DISCRETE TIME MARKOV CHAIN AND SIMULATION OF MARKOV CHAIN

by

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UNDER THE GUIDANCE OF

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Certificate

This is certified that the work incorporated in the Report entitled **Discrete Time Markov chain and Simulation of Markov chain** that is being submitted by **Mr. Avinash Desai** has been carried out under my supervision and guidance. The material in the Report is done by him and material that has been obtained from other sources has been duly acknowledge in the Report.

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Mr. Avinash Desai

Abstract

A Discrete-Time Markov Chain is a mathematical framework used to model systems that transition between states in a stochastic manner, where the probability of transitioning to a future state depends only on the current state and not on the sequence of previous states. This "memoryless" property, known as the Markov property, makes DTMCs particularly useful in analyzing and predicting the behavior of dynamic systems in various domains such as physics, biology, finance, and computer science.

This abstract provides an overview of the fundamental concepts, including state spaces, transition probability matrices, and classification of states (e.g., recurrent, transient, absorbing). Key properties such as stationarity, limiting distributions, and ergodicity are explored, highlighting their relevance in real-world applications. DTMCs are also employed in solving problems involving queuing theory, reliability analysis, and stochastic processes, demonstrating their versatility and computational efficiency. Advanced topics such as mixing times and convergence to steady states are briefly introduced, underscoring the richness of the DTMC framework in both theoretical and practical contexts.

Contents

Chapter 1
Discrete-Time Markov Chain

1.1 Stochastic Processes

A stochastic process is a mathematical model used to describe systems or phenomena that evolve over time in a probabilistic manner. It is essentially a collection of random variables indexed by time (or another parameter), where the randomness reflects the inherent uncertainty in the system's evolution. These processes are widely applied in fields such as physics, biology, finance, economics, and engineering.

Definition: Let $T = \{0, 1, 2, \ldots\}$ or $T = [0, \infty)$. A stochastic process $\{X_t\}_{t \in T}$ is a collection of real-valued random variables.
This means that for each $t \in T$, $\{X_t\}$ is a random variable.

We interpret t as **time**, and call X_t the **state of the process at time** t.

- When $T = \{0, 1, 2, 3, \ldots\}$,$\{X_t\}$is called a **discrete-time stochastic process**.
- When $T = [0, \infty)$, $\{X_t\}$ is called a **continuous-time stochastic process**.
- The set of all possible values taken by a Stochastic process $\{X_t\}$ is called state space.
- The **PMF** of X_0 is called the initial distribution.

Example 1: A sample path for total number of visitors in an amusement park up to time t of a day is given below.

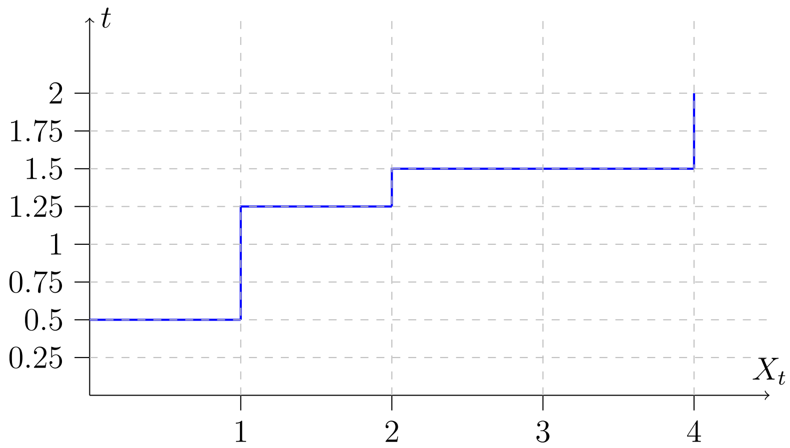

Figure 1.1

Suppose that the park open at 9 am (t=0), and suppose the first visitor arrive at 9:30 am (t=0.5) the second visitor arrive at 10:15 am (t=1.25). Two more visitors arrive at 10:30 am (t=1.5) and so on.

Example 2: Consider throwing a fair die. Let $X_0 = 0$, and let X_n denote the number of sixes in the first n throws, $n \geq 1$. Then, $\{X_n\}_{n\geq 0}$ is a stochastic process.

- The state space is $S = \{0, 1, 2, \ldots\}$.
- X_n's are dependent.
- The initial distribution is δ_0 (all probability mass at 0).

For $n \geq 1$ and $i \in S$:

$$P(X_n = j \mid X_{n-1} = i) = \begin{cases} \frac{1}{6}, & \text{for } j = i + 1, \\ \frac{5}{6}, & \text{for } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

1.2 Markov chain

Definition: A stochastic process $\{X_n\}_{n\geq 0}$ is said to be a **Markov chain (MC)** if:

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i),$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j$ and all $n \geq 0$.

Remark: For a Markov chain, the conditional distribution of X_{n+1} (future) given the past $X_0, X_1, \ldots, X_{n-1}$ and the present X_n, depends only on the present X_n and not on the past $X_0, X_1, \ldots, X_{n-1}$.

1.2.1 Time homogeneous Markov chain

Definition: A Markov chain $\{X_n\}_{n\geq 0}$ is said to be a **time-homogeneous Markov chain** if:

$$P(X_{n+1} = j \mid X_n = i) = P(X_n = j \mid X_{n-1} = i) = p_{ij}, \quad \text{for all } n \geq 1.$$

Properties:

- $p_{ij} \geq 0$ for all i and j.
- The row sums satisfy:

$$\sum_{j=0}^{\infty} p_{ij} = 1 \quad \text{for all } i.$$

- p_{ij} is called the **one-step transition probability** from state i to state j.
- The matrix $P = (p_{ij})_{i,j\geq 0}$ is called the **one-step transition probability matrix (TPM)**.

Example 3: Weather Conditions and Markov Chain
Suppose that the chance of rain tomorrow depends on the previous day's weather conditions only through whether or not it is raining today, and not on past weather conditions.
Let:

- 0 denote the state that it is **not raining**.

- 1 denote the state that it is **raining**.
- X_n denote the state of the weather on the n-th day.
- The state space is $\{0, 1\}$, where 0 represents "not raining" and 1 represents "raining".
- α is the probability that it will rain tomorrow given that it is raining today.
- β is the probability that it will rain tomorrow given that it is not raining today.

The one-step transition probability matrix, $P = (p_{ij})$, is given by:

$$P = \begin{pmatrix} 1-\beta & \beta \\ 1-\alpha & \alpha \end{pmatrix}$$

where:

- $p_{00} = 1-\beta$: The probability that it does not rain tomorrow given that it is not raining today.
- $p_{01} = \beta$: The probability that it rains tomorrow given that it is not raining today.
- $p_{10} = 1-\alpha$: The probability that it does not rain tomorrow given that it is raining today.
- $p_{11} = \alpha$: The probability that it rains tomorrow given that it is raining today.

Example 4: Suppose that the chance of rain tomorrow depends on the previous weather conditions through the last two days. The probabilities are given as:

- P(Rain tomorrow | Rain for past two days) $= 0.7$,
- P(Rain tomorrow | Rain today, no rain yesterday) $= 0.5$,
- P(Rain tomorrow | Rain yesterday, no rain today) $= 0.4$,
- P(Rain tomorrow | No rain for past two days) $= 0.2$.

If X_n is the state of the n-th day (rain or no rain), then $\{X_n\}_{n\geq 0}$ is **not a Markov chain**, because the transition probabilities depend on the weather conditions of the past two days, not just the current state.

To transform this into a Markov chain, let us define the states as follows:

$$Y_n = \begin{cases} 0 & \text{if it rained both today and yesterday,} \\ 1 & \text{if it rained today but not yesterday,} \\ 2 & \text{if it did not rain today but rained yesterday,} \\ 3 & \text{if it did not rain both today and yesterday.} \end{cases}$$

With this new definition, $\{Y_n\}_{n\geq 0}$ is a **Markov chain** with the following transition probability matrix (TPM):

$$P = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$$

Explanation of the TPM:

- Row 0 corresponds to the state where it rained on both today and yesterday ($Y_n = 0$).
- Row 1 corresponds to the state where it rained today but not yesterday ($Y_n = 1$).
- Row 2 corresponds to the state where it did not rain today but rained yesterday ($Y_n = 2$).
- Row 3 corresponds to the state where it did not rain on both today and yesterday ($Y_n = 3$).

Example 5:A Gambling Model Consider a gambler who, at each play of the game, either:

- Wins Re. 1 with probability p, or
- Loses Re. 1 with probability $1 - p$.

The gambler quits playing when either:

- They go broke (reach fortune 0), or
- They reach a target fortune of Rs. N.

Let X_n denote the gambler's fortune after the n-th game. The process $\{X_n\}_{n\geq 0}$ is a **Markov chain (MC)** with:

- **State space**: $\{0, 1, \ldots, N\}$.
- **One-step transition probabilities**:

$$P_{00} = 1, \quad P_{NN} = 1$$

$$P_{i,i+1} = p, \quad P_{i,i-1} = 1 - p \quad \text{for } i = 1, 2, \ldots, N-1$$

$$P_{ij} = 0 \quad \text{otherwise.}$$

This model can be interpreted as a simple random walk:

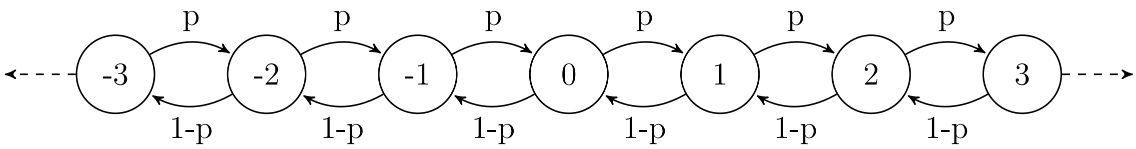

Figure 1.2: Simple random walk

- At each step, the gambler moves:
  - One step to the right ($i \to i + 1$) with probability p, or
  - One step to the left ($i \to i - 1$) with probability $1 - p$.
- States 0 and N are **absorbing states**, meaning that once the gambler reaches these states, the game ends.

Remark: Just as a random variable is probabilistically specified by its distribution, a **stochastic process** is specified by its **finite-dimensional distributions**, i.e., by:

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

for $n > 0$ and all $i_0, i_1, \ldots, i_n \in S$, where S is the **state space**.

Let $P(X_0 = i) = \mu_i$ for $i \in S$, where S is the state space. Then, the joint probability distribution of the states $\{X_k\}_{k=0}^{n}$ is given by:

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \left(\prod_{k=0}^{n-1} \rho_{i_k i_{k+1}} \right) \mu_{i_0},$$

where:

- μ_{i_0} is the initial distribution, i.e., the probability that the process starts in state i_0.
- $\rho_{i_k i_{k+1}}$ are the one-step transition probabilities from state i_k to state i_{k+1}.

Thus, a Markov Chain is probabilistically specified by:

- Its initial distribution $\mu = \{\mu_i : i \in S\}$, and
- Its one-step transition probability matrix (TPM).

{By conditional probability of an event A given another event B is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{where } P(B) > 0.$$

} we have,

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$
$$= P(X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) P(X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$
$$= P(X_n = i | X_{n-1} = i_{n-1}) P(X_{n-1} = i_{n-1} \ldots X_0 = i_0)$$
$$= (p_{i_{n-1}, i_n}) P(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2} \ldots X_0 = i_0) P(X_{n-2} = i_{n-2} \ldots X_0 = i_0)$$

we can conclude that,

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = p_{i_{n-1}, i_n} \ldots p_{i_0, i_1} P(X_0 = i_0)$$

1.2.2 Markov Property

Theorem: Let $\{X_n\}_{n\geq 0}$ be a Markov Chain (MC) with initial distribution μ and transition probability matrix (TPM) P. Then, conditional on $X_m = i$, the process $\{X_{m+n}\}_{n\geq 0}$ is also a Markov Chain with Initial distribution δ_i, where δ_i is defin as ,

$$\delta_i(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

and Transition probability matrix P. The Markov property ensures that the process $\{X_{m+n}\}_{n\geq 0}$, conditional on $X_m = i$, "restarts" at state i with the same transition dynamics defined by P. This theorem shows that a Markov chain retains its Markov property even when conditioned on reaching a specific state at some time m.

1.2.3 Chapman Kolmogorov Equation

The n-step transition probability is defined as:

$$p_{ij}^{(n)} = p(X_n = j \mid X_0 = i),$$

where X_n denotes the state at time n and X_0 is the initial state.
Note that:

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) = P(X_{n+k} = j \mid X_k = i).$$

For example, the two-step transition probability can be written as:

$$P(X_2 = j \mid X_0 = i) = P(X_{2+k} = j \mid X_k = i).$$

$$= \sum_{l\in S} P(X_{k+2} = j, X_{k+1} = l \mid X_k = i)$$

where l represents all possible intermediate states.

$$= \sum_{l\in S} P(X_{k+2} = j|X_{k+1} = l, X_k = i)P(X_{k+1} = l|X_k = i)$$

$$= \sum_{l\in S} p_{lj}p_{il} = P(X_{k+1} = j|X_k = i)$$

Theorem: Chapman-Kolmogorov equations
Consider a Markov Chain (MC) with state space $S = \{0, 1, \dots\}$ and one-step transition probabilities p_{ij} for $i, j \in S$.

