Sinusoidal Signals

Laboratory Report

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1 Lab

1.1 Sine and Cosine Functions

$$A\cos(\omega_0 t + \phi)$$

Equation 1-1

Sinusoidal signals are defined in terms of the familiar sine and cosine functions of trigonometry. The trigonometric functions sine and cosine take an angle as an argument. Angles are often thought of in terms of degrees, but where sine and cosine functions are concerned, angles must be dimensionless. Angles are therefore specified in terms of radians. If the angle θ is in the first quadrant, where $\left(0 \le \theta < \frac{\pi}{2}\text{ radians}\right)$.

Then the sin of θ is the length y of the side of the triangle opposite the angle θ divided by the length r of the hypotenuse of the right triangle. Similarly, the cosine of θ is the ratio of the length of the adjacent side x to the length of the hypotenuse.

Table 1-1 Basic properties of the sine and cosine functions

Property	Equation
Equivalence	$\sin\theta = \cos(\theta - \pi/2)$ or $\cos\theta = \sin(\theta + \pi/2)$
Periodicity	$\cos(\theta + \pi 2k) = \cos\theta$, when k is an integer.
Evenness of cosine	$\cos(-\theta) = \cos\theta$
Oddness of cosine	$\sin(-\theta) = -\sin\theta$
Zeros of sine	$\sin(\pi k) = 0$, when k is an integer.
Ones of cosine	$\cos(2\pi k) = 1$, when k is an integer.
Minus ones of cosine	$\cos\left[2\pi\left(k + \frac{1}{2}\right)\right] = -1$ when k is an integer.

(McClellan, Schafer, & Yoder, 2003)

Note that as θ increases from 0 to $\frac{\pi}{2}$, $\cos\theta$ decreases from 1 to 0 and $\sin\theta$ increases from 0 to 1. When the angle is greater than $\frac{\pi}{2}$, the algebraic signs of x and y being negative in the third and fourth quadrants. This is most easily shown by plotting the values of $\sin\theta$ and $\cos\theta$ as a function of θ as in Figure 2-5. Several features of these plots are worthy of comment. The two functions have exactly the same shape. In fact, the sine function is just a cosine function that is shifted to the right by $\frac{\pi}{2}$; that is, $\sin\theta = \cos(\theta - \frac{\pi}{2})$. Both functions oscillate between $+1$ and -1, and they repeat the same pattern periodically with period 2π. Furthermore the sine function is an odd function of its argument, and the cosine is an even function. A summary of these and other properties is presented in Table 1-2 .

Table 1-2 Basic trigonometric identities

Number	Equation
1	$\sin^2\theta + \cos^2\theta = 1$
2	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
3	$\sin 2\theta = 2\sin\theta\cos\theta$
4	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
5	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Clearly, the sine and cosine functions are very closely related. This often leads to opportunities for simplification of expressions involving both sine and cosine functions. In calculus, there is an interesting property that the sine and cosine functions are derivatives of each other:

$$\frac{d\sin\theta}{d\theta} = \frac{d\cos\theta}{d\theta} = -\sin\theta$$

That is, the cosine function gives the slope of the sine function, and the sine function gives the slope of the cosine function. In trigonometry, there are many identities that can be used in simplifying expressions involving combinations of sinusoidal signals. Table 1-2 lists *trigonometric identities* that will be useful. These identities are not independent; that is, identity 3 can be obtained from identity 4 by substituting $\alpha = \beta = 0$. Additionally, these identities can be combined to derive other identities. For example, combining identity 1 with identity 2 leads to the identities

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

1.2 Sinusoidal Signals

The most general mathematical formula for a sinusoidal signal is obtained by making the argument (that is, the angle) of the cosine be the function of t (time). The following equation gives two equivalent forms:

$$x(t) = A\cos(\omega_0 t + \phi) = A\cos(2\pi f_0 t + \phi)$$

Equation 1-2

The two forms are related by defining $\omega_0 = 2\pi f_0$. In either form given in Equation 1-2, there are three independent parameters, which are defined as follows:

