.*, vol. 13, pp. 2809– 2823, May 2014.

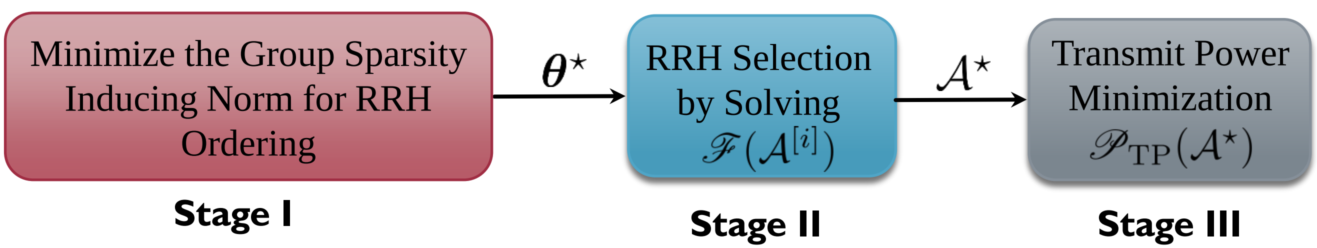
Proposed Algorithm

- **Proposition [1]:** The *tightest* convex positively homogeneous lower bound of the combinatorial composite objective function

$$\Omega(\mathbf{v}) = 2\sum_{l=1}^{L}\sqrt{\frac{P_l^c}{\eta_l}}\|\tilde{\mathbf{v}}_l\|_2 \quad \Longrightarrow \quad \underset{\mathbf{v}\in\mathcal{C}}{\text{minimize}} \;\; \Omega(\mathbf{v})$$

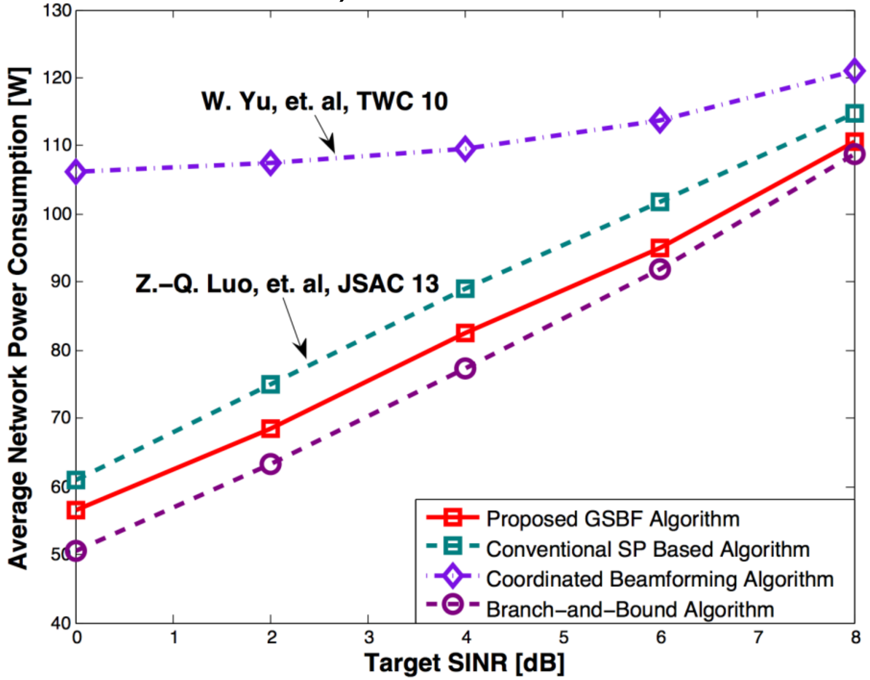
mixed ℓ_1/ℓ_2-norm — induce group sparsity

- Adaptive RRH selection: switch off the RRHs with smallest coefficients in the aggregative beamformers



The Power of Group Sparse Beamforming

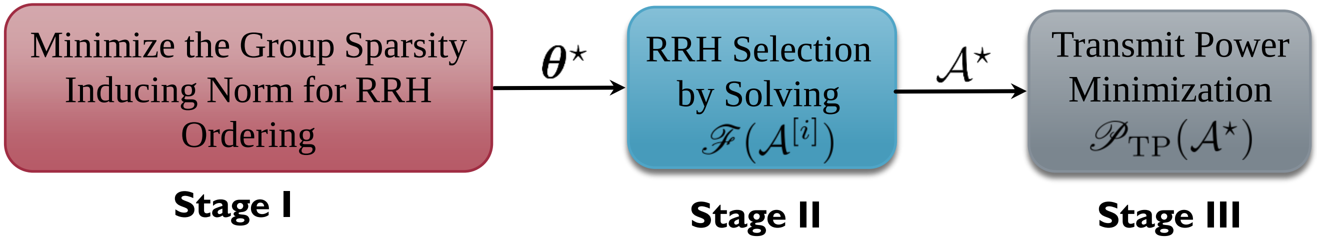
- **Example:** Group spare beamforming for green Cloud-RAN [1] (10 RRHs, 15 MUs)



Advantages:
1) Enabling flexible network adaptation;
2) Offering efficient algorithm design via convex programming
3) Empowering wide applications

[1] Y. Shi, J. Zhang, and K. B. Letaief, "Group sparse beamforming for green Cloud-RAN," *IEEE Trans. Wireless Commun.*, vol. 13, pp. 2809– 2823, May 2014.

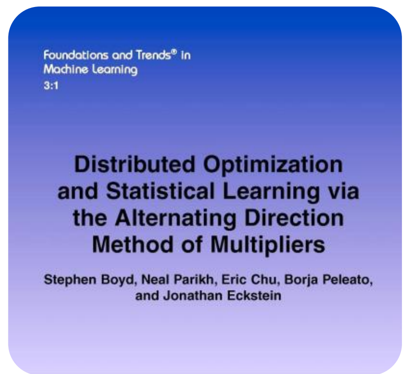
Scalability in Dense Cloud-RAN?



High computational complexity: a sequence of convex optimization and feasibility problems needs to be solved.

Solution: Large-Scale Convex Optimization for Dense Cloud-RAN

$$\xrightarrow{\mathscr{P}_{\text{Original}}} \boxed{\text{Matrix Stuffing}} \xrightarrow{\mathscr{P}_{\text{HSD}}} \boxed{\text{ADMM Solver}} \xrightarrow{\mathbf{x}^\star}$$


Foundations and Trends® in
Machine Learning
3:1

Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers

Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato, and Jonathan Eckstein

Large-Scale Convex Optimization

- **Large-scale convex optimization:** A powerful tool for system design in dense wireless networks

Beamforming, wireless caching, user admission control, etc.

IEEE TRANSACTIONS ON SIGNAL PROCESSING, VOL. 63, NO. 18, SEPTEMBER 15, 2015

Large-Scale Convex Optimization for Dense Wireless Cooperative Networks

Yuanming Shi, *Student Member, IEEE*, Jun Zhang, *Member, IEEE*, Brendan O'Donoghue, and Khaled B. Letaief, *Fellow, IEEE*

- **Prior works:** Mainly focus on small-size networks or well-structured problems
  - Limitations: scalability [Luo, et al., SPMag 10], parallelization [Yu and Lan, TWC 10], infeasibility detection [Liao, et al., TSP 14], …
- **Unique challenges in dense Cloud-RAN:**
  - Design problems: 1) A high dimension; 2) a large number of constraints; 3) complicated structures

Matrix Stuffing and Operator Splitting

- **Goal:** Design a unified framework for general large-scale convex optimization problem $\mathscr{P}_{\text{Original}}$?
- **Disciplined convex programming framework** [Grant & Boyd '08]

$$\xrightarrow{\mathscr{P}_{\text{Original}}} \boxed{\text{CVX}} \xrightarrow{\mathscr{P}_{\text{cone}}} \boxed{\text{Interior-Point Solver}} \xrightarrow{\mathbf{x}^{\star}}$$

CVX RESEARCH

Time consuming: modeling phase & solving phase

- **Proposal:** Two-stage approach for large-scale convex optimization

$$\xrightarrow{\mathscr{P}_{\text{Original}}} \boxed{\text{Matrix Stuffing}} \xrightarrow{\mathscr{P}_{\text{HSD}}} \boxed{\text{ADMM Solver}} \xrightarrow{\mathbf{x}^{\star}}$$

