Chapter 1

Set Theory

1.1 INTRODUCTION

The concept of a *set* appears in all mathematics. This chapter introduces the notation and terminology of set theory which is basic and used throughout the text.

Though logic is formally treated in Chapter 4, we introduce Venn diagram representation of sets here, and we show how it can be applied to logical arguments. The relation between set theory and logic will be further explored when we discuss Boolean algebra in Chapter 15.

This chapter closes with the formal definition of mathematical induction, with examples.

1.2 SETS AND ELEMENTS

A *set* may be viewed as a collection of objects, the *elements* or *members* of the set. We ordinarily use capital letters, $A, B, X, Y, \ldots$, to denote sets, and lowercase letters, $a, b, x, y, \ldots$, to denote elements of sets. The statement "p is an element of A", or, equivalently, "p belongs to A", is written

$$p \in A$$

The statement that p is not an element of A, that is, the negation of $p \in A$, is written

$$p \notin A$$

The fact that a set is completely determined when its members are specified is formally stated as the principle of extension.

Principle of Extension: Two sets A and B are equal if and only if they have the same members.

As usual, we write $A = B$ if the sets A and B are equal, and we write $A \neq B$ if the sets are not equal.

Specifying Sets

There are essentially two ways to specify a particular set. One way, if possible, is to list its members. For example,

$$A = \{a, e, i, o, u\}$$

denotes the set A whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces { }. The second way is to state those properties which characterized the elements in the set. For example,

$$B = \{x: x \text{ is an even integer, } x > 0\}$$

which reads "B is the set of x such that x is an even integer and x is greater than 0", denotes the set B whose elements are the positive integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and".

EXAMPLE 1.1

(*a*) The set A above can also be written as

$$A = \{x: x \text{ is a letter in the English alphabet, } x \text{ is a vowel}\}$$

Observe that $b \notin A$, $e \in A$, and $p \notin A$.

(*b*) We could not list all the elements of the above set B although frequently we specify the set by writing

$$B = \{2, 4, 6, \ldots\}$$

where we assume that everyone knows what we mean. Observe that $8 \in B$ but $-7 \notin B$.

(c) Let $E = \{x: x^2 - 3x + 2 = 0\}$. In other words, E consists of those numbers which are solutions of the equation $x^2 - 3x + 2 = 0$, sometimes called the *solution set* of the given equation. Since the solutions of the equation are 1 and 2, we could also write $E = \{1, 2\}$.

(d) Let $E = \{x: x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$ and $G = \{1, 2, 2, 1, \frac{6}{3}\}$. Then $E = F = G$. Observe that a set does not depend on the way in which its elements are displayed. A set remains the same if its elements are repeated or rearranged.

Some sets will occur very often in the text and so we use special symbols for them. Unless otherwise specified, we will let

$$\begin{aligned}
\mathbf{N} &= \text{the set of positive integers: } 1, 2, 3, \ldots \\
\mathbf{Z} &= \text{the set of integers: } \ldots, -2, -1, 0, 1, 2, \ldots \\
\mathbf{Q} &= \text{the set of rational numbers} \\
\mathbf{R} &= \text{the set of real numbers} \\
\mathbf{C} &= \text{the set of complex numbers}
\end{aligned}$$

Even if we can list the elements of a set, it may not be practical to do so. For example, we would not list the members of the ~~set of~~ people born in the world during the year 1976 although theoretically it is possible to compile such a list. That is, we describe a set by listing its elements only if the set contains a few elements; otherwise we describe a set by the property which characterizes its elements.

The fact that we can describe a set in terms of a property is formally stated as the *principle of abstraction*.

Principle of Abstraction: Given any set U and any property P, there is a set A such that the elements of A are exactly those members of U which have the property P.

1.3 UNIVERSAL SET AND EMPTY SET

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the *universal set*. For example, in plane geometry, the universal set consists of all the points in the plane, and in human population studies the universal set consists of all the people in the world. We will let the symbol

$$U$$

denote the universal set unless otherwise stated or implied.

For a given set U and a property P, there may not be any elements of U which have property P. For example, the set

$$S = \{x: x \text{ is a positive integer, } x^2 = 3\}$$

has no elements since no positive integer has the required property.

The set with no elements is called the *empty set* or *null set* and is denoted by

$$\varnothing$$

There is only one empty set. That is, if S and T are both empty, then $S = T$ since they have exactly the same elements, namely, none.

1.4 SUBSETS

If every element in a set A is also an element of a set B, then A is called a *subset* of B. We also say that A is *contained* in B or that B *contains* A. This relationship is written

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

{0} – the set {0} contain one element, the number zero
The set ∅ contain no elements; it is empty set
the set {∅} – also contain one element, the null set

If A is not a subset of B, i.e., if at least one element of A does not belong to B, we write $A \not\subseteq B$ or $B \not\supseteq A$.

EXAMPLE 1.2

(a) Consider the sets

$$A = \{1,3,4,5,8,9\} \qquad B = \{1,2,3,5,7\} \qquad C = \{1,5\}$$

Then $C \subseteq A$ and $C \subseteq B$ since 1 and 5, the elements of C, are also members of A and B. But $B \not\subseteq A$ since some of its elements, e.g., 2 and 7, do not belong to A. Furthermore, since the elements of A, B, and C must also belong to the universal set U, we have that U must at least contain the set $\{1,2,3,4,5,6,7,8,9\}$.

(b) Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ be defined as in Section 1.2. Then

$$\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$$

(c) The set $E = \{2,4,6\}$ is a subset of the set $F = \{6,2,4\}$, since each number 2, 4, and 6 belonging to E also belongs to F. In fact, $E = F$. In a similar manner it can be shown that every set is a subset of itself.

The following properties of sets should be noted:

(i) Every set A is a subset of the universal set U since, by definition, all the elements of A belong to U. Also the empty set $\varnothing$ is a subset of A.

(ii) Every set A is a subset of itself since, trivially, the elements of A belong to A.

(iii) If every element of A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(iv) If $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements, i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

We state these results formally.

Theorem 1.1: (i) For any set A, we have $\varnothing \subseteq A \subseteq U$.
(ii) For any set A, we have $A \subseteq A$.
(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

If $A \subseteq B$, then it is still possible that $A = B$. When $A \subseteq B$ but $A \neq B$, we say A is a *proper subset* of B. We will write $A \subset B$ when A is a proper subset of B. For example, suppose

$$A = \{1,3\} \qquad B = \{1,2,3\}, \qquad C = \{1,3,2\}$$

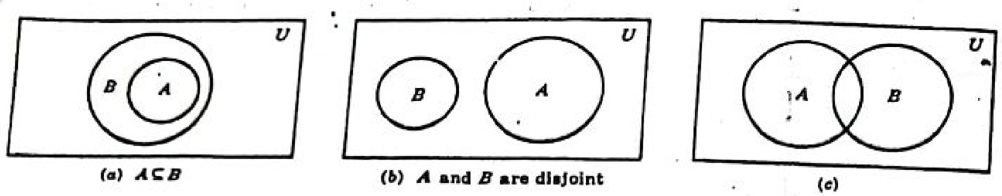
Then A and B are both subsets of C; but A is a proper subset of C, whereas B is not a proper subset of C since $B = C$.

1.5 VENN DIAGRAMS

A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane.

The universal set U is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B as in Fig. 1-1(a). If A and B are disjoint, i.e., if they have no elements in common, then the disk representing A will be separated from the disk representing B as in Fig. 1-1(b).

However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Fig. 1-1(c).



(a) $A \subseteq B$ (b) A and B are disjoint (c)

Fig. 1-1

Arguments and Venn Diagrams

Many verbal statements are essentially statements about sets and can therefore be described by Venn diagrams.

Hence Venn diagrams can sometimes be used to determine whether or not an argument is valid. Consider the following example.

EXAMPLE 1.3 Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid:

S_1: My saucepans are the only things I have that are made of tin.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin.

(The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.)

By S_1 the tin objects are contained in the set of saucepans and by S_3 the set of saucepans and the set of useful things are disjoint: hence draw the Venn diagram of Fig. 1-2.

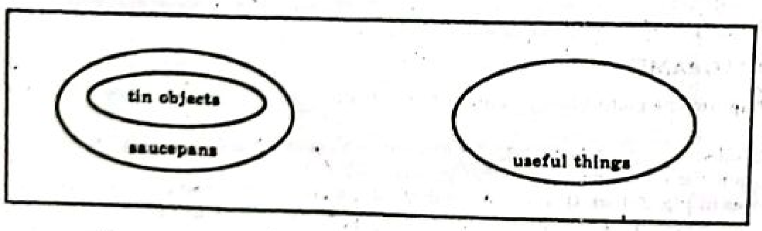


Fig. 1-2

By S_2 the set of "your presents" is a subset of the set of useful things; hence draw Fig. 1-3.

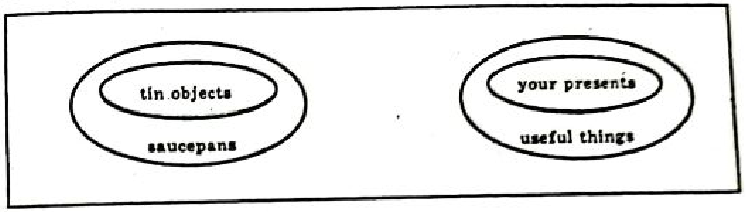


Fig. 1-3

The conclusion is clearly valid by the above Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

1.6 SET OPERATIONS

This section introduces a number of important operations on sets.

Union and Intersection

The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is,

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$

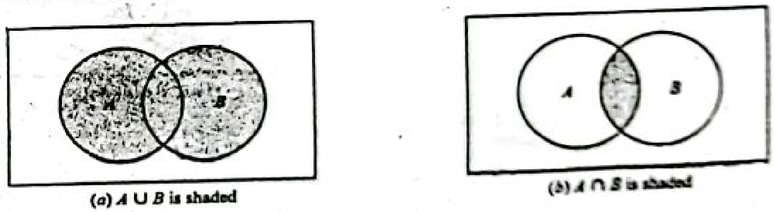
Here "or" is used in the sense of and/or. Figure 1-4(*a*) is a Venn diagram in which $A \cup B$ is shaded.

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,

$$A \cap B = \{x: x \in A \text{ and } x \in B\}$$

Figure 1-4(*b*) is a Venn diagram in which $A \cap B$ is shaded.

If $A \cap B = \varnothing$, that is, if A and B do not have any elements in common, then A and B are said to be *disjoint* or *nonintersecting*.



(a) $A \cup B$ is shaded (b) $A \cap B$ is shaded

Fig. 1-4

EXAMPLE 1.4

(*a*) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 5, 7\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\} \qquad A \cap B = \{3, 4\}$$
$$A \cup C = \{1, 2, 3, 4, 5, 7\} \qquad A \cap C = \{2, 3\}$$

(*b*) Let M denote the set of male students in a university C, and let F denote the set of female students in C. Then

$$M \cup F = C$$

since each student in C belongs to either M or F. On the other hand,

$$M \cap F = \varnothing$$

since no student belongs to both M and F.

The operation of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem.

Theorem 1.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

Note: This theorem is proved in Problem 1.27. Other conditions equivalent to $A \subseteq B$ are given in Problem 1.37.

Complements

Recall that all sets under consideration at a particular time are subsets of a fixed universal set U. The *absolute complement* or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to U but which do not belong to A; that is,

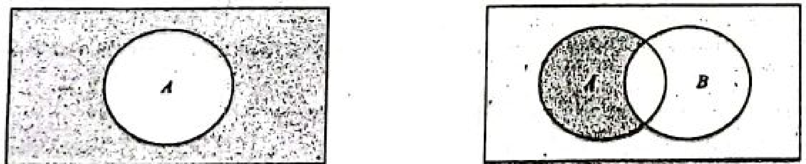
$$A^c = \{x: x \in U, x \notin A\}$$

Some texts denote the complement of A by A' or $\bar{A}$. Figure 1-5(*a*) is a Venn diagram in which A^c is shaded.

The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B; that is

$$A \backslash B = \{x: x \in A, x \notin B\}$$

The set $A \backslash B$ is read "A minus B". Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$. Figure 1-5(*b*) is a Venn diagram in which $A \backslash B$ is shaded.



(*a*) A^c is shaded (*b*) $A \backslash B$ is shaded

Fig. 1-5

EXAMPLE 1.5 Suppose $U = \mathbf{N} = \{1, 2, 3, \ldots\}$, the positive integers, is the universal set. Let

$$A = \{1, 2, 3, 4,\}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{6, 7, 8, 9\}$$

and let $E = \{2, 4, 6, 8, \ldots\}$, the even integers. Then

$$A^c = \{5, 6, 7, 8, \ldots\}, \quad B^c = \{1, 2, 8, 9, 10, \ldots\}, \quad C^c = \{1, 2, 3, 4, 5, 10, 11, \ldots\}$$

and

$$A \backslash B = \{1, 2\}, \quad B \backslash C = \{3, 4, 5\}, \quad B \backslash A = \{5, 6, 7\}, \quad C \backslash E = \{7, 9\}$$

Also, $E^c = \{1, 3, 5, \ldots\}$, the odd integers.

Fundamental Products

Consider n distinct sets $A_1, A_2, \ldots, A_n$. A *fundamental product* of the sets is a set of the form

$$A_1^* \cap A_2^* \cap \cdots \cap A_n^*$$

where A_i^* is either A_i or A_i^c. We note that (1) there are 2^n such fundamental products, (2) any two such fundamental products are disjoint, and (3) the universal set U is the union of all the fundamental products (Problem 1.64). There is a geometrical description of these sets which is illustrated below.

EXAMPLE 1.6 Consider three sets A, B, and C. The following lists the eight fundamental products of the three sets:

$$P_1 = A \cap B \cap C, \quad P_3 = A \cap B^c \cap C, \quad P_5 = A^c \cap B \cap C, \quad P_7 = A^c \cap B^c \cap C$$
$$P_2 = A \cap B \cap C^c, \quad P_4 = A \cap B^c \cap C^c, \quad P_6 = A^c \cap B \cap C^c, \quad P_8 = A^c \cap B^c \cap C^c$$

These eight products correspond precisely to the eight disjoint regions in the Venn diagram of sets A, B, C in Fig. 1-6 as indicated by the labeling of the regions.