Table 1-3 Parameters of a sinusoidal signal

Variable	Definition
A	**Amplitude**, which is a scaling factor that determines how large the cosine signal, will be in terms of magnitude. Since the function $\cos\theta$ oscillates between $+1$ and -1, the signal $x(t)$ in Equation 1-2 oscillates between $+A$ and $-A$.
ϕ	**Phase Shift,** represented in radians since the argument of cosine must be in radians. It is generally preferred to use the cosine function when defining the phase shift. If given a formula containing a sine function such as $x(t) = A\sin(\omega_0 t + \phi)$, it can be rewritten using the equivalence property in Table 1-2. The result is $x(t) = A\sin(\omega_0 t + \phi) = A\cos\left(\omega_0 t + \phi - \frac{\pi}{2}\right)$. The phase shift is therefore defined as $\phi - \frac{\pi}{2}$ in Equation 1-2. For simplicity and to prevent confusion, use of the sine function is avoided
ω_0	**Radian frequency**. Since the argument of the cosine function must be in radians, which is dimensionless, the ω_0 must have units of rad/sec if t has units of seconds. Likewise, $f_0 = \frac{\omega_0}{2\pi}$, which is the *cyclic frequency*, and must have units of $\sec^{-1}$.

Figure … gives a plot of the signal

$$x(t) = 20\cos(2\pi(40)t + 0.4\pi)$$

Equation 1-3

In terms of the variables previously defined, the signal parameters are $A = 20$, $\omega_0 = 2\pi(40)$, $f_0 = 40$, and $\phi = -0.4\pi$. The dependence of the signal on the amplitude parameter A is obvious; its maximum and minimum values are $+20$ and -20, respectively. The maxima occur at

$$t = \cdots, -0.02, 0.005, 0.03, \cdots$$

and the minima at

$$t = \cdots, -0.02, 0.005, 0.03, \cdots$$

The time interval between successive maxima of the signal is $1/f_0$ or / sec.

1.3 Relation of Frequency to Period

The sinusoid in Fig 2-6 is clearly a periodic signal. The *period* of the sinusoid, denoted as T_0 is the length of one cycle of the sinusoid. In general, the frequency of the sinusoid determines its period, and the relationship can be found by examining the following equation:

$$x(t) = A\cos(\omega_0(t + T_0) + \phi) = A\cos(\omega_0 t + \phi)$$

$$\cos(\omega_0 t + \omega_0 T_0 + \phi) = \cos(\omega_0 t + \phi)$$

Since the cosine function has a period of 2π, the equality above holds for all values of t if

$$\omega_0 T_0 = 2\pi \qquad \Longrightarrow \qquad T_0 = \frac{2\pi}{\omega_0}$$

$$(2\pi f_0)T_0 = 2\pi \quad \Longrightarrow \quad T_0 = \frac{1}{f_0}$$

Equation 1-4

Since T_0 is the period of the signal, $f_0 = 1/T_0$ is the number of periods (cycles) per second. Therefore, cycles per second is an appropriate unit for f_0. When dealing with ω_0 the unit of radian frequency is rad/sec. The units of f_0 are often more convenient when describing the sinusoid, because cycles per second (or Hertz) define the period.

It is very important to understand the effect of the frequency parameter, f_0. Figure … shows this effect for several choices of the f_0 in the signal

$$x(t) = 5\cos 2\pi f_0 t$$

The four plots in Figure 2-7 show the effect of changing f_0. As expected, the waveform shape is similar for each value of frequency. However, for the higher frequency, the signal varies more rapidly with time; that is the cycle length is a shorter time interval. This is true because the period of a cosine signal is the reciprocal of the frequency. When the frequency doubles ($100 \rightarrow 200$), the period is halved. This is an illustration of the general principal that the higher the frequency, the more rapid the signal waveform changes with time.

It can be observed that $f_0 = 0$ is a reasonable values, and when this value is used, the resulting signal is constant, since $5\cos(2\pi \cdot 0 \cdot t) = 5$ for all values of t. Therefore, the constant signal, called DC, is a sinusoid of zero frequency.

1.4 Phase Shift and Time Shift

The phase sift parameter ϕ (together with the frequency) determines the time locations of the maxima and minima of a cosine wave. Specifically, notice that the sinusoid defined by Equation 1-2 with $\phi = 0$ has a positive peak at $t = 0$.