  - Matrix stuffing: Fast homogeneous self-dual embedding (HSD) transformation
  - Operator splitting (ADMM): Large-scale homogeneous self-dual embedding

Stage One: Fast Transformation

- **Example:** Coordinated beamforming problem family (with transmit power constraints and QoS constraints)

$$\begin{aligned} \mathscr{P}_{\text{Original}} : \text{minimize} \quad & \|\mathbf{v}\|_2^2 \\ \text{subject to} \quad & \|\mathbf{D}_l \mathbf{v}\|_2 \le \sqrt{P_l}, \forall l, \\ & \|\mathbf{C}_k \mathbf{v} + \mathbf{g}_k\|_2 \le \beta_k \mathbf{r}_k^T \mathbf{v}, \forall k. \end{aligned}$$

- **Smith form reformulation** [Smith '96]
  - Key idea: Introduce a new variable for each subexpression in $\mathscr{P}_{\text{Original}}$

Smith form for (1) $\quad \mathcal{G}_1(l) : \begin{cases} (y_0^l, \mathbf{y}_1^l) \in \mathcal{Q}^{KN_l+1} \\ y_0^l = \sqrt{P_l} \in \mathbb{R} \\ \mathbf{y}_1^l = \mathbf{D}_l \mathbf{v} \in \mathbb{R}^{KN_l} \end{cases}$

Second-order cone

Linear constraint

The Smith form is ready for standard cone programming transformation

Stage One: Fast Transformation

- **HSD embedding** of the **primal-dual pair** of transformed standard cone program (based on KKT conditions)

$$\boxed{\begin{array}{ll} \underset{\boldsymbol{\nu},\boldsymbol{\mu}}{\text{minimize}} & \mathbf{c}^T\boldsymbol{\nu} \\ \text{subject to} & \mathbf{A}\boldsymbol{\nu}+\boldsymbol{\mu}=\mathbf{b} \\ & (\boldsymbol{\nu},\boldsymbol{\mu})\in\mathbb{R}^n\times\mathcal{K}. \end{array}} + \boxed{\begin{array}{ll} \underset{\boldsymbol{\eta},\boldsymbol{\lambda}}{\text{maximize}} & -\mathbf{b}^T\boldsymbol{\eta} \\ \text{subject to} & -\mathbf{A}^T\boldsymbol{\eta}+\boldsymbol{\lambda}=\mathbf{c} \\ & (\boldsymbol{\lambda},\boldsymbol{\eta})\in\{0\}^n\times\mathcal{K}^* \end{array}} \Longrightarrow \boxed{\begin{array}{l} \mathscr{F}_{\text{HSD}}: \text{find } (\mathbf{x},\mathbf{y}) \\ \text{subject to } \mathbf{y}=\mathbf{Q}\mathbf{x} \\ \qquad \mathbf{x}\in\mathcal{C},\mathbf{y}\in\mathcal{C}^* \end{array}}$$

Certificate of infeasibility: $\tau=0,\kappa>0$

$$\underbrace{\begin{bmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\\ \kappa\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix}\mathbf{0} & \mathbf{A}^T & \mathbf{c}\\ -\mathbf{A} & \mathbf{0} & \mathbf{b}\\ -\mathbf{c}^T & -\mathbf{b}^T & \mathbf{0}\end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix}\boldsymbol{\nu}\\ \boldsymbol{\eta}\\ \tau\end{bmatrix}}_{\mathbf{x}}$$

- **Matrix stuffing for fast transformation:**
  - Generate and keep the structure $\mathbf{Q}$
  - Copy problem instance parameters to the pre-stored structure $\mathbf{Q}$

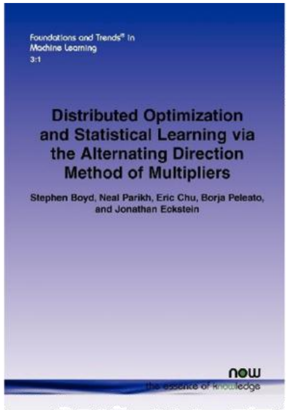
Stage Two: Parallel and Scalable Computing

- **HSD embedding in consensus form:**

$$\boxed{\begin{array}{rll} \mathscr{F}_{\mathrm{HSD}} : \text{find} & (\mathbf{x}, \mathbf{y}) \\ \text{subject to} & \mathbf{y} = \mathbf{Q}\mathbf{x} \\ & \mathbf{x} \in \mathcal{C}, \mathbf{y} \in \mathcal{C}^* \end{array}} \longrightarrow \boxed{\begin{array}{rll} \mathscr{P}_{\mathrm{ADMM}} : \underset{\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{y}, \tilde{\mathbf{y}}}{\text{minimize}} & I_{\mathcal{C} \times \mathcal{C}^*}(\mathbf{x}, \mathbf{y}) + I_{\mathbf{Q}\tilde{\mathbf{x}} = \tilde{\mathbf{y}}}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \\ \text{subject to} & (\mathbf{x}, \mathbf{y}) = (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \end{array}}$$

- **Final algorithm:** Apply the operating splitting method (ADMM) [Donoghue, Chu, Parikh, and Boyd '13]

$$\begin{aligned} \tilde{\mathbf{x}}^{[i+1]} &= (\mathbf{I} + \mathbf{Q})^{-1}(\mathbf{x}^{[i]} + \mathbf{y}^{[i]}) && \text{subspace projection} \\ \mathbf{x}^{[i+1]} &= \Pi_{\mathcal{C}}(\tilde{\mathbf{x}}^{[i+1]} - \mathbf{y}^{[i]}) && \text{parallel cone projection} \\ \mathbf{y}^{[i+1]} &= \mathbf{y}^{[i]} - \tilde{\mathbf{x}}^{[i+1]} + \mathbf{x}^{[i+1]} && \text{computationally trivial} \end{aligned}$$



Proximal Algorithms for Cone Projection

- Proximal algorithms for parallel cone projection [Parikn & Boyd, FTO 14]

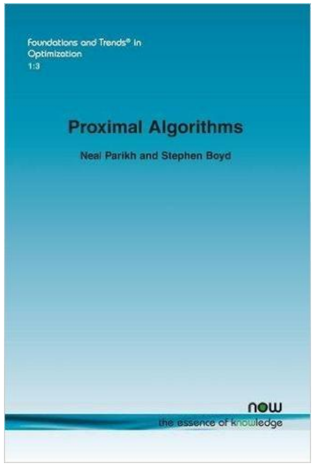
  - Projection onto the second-order cone: $\mathcal{C} = \{(y, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{p-1} | \|\mathbf{x}\| \leq y\}$

$$\Pi_{\mathcal{C}}(\boldsymbol{\omega}, \tau) = \begin{cases} 0, \|\boldsymbol{\omega}\|_2 \leq -\tau \\ (\boldsymbol{\omega}, \tau), \|\boldsymbol{\omega}\|_2 \leq \tau \\ (1/2)(1 + \tau/\|\boldsymbol{\omega}\|_2)(\boldsymbol{\omega}, \|\boldsymbol{\omega}\|_2), \|\boldsymbol{\omega}\|_2 \geq |\tau|. \end{cases}$$

  - Projection onto positive semidefinite cone: $\mathcal{C} = \mathbf{S}_+^n$

$$\Pi_{\mathcal{C}}(\mathbf{V}) = \sum_{i=1}^n (\lambda_i)_+ \mathbf{u}_i \mathbf{u}_i^T$$

SVD is computationally expensive



Numerical Results (I)

- **Example:** Power minimization coordinated beamforming problem [2]

Network Size (*L*=*K*)		20	50	100	150
CVX+SDPT3	Modeling Time [sec]	**0.7563**	4.4301	N/A	N/A
	Solving Time [sec]	4.2835	**326.2513**	N/A	N/A
	Objective [W]	12.2488	6.5216	N/A	N/A
Matrix Stuffing+ADMM	Modeling Time [sec]	**0.0128**	0.2401	2.4154	9.4167
	Solving Time [sec]	0.1009	**2.4821**	23.8088	81.0023
	Objective [W]	12.2523	6.5193	3.1296	2.0689