The Chapman-Kolmogorov equations are given by:

$$p_{ij}^{(m+n)} = \sum_{k=0}^{\infty} p_{ik}^{(m)} p_{kj}^{(n)},$$

for all $m, n \geq 0$ and all $i, j \in \{0, 1, \dots\}$.

Let P be the **one-step transition probability matrix** of a Markov chain, where

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}.$$

The n-step transition probability matrix $P^{(n)}$ is defined as:

$$P^{(n)} = P^n = P \times P \times \cdots \times P \quad \text{(n times)}.$$

Each entry $p_{ij}^{(n)}$ of $P^{(n)}$ represents the probability of transitioning from state i to state j in n steps:

$$P^{(n)} = \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} & \cdots & p_{1m}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & \cdots & p_{2m}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}^{(n)} & p_{m2}^{(n)} & \cdots & p_{mm}^{(n)} \end{bmatrix}.$$

The following properties hold:

- $P^{(0)} = I$, where I is the identity matrix.
- $P^{(1)} = P$.
- Rows of $P^{(n)}$ sum to 1, as they represent probability distributions.

Example 11: Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions.
Let:

- $a = 0.7$: Probability that it will rain tomorrow if it is raining today.

- $\beta = 0.4$: Probability that it will rain tomorrow if it is not raining today.

We define:

- State 0: It is raining.
- State 1: It is not raining.
- X_n: The state on the n-th day.

The process $\{X_n\}_{n\geq 0}$ is a Markov Chain with state space $\{0, 1\}$.
The transition probability matrix P is:

$$P = \begin{bmatrix} a & 1-a \\ \beta & 1-\beta \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}.$$

We want to calculate the probability that it will rain four days from today given that it is raining today, i.e., $p_{00}^{(4)}$.
Calculation: The n-step transition probabilities are given by:

$$P^{(n)} = P^n.$$

First, compute P^4:

$$P^4 = P \times P \times P \times P.$$

The resulting matrix P^4 is:

$$P^4 = \begin{bmatrix} p_{00}^{(4)} & p_{01}^{(4)} \\ p_{10}^{(4)} & p_{11}^{(4)} \end{bmatrix}.$$

The value of $p_{00}^{(4)}$ (top-left entry of P^4) gives the required probability.

$$P^4 = \begin{bmatrix} 0.5749 & 0.4241 \\ 0.5668 & 0.4332 \end{bmatrix}.$$

so we have, $p_{00}^{(4)} = 0.5749$

Example 12 Suppose that balls are successively distributed among 8 urns, with each ball being equally likely to be placed in any of these urns.
We aim to find the probability that there will be exactly 3 occupied urns after 9 balls have been distributed.

- X_n: Number of occupied urns after the distribution of n balls.
- $\{X_n\}_{n\geq 0}$: A Markov Chain with state space $S = \{0, 1, \ldots, 8\}$.

- Transition probabilities:

$$p_{i,i} = \frac{i}{8}, \quad p_{i,i+1} = \frac{8-i}{8}, \quad \text{for } i = 0, 1, \ldots, 8.$$

We want to compute $p_{03}^{(9)}$, which is the probability of having exactly 3 occupied urns after 9 balls have been distributed, starting from 0 occupied urns. Using the Chapman-Kolmogorov equations:

$$p_{03}^{(9)} = \sum_{k=0}^{8} p_{0k}^{(1)} p_{k3}^{(8)},$$

where:

- $p_{0k}^{(1)}$: Transition probabilities after one step.
- $p_{k3}^{(8)}$: Transition probabilities from k occupied urns to 3 occupied urns in 8 additional steps.

Calculating $p_{03}^{(9)}$ directly is difficult.
However, for specific cases (e.g., when starting from 1 occupied urn), the calculation simplifies. For example:

$$p_{13}^{(8)} = p_1 = 1,$$

We consider a Markov Chain $\{Y_n\}_{n\geq 0}$ where the states represent:

- i: i urns are occupied ($i = 1, 2, 3$).
- 4: At least four urns are occupied.

The state space of the new Markov Chain is:

$$\{1, 2, 3, 4\}.$$

We want to compute:

$$q_{13}^{(8)} = P(Y_8 = 3 \mid Y_0 = 1),$$

the probability that exactly 3 urns are occupied after 8 steps, starting from 1 occupied urn.
The transition probability matrix P for the new Markov Chain is:

$$Q = \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q^4 = \begin{bmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0 & 0.0079 & 0.0958 & 0.9009 \\ 0 & 0 & 0.0198 & 0.9802 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so we have, $q_{13}^{(8)} = 0.00756$

Chapter 2
Accessibility and Communication of states

2.1 Accessibility

Definition:

A state j is said to be accessible from state i if there exists $n \geq 0$ such that

$$p_{ij}^{(n)} > 0, \text{ where } p_{ij}^{(n)} = \delta_{ij}.$$

Notation:

$$P_i(.) = P(. \mid X_0 = i).$$

2.1.1 Equivalent Conditions

Theorem: For distinct states i and j, the following are equivalent:

1. j is accessible from i.
2. $P(\text{Ever be in } j \text{ starting from } i) > 0$.
3. There exists some sequence of states $i_0, i_1, \ldots, i_n$ with $i_0 = i$, $i_n = j$, and
$$p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i_{n-1},i_n} > 0,$$
for some $n \geq 0$.

Proof: Equivalence of 1 and 2
To prove the equivalence of conditions 1 and 2:

$$p_{ij}^{(n)} \leq P_i\left(\bigcup_{n=0}^{\infty} \{X_n = j\}\right) = P(\text{Ever be in } j \text{ starting from } i)$$

$$= P\left(X_n = j \text{ for some } n \geq 0\right).$$

On the other hand

$$P\left(X_n = j \text{ for some } n \geq 0\right) \leq \sum_{n=0}^{\infty} p_{ij}^{(n)}.$$

Thus, the equivalence of conditions 1 and 2 is established.

Equivalence of 1 and 3
From the definition of matrix multiplication, we have:

$$p_{ij}^{(n)} = \sum_{i_1, i_2, \ldots, i_{n-1}} p_{i,i_1} p_{i_1,i_2} \cdots p_{i_{n-1},j}.$$

If $p_{ij}^{(n)} > 0$, there exists at least one sequence of states $i_0 = i, i_1, i_2, \ldots, i_{n-1}, i_n = j$ such that:

$$p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i_{n-1},i_n} > 0.$$

Conversely, if such a sequence exists, it follows that:

$$p_{ij}^{(n)} > 0,$$

which implies that j is accessible from i.
Thus, the equivalence of conditions 1 and 3 is established.

2.2 Communication of States

Two states i and j are said to **communicate** if i and j are accessible from each other, i.e., there exist $m \geq 0$ and $n \geq 0$ such that:

$$P_{ij}^{(n)} > 0 \quad \text{and} \quad P_{ji}^{(m)} > 0.$$

Notation

- $i \rightarrow j$: j is accessible from i.
- $i \leftrightarrow j$: i and j communicate.

In Fig 2.1 state C and state B are communicating and state A is accessible form state B. As we goes into the state A from state B we always be in the state A and we cannot came out of state A, and hence State A and state B are not communicating, rather state A is accessible form state B. On opposite,

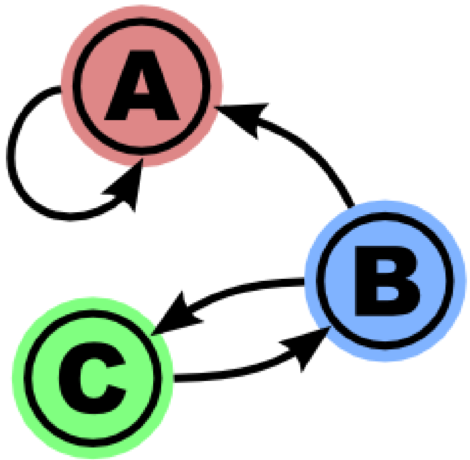

Figure 2.1

in state C and sate B we can see there is both sided path i.e. as we goes from state B to state C we are able to came back to state B from state C, that is state C is accessible from state B and state B is accessible from state C, hence state B and state C are communicating.

Properties of Communication
Communication satisfies the following properties:

1. **Reflexivity:** $i \leftrightarrow i$.
2. **Symmetry:** $i \leftrightarrow j$ if and only if $j \leftrightarrow i$.
3. **Transitivity:** If $i \leftrightarrow k$ and $k \leftrightarrow j$, then $i \leftrightarrow j$.

Since communication satisfies reflexivity, symmetry, and transitivity, it is an **equivalence relation**. Therefore, it partitions the state space into **communicating classes**. A communicating class is a set of states that communicate with each other.

Proof: Transitivity of Communication
Statement: If $i \leftrightarrow k$ and $k \leftrightarrow j$, then $i \leftrightarrow j$.

- $i \leftrightarrow k$: This means there exist $m \geq 0$ and $n \geq 0$ such that:

$$p_{ik}^{(m)} > 0 \quad \text{and} \quad p_{ki}^{(n)} > 0.$$

- $k \leftrightarrow j$: This means there exist $p \geq 0$ and $q \geq 0$ such that:

$$p_{kj}^{(p)} > 0 \quad \text{and} \quad p_{jk}^{(q)} > 0.$$

To prove: $i \leftrightarrow j$, i.e., there exist $r \geq 0$ and $s \geq 0$ such that

$$p_{ij}^{(r)} > 0 \quad \text{and} \quad p_{ji}^{(s)} > 0.$$

Consider the transition probabilities:

$$p_{ij}^{(m+p)} = \sum_{l \in S} p_{ik}^{(m)} p_{kj}^{(p)} > 0,$$

where the summation is over all states $l \in S$ (including k). Since $p_{ik}^{(m)} > 0$ and $p_{kj}^{(p)} > 0$, their product is positive. Hence

$$p_{ij}^{(m+p)} > 0.$$

Thus, $i \to j$.
Similarly, consider the transition probabilities

$$p_{ji}^{(q+n)} = \sum_{l} p_{jk}^{(q)} p_{ki}^{(n)} > 0.$$

Since $p_{jk}^{(q)} > 0$ and $p_{ki}^{(n)} > 0$, their product is positive. Hence

$$p_{ji}^{(q+n)} > 0.$$

Thus, $j \to i$.
Since $i \to j$ and $j \to i$, it follows that $i \leftrightarrow j$. Therefore, the transitivity property is proved.

2.3 Closed class and Irreducibility

1. Closed Class
A communicating class C is said to be **closed** if:

$$i \in C,\, i \to j \implies j \in C.$$

Thus, a closed class is one from which you cannot go out.

2. Absorbing State
A state i is called **absorbing** if $\{i\}$ is a closed class. This means:

$$P_{ii} = 1 \quad \text{and} \quad P_{ij} = 0 \, \text{for all } j \neq i.$$

3. Irreducibility
A Markov Chain is said to be **irreducible** if all states communicate with each other. That is, the entire state space forms a single communicating class:

$$\forall i, j \text{ in the state space, } i \leftrightarrow j.$$

Example 13:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

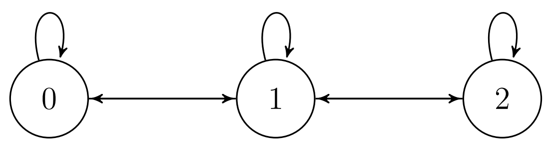

Figure 2.2

$\{0,1,2\}$ is closed communicating class and it is irreducible Markov Chain

Example 14:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

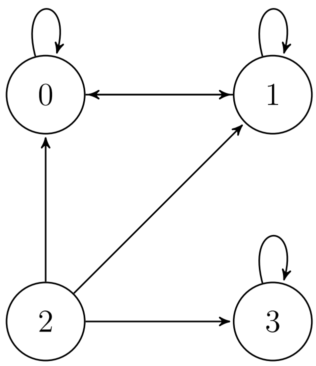

Figure 2.3

- $\{0,1\}$, $\{2\}$, $\{3\}$ are communicating classes.
- $\{0,1\}$ and $\{3\}$ are closed.
- $\{2\}$ is not closed.

It is not irreducible Markov Chain.

Chapter 3
Hitting Time

3.1 Hitting Time

Definition:
For any subset $A \subset S$ (state space), the **hitting time** T_A is defined as

$$T^A = \inf\{n \geq 0 \,|\, X_n \in A\},$$

with the convention that

$$\inf \emptyset = \infty.$$

- If $X_0 \in A$, then $T^A = 0$.
- If $X_0 \notin A$, then T^A is the first time after 0 when the chain enters A.