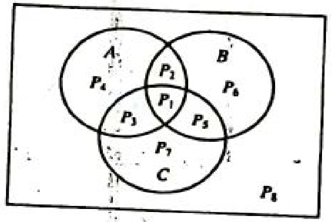
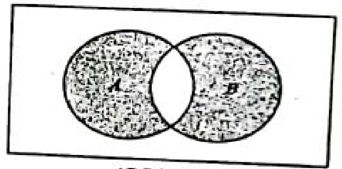


Fig. 1-6



$A \oplus B$ is shaded

Fig. 1-7

Symmetric Difference

The *symmetric difference* of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both; that is,

$$A \oplus B = (A \cup B) \backslash (A \cap B)$$

One can also show (Problem 1.18) that

$$A \oplus B = (A \backslash B) \cup (B \backslash A)$$

For example, suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8, 9\}$. Then

$$A \backslash B = \{1, 2, 3\}, \quad B \backslash A = \{7, 8, 9\} \quad \text{and so} \quad A \oplus B = \{1, 2, 3, 7, 8, 9\}$$

Figure 1-7 is a Venn diagram in which $A \oplus B$ is shaded.

1.7 ALGEBRA OF SETS AND DUALITY

Sets under the operations of union, intersection, and complement satisfy various laws or identities which are listed in Table 1-1. In fact, we formally state this:

Theorem 1.3: Sets satisfy the laws in Table 1-1.

There are two methods of proving equations involving set operations. One way is to use what it means for an object x to be an element of each side, and the other way is to use Venn diagrams. For example, consider the first of DeMorgan's laws.

$$(A \cup B)^c = A^c \cap B^c$$

Table 1-1 Laws of the algebra of sets

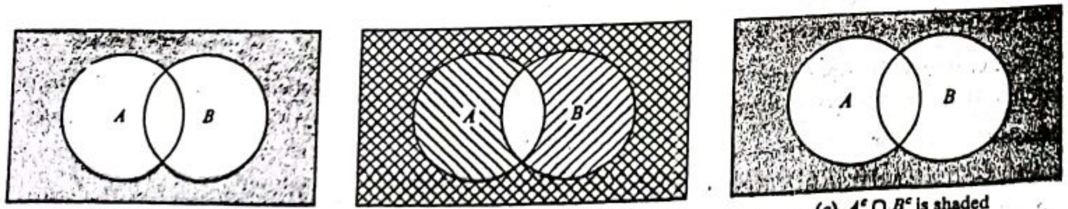
Idempotent laws	
(1a) $A \cup A = A$	(1b) $A \cap A = A$
Associative laws	
(2a) $(A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	
(3a) $A \cup B = B \cup A$	(3b) $A \cap B = B \cap A$
Distributive laws	
(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	
(5a) $A \cup \varnothing = A$	(5b) $A \cap U = A$
(6a) $A \cup U = U$	(6b) $A \cap \varnothing = \varnothing$
Involution laws	
(7) $(A^c)^c = A$	
Complement laws	
(8a) $A \cup A^c = U$	(8b) $A \cap A^c = \varnothing$
(9a) $U^c = \varnothing$	(9b) $\varnothing^c = U$
DeMorgan's laws	
(10a) $(A \cup B)^c = A^c \cap B^c$	(10b) $(A \cap B)^c = A^c \cup B^c$

Method 1: We first show that $(A \cup B)^c \subseteq A^c \cap B^c$. If $x \in (A \cup B)^c$, then $x \notin A \cup B$. Thus $x \notin A$ and $x \notin B$, and so $x \in A^c$ and $x \in B^c$. Hence $x \in A^c \cap B^c$.

Next we show that $A^c \cap B^c \subseteq (A \cup B)^c$. Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$, so $x \in (A \cup B)^c$.

We have proven that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together, these inclusions prove that the sets have the same elements, i.e., that $(A \cup B)^c = A^c \cap B^c$.

Method 2: From the Venn diagram for $A \cup B$ in Fig. 1-4, we see that $(A \cup B)^c$ is represented by the shaded area in Fig. 1-8(a). To find $A^c \cap B^c$, the area in both A^c and B^c, we shaded A^c with strokes in one direction and B^c with strokes in another direction as in Fig. 1-8(b). Then $A^c \cap B^c$ is represented by the crosshatched area, which is shaded in Fig. 1-8(c). Since $(A \cup B)^c$ and $A^c \cap B^c$ are represented by the same area, they are equal.



(a) $(A \cup B)^c$ is shaded
(b) A^c is shaded with ////; B^c is shaded with \\\\
(c) $A^c \cap B^c$ is shaded

Fig. 1-8

Duality

Note that the identities in Table 1-1 are arranged in pairs, as, for example, (2a) and (2b). We now consider the principle behind this arrangement. Suppose E is an equation of set algebra. The *dual* E^* of E is the equation obtained by replacing each occurrence of $\cup$, $\cap$, U, and $\varnothing$ in E by $\cap$, $\cup$, $\varnothing$ and U, respectively. For example, the dual of

$$(U \cap A) \cup (B \cap A) = A \quad \text{is} \quad (\varnothing \cup A) \cap (B \cup A) = A$$

Observe that the pairs of laws in Table 1-1 are duals of each other. It is a fact of set algebra, called the *principle of duality*, that, if any equation E is an identity, then its dual E^* is also an identity.

1.8 FINITE SETS, COUNTING PRINCIPLE

A set is said to be finite if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be infinite. For example, the empty set $\varnothing$ and the set of letters of the English alphabet are finite sets, whereas the set of even positive integers, $\{2, 4, 6, \ldots\}$, is infinite.

The notation $n(A)$ will denote the number of elements in a finite set A. Some texts use $\#(A)$, $|A|$ or card(A) instead of $n(A)$.

Lemma 1.4: If A and B are disjoint finite sets, then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

Proof. In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

We also have a formula for $n(A \cup B)$ even when they are not disjoint. This is proved in Problem 1.28.

Theorem 1.5: If A and B are finite sets, then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

We can apply this result to obtain a similar formula for three sets:

Corollary 1.6: If A, B, and C are finite sets, then so is $A \cup B \cup C$, and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.10) may be used to further generalize this result to any finite number of sets.

EXAMPLE 1.7 Consider the following data for 120 mathematics students at a college concerning the languages French, German, and Russian:

65 study French
45 study German
42 study Russian
20 study French and German
25 study French and Russian
15 study German and Russian
8 study all three languages.

Let F, G, and R denote the sets of students studying French, German and Russian, respectively. We wish to find the number of students who study at least one of the three languages, and to fill in the correct number of students in each of the eight regions of the Venn diagram shown in Fig. 1-9.

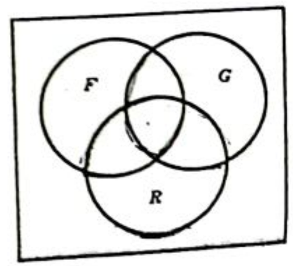


Fig. 1-9

By Corollary 1.6,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

That is, $n(F \cup G \cup R) = 100$ students study at least one of the three languages.

We now use this result to fill in the Venn diagram. We have:

8 study all three languages,
$20 - 8 = 12$ study French and German but not Russian
$25 - 8 = 17$ study French and Russian but not German
$15 - 8 = 7$ study German and Russian but not French
$65 - 12 - 8 - 17 = 28$ study only French
$45 - 12 - 8 - 7 = 18$ study only German
$42 - 17 - 8 - 7 = 10$ study only Russian
$120 - 100 = 20$ do not study any of the languages.

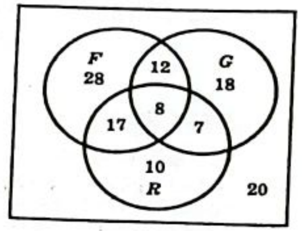


Fig. 1-10

Accordingly, the completed diagram appears in Fig. 1-10. Observe that $28 + 18 + 10 = 56$ students study only one of the languages.

1.9 CLASSES OF SETS, POWER SETS, PARTITIONS

Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a set of sets. Whenever such a situation occurs, to avoid confusion we will speak of a *class* of sets or *collection* of sets rather than a set of sets. If we wish to consider some of the sets in a given class of sets, then we speak of a *subclass* or *subcollection*.

EXAMPLE 1.8 Suppose $S = \{1, 2, 3, 4\}$. Let A be the class of subsets of S which contain exactly three elements of S. Then

$$A = [\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}]$$

The elements of A are the sets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$.

Let B be the class of subsets of S which contain 2 and two other elements of S. Then

$$B = [\{1,2,3\}, \{1,2,4\}, \{2,3,4\}]$$

The elements of B are the sets $\{1,2,3\}$, $\{1,2,4\}$, and $\{2,3,4\}$. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackets instead of braces.)

Power Sets

For a given set S, we may speak of the class of all subsets of S. This class is called the *power set* of S, and will be denoted by Power(S). If S is finite, then so is Power(S). In fact, the number of elements in Power(S) is 2 raised to the power of S; that is,

$$n(\text{Power}(S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S.)

EXAMPLE 1.9 Suppose $S = \{1, 2, 3\}$. Then

$$\text{Power}(S) = [\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S]$$

Note that the empty set $\varnothing$ belongs to Power(S) since $\varnothing$ is a subset of S. Similarly, S belongs to Power(S). As expected from the above remark, Power(S) has $2^3 = 8$ elements.

Partitions

Let S be a nonempty set. A partition of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i.

(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_i \neq A_j \quad \text{then} \quad A_i \cap A_j = \varnothing$$

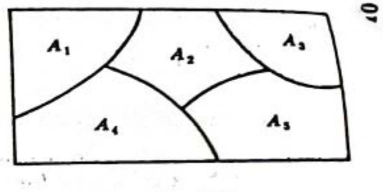


Fig. 1-11

The subsets in a partition are called *cells*. Figure 1-11 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, and A_5.

EXAMPLE 1.10 Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1,3,5\}, \{2,6\}, \{4,8,9\}]$
(ii) $[\{1,3,5\}, \{2,4,6,8\}, \{5,7,9\}]$
(iii) $[\{1,3,5\}, \{2,4,6,8\}, \{7,9\}]$

Then (i) is not a partition of S since 7 in S does not belong to any of the subsets. Furthermore, (ii) is not a partition of S since $\{1,3,5\}$ and $\{5,7,9\}$ are not disjoint. On the other hand, (iii) is a partition of S.

Generalized Set Operations

The set operations of union and intersection were defined above for two sets. These operations can be extended to any number of sets, finite or infinite, as follows.

Consider first a finite number of sets, say, $A_1, A_2, \ldots, A_m$. The union and intersection of these sets are denoted and defined, respectively, by

$$A_1 \cup A_2 \cup \cdots \cup A_m = \bigcup_{i=1}^{m} A_i = \{x : x \in A_i \text{ for some } A_i\}$$
$$A_1 \cap A_2 \cap \cdots \cap A_m = \bigcap_{i=1}^{m} A_i = \{x : x \in A_i \text{ for every } A_i\}$$

That is, the union consists of those elements which belong to at least one of the sets, and the intersection consists of those elements which belong to all the sets.

Now let A be any collection of sets. The union and the intersection of the sets in the collection A is denoted and defined, respectively, by

$$\cup(A : A \in A) = \{x : x \in A \text{ for some } A \in A\}$$
$$\cap(A : A \in A) = \{x : x \in A \text{ for every } A \in A\}$$

That is, the union consists of those elements which belong to at least one of the sets in the collection A, and the intersection consists of those elements which belong to every set in the collection A.

EXAMPLE 1.11 Consider the sets

$$A_1 = \{1, 2, 3, \ldots\} = \mathbf{N}, \quad A_2 = \{2, 3, 4, \ldots\}, \quad A_3 = \{3, 4, 5, \ldots\}, \quad A_n = \{n, n+1, n+2, \ldots\}$$

Then the union and intersection of the sets are as follows:

$$\cup(A_n : n \in \mathbf{N}) = \mathbf{N} \quad \text{and} \quad \cap(A_n : n \in \mathbf{N}) = \varnothing$$

DeMorgan's laws also hold for the above generalized operations. That is:

Theorem 1.7: Let A be a collection of sets. Then

(i) $(\cup(A : A \in A))^c = \cap(A^c : A \in A)$
(ii) $(\cap(A : A \in A))^c = \cup(A^c : A \in A)$

1.10 MATHEMATICAL INDUCTION

An essential property of the set

$$N = \{1, 2, 3, \ldots\}$$

which is used in many proofs, follows:

Principle of Mathematical Induction I: Let P be a proposition defined on the positive integers N, i.e., $P(n)$ is either true or false for each n in N. Suppose P has the following two properties:

(i) $P(1)$ is true.

(ii) $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every positive integer.

We shall not prove this principle. In fact, this principle is usually given as one of the axioms when N is developed axiomatically.

EXAMPLE 1.12 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n):\ 1 + 3 + 5 + \ldots + (2n-1) = n^2$$

The nth odd number is $2n - 1$, and the next odd number is $2n + 1$). Observe that $P(n)$ is true for $n = 1$, that is,

$$P(1):\ 1 = 1^2$$

Assuming $P(n)$ is true, we add $2n + 1$ to both sides of $P(n)$, obtaining

$$1 + 3 + 5 + \ldots + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

which is $P(n+1)$. That is, $P(n+1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true for all n.

There is a form of the principle of mathematical induction which is sometimes more convenient to use. Although it appears different, it is really equivalent to the principle of induction.

Principle of Mathematical Induction II: Let P be a proposition defined on the positive integers N such that:

(i) $P(1)$ is true.

(ii) $P(n)$ is true whenever $P(k)$ is true for all $1 \leq k < n$.

Then P is true for every positive integer.

Remark: Sometimes one wants to prove that a proposition P is true for the set of integers

$$\{a, a+1, a+2, \ldots\}$$

where a is any integer, possibly zero. This can be done by simply replacing 1 by a in either of the above Principles of Mathematical Induction.

Solved Problems

SETS AND SUBSETS

1.1. Which of these sets are equal: $\{r, t, s\}$, $\{s, t, r, s\}$, $\{t, s, t, r\}$, $\{s, r, s, t\}$?

They are all equal. Order and repetition do not change a set.

1.2. List the elements of the following sets; here $N = \{1, 2, 3, \ldots\}$.

(a) $A = \{x: x \in N, 3 < x < 12\}$
(b) $B = \{x: x \in N, x \text{ is even}, x < 15\}$
(c) $C = \{x: x \in N, 4 + x = 3\}$.

(a) A consists of the positive integers between 3 and 12; hence

$$A = \{4, 5, 6, 7, 8, 9, 10, 11\}$$

(b) B consists of the even positive integers less than 15; hence

$$B = \{2, 4, 6, 8, 10, 12, 14\}$$

(c) There are no positive integers which satisfy the condition $4 + x = 3$; hence C contains no elements. In other words, $C = \varnothing$, the empty set.