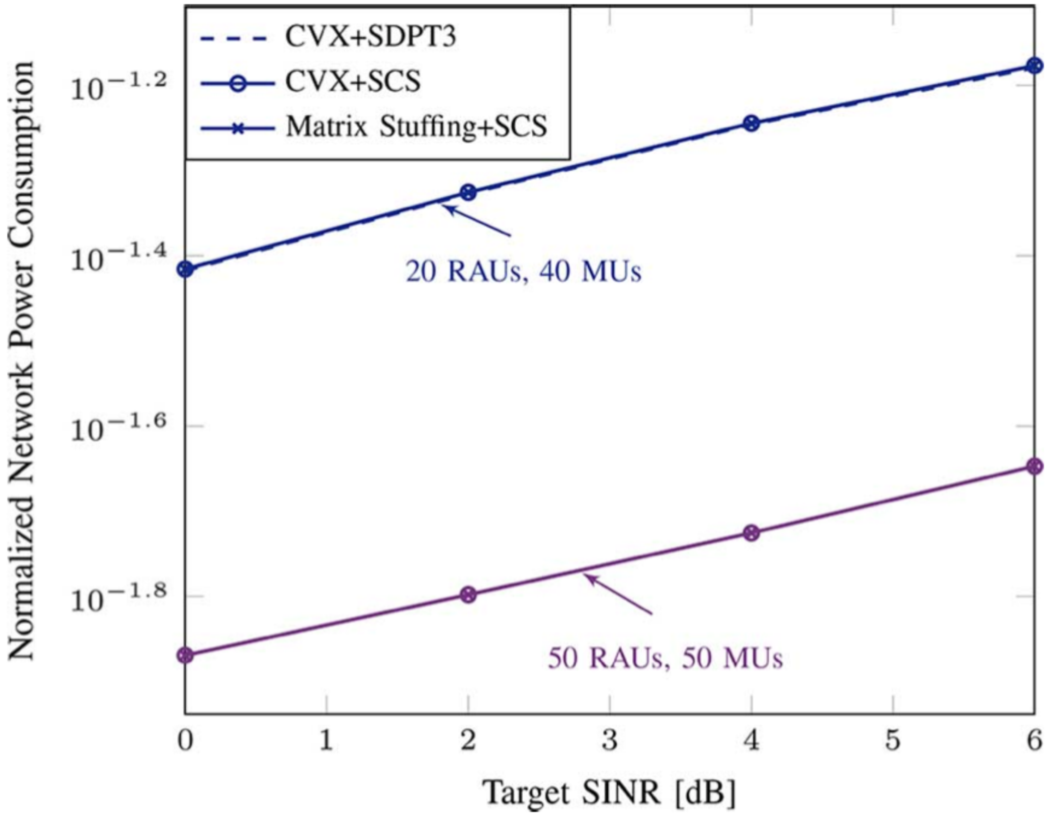
Matrix stuffing can speedup **60x** over CVX

ADMM can speedup **130x** over the interior-point method

[2] Y. Shi, J. Zhang, B. O'Donoghue, and K. B. Letaief, "Large-scale convex optimization for dense wireless cooperative networks," *IEEE Trans. Signal Process.*, vol. 63, no. 18, pp. 4729-4743, Sept. 2015.

Numerical Results (II)

- Group sparse beamforming for network power minimization [2]

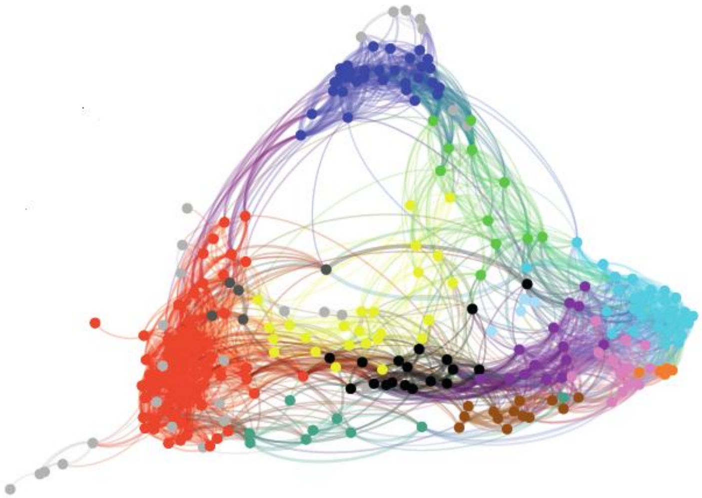


Can We do Better?

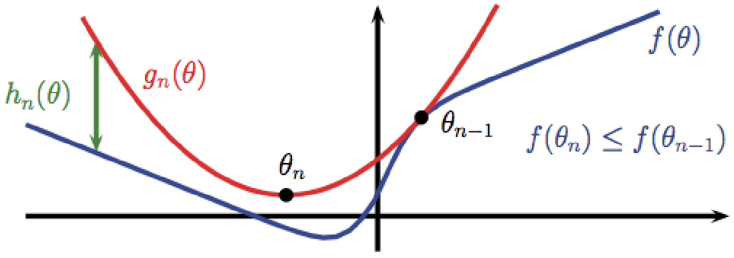
Stage I		Stage II		Stage III
Minimize the Group Sparsity Inducing Norm for RRH Ordering	$\xrightarrow{\boldsymbol{\theta}^\star}$	RRH Selection by Solving $\mathscr{F}(\mathcal{A}^{[i]})$	$\xrightarrow{\mathcal{A}^\star}$	Transmit Power Minimization $\mathscr{P}_{\mathrm{TP}}(\mathcal{A}^\star)$

1. **High computational complexity (computing depends on the instantaneous CSI)**
2. **Limited capability to enhance group sparsity**

Solution: Large System Analysis for Enhanced Group Sparse Beamforming



Proposed Algorithm: Iterative Reweighted-l2 Algorithm



Proposed Method

- Smoothed ℓ_p-minimization approach to induce group sparsity

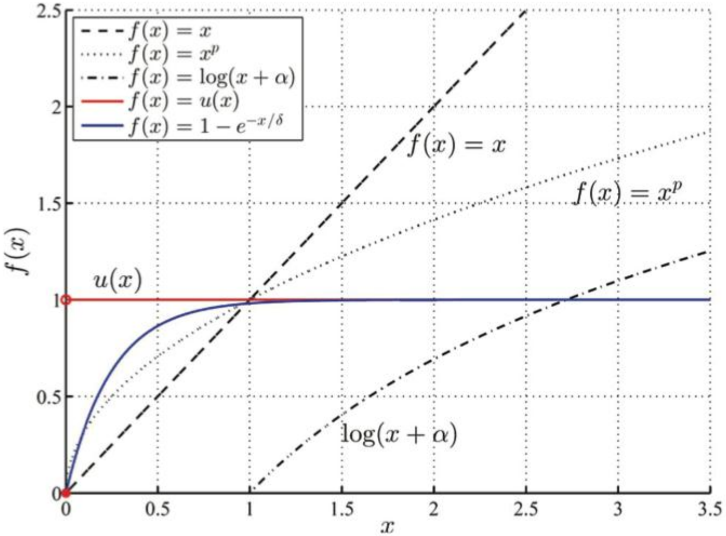
$$\underset{\mathbf{v}}{\text{minimize}} \quad g_p(\mathbf{v};\epsilon) := \sum_{l=1}^{L} \nu_l \left(\|\tilde{\mathbf{v}}_l\|_2^2 + \epsilon^2\right)^{p/2}$$

$$\text{subject to} \quad \frac{|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_k|^2}{\sum_{i\neq k}|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_i|^2 + \sigma_k^2} \geq \gamma_k, \forall k$$

Nonconvex!