It is useful to become familiar with the general concept of *time-shifting* a signal. Suppose that a signal $\sin(t)$is defined by a known formula. A simple example is the following triangularly shaped function:

$$s(t) = \begin{cases} 2t, & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{3}(4-2t), & \frac{1}{2} \leq t \leq 2 \\ 2t, & \text{elsewhere} \end{cases}$$

Equation 1-5

This simple function has a slop of 2 for $0 \leq t \leq \frac{1}{2}$ and a negative slope of $-\frac{2}{3}$ for $\frac{1}{2} \leq t \leq 2$. Now consider the function $x_1(t) = s(t-2)$. From the definition of $s(t)$,

It follows that $x_1(t)$ is nonzero for

$$0 \leq (t-2) \leq 2 \quad \Longrightarrow \quad 2 \leq t \leq 4$$

Within the time interval $[2, 4]$ the formula for the *time shifted* signal is now

$$s(t) = \begin{cases} 2(t-2), & 2 \leq t \leq 2\frac{1}{2} \\ \frac{1}{3}(8-2t), & 2\frac{1}{2} \leq t \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

Equation 1-6

In other words, $x_1(t)$ is simply the $s(t)$ function with its origin shifted (or delayed) to the *right* by 2 seconds. Similarly, $x_2(t+1)$ is the function shifted to the left (or advanced) by 1 second; its nonzero portion is located in the interval $-1 \leq t \leq 1$. The three signals $x(t) = s(t)$, $x_1(t) = s(t-2)$, $x_2(t) = s(t+1)$ are all shown in Figure 2-8.

There are occasions to consider time-shifted signals. Whenever a signal can be expressed in the form $x_1(t) = s(t - t_1)$, it can be said that $x_1(t)$ is a time shifted version of $s(t)$. If t_1 is a positive number, then the shift is to the right, and it can be said that the signal $s(t)$ has been *delayed* in time. Conversely, when t_1 is a negative number, it can be said that the signal $s(t)$ has been *advanced* in time. Time shift is essentially a redefinition of the time origin of the signal. In general, any function of the form $s(t - t_1)$ has its origin moved to the location of $t = t_1$.

One way to determine the time shift for a cosine signal would be to find the positive peak of the sinusoid that is closest to $t = 0$. In the plot of figure…., the time where is peak is closest to $t_1 = 0.005$ sec. Since the peak in this case occurs at a positive time (to the right of $t = 0$), it can be said that the time shift is a delay of the zero-phase cosine signal. Let $x_0(t) = A\cos(\omega_0 t)$ denote a cosine signal with a zero phase shift. A delay of $x_0(t)$ can be converted to a phase shift ϕ by making the following comparison:

$$x_0(t - t_1) = A\cos\big(\omega_0(t - t_1)\big) = A\cos(\omega_0 t + \phi)$$

$$\cos(\omega_0 t - \omega_0 t_1) = \cos(\omega_0 t + \phi)$$

Since this equation must hold for all t, it is necessary that

$$-\omega_0 t_1 = \phi$$

which leads to

$$t_1 = -\frac{\phi}{\omega_0} = -\frac{\phi}{2\pi f_0}$$

Notice that the phase shift is negative when the time shift is positive (a delay). In terms of the period ($T_0 = 1/f_0$), a more intuitive formula evolves

$$\phi = -2\pi f_0 t_1 = -2\pi\left(\frac{t_1}{T_0}\right)$$

Equation 1-7

which states that the phase shift 2π times the fraction of a cycle given by the ratio of the time shift to the period. Magnificent!

Since the positive peak nearest to $t = 0$ must always lie within $|t_1| \le T_0/2$, the phase shift can always be chosen to satisfy $-\pi < \phi \le \pi$, however the phase shift is also ambiguous because adding a multiple of 2π to the argument a cosine function does not change the value of the cosine. This is a direct consequence of the fact that the cosine is periodic with a period of 2π. Each different multiple of 2π corresponds to picking a different peak of the periodic waveform. Another way to compute the phase shift is to find any positive peak of the sinusoid and measure its corresponding time location. After the time location is converted to phase shift using Equation 1-7, an integer multiple of 2π can be added to or subtracted from the phase shift to produce the final result between $-\pi$ and $+\pi$.

This gives a final result identical to locating the peak that is within half a period of $t = 0$. The operation of adding or subtracting multiples of 2π is called *reducing modulo* 2π because it is similar to modulo reduction in mathematics, which amounts to dividing by 2π and taking the remainder. The value of phase shift that falls between $-\pi$ and $+\pi$ is called *principal value* of the phase shift.