1.3. Consider the following sets:

$$\varnothing,\quad A = \{1\},\quad B = \{1, 3\},\quad C = \{1, 5, 9\},\quad D = \{1, 2, 3, 4, 5\},$$
$$E = \{1, 3, 5, 7, 9\},\quad U = \{1, 2, \ldots, 8, 9\}$$

Insert the correct symbol $\subseteq$ or $\nsubseteq$ or between each pair of sets:

(a) $\varnothing, A$	(c) B, C	(e) C, D	(g) D, E
(b) A, B	(d) B, E	(f) C, E	(h) D, U

(a) $\varnothing \subseteq A$ because $\varnothing$ is a subset of every set.
(b) $A \subseteq B$ because 1 is the only element of A and it belongs to B.
(c) $B \nsubseteq C$ because $3 \in B$ but $3 \notin C$.
(d) $B \nsubseteq E$ because the elements of B also belong to E.
(e) $C \nsubseteq D$ because $9 \in C$ but $9 \notin D$.
(f) $C \subseteq E$ because the elements of C also belong to E.
(g) $D \nsubseteq E$ because $2 \in D$ but $2 \notin E$.
(h) $D \subseteq U$ because the elements of D also belong to U.

1.4. Show that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x: x \in N, x \text{ is even}\}$.

It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

1.5. Show that $A = \{2, 3, 4, 5\}$ is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

SET OPERATIONS

Problems 1.6 to 1.8 refer to the universal set $U = \{1, 2, \ldots, 9\}$ and the sets

$$A = \{1, 2, 3, 4, 5\},\quad C = \{5, 6, 7, 8, 9\},\quad E = \{2, 4, 6, 8\}$$
$$B = \{4, 5, 6, 7\},\quad D = \{1, 3, 5, 7, 9\},\quad F = \{1, 5, 9\}$$

1.6. Find:

(a) $A \cup B$ and $A \cap B$ (c) $A \cup C$ and $A \cap C$ (e) $E \cup E$ and $E \cap E$
(b) $B \cup D$ and $B \cap D$ (d) $D \cup E$ and $D \cap E$ (f) $D \cup F$ and $D \cap F$

Recall that the union $X \cup Y$ consists of those elements in either X or Y (or both), and that the intersection $X \cap Y$ consists of those elements in both X and Y.

(a) $A \cup B = \{1,2,3,4,5,6,7\}$ $A \cap B = \{4,5\}$
(b) $B \cup D = \{1,3,4,5,6,7,9\}$ $B \cap D = \{5,7\}$
(c) $A \cup C = \{1,2,3,4,5,6,7,8,9\} = U$ $A \cap C = \{5\}$
(d) $D \cap E = \{1,2,3,4,5,6,7,8,9\} = U$ $D \cap E = \varnothing$
(e) $E \cup E = \{2,4,6,8\} = E$ $E \cap E = \{2,4,6,8\} = E$
(f) $D \cup F = \{1,3,5,7,9\} = D$ $D \cap F = \{1,5,9\} = F$

Observe that $F \subseteq D$; so by Theorem 1.2 we must have $D \cup F = D$ and $D \cap F = F$.

1.7. Find: (a) A^c, B^c, D^c, E^c; (b) $A\backslash B$, $B\backslash A$, $D\backslash E$, $F\backslash D$; (c) $A \oplus B$, $C \oplus D$, $E \oplus F$.

Recall that:

(1) The complement X^c consists of those elements in the universal set U which do not belong to X.
(2) The difference $X\backslash Y$ consist of the elements in X which do not belong to Y.
(3) The symmetric difference $X \oplus Y$ consists of the elements in X or in Y but not in both X and Y.

Therefore:

(a) $A^c = \{6,7,8,9\}$; $B^c = \{1,2,3,8,9\}$; $D^c = \{2,4,6,8\} = E$; $E^c = \{1,3,5,7,9\} = D$.
(b) $A\backslash B = \{1,2,3\}$; $B\backslash A = \{6,7\}$; $D\backslash E = \{1,3,5,7,9\} = D$; $F\backslash D = \varnothing$.
(c) $A \oplus B = \{1,2,3,6,7\}$; $C \oplus D = \{1,3,8,9\}$; $E \oplus F = \{2,4,6,8,1,5,9\} = E \cup F$.

1.8. Find: (a) $A \cap (B \cup E)$; (b) $(A\backslash E)^c$;
(c) $(A \cap D)\backslash B$; (d) $(B \cap F) \cup (C \cap E)$.

(a) First compute $B \cup E = \{2,4,5,6,7,8\}$. Then $A \cap (B \cup E) = \{2,4,5\}$.
(b) $A\backslash E = \{1,3,5\}$. Then $(A\backslash E)^c = \{2,4,6,7,8,9\}$.
(c) $A \cap D = \{1,3,5\}$. Now $(A \cap D)\backslash B = \{1,3\}$.
(d) $B \cap F = \{5\}$ and $C \cap E = \{6,8\}$. So $(B \cap F) \cup (C \cap E) = \{5,6,8\}$.

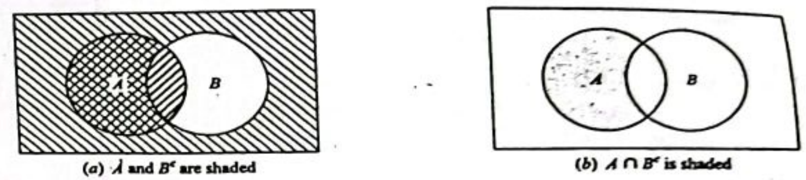
1.9. Show that we can have $A \cap B = A \cap C$ without $B = C$.

Let $A = \{1,2\}$, $B = \{2,3\}$, and $C = \{2,4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$. Thus $A \cap B = A \cap C$ but $B \neq C$.

VENN DIAGRAMS

1.10. Consider the Venn diagram of two arbitrary sets A and B in Fig. 1-1(c). Shade the sets: (a) $A \cap B^c$; (b) $(B\backslash A)^c$.

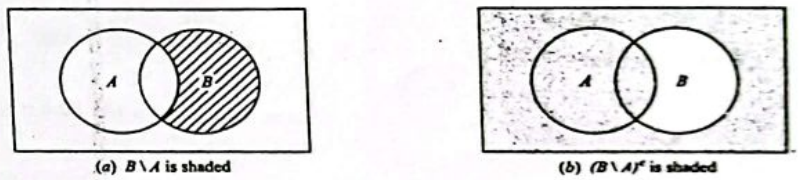
(a) First shade the area represented by A with strokes in one direction (///), and then shade the area represented by B^c (the area outside B), with strokes in another direction (\\\\\\). This is shown in Fig. 1-12(a). The cross-hatched area is the intersection of these two sets and represents $A \cap B^c$ and this is shown in Fig. 1-12(b). Observe that $A \cap B^c = A\backslash B$. In fact, $A\backslash B$ is sometimes defined to be $A \cap B^c$.



(a) A and B^c are shaded (b) $A \cap B^c$ is shaded

Fig. 1-12

(b) First shade the area represented by $B\backslash A$ (the area of B which does not lie in A) as in Fig. 1-13(a). Then the area outside this shaded region, which is shown in Fig. 1-13(b), represents $(B\backslash A)^c$.



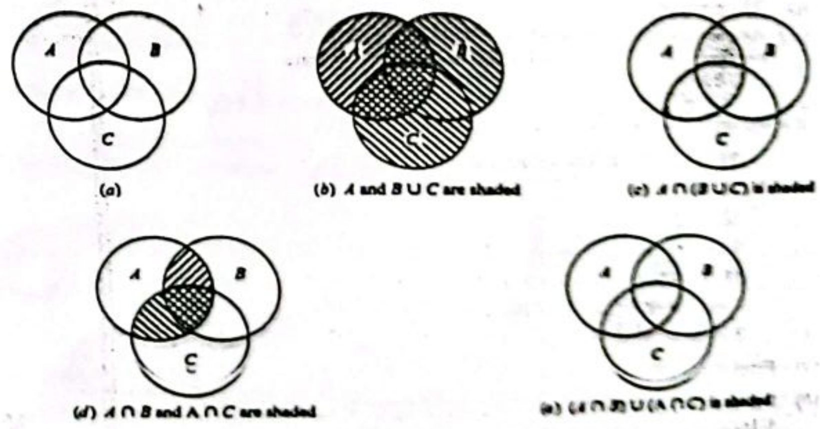
(a) $B\backslash A$ is shaded (b) $(B\backslash A)^c$ is shaded

Fig. 1-13

1.11. Illustrate the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ with Venn diagrams.

Draw three intersecting circles labeled A, B, C, as in Fig. 1-14(a). Now, as in Fig. 1-14(b) shade A with strokes in one direction and shade $B \cup C$ with strokes in another direction; the crosshatched area is $A \cap (B \cup C)$, as in Fig. 1-14(c). Next shade $A \cap B$ and then $A \cap C$, as in Fig. 1-14(d); the total area shaded is $(A \cap B) \cup (A \cap C)$, as in Fig. 1-14(e).

As expected by the distributive law, $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are both represented by the same set of points.



(a) (b) A and $B \cup C$ are shaded (c) $A \cap (B \cup C)$ is shaded

(d) $A \cap B$ and $A \cap C$ are shaded (e) $(A \cap B) \cup (A \cap C)$ is shaded

Fig. 1-14

Determine the validi

12. Determine the validity of the following argument:

S_1: All my friends are musicians.
S_2: John is my friend.
S_3: None of my neighbors are musicians.

S: John is not my neighbor.

The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-15. By S_2, John belongs to the set of friends which is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.

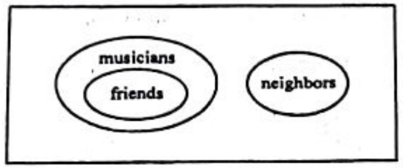


Fig. 1-15

FINITE SETS AND THE COUNTING PRINCIPLE

1.13. Determine which of the following sets are finite.

(a) $A = \{\text{seasons in the year}\}$ (d) $D = \{\text{odd integers}\}$
(b) $B = \{\text{states in the Union}\}$ (e) $E = \{\text{positive integral divisors of } 12\}$
(c) $C = \{\text{positive integers less than } 1\}$ (f) $F = \{\text{cats living in the United States}\}$

(a) A is finite since there are four seasons in the year, i.e., $n(A) = 4$.
(b) B is finite because there are 50 states in the Union, i.e. $n(B) = 50$.
(c) There are no positive integers less than 1; hence C is empty. Thus C is finite and $n(C) = 0$.
(d) D is infinite.
(e) The positive integer divisors of 12 are 1, 2, 3, 4, 6, and 12. Hence E is finite and $n(E) = 6$.
(f) Although it may be difficult to find the number of cats living in the United States, there is still a finite number of them at any point in time. Hence F is finite.

1.14. In a survey of 60 people, it was found that:

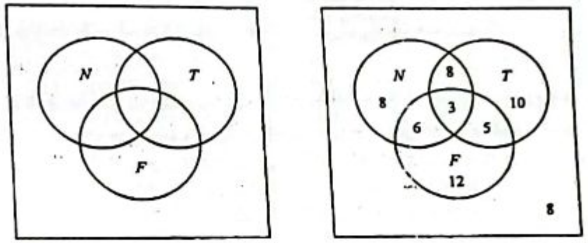
25 read *Newsweek* magazine
26 read *Time*
26 read *Fortune*
9 read both *Newsweek* and *Fortune*
11 read both *Newsweek* and *Time*
8 read both *Time* and *Fortune*
3 read all three magazines

(a) Find the number of people who read at least one of the three magazines.
(b) Fill in the correct number of people in each of the eight regions of the Venn diagram in Fig. 1-16(a) where N, T, and F denote the set of people who read *Newsweek*, *Time*, and *Fortune*, respectively.

(c) Find the number of people who read exactly one magazine.

(a) We want $n(N \cup T \cup F)$. By Corollary 1.6,

$$n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$$
$$= 25 + 26 + 26 - 11 - 9 - 8 + 3 = 52.$$



(a) (b)

Fig. 1-16

(b) The required Venn diagram in Fig. 1-16(b) is obtained as follows:

3 read all three magazines
$11 - 3 = 8$ read *Newsweek* and *Time* but not all three magazines
$9 - 3 = 6$ read *Newsweek* and *Fortune* but not all three magazines
$8 - 3 = 5$ read *Time* and *Fortune* but not all three magazines
$25 - 8 - 6 - 3 = 8$ read only *Newsweek*
$26 - 8 - 5 - 3 = 10$ read only *Time*
$26 - 6 - 5 - 3 = 12$ read only *Fortune*
$60 - 52 = 8$ read no magazine at all

(c) $8 + 10 + 12 = 30$ read only one magazine.

ALGEBRA OF SETS AND DUALITY

1.15. Write the dual of each set equation:

(a) $(U \cap A) \cup (B \cap A) = A$ (c) $(A \cap U) \cap (\varnothing \cup A^c) = \varnothing$
(b) $(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$ (d) $(A \cap U)^c \cap A = \varnothing$

Interchange $\cup$ and $\cap$ and also U and $\varnothing$ in each set equation:

(a) $(\varnothing \cup A) \cap (B \cup A) = A$ (c) $(A \cup \varnothing) \cup (U \cap A^c) = U$
(b) $(A \cap B \cap C)^c = (A \cap C)^c \cup (A \cap B)^c$ (d) $(A \cup \varnothing)^c \cup A = U$

1.16. Prove the Commutative laws: (a) $A \cup B = B \cup A$, and (b) $A \cap B = B \cap A$.

(a) $A \cup B = \{x: x \in A \text{ or } x \in B\} = \{x: x \in B \text{ or } x \in A\} = B \cup A$.
(b) $A \cap B = \{x: x \in A \text{ and } x \in B\} = \{x: x \in B \text{ and } x \in A\} = B \cap A$.

Prove the following identity: $(A \cup B) \cap (A \cup B^c) = A$.

Statement	Reason
1. $(A \cup B) \cap (A \cup B^c) = A \cup (B \cap B^c)$	Distributive law
2. $B \cap B^c = \varnothing$	Complement law
3. $(A \cup B) \cap (A \cup B^c) = A \cup \varnothing$	Substitution
4. $A \cup \varnothing = A$	Identity law
5. $(A \cup B) \cap (A \cup B^c) = A$	Substitution

Prove $(A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$. (Thus either one may be used to define $A \oplus B$.)

Using $X \backslash Y = X \cap Y^c$ and the laws in Table 1-1, including DeMorgan's laws, we obtain:

$$\begin{aligned}(A \cup B)\backslash(A \cap B) &= (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cup B^c)\\ &= (A \cap A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c)\\ &= \varnothing \cup (A \cap B^c) \cup (B \cap A^c) \cup \varnothing\\ &= (A \cap B^c) \cup (B \cap A^c) = (A \backslash B) \cup (B \backslash A).\end{aligned}$$

SES OF SETS

Find the elements of the set $A = [\{1,2,3\}, \{4,5\}, \{6,7,8\}]$.