For $A = \{i\}$ with $i \in S$, the hitting time is denoted by T^i.

- **Hitting Probability:** The probability that, starting from i, the Markov Chain will ever hit A is given by

  $$h_i^A = P_i(T^A < \infty).$$

  Informally, we write

  $$h_i^A = P_i(\text{hit } A).$$

- If A is a closed class, then h_i^A is called the **absorption probability**.

Example 15: Consider the MC with one step transition probability is specified by the following diagram

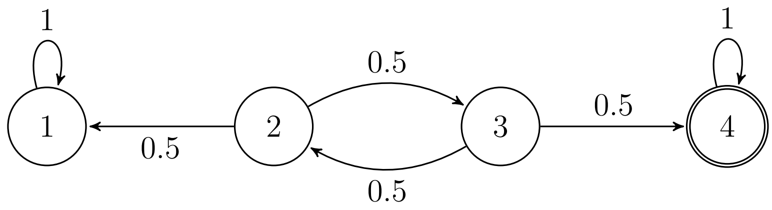

Figure 3.1

starting from 2 what is the probability of absorption in 4 ?

solution: Let $h_i = P_i(\text{hit } 4)$. We aim to find h_2, the probability of absorption in state 4 starting from state 2.

From the diagram:

$$h_1 = 0 \quad \text{(state 1 cannot reach state 4).}$$

$$h_4 = 1 \quad \text{(state 4 is absorbing).}$$

also we have,

$$T^4 = inf\{n \geq 0 \,|\, X_n = 4\}$$

and

$$T^4 = 0, \quad \text{if} \quad X_0 = 4$$

here, $h_2 = P(T^4 < \infty | X_0 = 2)$
For states $i = 1, 2, 3$, the equations are derived from the transition probabilities:

$$h_1 = 0.5h_1 + 0.5h_2,$$
$$h_2 = 0.5h_1 + 0.5h_3,$$
$$h_3 = 0.5h_2 + 0.5h_4.$$

Using $h_0 = 0$ and $h_4 = 1$, solve the system of equations to find h_2.

3.2 Expected Hitting Time

- Starting from i, the expected (mean) time taken by the Markov Chain (MC) to hit A is given by:

$$k_i^A = E_i(T^A) = \sum_{n \geq 1} n \cdot P_i(T^A = n) + \infty \cdot P_i(T^A = \infty)$$

- k_i^A is also called the expected (mean) hitting time to hit A starting from i.
- Informally, we will write:

$$k_i^A = E_i\,(\text{time to hit } A)$$

Example 16: Consider the Markov Chain (MC) with one-step transition probabilities specified by the following diagram:

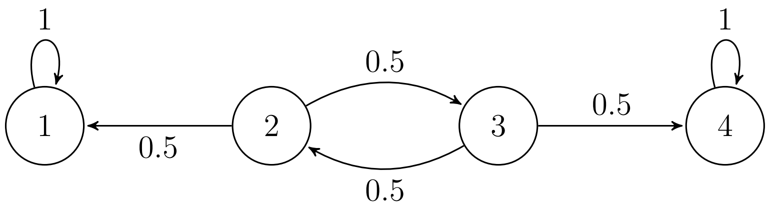

Figure 3.2

Starting from state 2, what is the mean time taken by the chain to get absorbed in either state 1 or 4?

Solution: Denote $k_i = E_i(\text{time to hit } \{1, 4\})$.

- Clearly, $k_1 = k_4 = 0$ because states 1 and 4 are absorbing.
- For state 2, the mean hitting time satisfies:

$$k_2 = 1 + 0.5 \cdot k_1 + 0.5 \cdot k_3$$

- For state 3, the mean hitting time satisfies:

$$k_3 = 1 + 0.5 \cdot k_2 + 0.5 \cdot k_4$$

Substituting $k_1 = 0$ and $k_4 = 0$ into the equations:

$$k_2 = 1 + 0.5 \cdot 0 + 0.5 \cdot k_3 = 1 + 0.5k_3$$

$$k_3 = 1 + 0.5 \cdot k_2 + 0.5 \cdot 0 = 1 + 0.5k_2$$

Solving these two linear equations:

$$k_2 = 1 + 0.5k_3$$

$$k_3 = 1 + 0.5k_2$$

Substitute k_3 into the first equation:

$$k_2 = 1 + 0.5 \cdot (1 + 0.5k_2)$$

$$k_2 = 1 + 0.5 + 0.25k_2$$

$$k_2 - 0.25k_2 = 1.5$$

$$0.75k_2 = 1.5 \implies k_2 = 2$$

Substitute k_2 back to find k_3:

$$k_3 = 1 + 0.5 \cdot 2 = 1 + 1 = 2$$

The mean time to absorption starting from state 2 is:

$$k_2 = 2$$

3.3 Mean Hitting Times

Theorem: The vector of mean hitting times $k^A = (k_i^A \mid i \in S)$ is the minimal non-negative solution of the system of linear equations:

$$k_j^A = 0 \quad \text{for } j \in A,$$

$$k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A \quad \text{for } i \notin A.$$

Explanation: Here, k_i^A represents the expected (mean) time taken by the Markov Chain to hit the set A starting from state i. The first equation states that the mean hitting time is zero for all states that are already in A. The second equation represents the recurrence relation for states not in A, where the hitting time depends on the transition probabilities p_{ij} and the mean hitting times of subsequent states.

Example 17: Snakes-and-Ladders Game
Consider a simple game of snakes-and-ladders played on a 3×3 board.

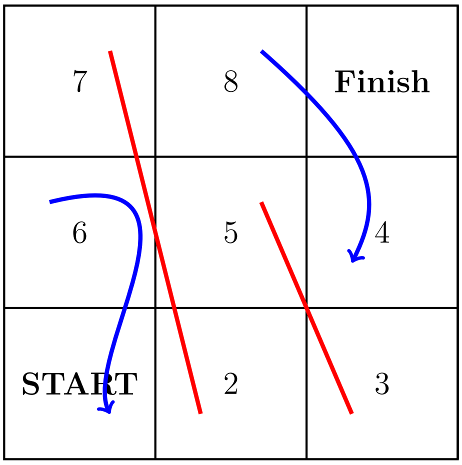

Figure 3.3

At each turn, a player tosses a fair coin and advances one or two places depending on whether the coin lands heads or tails. If the player lands at the foot of a ladder, they climb to the top of the ladder. If the player lands at the head of a snake, they slide down to the tail of the snake. The goal is to calculate the expected number of turns to complete the game, starting from square 1.

Solution: Let E_i denote the expected number of turns to finish the game starting from square i. The following equations describe the Markov chain:

$$E_9 = 0,$$

$$E_8 = 1 + \frac{1}{2}(E_9 + E_9),$$

$$E_7 = 1 + \frac{1}{2}(E_8 + E_4),$$

$$E_6 = 1 + \frac{1}{2}(E_7 + E_5),$$

$$E_5 = 1 + \frac{1}{2}(E_6 + E_4),$$

$$E_4 = 1 + \frac{1}{2}(E_5 + E_3),$$

$$E_3 = 1 + \frac{1}{2}(E_4 + E_2),$$

$$E_2 = 1 + \frac{1}{2}(E_3 + E_6),$$

$$E_1 = 1 + \frac{1}{2}(E_2 + E_7).$$

By solving these equations iteratively or using matrix methods, we can calculate the expected number of turns to complete the game. i.e. we know that, $k_i = E_i(\text{time to hit 9 })$ so,

$$k_1 = E_1(\text{time to hit 9 })$$

and the value of $k_1 = 6.625$ by solving all above inequalities

3.4 Strong Markov Property

We have already established that a Markov Chain $\{X_n\}_{n\geq 0}$ satisfies the Markov property, i.e., for each deterministic time m,

$$P(X_{m+1} = j \mid X_0, X_1, \ldots, X_m) = P(X_{m+1} = j \mid X_m).$$

This means that conditional on $X_m = i$, the process after time m starts afresh from state i.

What happens at a random time?
If we replace the deterministic time m with a *random time* T', the question arises:

Does the process after time T' also start afresh?

For example, consider a stopping time T', such as the first time the chain enters a specific state or set of states. The **Strong Markov Property** states that the process after T' behaves like the original Markov Chain starting from $X_{T'}$.

The Strong Markov Property
Let $\{X_n\}_{n\geq 0}$ be a Markov Chain and T' a stopping time. Then, for all states i, j and all $n \geq 0$, we have:

$$P(X_{T'+n} = j \mid T' = m, X_0, X_1, \ldots, X_m) = P(X_n = j \mid X_m),$$

whenever $T' = m$. This means that:

- The process $\{X_n\}$ starts afresh at the random time T'.
- What happens before T' does not affect the behavior of the chain after T', apart from the state $X_{T'}$ where the chain starts again.

3.4.1 Stopping time

A random variable T taking values in the set $\{0, 1, 2, \ldots, \infty\}$ is called a stopping time with respect to a Markov chain X_n (for $n \geq 0$) if the event $\{T = n\}$ depends only on the observations $X_0, X_1, \ldots, X_n$, i.e., for each n, the event can be determined by the process up to time n.
Intuitively, by observing the process, you know when to stop if you are asked to stop at T.

Example 18: For $i \in S$, let

$$L^i = \sup\{n \geq 0 : X_n = i\}.$$

Then L^i is not, in general, a stopping time because the event $\{L^i = n\}$ depends on whether the process X_{n+m} for $m \geq 1$ visits i or not.

3.4.2 Theorem

Let X_n for $n \geq 0$ be a Markov chain with transition probability matrix (TPM) P and some initial distribution. Let T be a stopping time for X_n with $n \geq 0$.
Then, conditional on $\{T < \infty\}$ and $\{X_T = i\}$, the process X_{T+n}, for $n \geq 0$, is a Markov chain with TPM P and initial distribution δ_i, where δ_i is the Dirac delta function at state i, and it is independent of $X_0, X_1, \ldots, X_T$.

Chapter 4
Passage Time and Excursions

4.1 Classification of states

Definition: For a Markov chain X_n, $n \geq 0$, the first passage time to state i is the random variable T_i defined as:

$$T_i = \inf\{n \geq 1 : X_n = i\},$$

where $\inf \emptyset = \infty$.

Definition: A state i is called **recurrent** if

$$P_i(T_i < \infty) = 1,$$

which means that starting at state i, the chain will return to i with probability 1.

Definition: A state i is called **transient** if

$$P_i(T_i < \infty) < 1,$$

which means that starting at state i, the chain will not return to i with probability 1.

Remark: By the strong Markov property, state i is recurrent if and only if

$$P_i(X_n = i \text{ for infinitely many } n) = 1.$$

This means that, if i is recurrent, the chain will visit state i infinitely often with probability 1.

4.2 k^{th} Passage time and Excursions

Definition: For a state i, define:

$$T_i^{(0)} = 0,$$

and for $k \geq 0$,

$$T_i^{(k+1)} = \begin{cases} \inf\{n \mid n > T_i^{(k)}, X_n = i\}, & \text{if } T_i^{(k)} < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

The random variable $T_i^{(k)}$ is called the k^{th}**passage time** to state i.