1.5 Sampling and Plotting Sinusoids

All of the plots of sinusoids in this laboratory procedures were created using MATLAB. This had to be done with care, because MATLAB deals only with discrete signals represented by row or column matrices, yet the continuous function $x(t)$ must be represented realistically. For example, if the following function is to be plotted:

$$x(t) = 20\cos(2\pi(40)t - 0.4\pi)$$

as shown in figure…., it is necessary to evaluate $x(t)$ at a discrete set of times,

$t_n = nT_s$, where n is an integer. If this is performed the following sequence of samples is obtained:

$$x(nT_s) = 20\cos(80\pi nT_s - 0.4\pi)$$

where T_s is called the *sampling spacing* or *sampling period*, and n is an integer that can be used as a index to identify a specific sample. When plotting the function using the `plot` function in MATLAB, we must provide a pair of row or column vectors, one containing the time values and the other the computed function values to be plotted. For example, the MATLAB statements

```
n  = -7.5;
Ts = 0.005;
tn = n*Ts
xn = 20*c0s(80*pi*tn-.4*pi);
plot (tn,xn)
```

would create a row vector `tn` of 13 numbers between -0.035 and 0.025 spaced by the sampling period 0.005 and a row vector of `xn` of samples of $x(t)$. Then the MATLAB function `plot` draws the corresponding points, connecting them with straight line segments. Constructing the curve between sample points ion this way is called *linear interpolation*. The solid gray curve in the upper plot of Fig. 2-9 shows the result of linear interpolation when the sample spacing is $T_s = 0.005$. Intuitively, it can be realized that if the points are very close together, a smooth curve will be evident. The important question is, how small must the sample spacing be so that the cosine can be accurately reconstructed between samples by linear interpolation?". A qualitative answer to this question is provided by Figure 2-9, which shows plots produced by three different sampling periods.

Obviously the sample spacing of $T_s = 0.005$ is not sufficiently close to create an accurate plot when the sample points are connected by straight line. Sample points are shown as dots in the upper two plots. With spacing of $T_s = 0.0025$, the plot starts to approximate a cosine, but it is still possible to see places where it is clear that the points are connected by straight lines rather than the smooth cosine function. Only in the lower plot of Figure 2-9 where spacing is $T_s = 0.0001$, does the sampling spacing become so dense that it can be visually perceived as a faithful representation of the cosine function. From this example, it can be observed that as the sampling period decreases, more samples are taken across one cycle of the periodic cosine signal. When $T_s = 0.005$, there are 5 samples per cycle, when $T_s = 0.0025$ there are 10 samples per cycle, and when $T_s = 0.0001$, there are 250 samples per cycle. It is apparent that 10 samples per cycle are not quite enough, and 250 samples is probably more than necessary, but in general the more samples per cycle, the smoother and more accurate is the linearly interpolated curve.

1.6 Complex Exponentials and Phasors

It has been shown that cosine signals are useful mathematical representations for signals that arise in a practical setting, and that they are simple to define and interpret. However, it turns out that the analysis and manipulation of sinusoidal signals is often greatly simplified by dealing with related signals called *complex exponential signals*.