- **Enhance sparsity:**

$$\|\boldsymbol{z}\|_0 = \lim_{p\to 0}\|\boldsymbol{z}\|_p^p = \lim_{p\to 0}\sum_i |z_i|^p$$



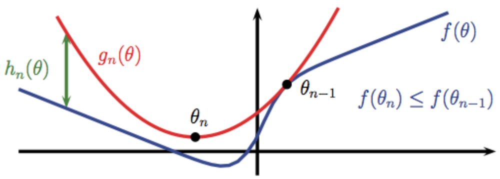
Majorization-Minimization Algorithm

- Solve the following (nonconvex) smoothed ℓ_p-minimization problem

$$\underset{\boldsymbol{z}\in\mathcal{C}}{\text{minimize}}\; f(\boldsymbol{z}) := \sum_{i=1}^{m}(z_i^2+\epsilon^2)^{p/2}$$

- **MM algorithm:** the successive upper-bound minimization method

1: Find a feasible point $\boldsymbol{z}^{[0]} \in \mathcal{C}$ and set $k=0$
2: **repeat**
3: $\boldsymbol{z}^{[k+1]} = \arg\min_{\boldsymbol{z}\in\mathcal{C}} g(\boldsymbol{z}|\boldsymbol{z}^{[k]})$ (**global minimum**)
4: $k \leftarrow k+1$
5: **until** some convergence criterion is met



- An upper bound for the objective function $f(\boldsymbol{z})$ can be constructed as

$$Q(\boldsymbol{z};\boldsymbol{\omega}^{[k]}) := \sum_{i=1}^{m}\omega_i^{[k]} z_i^2 \qquad \omega_i^{[k]} = \frac{p}{2}\left[\left(z_i^{[k]}\right)^2+\epsilon^2\right]^{\frac{p}{2}-1}, \forall i$$

Enhanced Group Sparse Beamforming

- **Final algorithm:** iterative reweighted-ℓ_2 algorithm

$$\begin{aligned} \underset{\mathbf{v}}{\text{minimize}} \quad & \sum_{l=1}^{L} \omega_l^{[n]} \|\tilde{\mathbf{v}}_l\|_2^2 \\ \text{subject to} \quad & \frac{|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_k|^2}{\sum_{i\neq k} |\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_i|^2 + \sigma_k^2} \geq \gamma_k, \forall k \end{aligned}$$

weights:

$$\omega_l^{[n]} = \frac{p\nu_l}{2}\left[\left\|\tilde{\mathbf{v}}_l^{[n]}\right\|_2^2 + \epsilon^2\right]^{\frac{p}{2}-1}, \forall l$$

Advantageous:

1. Enhance sparsity
2. Lead to closed form solution via duality theory

Simple Solution Structures

- Optimal beamforming vectors $\mathbf{v}_1^*, \ldots, \mathbf{v}_K^*$ are given by

$$\mathbf{v}_k^\star = \sqrt{\frac{p_k}{LN}} \frac{\left(\mathbf{Q}^{[n]} + \sum_{i=1}^K \frac{\lambda_i}{LN} \mathbf{h}_i \mathbf{h}_i^\mathsf{H}\right)^{-1} \mathbf{h}_k}{\left\| \left(\mathbf{Q}^{[n]} + \sum_{i=1}^K \frac{\lambda_i}{LN} \mathbf{h}_i \mathbf{h}_i^\mathsf{H}\right)^{-1} \mathbf{h}_k \right\|}, \forall k$$

beamforming direction

  - The K powers are given by

$$\begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_K^2 \end{bmatrix} \qquad [\mathbf{M}]_{ij} = \begin{cases} \frac{1}{\gamma_i LN} \frac{|\mathbf{h}_i^\mathsf{H} \bar{\mathbf{v}}_i|^2}{\|\bar{\mathbf{v}}_i\|_2^2} & , i = j, \\ -\frac{1}{LN} \frac{|\mathbf{h}_i^\mathsf{H} \bar{\mathbf{v}}_j|^2}{\|\bar{\mathbf{v}}_j\|_2^2} & , i \neq j, \end{cases}$$

  - The Lagrange multipliers can be computed from the fixed-point equations

$$\lambda_k = LN \left[\left(1 + \frac{1}{\gamma_k}\right) \mathbf{h}_k^\mathsf{H} \left(\mathbf{Q}^{[n]} + \sum_{i=1}^K \frac{\lambda_i}{LN} \mathbf{h}_i \mathbf{h}_i^\mathsf{H}\right)^{-1} \mathbf{h}_k \right]^{-1}$$

The first step to reduce computational complexity

Optimality

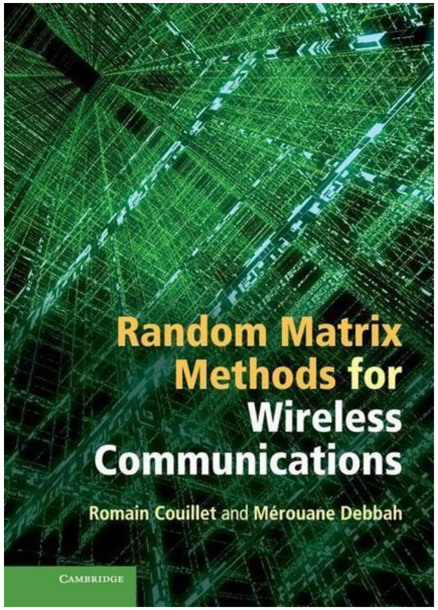
- ***Theorem 1:*** Let $\{\mathbf{v}^{[n]}\}_{n=1}^{\infty}$ be the sequence generated by the iterative reweighted- ℓ_2 algorithm. Then, every limit point $\bar{\mathbf{v}}$ of $\{\mathbf{v}^{[n]}\}_{n=1}^{\infty}$ has the following properties:
  - 1) $\bar{\mathbf{v}}$ is a KKT point of the smoothed L_p-minimization problem
  - 2) $g_p(\mathbf{v}^{[n]};\epsilon)$ converges monotonically to $g_p(\mathbf{v}^{\star};\epsilon)$ for some KKT point $\mathbf{v}^{\star}$

- RRH ordering criteria to determine which RRHs should be switched off

$$\theta_l = \kappa_l \|\tilde{\mathbf{v}}_l\|_2^2 = \kappa_l \sum_{k=1}^{K} \mathbf{v}_k^{\mathsf{H}} \mathbf{Q}_{lk} \mathbf{v}_k, \forall l = 1, \ldots, L$$

- **Challenges** to compute the ordering criteria
  - Massive instantaneous CSI
  - High computation cost

Random Matrix Theory: Large System Analysis



Modern Applications

Random Matrix Theory

Robust Statistics

Since 2012

- Detection in impulsive noises

Machine Learning

Since 2015

- Subspace clustering
- Community detection

Wireless Communication

Since 1990

- Performance analysis
- Optimal transceiver design

Signal Processing

Since 2007

- Estimation
- Detection

Deterministic Equivalent of Optimal Parameters (I)

- Channel models in Cloud-RAN with distributed RRHs:

$$\mathbf{h}_k = \mathbf{\Theta}_k^{1/2}\mathbf{g}_k, \mathbf{g}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{NL}), \mathbf{\Theta}_k = \mathrm{diag}\{d_{k1}, \ldots, d_{kL}\} \otimes \mathbf{I}_N$$

- Optimal Lagrange multipliers

$$\lambda_k = LN\left[\left(1+\frac{1}{\gamma_k}\right)\mathbf{h}_k^{\mathsf{H}}\left(\mathbf{Q}^{[n]} + \sum_{i=1}^{K}\frac{\lambda_i}{LN}\mathbf{h}_i\mathbf{h}_i^{\mathsf{H}}\right)^{-1}\mathbf{h}_k\right]^{-1}$$

- ***Lemma 1 (Deterministic Equivalent of the λ-Parameter):***

Assume $0 < \liminf_{N\to\infty} K/N \leq \limsup_{N\to\infty} K/N < \infty$. Let $\{d_{kl}\}$ and $\{\gamma_k\}$ satisfy $\limsup_N \max_{k,l}\{d_{kl}\} < \infty$ and $\limsup_N \max_k \gamma_k < \infty$, respectively. We have

$$\boxed{\max_{1\leq k\leq K}\left|\lambda_k - \lambda_k^\circ\right| \overset{N\to\infty}{\longrightarrow} 0 \quad \text{almost surely}}$$

where

$$\lambda_k^\circ = \gamma_k\left(\frac{1}{L}\sum_{l=1}^{L} d_{kl}\eta_l\right)^{-1} \qquad \eta_l = \left(\frac{1}{NL}\sum_{i=1}^{K}\frac{d_{il}}{\frac{1}{L}\sum_{j=1}^{L} d_{ij}\eta_j}\frac{\gamma_i}{1+\gamma_i} + \omega_l^{[n]}\right)^{-1}$$

Deterministic Equivalent of Optimal Parameters (II)