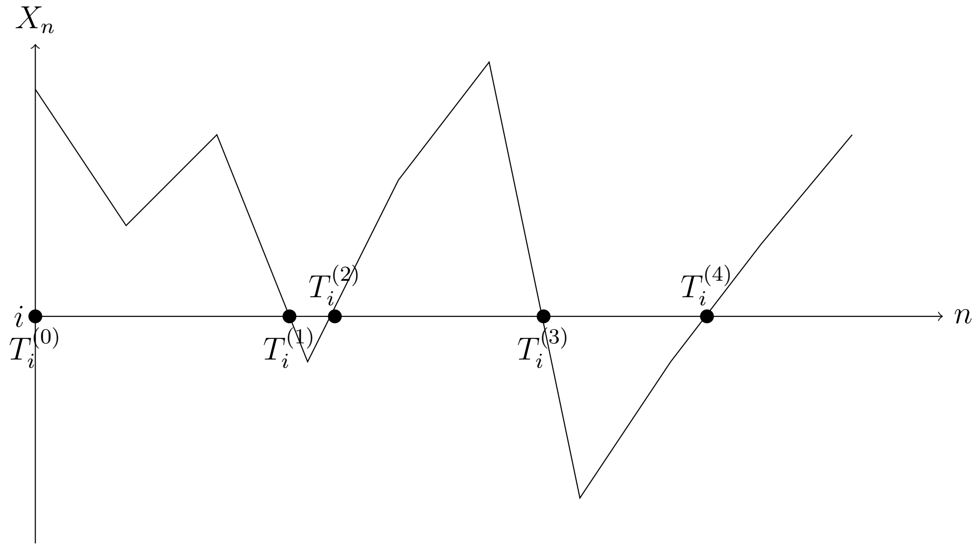

Figure 4.1

Remark: If i is recurrent state then, for all $k \geq 0$

$$P_i(T_i^{(k)} < \infty) = 1$$

Definition: The length of k^{th} Excursion to i is given by

$$S_i^{(k)} = \begin{cases} T_i^{(k)} - T_i^{(k-1)}, & \text{if } T_i^{(k-1)} < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

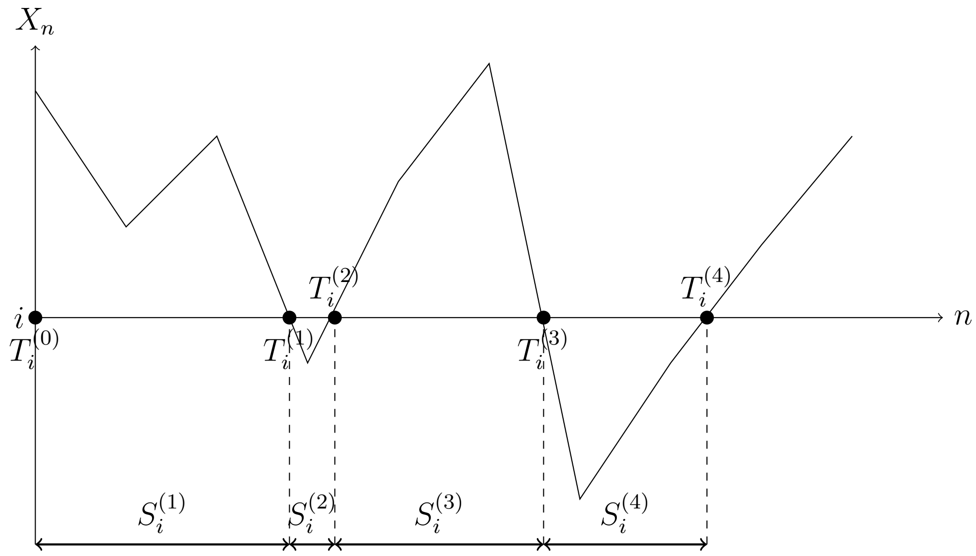

Figure 4.2

4.2.1 Distribution of $S_i^{(k)}$

Lemma: For $k = 2, 3, \ldots$, conditional on $T_i^{(k-1)} < \infty$, $S_i^{(k)}$ is independent of $\{X_m : m \leq T_i^{(k-1)}\}$ and

$$P(S_i^{(k)} = n \mid T_i^{(k-1)} < \infty) = P_i(T_i = n).$$

Proof:

- We will use the strong Markov property at the stopping time $T = T_i^{(k-1)}$.
- It is easy to see that $X_T = i$ on $T < \infty$.
- Thus, conditional on $T < \infty$, $\{X_{T+n} : n \geq 0\}$ is a Markov chain with the same transition probability matrix (TPM) and initial distribution δ_i, and it is independent of $\{X_0, X_1, \ldots, X_T\}$.
- From the definition,

$$S_i^{(k)} = \inf\{n \geq 1 : X_{T+n} = i\}.$$

- Therefore, $S_i^{(k)}$ is the first passage time of $\{X_{T+n} : n \geq 0\}$ to state i.

4.3 Number of Visits

The number of visits to a state i in a Markov chain can be expressed as:

$$V_i = \sum_{n=0}^{\infty} \delta_i(X_n),$$

where,

$$\delta_i(X_n) = \begin{cases} 1 & \text{if } X_n = i, \\ 0 & \text{otherwise.} \end{cases}$$

4.3.1 Distribution of V_i

Lemma: For $k = 0, 1, 2, \dots$, we have:

$$P_i(V_i > k) = f_i^k,$$

where $f_i = P_i(T_i < \infty)$.

Solution:

- Recall that V_i is the total number of visits to state i, expressed as:

$$V_i = \sum_{n=0}^{\infty} \delta_i(X_n),$$

where $\delta_i(X_n)$ is the indicator function:

$$\delta_i(X_n) = \begin{cases} 1 & \text{if } X_n = i, \\ 0 & \text{otherwise.} \end{cases}$$

- The event $V_i > k$ implies that the process visits state i at least $k+1$ times. This means the k-th return time to state i, $T_i^{(k)}$, must be finite:

$$P_i(V_i > k) = P_i(T_i^{(k)} < \infty).$$

- By definition, f_i^k represents the probability that the process returns to state i for the k-th time:

$$f_i^k = P_i(T_i^{(k)} < \infty).$$

- For $k = 1$, this corresponds to the first return to state i, so:

$$f_i^1 = P_i(T_i < \infty).$$

- Therefore, by induction and the strong Markov property, it follows that:

$$P_i(V_i > k) = f_i^k \quad \text{for all } k = 0, 1, 2, \dots.$$

4.4 Transient (Revisited)

Corollary: A state i is transient if and only if:

$$P_i(X_n = i \text{ for infinitely many } n) = 0.$$

Proof:

- Since i is transient, the probability of returning to state i in a single excursion, f_i, satisfies $f_i < 1$.
- The total number of visits to state i, V_i, is finite for transient states. Thus:
$$P_i(V_i = \infty) = \lim_{k\to\infty} P_i(V_i > k) = 0.$$
- Consequently, the probability of $X_n = i$ for infinitely many n (i.e., an infinite number of visits to i) is:
$$P_i(X_n = i \text{ for infinitely many } n) = 0.$$

Remark: Recall that a state i is recurrent if and only if:

$$P_i(X_n = i \text{ for infinitely many } n) = 1.$$

4.4.1 Equivalent Conditions for Recurrent and Transient

Theorem:

- A state i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.
- A state i is transient if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$.

Proof:

- We only prove part 1. Part 2 follows from part 1.
- Suppose that i is recurrent.
- Then $f_i = 1$.
- Hence:
$$P_i(V_i = \infty) = \lim_{k\to\infty} P_i(V_i > k) = 1.$$

- So,
$$\infty = E_i(V_i) = E_i\left(\sum_{n=0}^{\infty}\delta_i(X_n)\right) = \sum_{n=0}^{\infty}E_i(\delta_i(X_n)) = \sum_{n=0}^{\infty}P_i(X_n = i)$$
$$= \sum_{n=0}^{\infty}p_{ii}^{(n)}.$$
- Now, suppose that $\sum_{n=0}^{\infty}p_{ii}^{(n)} = \infty$.
- If possible, suppose i is transient.
- Then $f_i < 1$.
- So,
$$\sum_{n=0}^{\infty}p_{ii}^{(n)} = E_i(V_i) = \sum_{r=0}^{\infty}P_i(V_i > r) = \sum_{r=0}^{\infty}f_i^{(r)} = \frac{1}{1-f_i} < \infty,$$
which is a contradiction.
- Hence, i is recurrent.

Fact: For a non-negative integer-valued random variable X,
$$E(X) = \sum_{n=0}^{\infty}P(X > n).$$

4.4.2 Theorem

If $\{X_n\}_{n\geq 0}$ is a Markov Chain (MC) with finite state space S, then at least one state must be recurrent.

Proof: Let $S = \{1, 2, \ldots, N\}$.

- Note that
$$n = \sum_{i=1}^{N}\sum_{j=1}^{n}\delta_i(X_j).$$
- Thus, taking $n \to \infty$ on both sides, we get that there must exist some i_0 such that
$$\sum_{j=1}^{\infty}\delta_{i_0}(X_j) = \infty$$
with positive probability.
- This implies that i_0 must be recurrent.

4.5 Class Property

Theorem: Let C be a communicating class. Then either all states in C are recurrent or all are transient.

Proof:

- Consider any pair of states $i, j \in C$, $i \neq j$. Then there exist $n, m \geq 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$.
- Now, suppose that i is transient.
- For any $r \geq 0$,
$$p_{ii}^{(n+r+m)} \geq p_{ij}^{(n)} p_{jj}^{(r)} p_{ji}^{(m)}.$$
- So, By Chapman-kolmogorov equation we have,
$$p_{ij}^{(n+m)} = \sum_{k \in S} p_{ik}^{(n)} p_{kj}^{(m)}$$
- And hence,
$$\sum_{r=0}^{\infty} p_{jj}^{(r)} \leq \frac{1}{p_{ij}^{(n)} p_{ji}^{(m)}} \sum_{r=0}^{\infty} p_{ii}^{(n+r+m)} < \infty.$$
- Hence, j is also transient.
- Suppose that i is recurrent.
- If possible, assume that j is transient.
- As $j \to i$, i is transient, which is a contradiction.
- Therefore, j is a recurrent state.

Remark: A communicating class is called a **recurrent class** (or **transient class**) if all the states of the class are recurrent (or transient).

Remark: If a Markov chain is irreducible, then all its states are either recurrent or transient. Accordingly, we say that the Markov chain is **recurrent** or **transient**.

4.5.1 Theorem:

Every recurrent class is closed.

Proof:

- Suppose that C is a recurrent class.
- If possible, suppose that C is not closed. Then there exists $i \in C$, $j \notin C$, and $k \geq 1$ such that $P_i(X_k = j) > 0$.
- Since,

$$P_i(\{X_k = j\} \cap \{X_n = i \text{ for infinitely many } n\}) = 0,$$

  we must have $P_i(X_n = i \text{ for infinitely many } n) < 1$.

- Hence, i is not recurrent and thereby C is not recurrent, which is a contradiction.

Every finite closed class is recurrent.

Proof:

- Suppose that C is closed and finite and $\{X_n\}_{n\geq 0}$ starts in C.
- Then for some $i \in C$, we must have

$$0 < P(X_n = i \text{ for infinitely many } n)$$

$$= P(X_n = i \text{ for some } n) \cdot P_i(X_n = i \text{ for infinitely many } n).$$

- Hence,

$$P_i(X_n = i \text{ for infinitely many } n) > 0.$$

- Thus, i is not transient and thereby has to be recurrent.
- Since recurrence is a class property, C is recurrent.

Example 19: Recall from Ex. 14

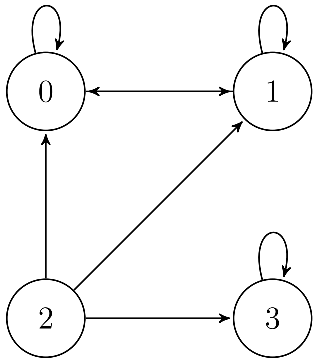

Figure 4.3

- $\{0,1\}, \{2\}$ & $\{3\}$ are communicating classes.
- $\{0,1\}$ & $\{3\}$ are closed and finite.
- $\{2\}$ is not closed.
- The classes $\{0,1\}$ & $\{3\}$ are recurrent.
- The class $\{2\}$ is transient.

Here, if we consider that $\{2\}$ is recurrent then $\{2\}$ has to be closed which is the contradiction and hence $\{2\}$ is transient.

Chapter 5
Stationary Distribution

5.1 Stationary Distribution

Definition: A row vector $\pi = (\pi_i \,|\, i \in S)$ of non-negative entries is said to be a **stationary** (or an **invariant**) measure for a Markov Chain with transition probability matrix $P = (p_{ij})_{i,j \in S}$ if

$$\pi_j = \sum_{i \in S} \pi_i p_{ij} \tag{1}$$

for all $j \in S$.

Remark: In matrix notation, equation (1) can be written as:

$$\pi P = \pi.$$

Remark: It is easy to see that for any $n \geq 1$, the following holds:

$$\pi P^n = \pi.$$

Definition: A stationary measure π is called a **stationary distribution** if:

$$\sum_{i \in S} \pi_i = 1.$$

5.1.1 Justification of Terminology

The following theorem justifies the use of the terminology "Stationary."

Theorem: Let $\{X_n\}_{n \geq 0}$ be a Markov Chain with transition probability matrix P and initial distribution $\pi = (\pi_i \,|\, i \in S)$. If π is a stationary distribution, then:

$$P(X_n = j) = \pi_j \quad \text{for all } n \geq 0 \text{ and for all } j \in S.$$

Proof: We will prove by induction.

- From the definition of the initial distribution, we have:

$$P(X_0 = j) = \pi_j.$$

- Assume that $P(X_n = j) = \pi_j$ for some $n \geq 0$.
- Now, by the Markov property:

$$P(X_{n+1} = j) = \sum_{i \in S} P(X_{n+1} = j \mid X_n = i) P(X_n = i).$$

Substituting the induction hypothesis $P(X_n = i) = \pi_i$, we get:

$$P(X_{n+1} = j) = \sum_{i \in S} \pi_i p_{ij}.$$

Since π is a stationary distribution, $\sum_{i \in S} \pi_i p_{ij} = \pi_j$ for all $j \in S$. Thus:

$$P(X_{n+1} = j) = \pi_j \quad \text{for all } j \in S.$$

By induction, the result follows.

5.1.2 Limiting Distribution (Finite State Space)

The following theorem states that the "limiting" distribution, if it exists, is a stationary distribution.