A is a class of sets; its elements are the sets $\{1,2,3\}$, $\{4,5\}$, and $\{6,7,8\}$.

Consider the class A of sets in Problem 1.19. Determine whether each of the following is true or false:

(a) $1 \in A$ (c) $\{6,7,8\} \in A$ (e) $\varnothing \in A$
(b) $\{1,2,3\} \subseteq A$ (d) $\{\{4,5\}\} \subseteq A$ (f) $\varnothing \subseteq A$

(a) False. 1 is not one of the elements of A.
(b) False. $\{1,2,3\}$ is not a subset of A; it is one of the elements of A.
(c) True. $\{6,7,8\}$ is one of the elements of A.
(d) True. $\{\{4,5\}\}$, the set consisting of the element $\{4,5\}$, is a subset of A.
(e) False. The empty set is not an element of A, i.e., it is not one of the three sets listed as elements of A.
(f) True. The empty set is a subset of every set; even a class of sets.

Determine the power set Power(A) of $A = \{a,b,c,d\}$.

The elements of Power(A) are the subsets of A. Hence

$$\text{Power}(A) = [A, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a\}, \{b\}, \{c\}, \{d\}, \varnothing]$$

As expected, Power(A) has $2^4 = 16$ elements.

Let $S = \{red, blue, green, yellow\}$. Determine which of the following is a partition of S:

(a) $P_1 = [\{red\}, \{blue, green\}]$. (c) $P_3 = [\varnothing, \{red, blue\}, \{green, yellow\}]$.
(b) $P_2 = [\{red, blue, green, yellow\}]$. (d) $P_4 = [\{blue\}\{red, yellow, green\}]$.

77

(a) No, since *yellow* does not belong to any cell.
(b) Yes, since P_2 is a partition of S whose only element is S itself.
(c) No, since the empty set $\varnothing$ cannot belong to a partition.
(d) Yes, since each element of S appears in exactly one cell.

1.23. Find all partitions of $S = \{1,2,3\}$.

Note that each partition of S contains either 1, 2, or 3 cells. The partitions for each number of cells are as follows:

(1): $[S]$
(2): $[\{1\}, \{2,3\}]$, $[\{2\}, \{1,3\}]$, $[\{3\}, \{1,2\}]$
(3): $[\{1\}, \{2\}, \{3\}]$

Thus we see that there are five different partitions of S.

MISCELLANEOUS PROBLEMS

1.24. Prove the proposition P that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$P(n): \quad 1+2+3+\ldots+n = \frac{1}{2}n(n+1)$$

The proposition holds for $n = 1$ since

$$P(1): 1 = \tfrac{1}{2}(1)(1+1)$$

Assuming $P(n)$ is true, we add $n+1$ to both sides of $P(n)$, obtaining

$$\begin{aligned}1+2+3+\ldots+n+(n+1) &= \tfrac{1}{2}n(n+1)+(n+1)\\ &= \tfrac{1}{2}[n(n+1)+2(n+1)]\\ &= \tfrac{1}{2}[(n+1)(n+2)]\end{aligned}$$

which is $P(n+1)$. That is, $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.25. Prove the following proposition (for $n \geq 0$):

$$P(n): \quad 1+2+2^2+2^3+\cdots+2^n = 2^{n+1}-1$$

$P(0)$ is true since $1 = 2^1 - 1$. Assuming $P(n)$ is true, we add 2^{n+1} to both sides of $P(n)$, obtaining

$$\begin{aligned}1+2^1+2^2+\cdots+2^n+2^{n+1} &= 2^{n+1}-1+2^{n+1}\\ &= 2(2^{n+1})-1\\ &= 2^{n+2}-1\end{aligned}$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all $n \geq 0$.

1.26. Prove: $(A \cap B) \subseteq A \subseteq (A \cup B)$ and $(A \cap B) \subseteq B \subseteq (A \cup B)$.

Since every element in $A \cap B$ is in both A and B, it is certainly true that if $x \in (A \cap B)$ then $x \in A$; hence $(A \cap B) \subseteq A$. Furthermore, if $x \in A$, then $x \in (A \cup B)$ (by the definition of $A \cup B$), so $A \subseteq (A \cup B)$. Putting these together gives $(A \cap B) \subseteq A \subseteq (A \cup B)$. Similarly, $(A \cap B) \subseteq B \subseteq (A \cup B)$.

Prove Theorem

Suppose $A \subseteq$
$(A \cap B) \subseteq A$. Ther
$x \in (A \cap B)$, hence
$A \cap B = A$.

.7. Prove Theorem 1.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

Suppose $A \subseteq B$ and let $x \in A$. Then $x \in B$, hence $x \in A \cap B$ and $A \subseteq A \cap B$. By Problem 1.26, $(A \cap B) \subseteq A$. Therefore $A \cap B = A$. On the other hand, suppose $A \cap B = A$ and let $x \in A$. Then $x \in (A \cap B)$, hence $x \in A$ and $x \in B$. Therefore, $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cap B = A$.

Suppose again that $A \subseteq B$. Let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$ because $A \subseteq B$. In either case, $x \in B$. Therefore $A \cup B \subseteq B$. By Problem 1.26, $B \subseteq A \cup B$. Therefore $A \cup B = B$. Now suppose $A \cup B = B$ and let $x \in A$. Then $x \in A \cup B$ by definition of union sets. Hence $x \in B = A \cup B$. Therefore $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cup B = B$.

Thus $A \subseteq B$, $A \cap B = A$ and $A \cup B = B$ are equivalent.

28. Prove Theorem 1.5: If A and B are finite sets, then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If A and B are finite, then clearly $A \cap B$ and $A \cup B$ are finite.

Suppose we count the elements of A and then count the elements of B. Then every element in $A \cap B$ would be counted twice, once in A and once in B. Hence

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Alternatively, (Problem 1.36) A is the disjoint union of $A \backslash B$ and $A \cap B$, B is the disjoint union of $B \backslash A$ and $A \cap B$, and $A \cup B$ is the disjoint union of $A \backslash B$, $A \cap B$, and $B \backslash A$. Therefore, by Lemma 1.4,

$$\begin{aligned} n(A \cup B) &= n(A \backslash B) + n(A \cap B) + n(B \backslash A) \\ &= n(A \backslash B) + n(A \cap B) + n(B \backslash A) + n(A \cap B) - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B) \end{aligned}$$

Supplementary Problems

S AND SUBSETS

. Which of the following sets are equal?

$A = \{x: x^2 - 4x + 3 = 0\}$ $C = \{x: x \in \mathbf{N}, x < 3\}$ $E = \{1, 2\}$ $G = \{3, 1\}$

$B = \{x: x^2 - 3x + 2 = 0\}$ $D = \{x: x \in \mathbf{N}, x \text{ is odd}, x < 5\}$ $F = \{1, 2, 1\}$ $H = \{1, 1, 3\}$

. List the elements of the following sets if the universal set is $U = \{a, b, c, \ldots, y, z\}$. Furthermore, identify which of the sets, if any, are equal.

$A = \{x: x \text{ is a vowel}\}$ $C = \{x: x \text{ precedes f in the alphabet}\}$

$B = \{x: x \text{ is a letter in the word "little"}\}$ $D = \{x: x \text{ is a letter in the word "title"}\}$

) Let $A = \{1, 2, \ldots, 8, 9\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 3, 5, 7, 9\}$, $D = \{3, 4, 5\}$, $E = \{3, 5\}$.

Which of the above sets can equal a set X under each of the following conditions?

(a) X and B are disjoint. (c) $X \subseteq A$ but $X \not\subseteq C$.

(b) $X \subseteq D$ but $X \not\subseteq B$. (d) $X \subseteq C$ but $X \not\subseteq A$.

T OPERATIONS

blems 1.32 to 1.34 refer to the sets $U = \{1, 2, 3, \ldots, 8, 9\}$ and $A = \{1, 2, 5, 6\}$, $B = \{2, 5, 7\}$, $C = \{1, 3, 5, 7, 9\}$.

1.32. Find: (a) $A \cap B$ and $A \cap C$; (b) $A \cup B$ and $B \cup C$; (c) A^c and C^c.

1.33. Find: (a) $A \backslash B$ and $A \backslash C$; (b) $A \oplus B$ and $A \oplus C$.

1.34. Find: (a) $(A \cup C) \backslash B$; (b) $(A \cup B)^c$; (c) $(B \oplus C) \backslash A$.

1.35. Let $A = \{a, b, c, d, e\}$, $B = \{a, b, d, f, g\}$, $C = \{b, c, e, y, h\}$, $D = \{d, e, f, g, h\}$.

Find:

(a) $A \cup B$	(d) $A \cap (B \cup D)$	(g) $(A \cup D) \backslash C$	(j) $A \oplus B$
(b) $B \cap C$	(e) $B \backslash (C \cup D)$	(h) $B \cap C \cap D$	(k) $A \oplus C$
(c) $C \backslash D$	(f) $(A \cap D) \cup B$	(i) $(C \backslash A) \backslash D$	(l) $(A \oplus D) \backslash B$

1.36. Let A and B be any sets. Prove:

(a) A is the disjoint union of $A \backslash B$ and $A \cap B$.

(b) $A \cup B$ is the disjoint union of $A \backslash B$, $A \cap B$, and $B \backslash A$.

1.37. Prove the following:

(a) $A \subseteq B$ if and only if $A \cap B^c = \varnothing$.

(b) $A \subseteq B$ if and only if $A^c \cup B = U$.

(c) $A \subseteq B$ if and only if $B^c \subseteq A^c$.

(d) $A \subseteq B$ if and only if $A \backslash B = \varnothing$.

(Compare results with Theorem 1.2.)

1.38. Prove the Absorption laws: (a) $A \cup (A \cap B) = A$; (b) $A \cap (A \cup B) = A$.

1.39. The formula $A \backslash B = A \cap B^c$ defines the difference operation in terms of the operations of intersection and complement. Find a formula that defines the union $A \cup B$ in terms of the operations of intersection and complement.

VENN DIAGRAMS

1.40. The Venn diagram in Fig. 1-17 shows sets A, B, C. Shade the following sets: (a) $A \backslash (B \cup C)$; (b) $A^c \cap (B \cup C)$; (c) $A^c \cap (C \backslash B)$.

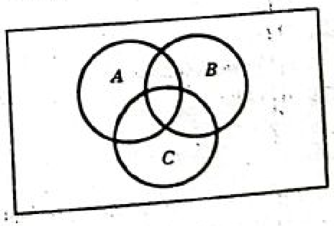


Fig. 1-17

1.41. Use the Venn diagram Fig. 1-6 and Example 1.6 to write each set as the (disjoint) union of fundamental products:

(*a*) $A \cap (B \cup C)$, (*b*) $A^c \cap (B \cup C)$, (*c*) $A \cup (B \setminus C)$.

1.42. Draw a Venn diagram of sets A, B, C where $A \subseteq B$, sets B and C are disjoint, but A and C have elements in common.

ALGEBRA OF SETS AND DUALITY

1.43. Write the dual of each equation:

(*a*) $A \cup B = (B^c \cap A^c)^c$ (*b*) $A = (B^c \cap A) \cup (A \cap B)$
(*c*) $A \cup (A \cap B) = A$ (*d*) $(A \cap B) \cup (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c) = U$

1.44. Use the laws in Table 1-1 to prove each set identity:

(*a*) $(A \cap B) \cup (A \cap B^c) = A$.
(*b*) $A \cup (A \cap B) = A$.
(*c*) $A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)$.

FINITE SETS AND THE COUNTING PRINCIPLE

1.45. Determine which of the following sets are finite:

(*a*) The set of lines parallel to the x axis
(*b*) The set of letters in the English alphabet
(*c*) The set of numbers which are multiples of 5
(*d*) The set of animals living on the earth
(*e*) The set of numbers which are solutions of the equation:

$$x^{27} + 26x^{18} - 17x^{11} + 7x^3 - 10 = 0$$

(*f*) The set of circles through the origin (0, 0)

1.46. Use Theorem 1.5 to prove Corollary 1.6: If A, B, and C are finite sets, then so is $A \cup B \cup C$ and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

1.47. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found:

15 had air-conditioning.
12 had radio.
11 had power windows.
5 had air-conditioning and power windows.
9 had air-conditioning and radio.
4 had radio and power windows.
3 had all three options.

Find the number of cars that had: (*a*) only power windows; (*b*) only air-conditioning; (*c*) only radio; (*d*) radio and power windows but not air-conditioning; (*e*) air-conditioning and radio, but not power windows; and (*f*) only one of the options; (*g*) at least one option; (*h*) none of the options.

CLASSES OF SETS

1.48. Find the power set Power(A) of $A = \{1, 2, 3, 4, 5\}$.

1.49. Given $A = [\{a, b\}, \{c\}, \{d, e, f\}]$.

(*a*) State whether each of the following is true or false:
(*i*) $a \in A$, (ii) $\{c\} \subseteq A$, (iii) $\{d, e, f\} \in A$, (iv) $\{\{a, b\}\} \subseteq A$, (v) $\varnothing \subseteq A$.

(*b*) Find the power set of A.

1.50. Suppose A is a finite set and $n(A) = m$. Prove that Power(A) has 2^m elements.

PARTITIONS

1.51. Let $X = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of X:

(*a*) $[\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}]$ (*c*) $[\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}]$
(*b*) $[\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}]$ (*d*) $[\{1, 2, 7\}, \{3, 5\}, \{4, 6, 8, 9\}, \{3, 5\}]$

1.52. Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not each of the following is a partition of S:

(*a*) $P_1 = [\{1, 2, 3\}, \{1, 4, 5, 6\}]$ (*c*) $P_3 = [\{1, 3, 5\}, \{2, 4\}, \{6\}]$
(*b*) $P_2 = [\{1, 2\}, \{3, 5, 6\}]$ (*d*) $P_4 = [\{1, 3, 5\}, \{2, 4, 6, 7\}]$

1.53. Determine whether or not each of the following is a partition of the set $\mathbf{N}$ of positive integers:

(*a*) $[\{n : n > 5\}, \{n : n < 5\}]$ (*b*) $[\{n : n > 5\}, \{0\}, \{1, 2, 3, 4, 5\}]$, (*c*) $[\{n : n^2 > 11\}, \{n : n^2 < 11\}]$

1.54. Let $[A_1, A_2, \ldots, A_m]$ and $[B_1, B_2, \ldots, B_n]$ be partitions of a set X. Show that the collection of sets

$$P = [A_i \cap B_j : i = 1, \ldots, m, \ j = 1, \ldots, n] \setminus \varnothing$$

is also a partition (called the *cross partition*) of X. (Observe that we have deleted the empty set $\varnothing$.)