- ***Lemma 2 (Asymptotic Result for the Optimal Powers):*** Let $\mathbf{\Delta} \in \mathbb{R}^{K \times K}$ be such that $[\mathbf{\Delta}]_{k,i} := \frac{1}{NL} \frac{\gamma_i}{(1+\gamma_i)^2} \frac{\psi'_{ik}}{\psi_i^2}$. If and only if $\limsup_K \|\mathbf{\Delta}\|_2 < 1$, then

$$\boxed{\max_k |p_k - p_k^\circ| \xrightarrow{N\to\infty} 0 \quad \text{almost surely}}$$

where $p_k^\circ = \gamma_k \frac{\psi'_k}{\psi_k^2} \left(\frac{\tau_k}{(1+\gamma_k)^2} + \sigma_k^2 \right)$

Here $\psi_k = \frac{1}{L} \sum_{l=1}^{L} d_{kl}\eta_l$, ψ'_k and ψ'_{ik} are given by

$$\psi'_k = \frac{1}{L} \sum_{l=1}^{L} d_{kl}\eta_l^2 + \frac{1}{NL} \sum_{j=1}^{K} \frac{\lambda_j^{\circ 2} \psi'_j}{(1+\gamma_j)^2} \frac{1}{L} \sum_{l=1}^{L} d_{il} d_{jl} \eta_l^2$$

$$\psi'_{ik} = \frac{1}{NL} \sum_{j=1}^{K} \frac{\lambda_j^{\circ 2} \psi'_{jk}}{(1+\gamma_j)^2} \frac{1}{L} \sum_{l=1}^{L} d_{il} d_{jl} \eta_l^2 + \frac{1}{L} \sum_{l=1}^{L} d_{il} d_{kl} \eta_l^2$$

$$\boldsymbol{\tau} = \sigma^2 \left(\mathbf{I}_K - \mathbf{\Delta} \right)^{-1} \boldsymbol{\delta} \qquad \delta_k = \frac{1}{NL} \sum_{i=1}^{K} \gamma_i \frac{\psi'_{ik}}{\psi_i^2}$$

Statistical Group Sparse Beamforming

- ***Theorem 2 (Asymptotic Result for RRH Ordering Criteria):***

$$\max_l |\theta_l - \theta_l^\circ| \xrightarrow{N \to \infty} 0 \quad \text{almost surely}$$

where

$$\theta_l^\circ = \frac{\kappa_l}{NL} \sum_{k=1}^{K} p_k^\circ \frac{\psi_{kl}}{\psi_k'}$$

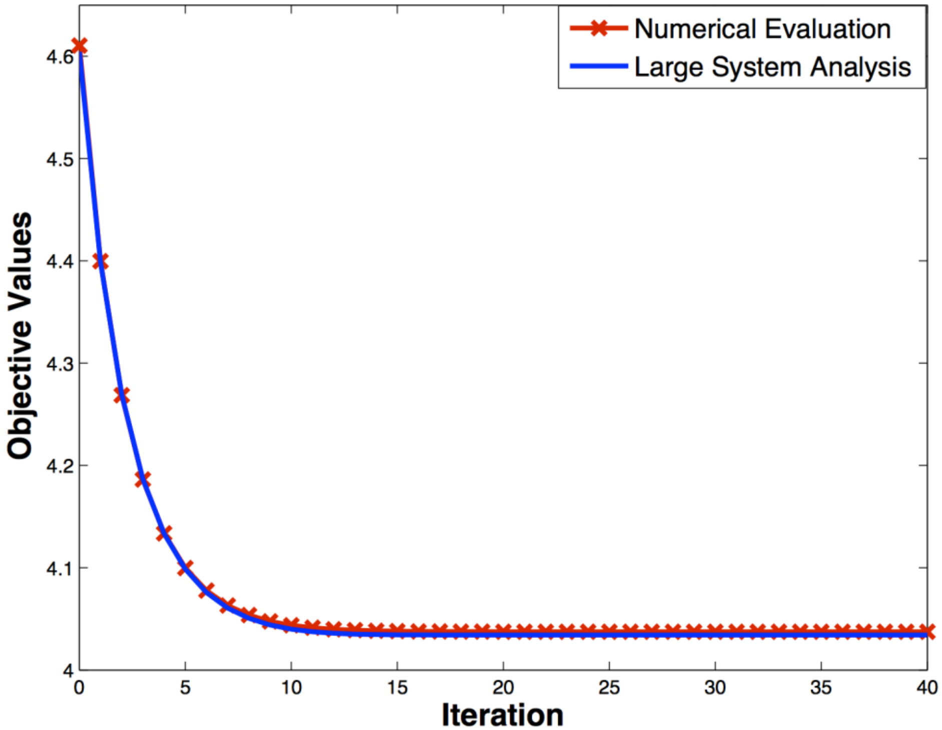
$$\psi_{kl} = \frac{1}{NL} d_{kl} \eta_l^2 + \frac{1}{NL} \sum_{j=1}^{K} \frac{\lambda_j^{\circ 2} \psi_{jl}}{(1+\gamma_j)^2} \frac{1}{L} \sum_{l=1}^{L} d_{il} d_{jl} \eta_l^2$$

The ordering criteria will change only when the long-term channel attenuation is updated!

The second step to reduce computational complexity

Simulation Results (I)

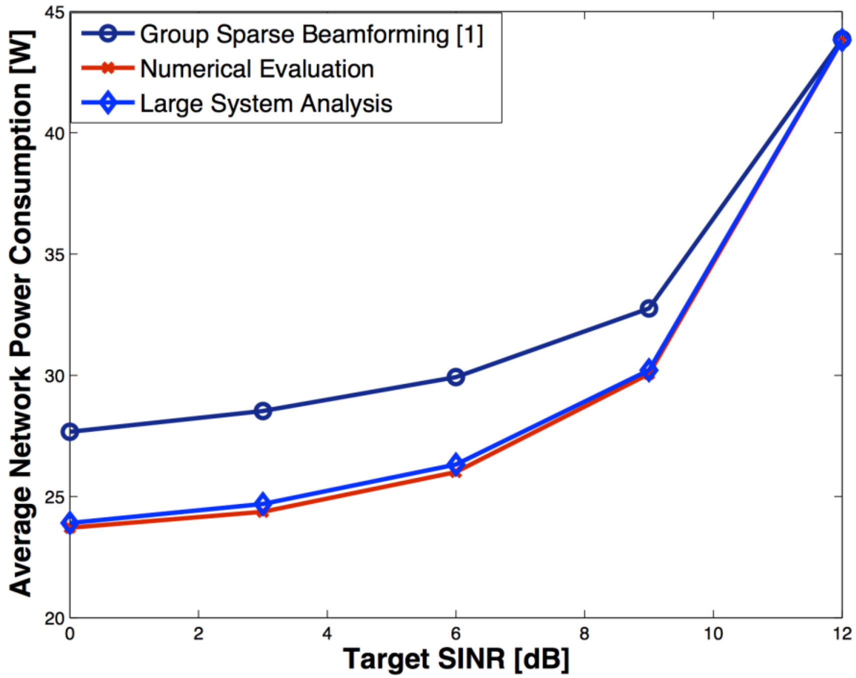
- Convergence results (5 30-antenna RRHs and 5 single antenna MUs) [3]



[3] Y. Shi, J. Zhang, and K. B. Letaief, "Scalable Group Sparse Beamforming for Dense Green Cloud-RAN: A Random Matrix Approach," submitted to *IEEE Trans. Signal Process.*, Jul. 2016.

Simulation Results (II)

- Network power minimization (5 10-antenna RRHs and 6 single antenna MUs) [3]



[1] Y. Shi, J. Zhang, and K. B. Letaief, "Group sparse beamforming for green Cloud-RAN," IEEE Trans. Wireless Commun., vol. 13, pp. 2809– 2823, May 2014.