Theorem: Let S be finite. Suppose that for some $i \in S$,

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j \quad \text{for all } j \in S.$$

Then $(\pi_j \mid j \in S)$ is a stationary distribution.

Proof: We have:

$$\sum_{j \in S} \pi_j = \sum_{j \in S} \lim_{n \to \infty} p_{ij}^{(n)} = \lim_{n \to \infty} \sum_{j \in S} p_{ij}^{(n)} = 1.$$

Also:

$$\pi_j = \lim_{n \to \infty} p_{ij}^{(n)} = \lim_{n \to \infty} \sum_{k \in S} p_{ik}^{(n-1)} p_{kj}.$$

Using the assumption that $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$, we can write:

$$\pi_j = \sum_{k\in S} \lim_{n\to\infty} p_{ik}^{(n-1)} p_{kj} = \sum_{k\in S} \pi_k p_{kj}.$$

Thus, $(\pi_j \mid j \in S)$ satisfies the stationary distribution equation.

Example 20: Consider the two-state Markov Chain with transition probability matrix (TPM):

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

Let (π_0, π_1) be a stationary distribution. Then, we have the equations:

$$\pi_0 = (1-\alpha)\pi_0 + \beta\pi_1,$$

$$\pi_1 = \alpha\pi_0 + (1-\beta)\pi_1,$$

and

$$\pi_0 + \pi_1 = 1.$$

Solving these equations, we get:

$$\pi_0 = \frac{\beta}{\alpha+\beta}, \quad \pi_1 = \frac{\alpha}{\alpha+\beta}.$$

Thus, for this example, the stationary distribution exists and is unique.

Example 21: Consider the Markov Chain (MC) with the following transition diagram ($p + q = 1, 0 < p < 1$):

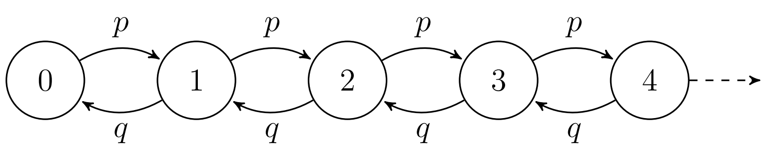

Figure 5.1

Let $(\pi_i : i \geq 0)$ be a stationary distribution. Then:

$$\pi_0 = \pi_1 q,$$

$$\pi_1 = \pi_0 + \pi_2 p = \pi_1 q + \pi_2 p \implies \pi_1 p = \pi_2 p \implies \pi_1 = \pi_2.$$

Similarly:

$$\pi_2 = \pi_3, \quad \pi_3 = \pi_4, \quad \pi_4 = \pi_5, \ldots.$$

Theorem: If $\{X_n\}_{n\geq 0}$ is a Markov Chain with a finite state space, then there exists at least one stationary distribution.

5.1.3 Theorems

Theorem: Let $\{X_n\}_{n\geq 0}$ be an irreducible Markov Chain (MC). If π is an invariant measure such that $\pi_i > 0$ for some $i \in S$, then $\pi_k > 0$ for all $k \in S$.

Proof:

- Consider any state $k \neq i$.
- By irreducibility, there exists a path from i to k, say $i \to j_1 \to j_2 \to \cdots \to k$, with $P(i \to j_1) > 0$, $P(j_1 \to j_2) > 0, \ldots$, and $P(j_m \to k) > 0$.
- Since π is an invariant measure, we have:

$$\pi_k = \sum_{j \in S} \pi_j P(j \to k).$$

- From the above equation, as $\pi_i > 0$ and $P(i \to k) > 0$, we conclude that $\pi_k > 0$.
- Therefore, $\pi_k > 0$ for all $k \in S$.

Note: For a fixed state k, consider for each $i \in S$ the expected time spent in i between visits to k:

$$\gamma_i^k = E_k \left(\sum_{n=0}^{T_k - 1} \delta_i(X_n) \right),$$

where $\delta_i(X_n)$ is the indicator function, T_k is the return time to k, and E_k denotes the expectation conditioned on starting at k.

Theorem: Let $\{X_n\}_{n\geq 0}$ be an irreducible and recurrent Markov chain (MC). Then:

- $\gamma_k^k = 1$.
- $\gamma_j^k = \sum_{i \in S} \gamma_i^k p_{ij}$, $\forall j \in S$.

- $0 < \gamma_i^k < \infty, \forall i \in S$.

Theorem: Let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain (MC). Fix a state k and let π be an invariant measure with $\pi_k = 1$. Then:

$$\pi \geq \gamma^k \quad \text{(entrywise)},$$

where γ^k represents the expected time spent in each state $i \in S$ between visits to k. If, in addition, the chain is recurrent, then:

$$\pi = \gamma^k.$$

Remark: Let $\{X_n\}_{n\geq 0}$ be an irreducible and recurrent Markov chain. If π and μ are two non-zero invariant measures, then there exists a constant $c > 0$ such that:

$$\mu = c\pi.$$

Proof

- Fix a state k.
- By the theorem:
$$\frac{\pi}{\pi_k} = \gamma^k = \frac{\mu}{\mu_k}.$$
- Hence,
$$\mu = \frac{\mu_k}{\pi_k}\pi = c\pi,$$
where $c = \frac{\mu_k}{\pi_k}$.

5.2 Positive and Null recurrent

Definition: A recurrent state i is said to be **positive recurrent** if:

$$m_i = E(T_i) < \infty.$$

If i is not positive recurrent, then it is called **null recurrent**.

Theorem Let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain (MC). Then the following are equivalent:

1. Every state is positive recurrent.

2. Some state i is positive recurrent.

3. The Markov chain has a stationary distribution π.

Moreover, when (3) holds, $\pi_i > 0$ for all $i \in S$.

Remark Positive recurrence and null recurrence are **class properties**.

Example 22: Consider the Markov chain (MC) with state space $S = \{0, 1, 2, \dots\}$ and transition probabilities:

$$p_{0,1} = 1, \quad p_{i,i+1} = p_{i,0} = \frac{1}{2} \quad \text{for all } i \geq 1.$$

Solution:

- It is easy to see that the MC is irreducible.
- Let $\pi = (\pi_i : i \in S)$ be a stationary distribution, then

$$\pi_0 = \frac{1}{2}(\pi_1 + \pi_2 + \dots),$$

$$\pi_1 = \pi_0, \quad \pi_2 = \frac{\pi_1}{2} = \frac{\pi_0}{2}, \quad \pi_3 = \frac{\pi_2}{2} = \frac{\pi_0}{2^2}, \quad \dots, \quad \pi_n = \frac{\pi_0}{2^{n-1}}, \quad \dots.$$

- As $\sum_{i=0}^{\infty} \pi_i = 1$, we have:

$$\pi_0 + \pi_0 + \frac{\pi_0}{2} + \frac{\pi_0}{2^2} + \dots = 1 \implies \pi_0(1+2) = 1 \implies \pi_0 = \frac{1}{3}.$$

Thus, the unique stationary distribution is:

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3 \cdot 2}, \frac{1}{3 \cdot 2^2}, \dots\right).$$

Chapter 6
Limiting Theorems

6.1 Limiting Distribution

Definition: For $i \in S$,

$$V_i^{(n)} = \#\{t : 0 \le t \le n, X_t = i\} = \sum_{k=0}^{n} \delta_i(X_k)$$

is the number of visits to state i during $\{0, 1, \dots, n\}$.

Definition: For $i \in S$, define

$$L_i^{(n)} = \frac{V_i^{(n)}}{n+1}.$$

Then, $\{L_i^{(n)} : i \in S\}$ is called the empirical distribution at time n.

Theorem: [Law of Large Numbers for Markov Chains]
Let $\{X_n\}_{n \ge 0}$ be an irreducible and recurrent Markov Chain (MC). Fix a state i. Suppose that $P(X_0 = i) = 1$. Then, for any state j,

$$L_j^{(n)} \to \begin{cases} \frac{\gamma^j}{E_i(T_i)} & \text{if } E_i(T_i) < \infty, \\ 0 & \text{if } E_i(T_i) = \infty \end{cases}$$

with probability 1. In particular, if i is positive recurrent, then

$$L_i^{(n)} \to \frac{1}{E_i(T_i)} = \pi_i,$$

with probability 1.

Theorem: Fix the state $i \in S$. Then:

- i is transient if and only if

$$\sum_{k=0}^{\infty} p_{ii}^{(k)} < \infty \quad \text{(Already proved).}$$

- i is null recurrent if and only if

$$\sum_{k=0}^{\infty} p_{ii}^{(k)} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ii}^{(k)} = 0.$$

- i is positive recurrent if and only if

$$\sum_{k=0}^{\infty} p_{ii}^{(k)} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ii}^{(k)} > 0.$$

Proof
Here, the proof is given assuming irreducibility of the MC.

Proof of (2) and (3):
Case 1: Suppose i is null recurrent. As i is recurrent,

$$\sum_{k=0}^{\infty} p_{ii}^{(k)} = \infty.$$

Now, by the previous theorem,

$$\lim_{n\to\infty} L_i^{(n)} = 0 \implies \lim_{n\to\infty} E_i(L_i^{(n)}) = 0 \implies \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ii}^{(k)} = 0.$$

Case 2: Now, suppose that i is positive recurrent. As i is recurrent,

$$\sum_{k=0}^{\infty} p_{ii}^{(k)} = \infty.$$

Again, by the previous theorem,

$$\lim_{n\to\infty} L_i^{(n)} = \frac{1}{E_i(T_i)} \implies \lim_{n\to\infty} E_i(L_i^{(n)}) = \frac{1}{E_i(T_i)} \implies \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ii}^{(k)}$$

$$= \frac{1}{E_i(T_i)} > 0.$$

6.2 Period

Definition: The period of a state i is defined by the greatest common divisor of all integers $n \geq 1$ for which $p_{ii}^{(n)} > 0$, i.e.,

$$d_i = \begin{cases} \gcd\{n \geq 1 : p_{ii}^{(n)} > 0\} & \text{if } \{n \geq 1 : p_{ii}^{(n)} > 0\} \neq \emptyset, \\ 0 & \text{if } \{n \geq 1 : p_{ii}^{(n)} > 0\} = \emptyset. \end{cases}$$

Further, i is called **aperiodic** if $d_i = 1$.

Example 23: Consider the Markov chain with the following transition probability matrix (TPM):

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

From this TPM:

- $p_{00}^{(2n-1)} = 0$ and $p_{00}^{(2n)} = 1$ for all $n \geq 1$.
- Thus, $d_0 = 2$.

Example 24: Let $S = \{0, \pm 1, \pm 2, \dots\}$. The transition probabilities are given by:

$$p_{i,i+1} = a, \quad p_{i,i-1} = b, \quad p_{ii} = c,$$

where $a + b + c = 1$, $a > 0$, $b > 0$, and $c \geq 0$.

- Fix a state i.
- Suppose that $p_{ii}^{(1)} = c > 0$.
  - Then, $1 \in \{n \geq 1 : p_{ii}^{(n)} > 0\}$.
  - Hence, $d_i = 1$ (aperiodic).
- Now, suppose that $p_{ii}^{(1)} = c = 0$.
  - Then, $p_{ii}^{(2m-1)} = 0$ and $p_{ii}^{(2m)} > 0$ for all $m \geq 1$.
  - Therefore, $\{n \geq 1 : p_{ii}^{(n)} > 0\} = \{2m : m \geq 1\}$.
  - Hence, $d_i = 2$.

Theorem: If $i \leftrightarrow j$, then $d(i) = d(j)$.

Remark: Thus, if $\{X_n\}_{n\geq 0}$ is irreducible, then all states have the same period. In particular, if all states have period 1, the Markov Chain (MC) is called **aperiodic**.

Theorem: Let $\{X_n\}_{n\geq 0}$ be an irreducible, positive recurrent, and aperiodic Markov Chain. Then, for any initial distribution μ:

$$\lim_{n\to\infty} P_\mu(X_n = i)$$

exists and is equal to π_i for all i, where $(\pi_i : i \in S)$ is the unique stationary distribution.
In particular, for any two states i and j:

$$\lim_{n\to\infty} P^{(n)}_{(ij)} = \pi_j.$$

Example 25: Consider the Markov Chain (MC) with the following Transition Probability Matrix (TPM):

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

We aim to find all stationary distributions.