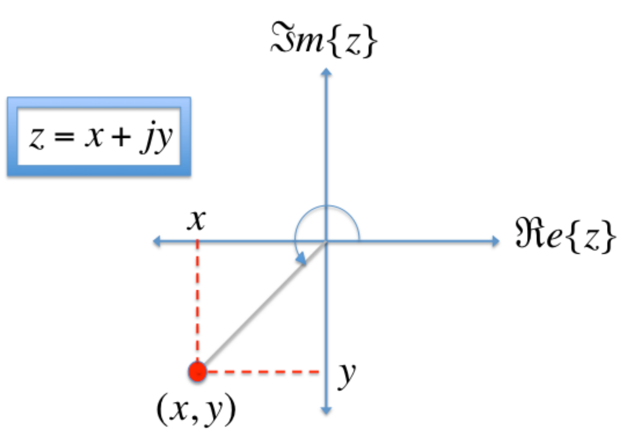

Figure 1-1 Cartesian representation of complex numbers in the complex plane

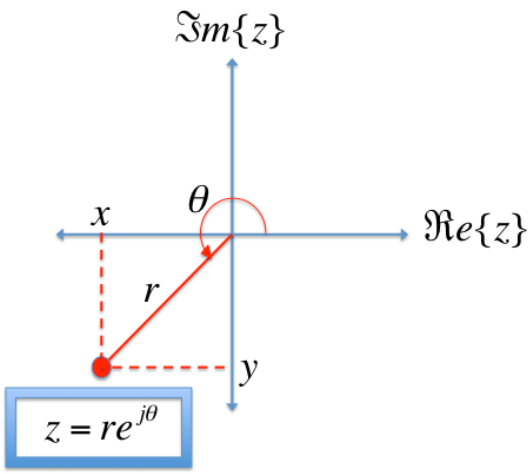

Figure 1-2 Polar representation of complex numbers in the complex plane

1.6.1 Complex numbers

A complex number z is an ordered pair of real numbers. Complex numbers may be represented by the notation $z = (x, y)$, where $x = \Re\{z\}$ is the *real part* and $x = \Im\{z\}$ is the imaginary part of z. In the electrical engineering field, the symbol j represents $\sqrt{-1}$ instead of i (used by pure mathematicians), so complex numbers can also be represented as $z = x + jy$. These two representations are called the *Cartesian form* of the complex number. Complex numbers are often represented as points in a *complex plane*, where the real and imaginary parts are the horizontal and vertical coordinates, respectively. With the Cartesian notation and the understanding that any number multiplied by j is included in the imaginary part, the operations of complex addition, complex subtraction, complex multiplication, and complex division can be defined in terms of real operations on the real and imaginary parts. For example, the sum of two complex numbers is defined as the complex number whose real part is the sum of the real parts and whose imaginary part is the sum of the imaginary parts.

Since complex numbers can be represented as points in the plain, it follows that complex numbers are analogous to vectors in a two-dimensional space. This leads to a useful geometric interpretation of a complex number as a vector. Vectors have length and direction so another way to represent a complex number is the *polar form* in which the complex number is represented by r, its vector length, together with θ, its angle with respect to the real axis. The length of the vector is also called the *magnitude* of z (denoted $|z|$), and the angle with the real axis called the *argument* of z, (denoted arg z). This us indicated by the descriptive notation $z \leftrightarrow r\angle\theta$, which is interpreted to mean that the vector representing z has length r and makes an angle θ with the real axis.

It is important to be able to convert between the Cartesian and polar forms of complex numbers. Figure 1-1 and Figure 1-2 shows a complex number z and the quantities involved in both the Cartesian and polar representations respectively. Using these figures, as well as simple trigonometry and the Pythagorean theorem, a method can be derived for computing the Cartesian coordinates (x, y) from the polar variables $r\angle\theta$:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Equation 1-8

and, likewise, for going from Cartesian to polar form

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Equation 1-9

The $r\angle\theta$ notation is clumsy, and does not lend itself to ordinary algebraic rules. A much better polar form is given by using Euler's formula for the complex exponential.

$$e^{j\theta} = \cos\theta + j\ sin\ \theta$$

Equation 1-10

The Cartesian pair $(\cos\theta, sin\ \theta)$ can represent any point on a circle of radius 1, so a slight generalization of Equation 1-10 gives a representation valid for any complex number z

$$z = re^{j\theta} = r\cos\theta + jr\ \sin\theta$$

Equation 1-11

The complex exponential polar form of a complex number is most convenient when calculating a complex multiplication or division. It also serves as the basis for the complex exponential signal.

1.7 Complex Exponential Signals

The *complex exponential signal* is defined as

$$z(t) = Ae^{j(\omega_o t + \phi)}$$

Equation 1-12

The complex exponential signal is a complex-value of t, where the magnitude of $z(t)$ is $|z(t)| = A$ and the angle of $z(t)$ is $\arg z(t) = (\omega_o t + \phi)$. Using Euler's formula

(Equation 1-10), the complex exponential signal can be expressed in Cartesian form as

$$z(t) = Ae^{j(\omega_o t+\phi)}$$

$$= A\cos(\omega_o t + \phi) + jA\cos(\omega_o t + \phi)$$

Equation 1-13

As with the real sinusoid, A is the *amplitude*, and should be a positive real number; ϕ is the *phase shift*; and ω_o is the *frequency* in rad/sec. Based on Equation 1-13, it is evident that the real part of the complex exponential signal is a real cosine signal as defined by Equation 1-2 and that its imaginary part is a real sine signal. Figure…. Shows a plot of the following complex exponential signal:

$$z(t) = 20e^{j(2\pi(40)t-0.4\pi)}$$

$$= 20e^{j(80\pi t-0.4\pi)}$$

$$= 20\cos(80\pi t - 0.4\pi) + j20\sin(80\pi t - 0.4\pi)$$

$$= 20\cos(80\pi t - 0.4\pi) + j20\sin(80\pi t - 0.9\pi)$$

Plotting a complex signal as a function of time requires two graphs, one for the real part, and another for the imaginary part. The real and imaginary parts of the complex exponential signal are both real sinusoidal signals, and they differ by only a phase shift of 0.4π rad.