Conclusions

- **Network power minimization:** A difficult non-convex mixed combinatorial optimization problem
- **Key techniques (scalable algorithms design):**
  - **GSBF:** convexify the combinatorial composite network power consumption function using the mixed ℓ_1/ℓ_2-norm
  - **Large-Scale Convex Optimization:**
    - Matrix stuffing: fast transformation
    - Operator splitting method (ADMM): large-scale HSD embedding
  - **Enhanced GSBF:**
    - Smoothed ℓ_p-minimization with iterative reweighted-ℓ_2 algorithm
    - Large random matrix theory: low computational complexity of RRH selection
- **Results: group sparse optimization** offers a principled way to design a dense green Cloud-RAN

Vignette B: Low-Rank Matrix Completion for Topological Interference Management

1		0	0	
	1	0	0	
0		1		0
0			1	0
	0			1

Issue B: Interference Management

- **Goal:** Interference mitigation in dense wireless networks

- **Prior works:** Abundant CSIT → Relaxed CSIT
  - **Perfect CSIT** [Cadambe and Jafar, TIT 08]
  - **Delayed CSIT** [Maddah-Ali and Tse, TIT 12]
  - **Alternating CSIT** [Tandon, et al., TIT 13], **partial and imperfect CSIT** [Shi, et al., TSP 14],…

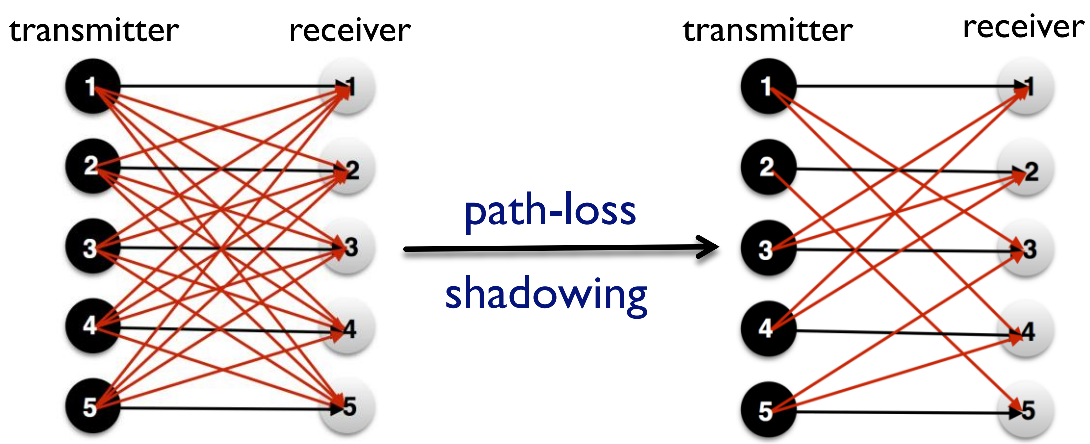
- **Curses:** CSIT is rarely abundant (due to training & feedback overhead)

Start here? → Applicable? ← Prior works

No CSIT CSIT → Perfect CSIT

Topological Interference Management

- **Blessings:** Partial connectivity in dense wireless networks



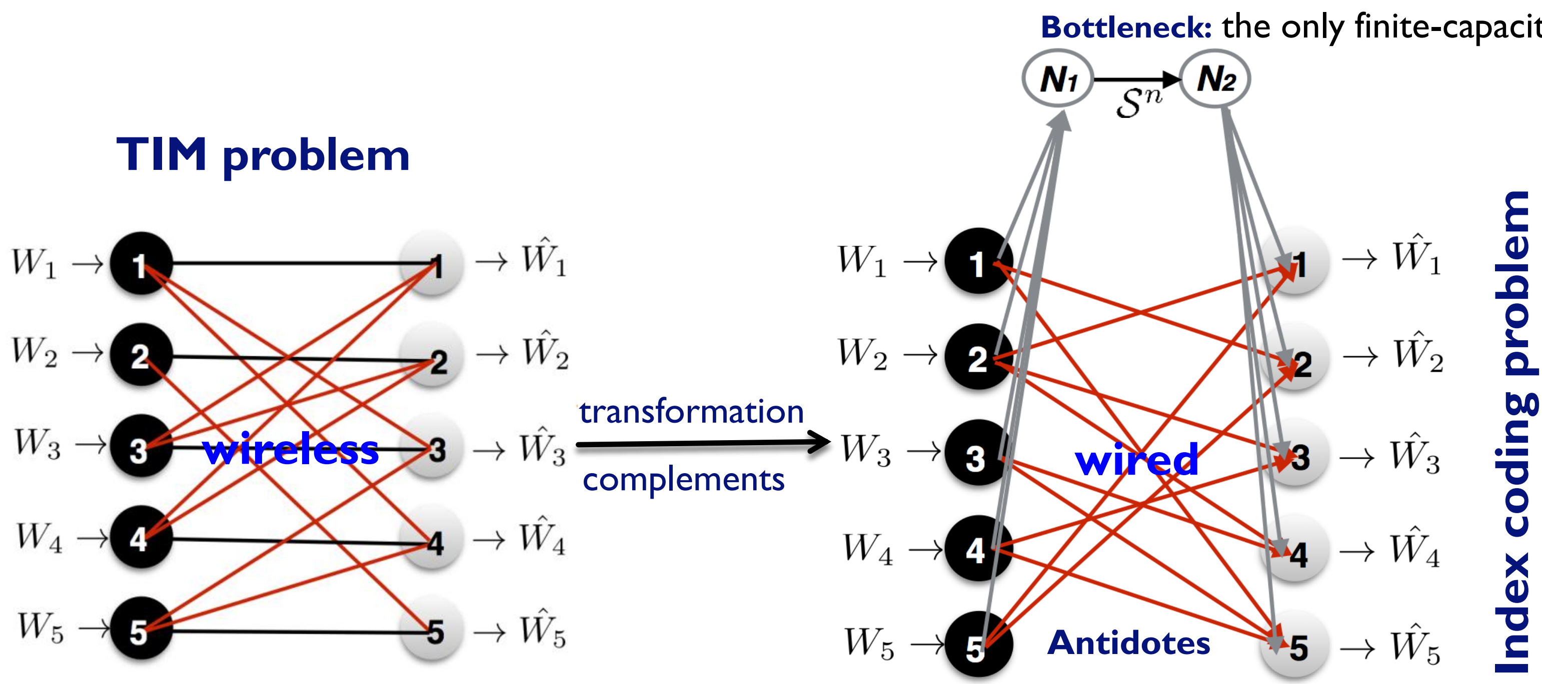
Degrees of Freedom?

$$\mathrm{DoF} = \lim_{\mathrm{SNR}\to\infty} \frac{C(\mathrm{SNR})}{\log(\mathrm{SNR})}$$

- **Approach:** Topological interference management (TIM) [Jafar, TIT 14]
  - **Maximize the achievable DoF:** Only based on the network topology information **(no CSIT)**

TIM via Index Coding

- **Theorem** [Jafar, TIT 14]: Under linear (vector space) solutions, TIM problem and index coding problem are ***equivalent***



Only a few index coding problems have been solved!

TIM via LRMC

- **Goal:** Deliver one data stream per user over N time slots
  - $\mathbf{v}_i \in \mathbb{C}^N$: tx. beamformer at the i-th tx.
  - $\mathbf{u}_j \in \mathbb{C}^N$: rx. beamformer at the j-th rx.

align interference

- **We need:** $X_{ij} = \begin{cases} \mathbf{u}_i^{\mathsf{H}}\mathbf{v}_i = 1, & \forall i, \\ \mathbf{u}_i^{\mathsf{H}}\mathbf{v}_j = 0, & \forall i \neq j, (i,j) \in \Omega, \\ \star, & \text{otherwise.} \end{cases}$ $\xrightarrow{\text{rewrite}}$ $\mathcal{P}_\Omega(\mathbf{X}) = \mathbf{I}_K$ $\Downarrow$ $1/N$ **DoF**

- **Approach:** Low-rank matrix completion (LRMC) [4]

$$\begin{array}{ll} \text{minimize} & \operatorname{rank}(\mathbf{X}) \\ \text{subject to} & \mathcal{P}_\Omega(\mathbf{X}) = \mathbf{I}_K \end{array}$$

Key conclusion: DoF $= 1/\operatorname{rank}(\mathbf{X})$

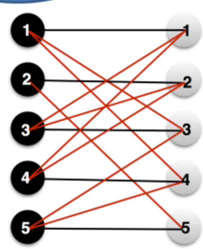
Any network topology: Ω

[4] Y. Shi, J. Zhang, and K. B. Letaief, "Low-rank matrix completion for topological interference management by Riemannian pursuit," *IEEE Trans. Wireless Commun.*, vol. 15, no. 7, Jul. 2016.