Solution: Let the stationary distribution be $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4)$, such that:

$$\pi P = \pi,$$

and

$$\sum_{i=0}^{4} \pi_i = 1.$$

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_4, \tag{6.1}$$

$$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_3, \tag{6.2}$$

$$\pi_2 = \pi_2 + \frac{1}{4}\pi_3, \tag{6.3}$$

$$\pi_3 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_3, \tag{6.4}$$

$$\pi_4 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_3 + \frac{1}{2}\pi_4. \tag{6.5}$$

To find the stationary distribution, we have to solve the system of equations (1)–(5) such as,

$$\pi_0 = \pi_4,$$
$$\pi_3 = 0,$$
$$\pi_1 = 0.$$

thus $(\alpha, 0, \beta, 0, \alpha)$ is the stationary distribution if

$$0 \leq \alpha, \beta \geq 1 \ \ \& \ \ 2\alpha + \beta = 1$$

Example 26: A particle performs random walk on the vertices of a tetrahedron. At each step, it remains where it is with probability 0.25 or moves to one of its neighboring vertices each with probability 0.25. Suppose that the particle starts at A (see the following figure). Find the mean number of steps until its first return to A.

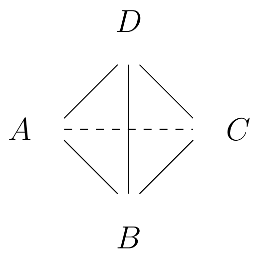

Figure 6.1

Let X_n be the position of the particle at time n. Then, $\{X_n\}_{n\geq 0}$ is a Markov Chain (MC) with transition probability matrix (TPM)

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Useful Fact: An $n \times n$ TPM is said to be doubly stochastic if its each column sum is also equal to 1. If $\{X_n\}_{n\geq 0}$ is an irreducible MC with state space S, with $|S| = n$ and a doubly stochastic TPM, then, the unique stationary distribution of the MC is given by

$$\pi_i = \frac{1}{n} : i \in S.$$

- Using the above fact, the stationary distribution of $\{X_n\}_{n\geq 0}$ is
$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$
Thus,
$$E_A(T_A) = \frac{1}{\frac{1}{4}} = 4.$$

Example 27: Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return, he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of 4 pairs of running shoes, what is the long run proportion of time he runs barefooted?

- Let X_n be the number of pairs of shoes at the front door on the nth morning.
- Then, $\{X_n\}_{n\geq 0}$ is a Markov chain (MC) with Transition Probability Matrix (TPM)
$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$
- By the Useful Fact, the unique stationary distribution is
$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$
- Thus, the long-run proportion of times he runs barefooted is
$$\frac{1}{2}\pi_0 + \frac{1}{2}\pi_4 = \frac{1}{5}.$$

6.3 Time reversal Markov chain

Theorem: Let $\{X_n\}_{0\leq n\leq N}$ be an irreducible Markov chain (MC) with stationary distribution π. Define $Y_n = X_{N-n}$. Then $\{Y_n\}_{0\leq n\leq N}$ is again a MC with initial distribution π and TPM $\hat{P} = (\hat{p}_{ij})$ is given by

$$\pi_j \hat{p}_{ji} = \pi_i p_{ij} \quad \text{for all } i, j.$$

Moreover, $\{Y_n\}_{0\leq n\leq N}$ is also irreducible and has invariant distribution π.

Remark: The chain $\{Y_n\}_{0\leq n\leq N}$ is called the time reversal of $\{X_n\}_{0\leq n\leq N}$.

Definition: Let $\{X_n\}_{n\geq 0}$ be an irreducible MC with TPM P and initial distribution π, where π is also a stationary distribution. If for all $N \geq 1$, the time-reversed chain $\{Y_n\}_{0\leq n\leq N}$ also has TPM P, then $\{X_n\}_{n\geq 0}$ is said to be reversible.

Example 28: Consider the Markov chain (MC) with the following TPM

$$P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

- The MC is irreducible and has unique stationary distribution

  $$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

  (since P is doubly stochastic).

- Thus, if $\hat{P} = (\hat{p}_{ij})$ satisfies

  $$\pi_j \hat{p}_{ji} = \pi_i p_{ij} \quad \text{for all } i, j,$$

  then $\hat{p}_{ij} = p_{ji}$ for all i, j.

- Since $p_{12} \neq p_{21}$, the chain is not reversible.

Example 29: Consider the Markov chain (MC) with the following TPM

$$P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

- The MC is irreducible and has unique stationary distribution $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (P is doubly stochastic).
- Thus, if $\hat{P} = (\hat{p}_{ij})$ satisfies
$$\pi_j \hat{p}_{ji} = \pi_i p_{ij} \quad \text{for all } i, j,$$
then $\hat{p}_{ij} = p_{ji}$ for all i, j.
- Since $p_{ij} = p_{ji}$ for all i, j, the chain is reversible.

6.4 Detailed Balance

Definition: Let $\{X_n\}_{n\geq 0}$ be a Markov chain (MC) with TPM P. A non-negative row vector $\lambda = (\lambda_i : i \in S)$ and P are said to be in **detailed balance** if
$$\lambda_j p_{ji} = \lambda_i p_{ij} \quad \text{for all } i, j.$$

Remark: If λ and P are in detailed balance, then λ is an invariant measure for the MC.

Proof:
$$\sum_{i\in S} \lambda_i p_{ij} = \sum_{i\in S} \lambda_j p_{ji} = \lambda_j.$$

Theorem: Let $\{X_n\}_{n\geq 0}$ be an irreducible MC with TPM P. Let $\pi = (\pi_i : i \in S)$ be a positive row vector such that $\sum_{i\in S} \pi_i = 1$. Then the following statements are equivalent:

1. π and P are in detailed balance.
2. π is the stationary distribution for $\{X_n\}_{n\geq 0}$ and $\{X_n\}_{n\geq 0}$ is reversible.

Proof:

(1) $\implies$ (2): By the previous remark, π is a stationary distribution.For time reversibility, by the detailed balance condition, we get
$$\hat{p}_{ji} = \frac{\pi_i p_{ij}}{\pi_j} = \frac{\pi_j p_{ji}}{\pi_j} = p_{ji} \quad \text{for all } i, j.$$
Hence, $\{X_n\}_{n\geq 0}$ is reversible.

(2) $\implies$ (1): Since $\hat{P} = P$, $\pi_j p_{ji} = \pi_i p_{ij}$ for all i, j.

Hence, π and P are in detailed balance.

Example 30: A random king makes each permissible move with equal probability. If it starts in the square marked '1', how long on average will it take to return to '1'?

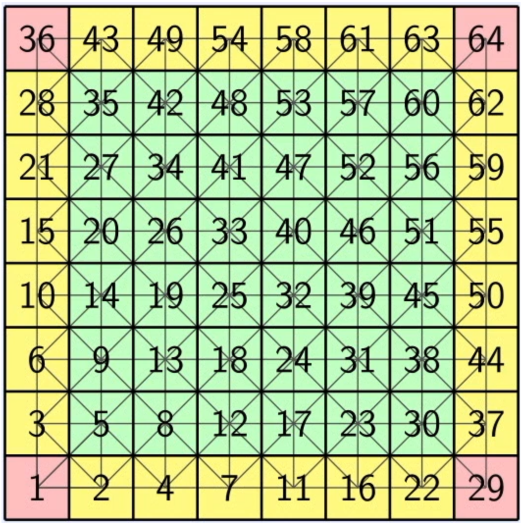

Figure 6.2

- Let X_n denote the position of the king after the nth move.
- Then, $\{X_n\}_{n\geq 0}$ is a Markov Chain (MC), where squares of the chessboard are the states.
- From the pink-colored squares, the king can move to 3 adjacent squares each with probability $\frac{1}{3}$.
- From the yellow-colored squares, the king can move to 5 adjacent squares each with probability $\frac{1}{5}$.
- From the green-colored squares, the king can move to 8 adjacent squares each with probability $\frac{1}{8}$.
- Thus,

$$p_{ij} = \begin{cases} \frac{1}{3}, & \text{if } i \text{ is pink and } j \text{ is adjacent to } i, \\ \frac{1}{5}, & \text{if } i \text{ is yellow and } j \text{ is adjacent to } i, \\ \frac{1}{8}, & \text{if } i \text{ is green and } j \text{ is adjacent to } i. \end{cases}$$

- The MC is irreducible.
- Let
$$\lambda_i = \begin{cases} 3 & \text{if } i \text{ is pink,} \\ 5 & \text{if } i \text{ is yellow,} \\ 8 & \text{if } i \text{ is green.} \end{cases}$$
- Then λ and P satisfy the detailed balance equations:
$$\lambda_i p_{ij} = \lambda_j p_{ji}, \quad \text{for all } i, j.$$
- Thus, $\left(\frac{\lambda_i}{\sum_j \lambda_j} : i \in S\right)$ is the unique stationary distribution.
- Therefore,
$$E_1(T_1) = \frac{\sum_j \lambda_j}{\lambda_1} = \frac{4 \times 3 + 24 \times 5 + 36 \times 8}{3} = 140.$$

Chapter 7
Simulation of Markov chain

7.1 Simulation on Financial conditions

7.1.1 State simulation

In financial markets, Markov chains can be used to model the evolution of asset prices, interest rates, or market states over time. The concept of initial distributions (u) and transition probability matrices (TPM) applies in similar ways, though with domain-specific interpretations.
Here's how Markov chains might work in a financial context, step by step:

- **State Space:** The state space represents the possible conditions of the financial market or asset.
  $S = \{1, 2, 3\}$ could represent:

  1. Bull market
  2. Bear market
  3. Stagnant market

- **Initial Distribution (u):** The initial distribution u reflects the probabilities of starting in a particular market state at time $t = 0$. It might be determined as follows:

  1. Historical Data: Use historical market data to estimate the likelihood of starting in each state. Example: If the market spends 60% of its time in a bull state, 30% in bear, and 10% in stagnant, set $u = [0.6, 0.3, 0.1]$.
  2. Assumptions: Start with a uniform distribution if no prior knowledge is available: $u = [1/3, 1/3, 1/3]$.

3. Market Conditions: Use current indicators (e.g., economic reports, sentiment analysis) to assign probabilities:

   High optimism $\rightarrow$higher u for bull markets.
   Poor outlook $\rightarrow$higher u for bear markets.

- **Transition Probability Matrix (TPM):** The TPM models the probabilities of moving between market states. Each row corresponds to the current state, and each column gives the probabilities of transitioning to other states. Example:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

  Here, if the current state is Bull market then there is 60% chance to remain in Bull market, 30% to transition in Bear market and 10% to move in Stagnant state.

- **Initial Distribution $X_0 \sim u$:** The initial state X_0 of the Markov chain is distributed according to a probability vector $u = \{u(1), u(2), \ldots, u(k)\}$:

  - $u(i)$: Probability that $X_0 = i$.
  - The vector u satisfies:

$$\sum_{i=1}^{k} u(i) = 1, \quad u(i) \geq 0 \quad \forall i.$$

- **Uniformly Distributed Random Variables $U_i \sim \mathbf{U}[0, 1]$:** The U_i are independent and identically distributed (i.i.d.) random variables drawn from the uniform distribution on $[0, 1]$.

  - **I.i.d.**: Each U_i is independent of the others and has the same distribution.
  - **Uniform on** $[0, 1]$: Any number in $[0, 1]$ is equally likely.

  In practical terms, U_i are random numbers generated using a library function such as `random.uniform(0, 1)` in Python.

- **How Does It All Fit Together?**

  The setup describes how to simulate or analyze the Markov chain $\{X_n\}$:

1. **Initial State** X_0: The Markov chain starts in state i with probability $u(i)$.
   - Generate a random number $U_0 \sim \mathrm{U}[0, 1]$.
   - Use u to determine X_0 based on cumulative probabilities:

$$X_0 = i \quad \text{if} \quad \sum_{j=1}^{i-1} u(j) \leq U_0 < \sum_{j=1}^{i} u(j).$$

2. **Transitions Using TPM** P: After determining X_0, the state evolves according to P:
   - At each step n, generate a random number $U_n \sim \mathrm{U}[0, 1]$.
   - Use P to decide the next state X_{n+1} based on cumulative probabilities in the current row of P:

$$X_{n+1} = j \quad \text{if} \quad \sum_{k=1}^{j-1} P(X_n, k) \leq U_n < \sum_{k=1}^{j} P(X_n, k).$$

- **Python code implementation**

```python

import numpy as np

def simulate_financial_market(P, u, N):
    """
    Simulates a financial market using a Markov chain of length N.

    Parameters:
    - P: Transition probability matrix (3x3 for Bull, Bear, and Stagnant).
    - u: Initial distribution vector ([P(Bull), P(Bear), P(Stagnant)]).
    - N: Length of the Markov chain.

    Returns:
    - chain: List of states in the financial market (1 = Bull, 2 = Bear, 3 = Stagnant).
    """
    states = ['Bull', 'Bear', 'Stagnant']  # Labels for the states
    chain = []

    # Step 1: Generate the initial state
```

```python
    U0 = np.random.uniform(0, 1)
    cumulative_u = np.cumsum(u)
    initial_state = np.where(U0 < cumulative_u)[0][0]
    chain.append(initial_state)

    # Step 2: Simulate transitions
    for _ in range(N - 1):
        current_state = chain[-1]
        U_next = np.random.uniform(0, 1)
        cumulative_p = np.cumsum(P[current_state])
        next_state = np.where(U_next < cumulative_p)[0][0]
        chain.append(next_state)

    # Convert state indices to labels
    labeled_chain = [states[state] for state in chain]
    return labeled_chain

# Example Usage
P = np.array([
    [0.6, 0.3, 0.1],  # Bull to Bull, Bear, Stagnant
    [0.2, 0.7, 0.1],  # Bear to Bull, Bear, Stagnant
    [0.3, 0.3, 0.4]   # Stagnant to Bull, Bear, Stagnant
])
u = [0.5, 0.1, 0.4]  # Initial probabilities for Bull, Bear, Stagnant
N = 300  # Length of the simulation

simulated_states = simulate_financial_market(P, u, N)
print("Simulated Financial Market States:", simulated_states)
```