The main reason that a complex exponential signal is used is that it is an alternative *representation* for the real cosine signal. This is because

$$x(t) = \Re\left\{Ae^{j(\omega_o t+\phi)}\right\} = A\cos(\omega_o t + \phi)$$

Equation 1-14

In fact, the real part of the complex exponential signal shown in Fig…. is identical to the cosine signal plotted in Fig…. Although is may seam that the problem has been complicated by first introducing the imaginary part to obtain the complex exponential signal and then throwing it away by taking only the real part, many calculations are actually simplified by using properties of the exponents. For example, it is possible to replace all trigonometric manipulations with algebraic operations of the exponents.

1.8 The Rotating Phasor Interpretation

When two complex numbers are multiplied, it is best to use the polar form for both numbers. To illustrate this, consider

$$z_3 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 \cdot r_2 \cdot e^{j\theta_1} \cdot e^{j\theta_2} = r_1 \cdot r_2 \cdot e^{j(\theta_1+\theta_2)}$$

The law of exponents has been used to combine the two complex exponentials. From this result it can be concluded that to multiply two complex numbers, the magnitudes are multiplied and the angles are added. If one of the complex numbers is represented by a fixed vector in the complex number, then multiplication by a second complex number scales the length of the first complex number and rotates it by the angle of the complex number as shown in Figure…., where it is assumed at

$r_1 > r_2$ so that $r_1 \cdot r_2 > r_2$.

This geometric view of complex multiplication leads to a useful interpretation of the complex exponential signal as a complex vector that rotates as time increases. If a complex number is defined as

$$X = Ae^{j\phi}$$

Equation 1-15

then Equation 1-12 can be expressed as

$$z(t) = Xe^{j\omega_0 t}$$

Equation 1-16

That is, $z(t)$ is a product of the complex number X and the complex-valued time function $e^{j\omega_0 t}$. The complex number X, which is aptly called the *complex amplitude*, is a polar representation created from the amplitude and the phase shift of the complex exponential signal. Taken together, the complex amplitude $X = Ae^{j\phi}$ and the frequency ω_0 are sufficient to represent $z(t)$, as well as the real cosine signal, $z(t) = A\cos(\omega_0 t + \phi)$, using Equation 1-14. The complex amplitude is also called a *phasor*. Use of this terminology is common in electrical circuit theory, where complex exponential signals are used to greatly simplify the analysis of design circuits. Since it is a complex number, X can be represented graphically as a vector in the complex plane, where the vector's magnitude ($|X| = A$) is the amplitude and the vector's angle ($\angle A = \phi$) is the phase shift of a complex exponential defined by (2.16). The terms *phasor* and *complex amplitude* can be used interchangeably, because they refer to the same quantity defined by Equation 1-16.

The complex exponential signal defined by Equation 1-15 can also be written as

$$z(t) = Xe^{j\omega_0 t} = Ae^{j\phi}e^{j\phi\omega_0 t} = Ae^{j\theta(t)}$$

where

$$\theta(t) = \omega_0 t + \phi \text{ (Radians)}$$

At a given instant in time, t, the value of the complex signal, $z(t)$ is a complex number whose argument is $\theta(t)$. Like any complex number, $z(t)$ can be represented as a vector in a complex plane. In this case, the tip of the vector always lies on the perimeter of a circle of radius A. If t increases, the complex vector $z(t)$ will simply rotate at a constant rate, determined by the radian frequency ω_0. Multiplying the phasor X by $e^{j\omega_0 t}$ as in Equation 1-16 causes the fixed phasor X to rotate. Since $\left|e^{j\omega_0 t}\right| = 1$, no scaling occurs. Therefore, another name for the complex exponential signal is *rotating phasor*.