LRMC & TIM & Index Coding

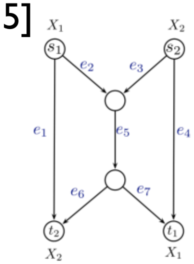
TIM Problem

[Jafar '14]



Network Coding

[Rouayheb et al. '10, '15]

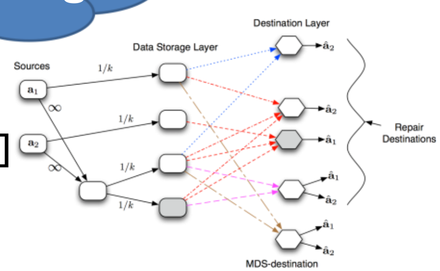


Index Coding

[Birk, Kol, INFOCOM'98]

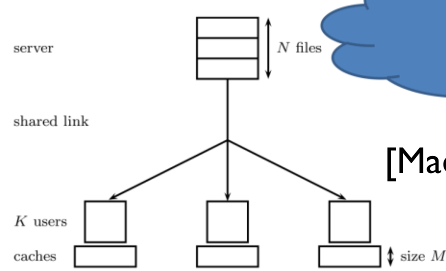
Distributed Storage

[Cadambe et al. '13]
[Shanmugam et al. '14]



Caching

[Maddah-Ali & Niesen '13]



LRMC offers a new way to investigate these problems!

Riemannian Pursuit Algorithm

- **NP-hard:** Non-convex rank objective function

$$\begin{aligned} &\text{minimize} \quad \operatorname{rank}(\mathbf{X}) \\ &\text{subject to} \quad \mathcal{P}_\Omega(\mathbf{X}) = \mathbf{I}_K \end{aligned}$$

$$|\operatorname{Tr}(\mathbf{X})| \leq \|\mathbf{X}\|_*$$

- **Poorly structured affine constraint:**
  - Nuclear-norm relaxation [Candes & Recht, FCM 09]: $\mathbf{X}^\star = \mathbf{I}_K$ **(full rank)**

- **Riemannian pursuit [4]:** Alternatively perform the fixed-rank optimization and rank increase

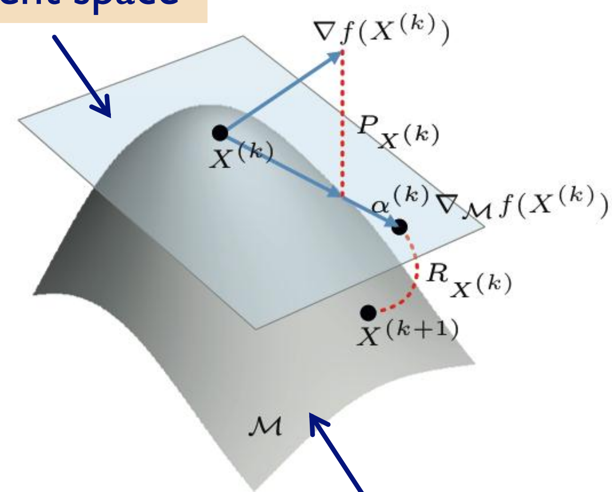
$$\begin{aligned} &\text{minimize} \quad \|\mathcal{P}_\Omega(\mathbf{X}) - \mathbf{I}_K\|_F^2 \\ &\text{subject to} \quad \operatorname{rank}(\mathbf{X}) = r \end{aligned}$$

  - Riemannian optimization: address convergence issues in fixed-rank methods

Riemannian Optimization for Fixed-Rank Problems

- Solve ***fixed-rank problems*** by Riemannian optimization [Absil, et al., 08]
  - Generalize Euclidean gradien (Hessian) to Riemannian gradient (Hessian)

Tangent space



Quotient manifold geometry of fixed rank matrices

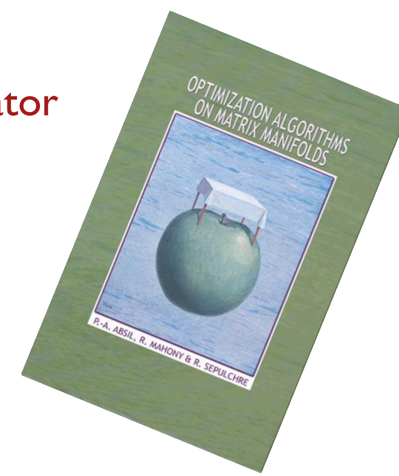
$$[\mathbf{X}] = \{(\mathbf{U}\mathbf{Q}_U, \mathbf{Q}_U^T \mathbf{\Sigma} \mathbf{Q}_V, \mathbf{V}\mathbf{Q}_V) : \mathbf{Q}_U, \mathbf{Q}_V \in \mathcal{Q}(r)\}$$

$$\nabla_{\mathcal{M}} f(\mathbf{X}^{(k)}) = P_{\mathbf{X}^{(k)}}(\nabla f(\mathbf{X}^{(k)}))$$

Riemannian Gradient Euclidean Gradient

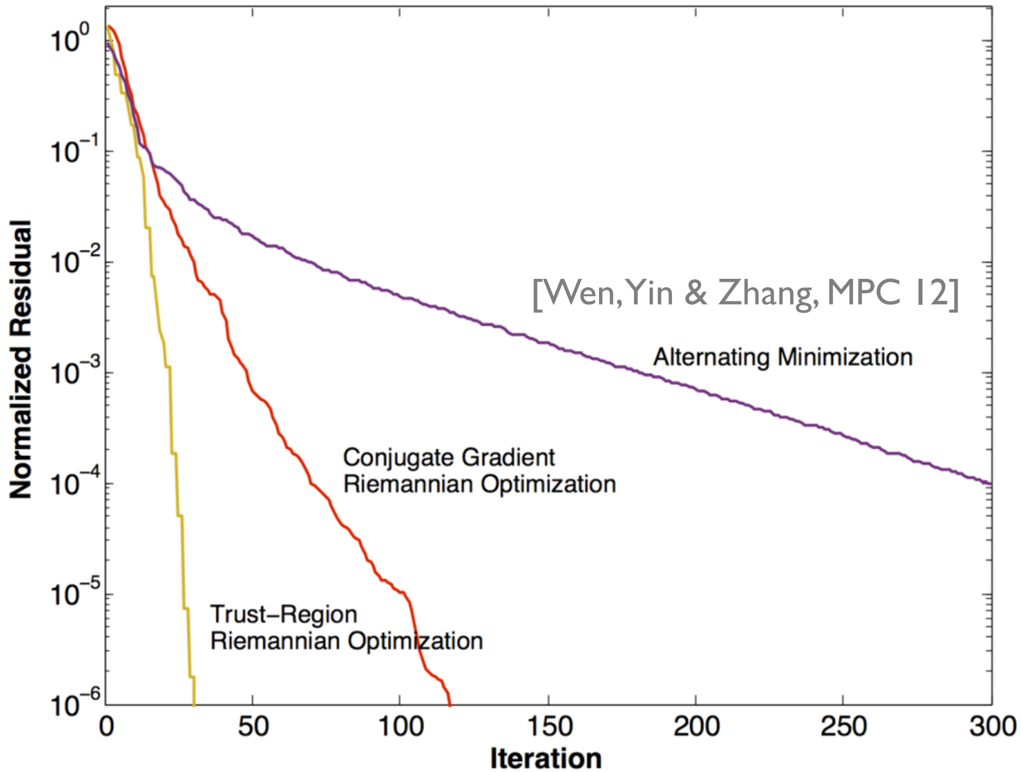
$$\mathbf{X}^{(k+1)} = \mathcal{R}_{\mathbf{X}^{(k)}}(-\alpha^{(k)} \nabla_{\mathcal{M}} f(\mathbf{X}^{(k)}))$$

Retraction Operator



Numerical Results (1): Convergence Rate

- Riemannian optimization over the quotient matrix manifold [4].