- **Output**

```
Simulated Financial Market States: ['Stagnant', 'Stagnant', 'Bull', 'Bear', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear',
'Bear', 'Stagnant', 'Stagnant', 'Stagnant', 'Stagnant', 'Stagnant', 'Stagnant', 'Bear', 'Bear', 'Bear', 'Bull', 'Bear',
'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Stagnant', 'Bear', 'Bear', 'Bull', 'Bull', 'Bull', 'Bull', 'Bull', 'Bear',
'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Bear', 'Bull', 'Stagnant',
'Stagnant', 'Stagnant', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull',
```

```
'Bear', 'Bear', 'Bull', 'Bear', 'Bear', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear',
'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Bear', 'Bear', 'Bull', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear',
'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Stagnant', 'Stagnant', 'Stagnant', 'Stagnant', 'Bear',
'Bull', 'Bear', 'Bull', 'Stagnant', 'Bear', 'Stagnant', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull',
'Bull', 'Bear', 'Bear', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull', 'Bull', 'Bull',
'Bull', 'Bull', 'Bear', 'Bear', 'Stagnant', 'Bear', 'Bull', 'Bear', 'Stagnant', 'Stagnant', 'Bull', 'Bear', 'Bull', 'Bull',
'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Bear', 'Bear', 'Bull',
'Stagnant', 'Bull', 'Bull', 'Bull', 'Stagnant', 'Bull', 'Bull', 'Stagnant', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull',
'Bear', 'Bear', 'Stagnant', 'Stagnant', 'Stagnant', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Stagnant',
'Bull', 'Stagnant', 'Bear', 'Bear', 'Bull', 'Stagnant', 'Bear', 'Bull', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull', 'Bull',
'Bull', 'Bear', 'Bear', 'Stagnant', 'Bear', 'Bear', 'Bull', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull',
'Bull', 'Bull', 'Bear', 'Bull', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull', 'Stagnant', 'Stagnant',
'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Stagnant', 'Bear', 'Bull', 'Bull', 'Stagnant', 'Stagnant', 'Stagnant',
'Stagnant', 'Stagnant', 'Bear', 'Stagnant', 'Bull', 'Stagnant', 'Bull', 'Bull', 'Bull', 'Bull', 'Bull', 'Bull', 'Bull',
'Bull', 'Bull', 'Bull', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Bear', 'Bull', 'Bear', 'Bear', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Stagnant', 'Stagnant', 'Stagnant', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull', 'Bull', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bear', 'Bull', 'Bull']
```

7.1.2 Empirical Transition Probability Matrix

The Empirical Transition Probability Matrix (TPM) is a matrix that represents the probabilities of transitioning from one state to another, based on observed data from a Markov chain simulation. It is estimated from the sequence of observed states by counting the transitions between states.

- **Definition:** Let $X_1, X_2, \ldots, X_N$ be a sequence of states observed from a Markov chain. The empirical transition probability from state i to state j is given by:

  $$P_{ij}^{\text{empirical}} = \frac{C_{ij}}{\sum_k C_{ik}}$$

  where:

  - C_{ij} is the number of observed transitions from state i to state j.
  - $\sum_k C_{ik}$ is the total number of transitions from state i.

  If there are no transitions from a state i, we assign equal probabilities to all possible transitions from i.

- **Example for illustration:** Consider a Markov chain with three states $\{1, 2, 3\}$ and the following observed sequence of states:

  $$[1, 1, 2, 2, 3, 1, 2, 2, 3, 3, 3, 1, 1, 2, 3]$$

  The counts of transitions between states are:

  $$C = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix}$$

  The total number of transition for state 1 and 2 is 5 and for 3 is 4.

  The empirical TPM is calculated as:

  $$P^{\text{empirical}} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 \\ 0 & \frac{2}{5} & \frac{3}{5} \\ \frac{2}{4} & 0 & \frac{2}{4} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0 & 0.4 & 0.6 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

- **Python code implementation**

```python
import numpy as np
import pandas as pd
from collections import defaultdict

def calculate_empirical_tpm(chain, states):
    """
    Calculates the Empirical Transition Probability Matrix (ETPM) from a simulated Markov chain.

    Parameters:
    - chain: List of states in the simulated Markov chain.
    - states: List of unique state labels (e.g., ['Bull', 'Bear', 'Stagnant']).

    Returns:
    - ETPM: Pandas DataFrame representing the empirical transition probabilities.
    """
    num_states = len(states)
    state_index = {state: i for i, state in enumerate(states)}  # Map state labels to indices
    transition_counts = np.zeros((num_states, num_states))

    # Count transitions
    for i in range(len(chain) - 1):
        current_state = state_index[chain[i]]
        next_state = state_index[chain[i + 1]]
        transition_counts[current_state, next_state] += 1

    # Calculate empirical probabilities
    ETPM = np.zeros((num_states, num_states))
    for i in range(num_states):
        row_sum = np.sum(transition_counts[i])
        if row_sum > 0:
            ETPM[i] = transition_counts[i] / row_sum

    # Convert to DataFrame for better readability
    ETPM_df = pd.DataFrame(ETPM, index=states, columns=states)
    return ETPM_df

# Example usage
states = ['Bull', 'Bear', 'Stagnant']
etpm = calculate_empirical_tpm(simulated_states, states)
print("Empirical Transition Probability Matrix (ETPM):")
print(etpm)
```

- **Output**

```
Empirical Transition Probability Matrix (ETPM):
               Bull      Bear   Stagnant
Bull       0.550562  0.314607  0.134831
Bear       0.207547  0.698113  0.094340
Stagnant   0.156863  0.392157  0.450980
```

7.2 Simulation on Machine conditions

- **Python code implementation**

```
import numpy as np
import pandas as pd
from collections import defaultdict

def simulate_markov_chain(P, u, N):
    """
    Simulates a Markov chain of length N for the machine
   model.

    Parameters:
    - P: Transition probability matrix (2x2).
    - u: Initial distribution vector (length 2).
    - N: Length of the Markov chain.

    Returns:
    - chain: List of states in the Markov chain.
    """
    k = len(u)  # Number of states (2 states: working,
   not working)
    chain = []

    # Step 1: Generate the initial state
    U0 = np.random.uniform(0, 1)
    cumulative_u = np.cumsum(u)
    X0 = np.where(U0 < cumulative_u)[0][0]
    chain.append(X0)

    # Step 2: Simulate transitions
    for _ in range(N - 1):
        current_state = chain[-1]
        U_next = np.random.uniform(0, 1)
        cumulative_p = np.cumsum(P[current_state])
        next_state = np.where(U_next < cumulative_p)
   [0][0]
```

```
        chain.append(next_state)

    return chain

def calculate_empirical_tpm(chain, states):
    """
    Calculates the Empirical Transition Probability Matrix (ETPM) from a simulated Markov chain.

    Parameters:
    - chain: List of states in the simulated Markov chain.
    - states: List of unique state labels (e.g., ['Working', 'Not Working']).

    Returns:
    - ETPM: Pandas DataFrame representing the empirical transition probabilities.
    """
    num_states = len(states)
    transition_counts = np.zeros((num_states, num_states))

    # Count transitions
    for i in range(len(chain) - 1):
        current_state = chain[i]
        next_state = chain[i + 1]
        transition_counts[current_state, next_state] += 1

    # Calculate empirical probabilities
    ETPM = np.zeros((num_states, num_states))
    for i in range(num_states):
        row_sum = np.sum(transition_counts[i])
        if row_sum > 0:
            ETPM[i] = transition_counts[i] / row_sum

    # Convert to DataFrame for better readability
    ETPM_df = pd.DataFrame(ETPM, index=states, columns=states)
    return ETPM_df

# Example usage
P = np.array([[0.8, 0.2],
              [0.3, 0.7]])
u = [0.5, 0.5]  # Initial distribution (50% chance for each state)
N = 100  # Number of steps

# Simulate the Markov chain for the machine model
```

```
simulated_states = simulate_markov_chain(P, u, N)
print("Simulated Machine States:", simulated_states)

# Calculate and print the Empirical Transition
    Probability Matrix (ETPM)
states = ['Working', 'Not Working']  # State labels
etpm = calculate_empirical_tpm(simulated_states, states)
print("\nEmpirical Transition Probability Matrix (ETPM):"
    )
print(etpm)
```

- **Output**

```
Simulated Machine States: [1, 0, 0, 1, 1, 1, 1, 0, 1, 1,
    0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0,
    0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1,
    1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1]

Empirical Transition Probability Matrix (ETPM):
              Working  Not Working
Working      0.819672     0.180328
Not Working  0.289474     0.710526
```

7.3 Simulation on Weather conditions

- **Python code implementation**

```
import numpy as np
import pandas as pd
from collections import defaultdict

# Step 1: Simulating the Weather Model
def simulate_weather_model(P, u, N):
    """
    Simulates the weather model using a Markov chain.

    Parameters:
    - P: Transition probability matrix (3x3).
    - u: Initial distribution vector (length 3).
    - N: Length of the Markov chain (number of steps).

    Returns:
    - chain: List of states in the Markov chain
    representing the weather.
```

```
    """
    k = len(u)  # Number of states (3: Sunny, Rainy, Winter)
    chain = []

    # Step 1: Generate the initial state
    U0 = np.random.uniform(0, 1)
    cumulative_u = np.cumsum(u)
    X0 = np.where(U0 < cumulative_u)[0][0]  # Initial state: 0 (Sunny), 1 (Rainy), or 2 (Winter)
    chain.append(X0)

    # Step 2: Simulate transitions
    for _ in range(N - 1):
        current_state = chain[-1]
        U_next = np.random.uniform(0, 1)
        cumulative_p = np.cumsum(P[current_state])
        next_state = np.where(U_next < cumulative_p)[0][0]  # Next state: 0 (Sunny), 1 (Rainy), 2 (Winter)
        chain.append(next_state)

    return chain


# Step 2: Calculate Empirical Transition Probability Matrix (ETPM)
def infer_empirical_tpm(simulated_chain):
    """
    Infers the Empirical Transition Probability Matrix (ETPM) from observed states.

    Parameters:
    - simulated_chain: Sequence of observed states.

    Returns:
    - pd.DataFrame: Empirical TPM.
    """
    # Step 1: Identify unique states
    unique_states = sorted(list(set(simulated_chain)))

    # Step 2: Count transitions
    transition_counts = defaultdict(lambda: defaultdict(int))
    for (current_state, next_state) in zip(simulated_chain[:-1], simulated_chain[1:]):
        transition_counts[current_state][next_state] += 1

    # Convert counts to DataFrame
```

```
    transition_df = pd.DataFrame(transition_counts).fillna(0).astype(int)

    # Step 3: Calculate TPM
    TPM = pd.DataFrame(columns=unique_states, index=unique_states).fillna(0.0)
    for state in unique_states:
        total_transitions = sum(transition_counts[state].values())
        if total_transitions > 0:
            for next_state in unique_states:
                TPM.loc[state, next_state] = transition_counts[state][next_state] / total_transitions
        else:
            # If no outgoing transitions, assign equal probabilities
            TPM.loc[state] = 1.0 / len(unique_states)

    return TPM


# Example usage for the Weather Model simulation
P = np.array([
    [0.6, 0.3, 0.1],  # Sunny (0): 60% Sunny, 30% Rainy, 10% Winter
    [0.5, 0.4, 0.1],  # Rainy (1): 50% Sunny, 40% Rainy, 10% Winter
    [0.3, 0.2, 0.5],  # Winter (2): 30% Sunny, 20% Rainy, 50% Winter
])

u = [0.5, 0.3, 0.2]  # Initial distribution: 50% Sunny, 30% Rainy, 20% Winter
N = 1000  # Number of steps in the Markov chain

# Simulate the weather states
simulated_weather = simulate_weather_model(P, u, N)
print("Simulated Weather States:", simulated_weather)

# Infer the Empirical Transition Probability Matrix (ETPM)
empirical_TPM = infer_empirical_tpm(simulated_weather)
print("\nEmpirical Transition Probability Matrix (ETPM):")
print(empirical_TPM)
```