1.55. Let $X = \{1, 2, 3, \ldots, 8, 9\}$. Find the cross partition P of the following partitions of X:

$$P_1 = [\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}] \quad \text{and} \quad P_2 = [\{1, 2, 3, 4\}, \{5, 7\}, \{6, 8, 9\}]$$

ARGUMENTS AND VENN DIAGRAMS

1.56. Use a Venn diagram to show that the following argument is valid:

S_1: Babies are illogical.
S_2: Nobody is despised who can manage a crocodile.
S_3: Illogical people are despised.

S: Babies cannot manage crocodiles.

(This argument is adopted from Lewis Carroll, *Symbolic Logic*; he is also the author of *Alice in Wonderland*.)

1.57. Consider the following assumptions:

S_1: All dictionaries are useful.
S_2: Mary owns only romance novels.
S_3: No romance novel is useful.

Determine the validity of each of the following conclusions: (*a*) Romance novels are not dictionaries. (*b*) Mary does not own a dictionary. (*c*) All useful books are dictionaries.

INDUCTION

1.58. Prove: $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

1.59. Prove: $1 + 4 + 7 + \cdots + (3n - 2) = 2n(3n - 1)$.

1.60. Prove: $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{1}{2n + 1}$.

1.61. Prove: $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$.

MISCELLANEOUS PROBLEMS

1.62. Suppose $\mathbf{N} = \{1, 2, 3, \ldots\}$ is the universal set and

$A = \{x: x \le 6\}$, $B = \{x: 4 \le x \le 9\}$, $C = \{1, 3, 5, 7, 9\}$, $D = \{2, 3, 5, 7, 8\}$

Find: (*a*) $A \oplus B$; (*b*) $B \oplus C$; (*c*) $A \cap (B \oplus D)$; (*d*) $(A \cap B) \oplus (A \cap D)$.

1.63. Prove the following properties of the symmetric difference:

(i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ (Associative law).
(ii) $A \oplus B = B \oplus A$ (Commutative law).
(iii) If $A \oplus B = A \oplus C$, then $B = C$ (Cancellation law).
(iv) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ (Distribution law).

1.64. Consider n distinct sets $A_1, A_2, \ldots, A_n$ in a universal set U. Prove:

(*a*) There are 2^n fundamental products of the n sets.
(*b*) Any two fundamental products are disjoint.
(*c*) U is the union of all the fundamental products.

Answers to Supplementary Problems

1.29. $B = C = E = F$; $A = D = G = H$.

1.30. $A = \{a, e, i, o, u\}$; $B = D = \{1, i, t, e\}$; $C = \{a, b, c, d, e\}$.

1.31. (*a*) C and E; (*b*) D and E; (*c*) A, B, D; (*d*) None.

1.32. (*a*) $A \cap B = \{2, 5\}$; $A \cap C = \{1, 5\}$. (*b*) $A \cup B = \{1, 2, 5, 6, 7\}$; $B \cup C = \{1, 2, 3, 5, 7, 9\}$.
(*c*) $A^c = \{3, 4, 7, 8, 9\}$; $C^c = \{2, 4, 6, 8\}$.

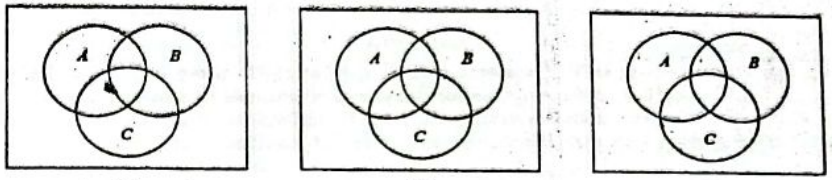
1.33. (*a*) $A \setminus B = \{1, 6\}$; $A \setminus C = \{2, 6\}$. (*b*) $A \oplus B = \{1, 6, 7\}$; $A \oplus C = \{2, 3, 6, 7, 9\}$.

1.34. (*a*) $(A \cup C) \setminus B = \{1, 3, 6, 9\}$. (*b*) $(A \cup B)^c = \{3, 4, 8, 9\}$. (*c*) $(B \oplus C) \setminus A = \{3, 9\}$.

1.35. (*a*) {a, b, c, d, e, f, g}; (*b*) {b, g}; (*c*) {b, c}; (*d*) {a, b, d, e}; (*e*) {a};
(*f*) {a, b, d, e, f, g}; (*g*) {a, d, f}; (*h*) {g}; (*i*) $\varnothing$; (*j*) {c, e, f, g}; (*k*) {a, d, y, h};
(*l*) {c, h}.

1.39. $A \cup B = (A^c \cap B^c)^c$

1.40. See Fig. 1-18.



(*a*) (*b*) (*c*)

Fig. 1-18

1.41. (*a*) $(A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C)$
(*b*) $(A^c \cap B \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B^c \cap C)$
(*c*) $(A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c) \cup (A \cap B^c \cap C^c)$

1.42. No such Venn diagram exists. If A and C have an element in common, say x, and $A \subseteq B$, then x must also belong to B. Thus B and C must also have an element in common.

1.43. (*a*) $A \cap B = (B^c \cup A^c)^c$; (*b*) $A = (B^c \cup A) \cap (A \cup B)$; (*c*) $A \cap (A \cup B) = A$;
(*d*) $(A \cup B) \cap (A^c \cup B) \cap (A \cup B^c) \cap (A^c \cup B^c) = \varnothing$.

1.45. (*a*) Infinite; (*b*) finite; (*c*) infinite; (*d*) finite; (*e*) finite; (*f*) infinite.

1.47. Use the data to first fill in the Venn diagram of A (air-conditioning), R (radio), and W (power windows) in Fig. 1-19. Then: (*a*) 5; (*b*) 4; (*c*) 2; (*d*) 4; (*e*) 6; (*f*) 11; (*g*) 23; (*h*) 2.

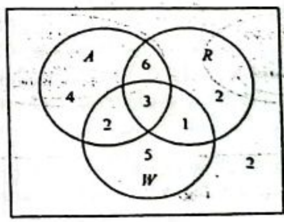


Fig. 1-19

Power(A) has $2^5 = 32$ elements as follows:
$[\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{2,3,4\}, \{2,3,5\}, \{3,4,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, A]$.

1.49. (*a*) (i) False; (ii) False; (iii) True; (iv) True; (v) True.
(*b*) Note $n(A) = 3$; hence Power(A) has $2^3 = 8$ elements:

$$\text{Power}(A) = \{A, [\{a,b\}, \{c\}], [\{a,b\}, \{d,e,f\}], [\{c\}, \{d,e,f\}], [\{a,b\}], [\{c\}], [\{d,e,f\}], \varnothing\}$$

1.50. Let x be an arbitrary element in Power(A). For each $a \in A$, there are two possibilities: $a \in A$ or $a \notin A$. But there are m elements in A; hence there are $2 \cdot 2 \cdot \ldots \cdot 2 = 2^m$ different sets X. That is, Power(A) has 2^m elements.

1.51. (*a*) No, (*b*) no, (*c*) yes, (*d*) yes.

1.52. (*a*) No, (*b*) no, (*c*) yes, (*d*) no.

1.53. (*a*) No, (*b*) no, (*c*) yes.

1.55. $P = [\{1,3\}, \{5,7\}, \{9\}, \{2,4\}, \{8\}]$.

1.56. The three premises lead to the Venn diagram in Fig. 1-20. The set of babies and the set of people who can manage crocodiles are disjoint. In other words, the conclusion S is valid.

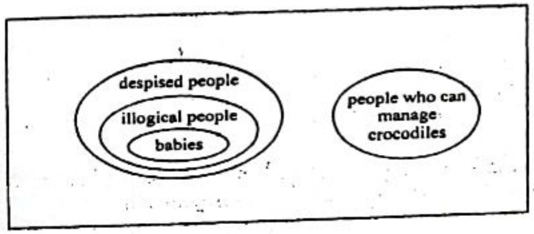


Fig. 1-20

1.57. The three premises lead to the Venn diagram in Fig. 1-21. From this diagram it follows that (*a*) and (*b*) are valid conclusions. However, (*c*) is not a valid conclusion since there may be useful books which are not dictionaries.

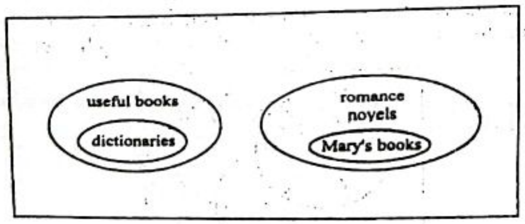


Fig. 1-21

1.62. (*a*) $\{1,2,3,7,8,9\}$; (*b*) $\{1,3,4,6,8\}$; (*c*) $\{2,3,4,6\}$; (*d*) $\{2,3,4,6\}$. [Note (*c*) = (*d*).]

Chapter 2

Relations

2.1 INTRODUCTION

The reader is familiar with many relations which are used in mathematics and computer science, e.g., "less than", "is parallel to", "is a subset of", and so on. In a certain sense, these relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs".

There are three kinds of relations which play a major role in our theory: (i) equivalence relations, (ii) order relations, (iii) functions. Equivalence relations are mainly covered in this chapter. Order relations are introduced here, but will also be discussed in Chapter 14. Functions are covered in the next chapter.

Relations, as noted above, will be defined in terms of ordered pairs (a, b) of elements, where a is designated as the first element and b as the second element. In particular,

$$(a,b) = (c,d)$$

if and only if $a = c$ and $b = d$. Thus $(a,b) \neq (b,a)$ unless $a = b$. This contrasts with sets studied in Chapter 1, where the order of elements is irrelevant; for example, $\{3,5\} = \{5,3\}$.

Although matrices will be covered in Chapter 5, we have included their connection with relations here for completeness. These sections, however, can be ignored at a first reading by those with no previous knowledge of matrix theory.

2.2 PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B". By definition,

$$A \times B = \{(a,b) \colon a \in A \text{ and } b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

EXAMPLE 2.1 $\mathbf{R}$ denotes the set of real numbers and so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of $\mathbf{R}^2$ as points in the plane as in Fig. 2-1. Here each point P represents an ordered pair (a,b) of real numbers and vice versa; the vertical line through P meets the x axis at a, and the horizontal line through P meets the y axis at b. $\mathbf{R}^2$ is frequently called the *Cartesian plane*.

EXAMPLE 2.2 Let $A = \{1,2\}$ and $B = \{a,b,c\}$. Then

$$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$$
$$B \times A = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$$

Also $$A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$$

There are two things worth noting in the above example. First of all $A \times B \neq B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly, using $n(S)$ for the number of elements in a set S, we have

$$n(A \times B) = 6 = 2 \cdot 3 = n(A) \cdot n(B)$$

In fact, $n(A \times B) = n(A) \cdot n(B)$ for any finite sets A and B. This follows from the observation that, for an ordered pair (a,b) in $A \times B$, there are $n(A)$ possibilities for a, and for each of these there are $n(B)$ possibilities for b.

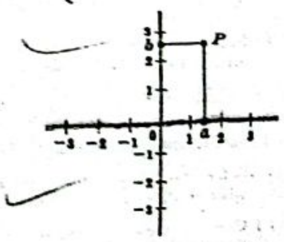


Fig. 2-1

The idea of a product of sets can be extended to any finite number of sets. For any sets $A_1, A_2, \ldots, A_n$, the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$ is called the *product* of the sets $A_1, \ldots, A_n$ and is denoted by

$$A_1 \times A_2 \times \cdots \times A_n \quad \text{or} \quad \prod_{i=1}^{n} A_i$$

Just as we write A^2 instead of $A \times A$, so we write A^n instead of $A \times A \times \cdots \times A$, where there are n factors all equal to A. For example, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the usual three-dimensional space.

RELATIONS

We begin with a definition.

Definition. Let A and B be sets. A *binary relation* or, simply, *relation* from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "a is *R-related* to b", written aRb.

(ii) $(a, b) \notin R$; we then say "a is not R-related to b", written $a\not R b$.

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation on A.

The *domain* of a relation R is the set of all first elements of the ordered pairs which belong to R, and the *range* of R is the set of second elements.

Although n-ary relations, which involve ordered n-tuples, are introduced in Section 2.12, the term relation shall mean binary relation unless otherwise stated or implied.

EXAMPLE 2.3

(a) Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry, \quad 1Rz, \quad 3Ry \quad \text{but} \quad 1\not R x, \quad 2\not R x, \quad 2\not R y, \quad 2\not R z, \quad 3\not R x, \quad 3\not R z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(b) Let $A = \{$eggs, milk, corn$\}$ and $B = \{$cows, goats, hens$\}$. We can define a relation R from A to B by $(a, b) \in R$ if a is produced by b. In other words,

$$R = \{(\text{eggs, hens}), (\text{milk, cows}), (\text{milk, goats})\}$$

With respect to this relation,

$$\text{eggs } R \text{ hens}, \quad \text{milk } R \text{ cows}, \quad \text{etc.}$$

(c) Suppose we say that two countries are *adjacent* if they have some part of their boundaries in common. Then "is adjacent to" is a relation R on the countries of the earth. Thus

$$(\text{Italy, Switzerland}) \in R \quad \text{but} \quad (\text{Canada, Mexico}) \notin R$$

(d) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of sets A and B, either $A \subseteq B$ or $A \not\subseteq B$.

(e) A familiar relation on the set $\mathbf{Z}$ of integers is "m divides n". A common notation for this relation is to write $m|n$ when m divides n. Thus $6|30$ but $7\nmid 25$.

(f) Consider the set L of lines in the plane. Perpendicularity, written $\perp$, is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to", written $\|$, is a relation on L since either $a \| b$ or $a \nparallel b$.

(g) Let A be any set. An important relation on A is that of *equality*,

$$\{(a, a) : a \in A\}$$

which is usually denoted by "=". This relation is also called the *identity* or *diagonal relation* on A and it will also be denoted by Δ_A or simply Δ.

(h) Let A be any set. Then $A \times A$ and $\varnothing$ are subsets of $A \times A$ and hence are relations on A called the *universal relation* and *empty relation*, respectively.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

For example, the inverse of the relation $R = \{(1, y), (1, z), (3, y)\}$ from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$ follows:

$$R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

2.4 PICTORIAL REPRESENTATIONS OF RELATIONS

First we consider a relation S on the set $\mathbf{R}$ of real numbers; that is, S is a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Since $\mathbf{R}^2$ can be represented by the set of points in the plane, we can picture S by emphasizing those points in the plane which belong to S. The pictorial representation of the relation is sometimes called the *graph* of the relation.