If the frequency ω_0 is positive, the direction is counterclockwise, because $\theta(t)$ will increase with increasing time. Similarly, when ω_0 is negative, the angle $\theta(t)$ changes in the negative direction as time increases, so the complex phasor rotates clockwise. Therefore, rotating phasors are said to have *positive frequency* if they rotate counterclockwise, ad *negative frequency* if they rotate clockwise.

A rotating phasor makes one complete revolution every time the angle $\theta(t)$ changes by 2π radians. The time it takes to make one revolution is also equal to the period T_0, of the complex exponential signal so,

$$\omega_0 T_0 = (2\pi f_0)T_0 = 2\pi \quad \Longrightarrow \quad T_0 = \frac{1}{f_0}$$

The phase shift ϕ defines where the phasor is pointing when $t = 0$. For example, if $\phi = 2\pi$, then the phasor is pointing straight up when $t = 0$, where as if $\phi = 0$, the phasor is pointing to the right when $t = 0$.

The plots in Fig… illustrates the relationship between a single complex rotating phasor and the cosine signal waveform. The upper left plot shows the complex plane with two vectors. The vector at an angle in the third quadrant represents the signal

$$z(t) = e^{j(t-\pi/4)}$$

at the specific time $t = 1.5\pi$. The horizontal vector pointing to the left represents the real part of the vector $z(t)$ at the particular time $t = 1.5\pi$. That is

$$x(1.5\pi) = \Re e\{z(1.5\pi)\} = \cos(1.5\pi - \pi/4) = -\frac{\sqrt{2}}{2}$$

As t increases, the rotating phasor $z(t)$ rotates in the counterclockwise direction, and its real part $x(t)$ oscillates left and right along the real axis. This is shown is the lower left plot, which shows how the real part of the phasor has varied over one period, that is, $0 \leq t \leq 2\pi$.

1.9 Inverse Euler Formulas

The inverse Euler formulas allow us to write the cosine function in terms of complex exponentials as

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Equation 1-17

and also the sine function can be expressed as

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

Equation 1-18

Equation 1-17 can be used to express $\cos(\omega_0 t + \phi)$ in terms of a positive and a negative frequency complex exponential as follows:

$$A\cos(\omega_0 t + \phi) = A\left(\frac{e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}}{2}\right)$$

$$= \frac{1}{2}Xe^{j\omega_0 t} + \frac{1}{2}X^* e^{-j\omega_0 t}$$

$$= \frac{1}{2}z(t) + \frac{1}{2}z^*(t) = \Re e\{z(t)\}$$

where * denotes the complex conjugate.

This formula has an interesting interpretation. The real cosine signal with frequency ω_0 is actually composed of two complex signals: one with positive frequency (ω_0) and the other with negative frequency ($-\omega_0$). The complex amplitude of the positive frequency complex exponential signal is $\frac{1}{2}X = \frac{1}{2}Ae^{j\phi}$, and the complex amplitude of the negative frequency complex exponential is $\frac{1}{2}X^* = \frac{1}{2}Ae^{-j\phi}$. In other words, the real cosine signal can be represented as the sum of two complex rotating phasors that are complex conjugates of each other.

Figure 2-13(b) illustrates how the sum of the two half-amplitude complex conjugates rotating phasors becomes the real cosine signal. In this case, the vector at an angle in the third quadrant is the complex phasor $\frac{1}{2}z(t)$ at time $t = 1.5\pi$. As t increases after that time, the angle would increase in the counterclockwise direction. Similarly, the vector in the second quadrant (plotted with a light orange line) is the complex rotating phasor $\frac{1}{2}z^*(t)$ at time $t = 1.5\pi$. As t increases after that time, the angle would increase in the clockwise direction. The horizontal vector pointing to the right is the sum of these two complex conjugate rotating phasors. The result is the same as the real vector in the plot on the left, and therefore the real cosine wave traced out, as a function of time is the same in both cases. The lower right shows the variation of the real values of $\cos(t - \pi/4)$ for $0 \leq t \leq 2\pi$.

This representation of real sinusoidal signals in terms of their positive and negative frequencies is a remarkably useful concept. The negative frequencies, which arise due to the complex exponential representation, turn out to lead many simplifications in the analysis of signal and systems problems.

1.10 MATLAB Demonstration of Phasors

2 Works Cited

McClellan, J. H., Schafer, R. W., & Yoder, M. A. (2003). *Signal Processing First.* Upper Saddle River, NJ, USA: Pearson Education, Inc.