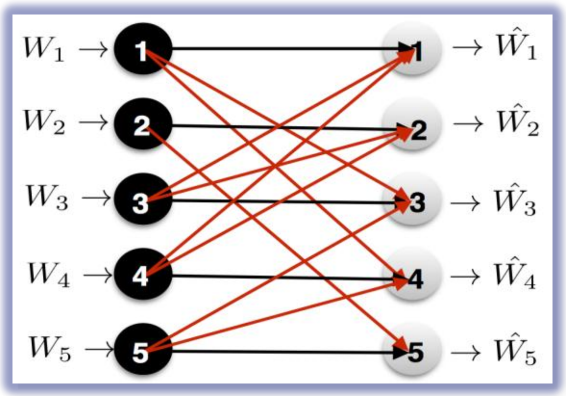


Riemannian algorithms:

1. Exploit the rank structure in a principled way
2. Develop second-order algorithms systematically
3. Scalable, SVD-free

[4] Y. Shi, J. Zhang, and K. B. Letaief, "Low-rank matrix completion for topological interference management by Riemannian pursuit," *IEEE Trans. Wireless Commun.*, vol. 15, no. 7, Jul. 2016.

Numerical Results (II): Symmetric DoF



Optimal DoF=1/2

transmitters

receivers

1		0	0	
	1	0	0	
0		1		0
0			1	0
	0			1

associated incomplete matrix

LRMC

1	.1	0	0	9.5
6.8	1	0	0	64
0	.1	1	-1	0
0	-.1	-1	1	0
.1	0	-.1	.1	1

Riemannian pursuit: Rank=2

Advantages:

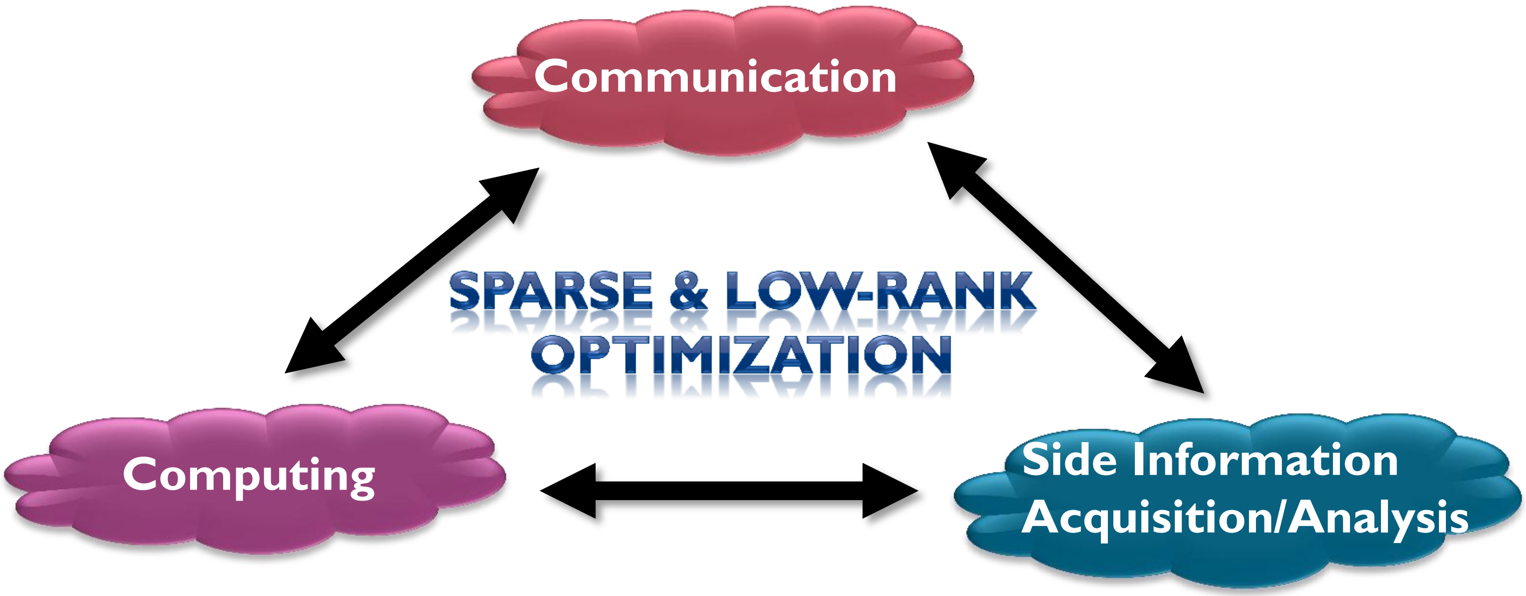
1. Recover all the optimal DoF results for the special TIM problems in [Jafar '14]
2. Provide numerical insights (optimal/lower-bound) for the general TIM problems

Conclusions

- **Topological interference management** significantly improves DoFs only based on the network topology information
- **Key techniques:**
  - Low-rank matrix completion
  - Riemannian optimization
- **Results: Low-rank matrix completion** provides a first algorithmic and systematic approach to investigate the TIM problem for ***any network topology***.
- **Extensions:**
  - ***User admission control, network topology design, finite SNR, …***
  - More applications: index coding, distributed storage and caching,…
  - Optimality: Riemannian pursuit algorithm, LRMC approach

Concluding Remarks

- **Future network design: dense, cooperative, scalable, unified**



1. **Structured models:** Sparsity, low-rankness
2. **Scalable algorithms:** Convex optimization, Riemannian optimization, ADMM
3. **Theory: Global optimality?**

Further Information: Sparse Optimization

- **Y. Shi**, J. Zhang, and K. B. Letaief, "Enhanced Group Sparse Beamforming for Dense Green Cloud-RAN: A Random Matrix Approach," submitted to *IEEE Trans. Signal Process.*, Jul. 2016.
- **Y. Shi**, J. Cheng, J. Zhang, B. Bai, W. Chen and K. B. Letaief, "Smoothed L_p-minimization for green Cloud-RAN with user admission control," *IEEE J. Select. Areas Commun.*, vol. 34, no. 4, Apr. 2016.
- **Y. Shi**, J. Zhang, B. O'Donoghue, and K. B. Letaief, "Large-scale convex optimization for dense wireless cooperative networks," *IEEE Trans. Signal Process.*, vol. 63, no. 18, pp. 4729-4743, Sept. 2015.
- **Y. Shi**, J. Zhang, and K. B. Letaief, "Robust group sparse beamforming for multicast green Cloud- RAN with imperfect CSI," *IEEE Trans. Signal Process.*, vol. 63, no. 17, pp. 4647-4659, Sept. 2015.
- **Y. Shi**, J. Zhang, K. B. Letaief, B. Bai and W. Chen,"Large-scale convex optimization for ultra-dense Cloud-RAN," *IEEE Wireless Commun. Mag.*, pp. 84-91, Jun. 2015.
- **Y. Shi**, J. Zhang, and K. B. Letaief, "Optimal stochastic coordinated beamforming for wireless cooperative networks with CSI uncertainty," *IEEE Trans. Signal Process.*, vol. 63,, no. 4, pp. 960-973, Feb. 2015.
- **Y. Shi**, J. Zhang, and K. B. Letaief, "Group sparse beamforming for green Cloud-RAN," *IEEE Trans. Wireless Commun.*, vol. 13, no. 5, pp. 2809-2823, May 2014. (The 2016 Marconi Prize Paper Award)

Further Information: Low-Rank Optimization

- **Y. Shi**, J. Zhang, and K. B. Letaief, "Low-rank matrix completion for topological interference management by Riemannian pursuit," *IEEE Trans. Wireless Commun.*, vol. 15, no. 7, Jul. 2016.
- **Y. Shi**, and B. Mishra, "Topological interference management with user admission control via Riemannian optimization," submitted to *IEEE Trans. Signal Process.*, Jul. 2016.
- **Y. Shi**, and B. Mishra, "Sparse and low-rank decomposition for wireless network densification by Riemannian optimization," to be submitted to *IEEE Trans. Signal Process.*
- K. Yang, **Y. Shi**, and Z. Ding, "Low-rank matrix completion for mobile edge caching in Fog-RAN via Riemannian optimization," accepted to *IEEE Global Communications Conf. (GLOBECOM),* Washington, DC, Dec. 2016.
- K. Yang, **Y. Shi**, J. Zhang, Z. Ding and K. B. Letaief, "A low-rank approach for interference management in dense wireless networks," submitted to *IEEE Global Conf. Signal and Inf. Process. (GlobalSIP),* Washington, DC, Dec. 2016

Personal Webpage: http://shiyuanming.github.io/home.html

Thanks