Frequently, the relation S consists of all ordered pairs of real numbers which satisfy some given equation

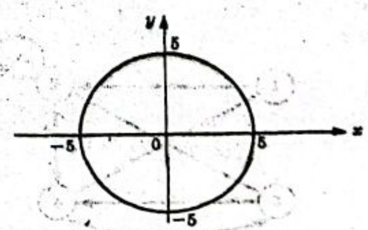
$$E(x, y) = 0$$

We usually identify the relation with the equation; that is, we speak of the relation $E(x, y) = 0$.

EXAMPLE 2.4 Consider the relation S defined by the equation

$$x^2 + y^2 = 25$$

That is, S consists of all ordered pairs (x, y) which satisfy the given equation. The graph of the equation is a circle having its center at the origin and radius 5. See Fig. 2-2.



$x^2 + y^2 = 25$

Fig. 2-2

Representations of Relations on Finite Sets

Suppose A and B are finite sets. The following are two ways of picturing a relation R from A to B.

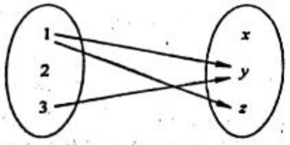
(i) Form a rectangular array whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation*.

(ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. This picture will be called the *arrow diagram* of the relation.

Figure 2-3 pictures the first relations in Example 2.3 by the above two ways.

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y), (1, z), (3, y)\}$

Fig. 2-3

Directed Graphs of Relations on Sets

There is another way of picturing a relation R when R is a relation from a finite set to itself. First we write down the elements of the set, and then we drawn an arrow from each element x to each element y whenever x is related to y. This diagram is called the *directed graph* of the relation. Figure 2-4, for example, shows the directed graph of the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$$

Observe that there is an arrow from 2 to itself, since 2 is related to 2 under R.

These directed graphs will be studied in detail as a separate subject in Chapter 8. We mention it here mainly for completeness.

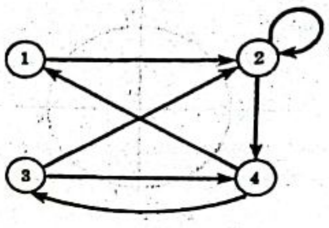


Fig. 2-4

2.5 COMPOSITION OF RELATIONS

Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } aRb \text{ and } bSc$$

That is,

$$R \circ S = \{(a, c)\text{: there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

Suppose R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself is always defined, and $R \circ R$ is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

Warning: Many texts denote the composition of relations R and S by $S \circ R$ rather than $R \circ S$. This is done in order to conform with the usual use of $g \circ f$ to denote the composition of f and g where f and g are functions. Thus the reader may have to adjust his notation when using this text as a supplement with another text. However, when a relation R is composed with itself, then the meaning of $R \circ R$ is unambiguous.

The arrow diagrams of relations give us a geometrical interpretation of the composition $R \circ S$ as seen in the following example.

Important

EXAMPLE 2.5 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let

$$R = \{(1, a), (2, d), (3, a)\ (3, b), (3, d)\} \quad \text{and} \quad S = \{(b, x), (b, z), (c, y), (d, z)\}$$

Consider the arrow diagrams of R and S as in Fig. 2-5. Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus

$$2(R \circ S)z \quad \text{since} \quad 2Rd \text{ and } dSz$$

Similarly there is a path from 3 to x and a path from 3 to z. Hence

$$3(R \circ S)x \quad \text{and} \quad 3(R \circ S)z$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

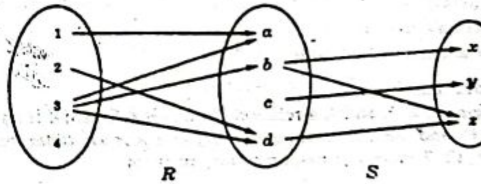


Fig. 2-5

Composition of Relations and Matrices

There is another way of finding $R \circ S$. Let M_R and M_S denote respectively the matrices of the relations R and S. Then

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Aultiplying M_R and M_S we obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

he nonzero entries in this matrix tell us which elements are related by $R \circ S$. Thus $M = M_R M_S$ and $'_{R \circ S}$ have the same nonzero entries.

Our first theorem tells us that the composition of relations is associative.

ieorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

e prove this theorem in Problem 2.11.

5 TYPES OF RELATIONS

Consider a given set A. This section discusses a number of important types of relations which are :fined on A.

eflexive Relations

A relation R on a set A is *reflexive* if aRa for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus : is not reflexive if there exists an $a \in A$ such that $(a, a) \notin R$.

:XAMPLE 2.6 Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
$R_3 = \{(1,3), (2,1)\}$
$R_4 = \varnothing$, the empty relation
$R_5 = A \times A$, the universal relation

Determine which of the relations are reflexive.

Since A contain the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs (1, 1), (2, 2), (3, 3), and (4, 4). Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, (2, 2) does not belong to any of them.

EXAMPLE 2.7 Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers
(2) Set inclusion $\subseteq$ on a collection C of sets
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.
(5) Relation $|$ of divisibility on the set $\mathbf{N}$ of positive integers. (Recall $x|y$ if there exists z such that $xz = y$.)

Determine which of the relations are reflexive.

The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every integer x in $\mathbf{Z}$, $A \subseteq A$ for any set A in C, and $n|n$ for every positive integer n in $\mathbf{N}$.

Symmetric and Antisymmetric Relations

A relation R on a set A is *symmetric* if whenever aRb then bRa, that is, if whenever $(a, b) \in R$ then $(b, a) \in R$. Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

EXAMPLE 2.8

(*a*) Determine which of the relations in Example 2.6 are symmetric.

R_1 is not symmetric since $(1, 2) \in R_1$ but $(2, 1) \notin R_1$. R_3 is not symmetric since $(1, 3) \in R_3$ but $(3, 1) \notin R_3$. The other relations are symmetric.

(*b*) Determine which of the relations in Example 2.7 are symmetric.

The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a. Also, $\parallel$ is symmetric since if line a is parallel to line b then b is parallel to a. The other relations are not symmetric. For example, $3 \leq 4$ but $4 \nleq 3$; $\{1, 2\} \subseteq \{1, 2, 3\}$ but $\{1, 2, 3\} \nsubseteq \{1, 2\}$, and $2|6$ but $6 \nmid 2$.

A relation R on a set A is *antisymmetric* if whenever aRb and bRa then $a = b$, that is, if whenever $(a, b), (b, a) \in R$ then $a = b$. Thus R is not antisymmetric if there exist $a, b \in A$ such that (a, b) and (b, a) belong to R, but $a \neq b$.

EXAMPLE 2.9

(*a*) Determine which of the relations in Example 2.6 are antisymmetric.

R_2 is not antisymmetric since (1, 2) and (2, 1) belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_5 is not antisymmetric. All the other relations are antisymmetric.

(*b*) Determine which of the relations in Example 2.7 are antisymmetric.

The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$. Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on $\mathbf{N}$ is antisymmetric since whenever $m|n$ and $n|m$ then $m = n$. (Note that divisibility on $\mathbf{Z}$ is not antisymmetric since $3|-3$ and $-3|3$ but $3 \neq -3$.) The relation $\perp$ is not antisymmetric since we can have distinct lines a and b such that $a \perp b$ and $b \perp a$. Similarly, $\parallel$ is not antisymmetric.

Remark: The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

Transitive Relations

A relation R on a set A is *transitive* if whenever aRb and bRc then aRc, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. Thus R is not transitive if there exist $a, b, c \in A$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

EXAMPLE 2.10

(*a*) Determine which of the relations in Example 2.6 are transitive.

The relation R_3 is not transitive since $(2, 1), (1, 3) \in R_3$ but $(2, 3) \notin R_3$. All the other relations are transitive.

(*b*) Determine which of the relations in Example 2.7 are transitive.

The relations $\leq$, $\subseteq$, and $|$ are transitive. That is: (i) If $a \leq b$ and $b \leq c$, then $a \leq c$. (ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (iii) If $a|b$ and $b|c$, then $a|c$.

On the other hand the relation $\perp$ is not transitive. If $a \perp b$ and $b \perp c$, then it is not true that $a \perp c$. Since no line is parallel to itself, we can have $a \parallel b$ and $b \parallel a$, but $a \nparallel a$. Thus $\parallel$ is not transitive. (We note that the relation "is parallel or equal to" is a transitive relation on the set L of lines in the plane.)

The property of transitivity can also be expressed in terms of the composition of relations. For a ation R on A we define

$$R^2 = R \circ R \quad \text{and, more generally,} \quad R^n = R^{n-1} \circ R$$

en we have the following result.

eorem 2.2: A relation R is transitive if and only if $R^n \subseteq R$ for $n \geq 1$.

CLOSURE PROPERTIES

Consider a given set A and the collection of all relations on A. Let P be a property of such relations, h as being symmetric or being transitive. A relation with property P will be called a P-relation. The P-*ure* of an arbitrary relation R on A, written $P(R)$, is a P-relation such that

$$R \subseteq P(R) \subseteq S$$

every P-relation S containing R. We will write

$$\text{reflexive}(R), \qquad \text{symmetric}(R), \qquad \text{and transitive}(R)$$

he reflexive, symmetric, and transitive closures of R.

Generally speaking, $P(R)$ need not exist. However, there is a general situation where $P(R)$ will ys exist. Suppose P is a property such that there is at least one P-relation containing R and that ntersection of any P-relations is again a P-relation. Then one can prove (Problem 2.16) that

$$P(R) = \cap(S: S \text{ is a } P\text{-relation and } R \subseteq S)$$

s one can obtain $P(R)$ from the "top-down", that is, as the intersection of relations. However, one lly wants to find $P(R)$ from the "bottom-up", that is, by adjoining elements to R to obtain $P(R)$. we do below.

exive and Symmetric Closures

The next theorem tells us how to easily obtain the reflexive and symmetric closures of a relation. $\Delta_A = \{(a, a): a \in A\}$ is the *diagonal* or *equality* relation on A.

orem 2.3: Let R be a relation on a set A. Then:

(i) $R \cup \Delta_A$ is the reflexive closure of R.

(ii) $R \cup R^{-1}$ is the symmetric closure of R.

In other words, reflexive(R) is obtained by simply adding to R those elements (a, a) in the diagonal ch do not already belong to R, and symmetric(R) is obtained by adding to R all pairs (b, a) whenever) belongs to R.

MPLE 2.11

Consider the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3)\}$$

Then

$$\text{reflexive}(R) = R \cup \{(2,2), (4,4)\} \quad \text{and} \quad \text{symmetric}(R) = R \cup \{(4,2), (3,4)\}$$

(b) Consider the relation $<$ (less than) on the set $\mathbf{N}$ of positive integers. Then

$$\text{reflexive}(<) = < \cup \Delta = \leq = \{(a, b): a \leq b\}$$
$$\text{symmetric}(<) = < \cup > = \{(a, b): a \neq b\}$$

Transitive Closure

Let R be a relation on a set A. Recall that $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$. We define

$$R^* = \bigcup_{i=1}^{\infty} R^i$$

The following theorem applies.

Theorem 2.4: R^* is the transitive closure of a relation R.

Suppose A is a finite set with n elements. Then we show in Chapter 8 on directed graphs that

$$R^* = R \cup R^2 \cup \cdots \cup R^n$$

This gives us the following result.

Theorem 2.5: Let R be a relation on a set A with n elements. Then

$$\text{transitive}(R) = R \cup R^2 \cup \cdots \cup R^n$$

Finding transitive(R) can take a lot of time when A has a large number of elements. An efficient way for doing this will be described in Chapter 8. Here we give a simple example where A has only three elements.

EXAMPLE 2.12 Consider the following relation R on $A = \{1, 2, 3\}$:

$$R = \{(1,2), (2,3), (3,3)\}$$

Then

$$R^2 = R \circ R = \{(1,3), (2,3)(3,3)\} \quad \text{and} \quad R^3 = R^2 \circ R = \{(1,3), (2,3), (3,3)\}$$

Accordingly,

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(1,2), (2,3), (3,3), (1,3)\}$$

2.8 EQUIVALENCE RELATIONS

Consider a nonempty set S. A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, aRa.

(2) If aRb, then bRa.

(3) If aRb and bRc, then aRc.

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike". In fact, the relation "=" of equality on any set S is an equivalence relation; that is:

(1) $a = a$ for every $a \in S$.

(2) If $a = b$, then $b = a$.

(3) If $a = b$ and $b = c$, then $a = c$.

Other equivalence relations follow.

(AMPLE 2.13

Consider the set L of lines and the set T of triangles in the Euclidean plane. The relation "is parallel to or identical to" is an equivalence relation on L, and congruence and similarity are equivalence relations on T.

The classification of animals by species, that is, the relation "is of the same species as", is an equivalence relation on the set of animals.

The relation $\subseteq$ of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

Let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m*, written

$$a \equiv b \pmod m$$

if m divides $a - b$. For example, for $m = 4$ we have $11 \equiv 3 \pmod 4$ since 4 divides $11 - 3$, and $22 \equiv 6 \pmod 4$ since 4 divides $22 - 6$. This relation of congruence modulo m is an equivalence relation.

valence Relations and Partitions

This subsection explores the relationship between equivalence relations and partitions on a non-y set S. Recall first that a partition P of S is a collection $\{A_i\}$ of nonempty subsets of S with the wing two properties:

(1) Each $a \in S$ belongs to some A_i.
(2) If $A_i \neq A_j$, then $A_i \cap A_j = \varnothing$.

her words, a partition P of S is a subdivision of S into disjoint nonempty sets. (See Section 1.9.)
Suppose R is an equivalence relation on a set S. For each a in S, let $[a]$ denote the set of elements of which a is related under R; that is,

$$[a] = \{x: (a, x) \in R\}$$

call $[a]$ the *equivalence class* of a in S; any $b \in [a]$ is called a *representative* of the equivalence class. The collection of all equivalence classes of elements of S under an equivalence relation R is denoted S/R, that is,

$$S/R = \{[a]: a \in S\}$$

called the *quotient* set of S by R. The fundamental property of a quotient set is contained in the owing theorem.

eorem 2.6: Let R be an equivalence relation on a set S. Then the quotient set S/R is a partition of S. Specifically:

(i) For each a in S, we have $a \in [a]$.
(ii) $[a] = [b]$ if and only if $(a, b) \in R$.
(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

Conversely, given a partition $\{A_i\}$ of the set S, there is an equivalence relation R on S such that the sets A_i are the equivalence classes.

This important theorem will be proved in Problem 2.21.