- **Output**

```
Simulated Weather States: [1, 0, 1, 0, 0, 1, 2, 2, 2, 0,
    2, 2, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 2,
    0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2,
    0, 1, 1, 1, 2, 1, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0,
    2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0,
    0, 1, 1, 2, 2, 0, 0, 0, 0, 0, 0, 1, 1, 0, 2, 2, 2, 0,
    0, 2, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0,
    2, 2, 2, 2, 0, 2, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0,
    2, 0, 1, 1, 0, 2, 2, 1, 2, 2, 2, 2, 1, 0, 0, 1, 1, 0,
    1, 1, 0, 0, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 2, 2, 2,
    1, 1, 0, 0, 0, 0, 2, 2, 2, 2, 1, 0, 1, 0, 2, 2, 2, 2,
    0, 0, 0, 2, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1,
    0, 1, 1, 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 1, 0,
    0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 2,
    0, 0, 0, 1, 1, 1, 2, 1, 0, 0, 0, 1, 0, 0, 2, 1, 2, 2,
    2, 1, 0, 0, 2, 2, 2, 1, 1, 0, 0, 0, 0, 1, 2, 1, 1, 2,
    2, 2, 1, 1, 2, 2, 2, 2, 2, 2, 1, 1, 0, 0, 0, 1, 1, 1,
    1, 0, 2, 0, 0, 2, 2, 2, 1, 2, 2, 1, 1, 1, 1, 0, 0, 0,
    1, 0, 2, 2, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 2, 2, 2, 1, 0,
    2, 2, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 2, 0, 0, 2,
    0, 0, 1, 1, 1, 0, 0, 2, 0, 1, 0, 1, 0, 1, 0, 1, 2, 2,
    0, 0, 1, 1, 2, 2, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0,
    0, 1, 0, 0, 0, 0, 2, 0, 0, 1, 0, 1, 0, 0, 0, 2, 0, 0,
    0, 0, 0, 0, 2, 0, 0, 1, 0, 1, 0, 0, 1, 0, 2, 2, 2, 1,
    0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 2, 2, 2, 0, 1, 1, 1, 2,
    0, 0, 0, 0, 0, 1, 1, 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 2,
    1, 2, 0, 0, 0, 0, 2, 1, 1, 1, 0, 1, 0, 0, 2, 2, 1, 2,
    1, 2, 0, 0, 1, 1, 0, 2, 2, 0, 1, 0, 1, 0, 0, 0, 0, 0,
    0, 0, 2, 2, 2, 2, 2, 2, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0,
    0, 1, 1, 0, 2, 1, 0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 0, 0,
    0, 1, 1, 0, 2, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1,
    0, 2, 2, 1, 0, 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0,
    0, 0, 2, 0, 0, 0, 0, 0, 0, 2, 1, 1, 1, 0, 1, 1, 2, 2,
    2, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1,
    0, 0, 0, 1, 2, 2, 2, 2, 1, 1, 0, 0, 2, 1, 0, 0, 0, 0,
    0, 1, 1, 0, 2, 0, 2, 2, 2, 2, 2, 0, 1, 0, 1, 1, 1, 1,
    1, 0, 2, 2, 2, 0, 0, 1, 0, 2, 1, 1, 1, 0, 0, 1, 0, 1,
    1, 0, 1, 0, 0, 2, 0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0,
    0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 2, 2, 2, 1, 1,
    1, 1, 1, 1, 1, 0, 2, 2, 2, 0, 1, 1, 1, 1, 0, 0, 1, 1,
    1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 2,
    1, 1, 2, 0, 0, 1, 0, 2, 2, 2, 0, 0, 0, 1, 0, 0, 0, 1,
    2, 0, 0, 0, 1, 1, 2, 1, 1, 0, 2, 1, 0, 2, 2, 0, 1, 2,
    2, 2, 0, 0, 1, 1, 1, 0, 0, 1, 2, 1, 1, 0, 2, 0, 0, 0,
    0, 0, 1, 1, 1, 1, 1, 0, 2, 0, 0, 1, 1, 0, 1, 1, 0, 0,
    0, 0, 2, 1, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0,
    0, 1, 1, 0, 1, 0, 0, 0, 2, 2, 1, 1, 1, 2, 2, 2, 2, 1,
    1, 1, 2, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1,
```

```
    2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1,
    0, 0, 0, 2, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0,
    1, 0, 0, 0, 1, 2, 2, 0, 1, 1, 1, 1, 0, 0, 0, 2, 1, 1,
    1, 0, 0, 0, 0, 2, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0,
    1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 2, 2, 2, 2, 1,
    1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2,
    2, 0, 2, 1, 1, 2, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1]

Empirical Transition Probability Matrix (ETPM):
          0         1         2
0   0.584222  0.292111  0.123667
1   0.440120  0.449102  0.110778
2   0.244898  0.239796  0.515306
```

7.4 Convergence of Empirical Distribution

- **Definition:** The empirical distribution of a Markov chain observed over N steps is defined as:

$$\pi_N(i) = \frac{1}{N} \sum_{t=1}^{N} 1\{X_t = i\}$$

  where:

  - X_t is the state of the chain at time t.
  - $1\{X_t = i\}$ is an indicator function that equals 1 if $X_t = i$ and 0 otherwise.

- **Convergence to Stationary Distribution:** For an irreducible and aperiodic Markov chain with stationary distribution π, the empirical distribution π_N converges to π as $N \to \infty$:

$$\lim_{N\to\infty} \pi_N(i) = \pi(i), \quad \forall i.$$

  This result follows from the **Law of Large Numbers** applied to the time averages of Markov chains, under the assumption that the chain is ergodic.

- **Python Code implementation**

```python
import numpy as np
import matplotlib.pyplot as plt

```

```
def simulate_markov_chain(u, P, N):
    states = np.arange(len(u))
    current_state = np.random.choice(states, p=u)
    simulated_chain = [current_state]
    for _ in range(N - 1):
        current_state = np.random.choice(states, p=P[
    current_state])
        simulated_chain.append(current_state)
    return simulated_chain

def empirical_distribution(simulated_chain, num_states):
    counts = np.zeros(num_states)
    for state in simulated_chain:
        counts[state] += 1
    return counts / len(simulated_chain)

# Parameters
u = [0.5, 0.1, 0.4]  # Initial distribution
P = [[0.6, 0.3, 0.1],
     [0.2, 0.7, 0.1],
     [0.3, 0.3, 0.4]]  # TPM
N = 1000  # Length of the chain

# Simulate Markov chain
simulated_chain = simulate_markov_chain(u, P, N)

# Compute empirical distribution at different steps
empirical_distributions = []
steps = [10, 50, 100, 500, 1000]
for step in steps:
    empirical_distributions.append(empirical_distribution
    (simulated_chain[:step], len(u)))

# Display results
for i, dist in zip(steps, empirical_distributions):
    print(f"Empirical distribution after {i} steps: {dist
    }")

# Plotting convergence
stationary_distribution = np.linalg.matrix_power(P, 1000)
    [0]  # Approximation
plt.figure(figsize=(8, 6))
for i in range(len(u)):
    plt.plot(steps, [dist[i] for dist in
    empirical_distributions], label=f"State {i+1}")
plt.axhline(y=stationary_distribution[0], color='r',
    linestyle='--', label="Stationary Dist. (State 1)")
plt.axhline(y=stationary_distribution[1], color='g',
    linestyle='--', label="Stationary Dist. (State 2)")
```

```
plt.axhline(y=stationary_distribution[2], color='b', linestyle='--', label="Stationary Dist. (State 3)")
plt.xlabel("Number of Steps")
plt.ylabel("Empirical Probability")
plt.title("Convergence of Empirical Distribution")
plt.legend()
plt.show()
```

- **Output**

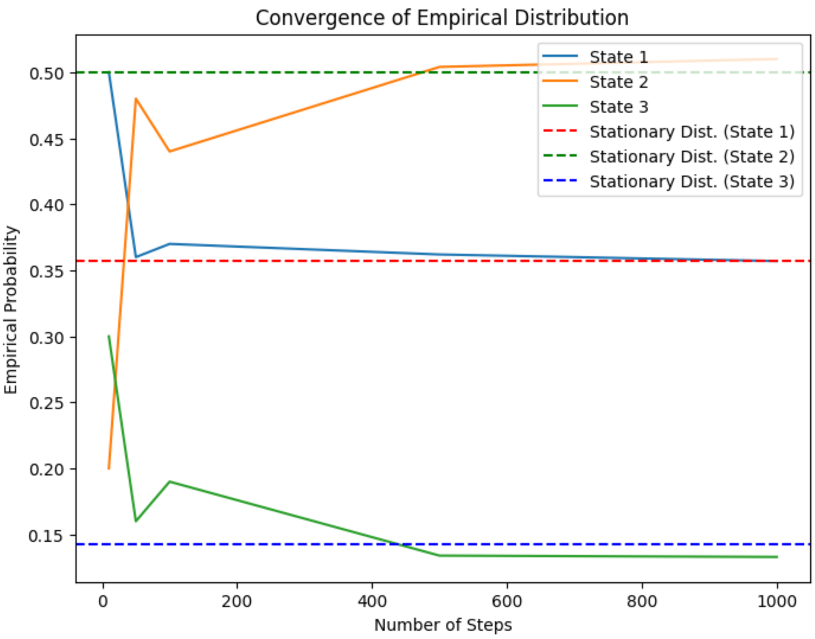

Figure 7.1

```
Empirical distribution after 10 steps: [0.5 0.2 0.3]
Empirical distribution after 50 steps: [0.36 0.48 0.16]
Empirical distribution after 100 steps: [0.37 0.44 0.19]
Empirical distribution after 500 steps: [0.362 0.504 0.134]
Empirical distribution after 1000 steps: [0.357 0.51 0.133]
```

As the number of steps N increases:

- The empirical distribution π_N for each state stabilizes and converges to the stationary distribution π.
- The rate of convergence depends on the properties of the TPM, such as the mixing time of the Markov chain.

7.5 Verifying Stationary Distribution

The stationary distribution π satisfies the equation:

$$\pi = \pi P$$

where π is the stationary distribution and P is the Transition Probability Matrix (TPM).

- **Transition Probability Matrix** P

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

- **Stationary Distribution** π**:** The stationary distribution, computed as the left eigenvector of P corresponding to the eigenvalue 1, is:

$$\pi = \begin{bmatrix} 0.35714286 & 0.5 & 0.14285714 \end{bmatrix}.$$

- **Verification:** $\pi = \pi P$ To verify, compute:

$$\pi \cdot P = \begin{bmatrix} 0.35714286 & 0.5 & 0.14285714 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}.$$

The resulting vector is:

$$\pi \cdot P = \begin{bmatrix} 0.35714286 & 0.5 & 0.14285714 \end{bmatrix}.$$

Since $\pi \cdot P = \pi$, the distribution π is indeed stationary.

- **Python code implementation**

```
import numpy as np

def check_stationary_distribution(pi, P, tolerance=1e-6):
    """
    Verifies if pi is a stationary distribution for the
    given TPM P.

    Parameters:
        pi (np.array): Candidate stationary distribution.
        P (2D np.array): Transition Probability Matrix (
    TPM).
```

```
        tolerance (float): Numerical tolerance for equality check.

    Returns:
        bool: True if pi is a stationary distribution, False otherwise.
    """
    # Compute pi * P
    pi_dot_P = np.dot(pi, P)

    # Check if pi * P is approximately equal to pi
    return np.allclose(pi_dot_P, pi, atol=tolerance), pi_dot_P

# Transition Probability Matrix
P = np.array([
    [0.6, 0.3, 0.1],
    [0.2, 0.7, 0.1],
    [0.3, 0.3, 0.4]
])

# Revised stationary distribution
pi = np.array([0.35714286, 0.5, 0.14285714])

# Check if pi satisfies pi = pi * P
is_stationary, pi_dot_P = check_stationary_distribution(pi, P)
print("Is pi a stationary distribution?", is_stationary)
print("Computed value of pi * P:", pi_dot_P)
```

- **Output**

```
Is pi a stationary distribution? True
Computed value of pi * P: [0.35714286  0.5  0.14285714]
```

References and Links

1. Notes provided by Dr. Anindya Goswam,
2. Video Lectures of Dr. Anindy Goswami,
3. Video lectures of Prof. Ayon Ganguly and Prof. Subhamay Saha, Department of Mathematics Indian Institute of Technology, Guwahati
4. Video lectures of Prof. Choongbum Lee, MIT,
5. A first course in Probability by Sheldon Ross,
6. Course notes on Stochastic processes by Department of Statistics University of Auckland,
7. Introduction to Stochastic Processes with R by Robert P. Dobrow,
8. Stochastic Processes by Sheldon Ross,
9. $https://youtu.be/G7FIQ9fXl6U?si = ieNz4YFSHp - 3UF7x$
10. $https://youtu.be/a1Gm3F4yykg?si = JvqimHMEafWS0WAc$