(AMPLE 2.14

Consider the following relation R on $S = \{1, 2, 3\}$:

$$R = \{(1,1), (1,2), (2,1), (3,3)\}$$

One can show that R is reflexive, symmetric, and transitive, that is, that R is an equivalence relation. Under the relation R,

$$[1] = \{1, 2\}, \qquad [2] = \{1, 2\}, \qquad [3] = \{3\}$$

Observe that $[1] = [2]$ and that $S/R = \{[1], [3]\}$ is a partition of S. One can choose either $\{1, 3\}$ or $\{2, 3\}$ as a set of representatives of the equivalence classes.

(*b*) Let R_5 be the relation on the set $\mathbf{Z}$ of integers defined by

$$x \equiv y \pmod 5$$

which reads "x is congruent to y modulo 5" and which means that the difference $x - y$ is divisible by 5. Then R_5 is an equivalence relation on $\mathbf{Z}$. There are exactly five equivalence classes in the quotient set $\mathbf{Z}/R_5$ as follows:

$$A_0 = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$$
$$A_1 = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$$
$$A_2 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$$
$$A_3 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$
$$A_4 = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$$

Observe that any integer x, which can be uniquely expressed in the form $x = 5q + r$ where $0 \le r < 5$, is a member of the equivalence class A_r where r is the remainder. As expected, the equivalence classes are disjoint and

$$\mathbf{Z} = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$$

Usually one chooses $\{0, 1, 2, 3, 4\}$ or $\{-2, -1, 0, 1, 2\}$ as a set of representatives of the equivalence classes.

2.9 PARTIAL ORDERING RELATIONS

This section defines another important class of relations. A relation R on a set S is called a *partial ordering* or a *partial order* if R is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set* or *poset*. Partially ordered sets will be studied in more detail in Chapter 14, so here we simply give some examples.

EXAMPLE 2.15

(*a*) The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three desired properties. That is,

(1) $A \subseteq A$ for any set A.
(2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
(3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(*b*) The relation $\le$ on the set $\mathbf{R}$ of real numbers is reflexive, antisymmetric, and transitive. Thus $\le$ is a partial ordering.

(*c*) The relation "a divides b" is a partial ordering on the set $\mathbf{N}$ of positive integers. However, "a divides b" is not a partial ordering on the set $\mathbf{Z}$ of integers since $a|b$ and $b|a$ does not imply $a = b$. For example, $3|-3$ and $-3|3$ but $3 \neq -3$.

2.10 *n*-ARY RELATIONS

All the relations discussed above were binary relations. By an *n-ary relation*, we mean a set of ordered n-tuples. For any set S, a subset of the product set S^n is called an n-ary relation on S. In particular, a subset of S^3 is called a *ternary relation* on S.

2.16

L be a line in the plane. Then "betweenness" is a ternary relation R on the points of L: that is, $(a,b,c) \in R$ b lies between a and c on L.

The equation $x^2+y^2+z^2=1$ determines a ternary relation T on the set $\mathbf{R}$ of real numbers. That is, a triple (x,y,z) belongs to T if (x,y,z) satisfies the equation, which means (x,y,z) is the coordinates of a point in $\mathbf{R}^3$ on the sphere S with radius 1 and center at the origin $\mathbf{O}=(0,0,0)$.

Solved Problems

ORDERED PAIRS AND PRODUCT SETS

Given: $A=\{1,2,3\}$ and $B=\{a,b\}$. Find: (a) $A\times B$; (b) $B\times A$; (c) $B\times B$.

(a) $A\times B$ consists of all ordered pairs (x,y) where $x\in A$ and $y\in B$. Hence

$$A\times B=\{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

(b) $B\times A$ consists of all ordered pairs (y,x) where $y\in B$ and $x\in A$. Hence

$$B\times A=\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$

(c) $B\times B$ consists of all ordered pairs (x,y) where $x,y\in B$. Hence

$$B\times B=\{(a,a),(a,b),(b,a),(b,b)\}$$

As expected, the number of elements in the product set is equal to the product of the numbers of the elements in each set.

Given: $A=\{1,2\}$, $B=\{x,y,z\}$, and $C=\{3,4\}$. Find: $A\times B\times C$.

$A\times B\times C$ consists of all ordered triplets (a,b,c) where $a\in A$, $b\in B$, $c\in C$. These elements of $A\times B\times C$ can be systematically obtained by a so-called tree diagram (Fig. 2-6). The elements of $A\times B\times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

Observe that $n(A)=2$, $n(B)=3$, and $n(C)=2$ and, as expected,

$$n(A\times B\times C)=12=n(A)\cdot n(B)\cdot n(C)$$

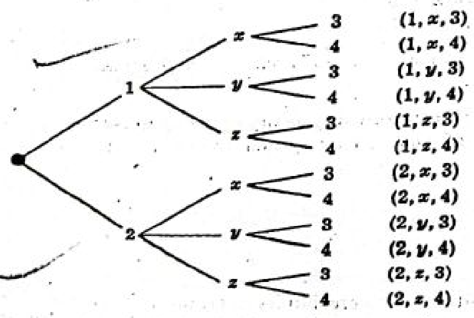


Fig. 2-6

2.3. Let $A=\{1,2\}$, $B=\{a,b,c\}$ and $C=\{c,d\}$. Find: $(A\times B)\cap(A\times C)$ and $(B\cap C)$.

We have

$$A\times B=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$$
$$A\times C=\{(1,c),(1,d),(2,c),(2,d)\}$$

Hence

$$(A\times B)\cap(A\times C)=\{(1,c),(2,c)\}$$

Since $B\cap C=\{c\}$,

$$A\times(B\cap C)=\{(1,c),(2,c)\}$$

Observe that $(A\times B)\cap(A\times C)=A\times(B\cap C)$. This is true for any sets A, B and C (see Problem 2.4).

2.4. Prove $(A\times B)\cap(A\times C)=A\times(B\cap C)$.

$$\begin{aligned}(A\times B)\cap(A\times C)&=\{(x,y)\colon (x,y)\in A\times B \text{ and } (x,y)\in A\times C\}\\&=\{(x,y)\colon x\in A, y\in B \text{ and } x\in A, y\in C\}\\&=\{(x,y)\colon x\in A, y\in B\cap C\} = A\times(B\cap C)\end{aligned}$$

2.5. Find x and y given $(2x, x+y)=(6,2)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations

$$2x=6 \quad\text{and}\quad x+y=2$$

from which we derive the answers $x=3$ and $y=-1$.

RELATIONS AND THEIR GRAPHS

2.6. Find the number of relations from $A=\{a,b,c\}$ to $B=\{1,2\}$.

There are $3(2)=6$ elements in $A\times B$, and hence there are $m=2^6=64$ subsets of $A\times B$. Thus there are $m=64$ relations from A to B.

2.7. Given $A=\{1,2,3,4\}$ and $B=\{x,y,z\}$. Let R be the following relation from A to B:

$$R=\{(1,y),(1,z),(3,y),(4,x),(4,z)\}$$

(a) Determine the matrix of the relation.
(b) Draw the arrow diagram of R.
(c) Find the inverse relation R^{-1} of R.
(d) Determine the domain and range of R.

(a) See Fig. 2-7(a). Observe that the rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that the entry in the matrix corresponding to $a\in A$ and $b\in B$ is 1 if a is related to b and 0 otherwise.

(b) See Fig. 2-7(b). Observe that there is an arrow from $a\in A$ to $b\in B$ iff a is related to b, i.e., iff $(a,b)\in R$.

(c) Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1}=\{(y,1),(z,1),(y,3),(x,4),(z,4)\}$$

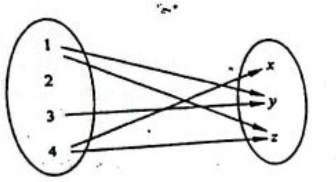
Observe that by reversing the arrows in Fig. 2.7(b) we obtain the arrow diagram of R^{-1}.

(d) The domain of R, Dom(R), consists of the first elements of the ordered pairs of R, and the range of R, Ran(R), consists of the second elements. Thus,

$$\text{Dom}(R) = \{1, 3, 4\} \quad \text{and} \quad \text{Ran}(R) = \{x, y, z\}$$

$$\begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a)



(b)

Fig. 2-7

2.8. Let $A = \{1, 2, 3, 4, 6\}$, and let R be the relation on A defined by "x divides y", written $x|y$. (Note $x|y$ iff there exists an integer z such that $xz = y$.)

(a) Write R as a set of ordered pairs.

(b) Draw its directed graph.

(c) Find the inverse relation R^{-1} of R. Can R^{-1} be described in words?

(a) Find those numbers in A divisible by 1, 2, 3, 4, and then 6. These are:

$$1|1, \quad 1|2, \quad 1|3, \quad 1|4, \quad 1|6, \quad 2|2, \quad 2|4, \quad 2|6, \quad 3|3, \quad 3|6, \quad 4|4, \quad 6|6$$

Hence

$$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\}$$

(b) See Fig. 2-8.

(c) Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)\}$$

R^{-1} can be described by the statement "x is a multiple of y".

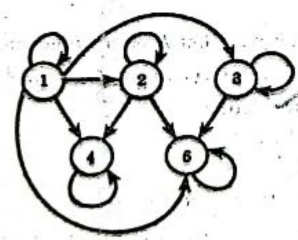


Fig. 2-8

2.9. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R and S from A to B and from B to C, respectively.

$$R = \{(1,b), (2,a), (2,c)\} \quad \text{and} \quad S = \{(a,y), (b,x), (c,y), (c,z)\}$$

$R = A \times B$
$S = B \times C$

(a) Find the composition relation $R \circ S$.

(b) Find the matrices M_R, M_S, and $M_{R \circ S}$ of the respective relations R, S, and $R \circ S$, and compare $M_{R \circ S}$ to the product $M_R M_S$.

(a) Draw the arrow diagram of the relations R and S as in Fig. 2-9. Observe that 1 in A is "connected" to x in C by the path $1 \to b \to x$; hence $(1, x)$ belongs to $R \circ S$. Similarly, $(2, y)$ and $(2, z)$ belong to $R \circ S$. We have

$$R \circ S = \{(1,x), (2,y), (2,z)\}$$

(See Example 2.5.)

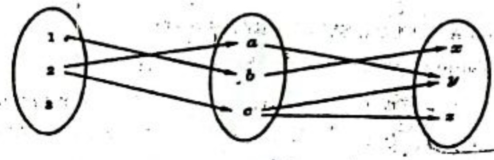


Fig. 2-9

(b) The matrices of M_R, M_S, and $M_{R \circ S}$ follow:

$$M_R = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad M_S = \begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad M_{R \circ S} = \begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Multiplying M_R and M_S we obtain

$$M_R M_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Observe that $M_{R \circ S}$ and $M_R M_S$ have the same zero entries.

2.10. Let R and S be the following relations on $A = \{1, 2, 3\}$:

$$R = \{(1,1), (1,2), (2,3), (3,1), (3,3)\}, \qquad S = \{(1,2), (1,3), (2,1), (3,3)\}$$

Find (a) $R \cap S$, $R \cup S$, R^c; (b) $R \circ S$; (c) $S^2 = S \circ S$.

(a) Treat R and S simply as sets, and take the usual intersection and union. For R^c, use the fact that $A \times A$ is the universal relation on A.

$$R \cap S = \{(1,2), (3,3)\}$$
$$R \cup S = \{(1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,3)\}$$
$$R^c = \{(1,3), (2,1), (2,2), (3,2)\}$$

(b) For each pair $(a, b) \in R$, find all pairs $(b, c) \in S$. Then $(a, c) \in R \circ S$. For example, $(1,1) \in R$ and $(1,2), (1,3) \in S$; hence $(1,2)$ and $(1,3)$ belong to $R \circ S$. Thus,

$$R \circ S = \{(1,2), (1,3), (1,1), (2,3), (3,2), (3,3)\}$$

(c) Following the algorithm in (b), we get

$$S^2 = S \circ S = \{(1,1), (1,3), (2,2), (2,3), (3,3)\}$$

4 - A

11. Prove Theo...
from B to C...
We need...
Suppose...
Since...

2.11. Prove Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C and T is a relation from C to D. Then $(R \circ S) \circ T = R \circ (S \circ T)$.

We need to show that each ordered pair in $(R \circ S) \circ T$ belongs to $R \circ (S \circ T)$, and vice versa.

Suppose (a,d) belongs to $(R \circ S) \circ T$. Then there exists a c in C such that $(a,c) \in R \circ S$ and $(c,d) \in T$. Since $(a,c) \in R \circ S$, there exists a b in B such that $(a,b) \in R$ and $(b,c) \in S$. Since $(b,c) \in S$ and $(c,d) \in T$, we have $(b,d) \in S \circ T$; and since $(a,b) \in R$ and $(b,d) \in S \circ T$, we have $(a,d) \in R \circ (S \circ T)$. Therefore, $(R \circ S) \circ T \subseteq R \circ (S \circ T)$. Similarly $R \circ (S \circ T) \subseteq (R \circ S) \circ T$. Both inclusion relations prove $(R \circ S) \circ T = R \circ (S \circ T)$.

TYPES OF RELATIONS AND CLOSURE PROPERTIES

2.12. Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$R = \{(1,1), (1,2), (1,3), (3,3)\} \qquad \varnothing = \text{empty relation}$$
$$S = \{(1,1), (1,2), (2,1), (2,2), (3,3)\} \qquad A \times A = \text{universal relation}$$
$$T = \{(1,1), (1,2), (2,2), (2,3)\}$$

Determine whether or not each of the above relations on A is: (*a*) reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

(*a*) R is not reflexive since $2 \in A$ but $(2,2) \notin R$. T is not reflexive since $(3,3) \notin T$ and, similarly, $\varnothing$ is not reflexive. S and $A \times A$ are reflexive.

(*b*) R is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$, and similarly T is not symmetric. S, $\varnothing$, and $A \times A$ are symmetric.

(*c*) T is not transitive since $(1,2)$ and $(2,3)$ belong to T, but $(1,3)$ does not belong to T. The other four relations are transitive.

(*d*) S is not antisymmetric since $1 \neq 2$, and $(1,2)$ and $(2,1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.13. Given $A = \{1, 2, 3, 4\}$. Consider the following relation in A:

$$R = \{(1,1), (2,2), (2,3), (3,2), (4,2), (4,4)\}$$

(*a*) Draw its directed graph.
(*b*) Is R (i) reflexive, (ii) symmetric, (iii) transitive, or (iv) antisymmetric?
(*c*) Find $R^2 = R \circ R$.

(*a*) See Fig. 2-10.

(*b*) (i) R is not reflexive because $3 \in A$ but $3 \not R 3$, i.e., $(3,3) \notin R$.
(ii) R is not symmetric because $4R2$ but $2 \not R 4$, i.e., $(4,2) \in R$ but $(2,4) \notin R$.
(iii) R is not transitive because $4R2$ and $2R3$ but $4 \not R 3$, i.e., $(4,2) \in R$ and $(2,3) \in R$ but $(4,3) \notin R$.
(iv) R is not antisymmetric because $2R3$ and $3R2$ but $2 \neq 3$.

(*c*) For each pair $(a,b) \in R$, find all $(b,c) \in R$. Since $(a,c) \in R^2$,

$$R^2 = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,2), (4,3), (4,4)\}$$

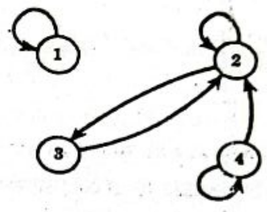


Fig. 2-10

2.14. Give examples of relations R on $A = \{1, 2, 3\}$ having the stated property.

(*a*) R is both symmetric and antisymmetric.
(*b*) R is neither symmetric nor antisymmetric.
(*c*) R is transitive but $R \cup R^{-1}$ is not transitive.

There are several possible examples for each answer. One possible set of examples follows:

(*a*) $R = \{(1,1), (2,2)\}$.
(*b*) $R = \{(1,2), (2,1), (2,3)\}$.
(*c*) $R = \{1,2)\}$.

2.15. Suppose C is a collection of relations S on a set A and let T be the intersection of the relation S, that is, $T = \cap(S: S \in C)$. Prove:

(*a*) If every S is symmetric, then T is symmetric.
(*b*) If every S is transitive, then T is transitive.

(*a*) Suppose $(a,b) \in T$. Then $(a,b) \in S$ for every S. Since each S is symmetric, $(b,a) \in S$ for every S. Hence $(b,a) \in T$ and T is symmetric.

(*b*) Suppose (a,b) and (b,c) belong to T. Then (a,b) and (b,c) belong to S for every S. Since each S is transitive, (a,c) belongs to S for every S. Hence, $(a,c) \in T$ and T is transitive.

2.16. Let R be a relation on a set A, and let P be a property of relations, such as, symmetry and transitivity. Then P will be called R-closable if P satisfies the following two conditions:

(1) There is a P-relation S containing R.
(2) The intersection of P-relations is a P-relation.

(*a*) Show that symmetry and transitivity are R-closable for any relation R.
(*b*) Suppose P is R-closable. Then $P(R)$, the P-closure of R, is the intersection of all P-relations S containing R, that is,

$$P(R) = \cap(S: S \text{ is a } P\text{-relation and } R \subseteq S)$$

(*a*) The universal relation $A \times A$ is symmetric and transitive and $A \times A$ contains any relation R on A. Thus (1) is satisfied. By Problem 2.15, symmetry and transitivity satisfy (2). Thus symmetry and transivity are R-closable for any relation R.

(*b*) Let $T = \cap(S: S \text{ is a } P\text{-relation and } R \subseteq S)$. Since P is R-closable, T is nonempty by (1) and T is a P-relation by (2). Since each relation S contains R, the intersection T contains R. Thus, T is a P-relation containing R. By definition, $P(R)$ is the smallest P-relation containing R; hence $P(R) \subseteq T$. On the

2.11. Prove Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C and T is a relation from C to D. Then $(R \circ S) \circ T = R \circ (S \circ T)$.

We need to show that each ordered pair in $(R \circ S) \circ T$ belongs to $R \circ (S \circ T)$, and vice versa. Suppose (a, d) belongs to $(R \circ S) \circ T$. Then there exists a c in C such that $(a,c) \in R \circ S$ and $(c,d) \in T$. Since $(a,c) \in R \circ S$, there exists a b in B such that $(a,b) \in R$ and $(b,c) \in S$. Since $(b,c) \in S$ and $(c,d) \in T$, we have $(b,d) \in S \circ T$; and since $(a,b) \in R$ and $(b,d) \in S \circ T$, we have $(a,d) \in R \circ (S \circ T)$. Therefore, $(R \circ S) \circ T \subseteq R \circ (S \circ T)$. Similarly $R \circ (S \circ T) \subseteq (R \circ S) \circ T$. Both inclusion relations prove $(R \circ S) \circ T = R \circ (S \circ T)$.

TYPES OF RELATIONS AND CLOSURE PROPERTIES

2.12. Consider the following five relations on the set $A = \{1, 2, 3\}$:

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$$T = \{(1,1), (1,2), (2,2), (2,3)\}$$

Determine whether or not each of the above relations on A is: (*a*) reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

(*a*) R is not reflexive since $2 \in A$ but $(2,2) \notin R$. T is not reflexive since $(3,3) \notin T$ and, similarly, $\varnothing$ is not reflexive. S and $A \times A$ are reflexive.

(*b*) R is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$, and similarly T is not symmetric. S, $\varnothing$, and $A \times A$ are symmetric.

(*c*) T is not transitive since $(1,2)$ and $(2,3)$ belong to T, but $(1,3)$ does not belong to T. The other four relations are transitive.

(*d*) S is not antisymmetric since $1 \neq 2$, and $(1,2)$ and $(2,1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.13. Given $A = \{1, 2, 3, 4\}$. Consider the following relation in A:

$$R = \{(1,1), (2,2), (2,3), (3,2), (4,2), (4,4)\}$$

(*a*) Draw its directed graph.
(*b*) Is R (i) reflexive, (ii) symmetric, (iii) transitive, or (iv) antisymmetric?
(*c*) Find $R^2 = R \circ R$.

(*a*) See Fig. 2-10.
(*b*) (i) R is not reflexive because $3 \in A$ but $3 \not R 3$, i.e., $(3,3) \notin R$.
(ii) R is not symmetric because $4R2$ but $2 \not R 4$, i.e., $(4,2) \in R$ but $(2,4) \notin R$.
(iii) R is not transitive because $4R2$ and $2R3$ but $4 \not R 3$, i.e., $(4,2) \in R$ and $(2,3) \in R$ but $(4,3) \notin R$.
(iv) R is not antisymmetric because $2R3$ and $3R2$ but $2 \neq 3$.
(*c*) For each pair $(a,b) \in R$, find all $(b,c) \in R$. Since $(a,c) \in R^2$,

$$R^2 = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,2), (4,3), (4,4)\}$$

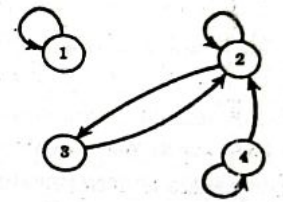


Fig. 2-10

2.14. Give examples of relations R on $A = \{1, 2, 3\}$ having the stated property.

(*a*) R is both symmetric and antisymmetric.
(*b*) R is neither symmetric nor antisymmetric.
(*c*) R is transitive but $R \cup R^{-1}$ is not transitive.

There are several possible examples for each answer. One possible set of examples follows:

(*a*) $R = \{(1,1), (2,2)\}$.
(*b*) $R = \{(1,2), (2,1), (2,3)\}$.
(*c*) $R = \{1,2)\}$.

2.15. Suppose C is a collection of relations S on a set A and let T be the intersection of the relation S, that is, $T = \cap(S: S \in C)$. Prove:

(*a*) If every S is symmetric, then T is symmetric.
(*b*) If every S is transitive, then T is transitive.

(*a*) Suppose $(a,b) \in T$. Then $(a,b) \in S$ for every S. Since each S is symmetric, $(b,a) \in S$ for every S. Hence $(b,a) \in T$ and T is symmetric.

(*b*) Suppose (a,b) and (b,c) belong to T. Then (a,b) and (b,c) belong to S for every S. Since each S is transitive, (a,c) belongs to S for every S. Hence, $(a,c) \in T$ and T is transitive.

2.16. Let R be a relation on a set A, and let P be a property of relations, such as, symmetry and transitivity. Then P will be called R-closable if P satisfies the following two conditions:

(1) There is a P-relation S containing R.
(2) The intersection of P-relations is a P-relation.

(*a*) Show that symmetry and transitivity are R-closable for any relation R.
(*b*) Suppose P is R-closable. Then $P(R)$, the P-closure of R, is the intersection of all P-relations S containing R, that is,

$$P(R) = \cap(S: S \text{ is a } P\text{-relation and } R \subseteq S)$$

(*a*) The universal relation $A \times A$ is symmetric and transitive and $A \times A$ contains any relation R on A. Thus (1) is satisfied. By Problem 2.15, symmetry and transitivity satisfy (2). Thus symmetry and transivity are R-closable for any relation R.

(*b*) Let $T = \cap(S: S$ is a P-relation and $R \subseteq S)$. Since P is R-closable, T is nonempty by (1) and T is a P-relation by (2). Since each relation S contains R, the intersection T contains R. Thus, T is a P-relation containing R. By definition, $P(R)$ is the smallest P-relation containing R; hence $P(R) \subseteq T$. On the other hand, $P(R)$ is one of the sets S defining T, that is, $P(R)$ is a P-relation and $R \subseteq P(R)$. Therefore, $T \subseteq P(R)$. Accordingly, $P(R) = T$.

2.17. Consider a set $A = \{a, b, c\}$ and the relation R on A defined by

$$R = \{(a, a), (a, b), (b, c), (c, c)\}$$

Find (*a*) reflexive(R); (*b*) symmetric(R); and (*c*) transitive(R).

(*a*) The reflexive closure on R is obtained by adding all diagonal pairs of $A \times A$ to R which are not currently in R. Hence,

$$\text{reflexive}(R) = R \cup \{(b, b)\} = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$$

(*b*) The symmetric closure on R is obtained by adding all the pairs in R^{-1} to R which are not currently in R. Hence,

$$\text{symmetric}(R) = R \cup \{(b, a), (c, b)\} = \{(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)\}$$

(*c*) The transitive closure on R, since A has three elements, is obtained by taking the union of R with $R^2 = R \circ R$ and $R^3 = R \circ R \circ R$. Note that

$$R^2 = R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$
$$R^3 = R \circ R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

Hence

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

EQUIVALENCE RELATIONS AND PARTITIONS

2.18. Consider the $\mathbf{Z}$ of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written

$$x \equiv y \pmod{m}$$

if $x - y$ is divisible by m. Show that this defines an equivalence relation on $\mathbf{Z}$.

We must show that the relation is reflexive, symmetric, and transitive.

(i) For any x in $\mathbf{Z}$ we have $x \equiv x \pmod{m}$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

(ii) Suppose $x \equiv y \pmod{m}$, so $x - y$ is divisible by m. Then $-(x - y) = y - x$ is also divisible by m, so $y \equiv x \pmod{m}$. Thus the relation is symmetric.

(iii) Now suppose $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod{m}$. Thus the relation is transitive.

Accordingly, the relation of congruence modulo m on $\mathbf{Z}$ is an equivalence relation.

2.19. Let A be a set of nonzero integers and let $\approx$ be the relation on $A \times A$ defined by

$$(a, b) \approx (c, d) \quad \text{whenever} \quad ad = bc$$

Prove that $\approx$ is an equivalence relation.

We must show that $\approx$ is reflexive, symmetric, and transitive.

(i) *Reflexivity*: We have $(a, b) \approx (a, b)$ since $ab = ba$. Hence $\approx$ is reflexive.

(ii) *Symmetry*: Suppose $(a, b) \approx (c, d)$. Then $ad = bc$. Accordingly, $cb = da$ and hence $(c, d) \approx (a, b)$. Thus, $\approx$ is symmetric.

(iii) *Transitivity*: Suppose $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f)$. Then $ad = bc$ and $cf = de$. Multiplying corresponding terms of the equations gives $(ad)(cf) = (bc)(de)$. Canceling $c \neq 0$ and $d \neq 0$ from both sides of the equation yields $af = be$, and hence $(a, b) \approx (e, f)$. Thus $\approx$ is transitive. Accordingly, $\approx$ is an equivalence relation.

2.20. Let R be the following equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$:

$$R = \{(1,1), (1,5), (2,2), (2,3), (2,6), (3,2), (3,3), (3,6), (4,4), (5,1), (5,5), (6,2), (6,3), (6,6)\}$$

Find the partition of A *induced by* R, i.e., find the equivalence classes of R.

Those elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

We pick an element which does not belong to [1], say 2. Those elements related to 2 are 2, 3 and 6, hence

$$[2] = \{2, 3, 6\}$$

The only element which does not belong to [1] or [2] is 4. The only element related to 4 is 4. Thus

$$[4] = \{4\}$$

Accordingly,

$$[\{1, 5\}, \{2, 3, 6\}, \{4\}]$$

is the partition of A induced by R.

2.21. Prove Theorem 2.6: Let R be an equivalence relation in a set A. Then the quotient set A/R is a partition of A. Specifically,

(i) $a \in [a]$, for every $a \in A$.

(ii) $[a] = [b]$ if and only if $(a, b) \in R$.

(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

Proof of (i): Since R is reflexive, $(a, a) \in R$ for every $a \in A$ and therefore $a \in [a]$.

Proof of (ii): Suppose $(a, b) \in R$. We want to show that $[a] = [b]$. Let $x \in [b]$; then $(b, x) \in R$. But by hypothesis $(a, b) \in R$ and so, by transitivity, $(a, x) \in R$. Accordingly $x \in [a]$. Thus $[b] \subseteq [a]$. To prove that $[a] \subseteq [b]$, we observe that $(a, b) \in R$ implies, by symmetry, that $(b, a) \in R$. Then, by a similar argument, we obtain $[a] \subseteq [b]$. Consequently, $[a] = [b]$.

On the other hand, if $[a] = [b]$, then, by (i), $b \in [b] = [a]$; hence $(a, b) \in R$.

Proof of (iii): We prove the equivalent contrapositive statement:

$$\text{If } [a] \cap [b] \neq \varnothing \quad \text{then} \quad [a] = [b]$$

If $[a] \cap [b] \neq \varnothing$, then there exists an element $x \in A$ with $x \in [a] \cap [b]$. Hence $(a, x) \in R$ and $(b, x) \in R$. By symmetry, $(x, b) \in R$ and by transitivity, $(a, b) \in R$. Consequently by (ii), $[a] = [b]$.

2.22. Consider the set of words W = {sheet, last, sky, wash, wind, sit}. Find W/R where R is the equivalence relation on W defined by either (*a*) "has the same number of letters as"; (*b*) "begins with the same letter as".

(*a*) Those words with the same number of letters belong to the same cell; hence

$$W/R = [\{\text{sheet}\}, \{\text{last, wash, wind}\}, \{\text{sky, sit}\}].$$

(*b*) Those words beginning with the same letter belong to the same cell; hence

$$W/R = \{\{\text{sheet, sky, sit}\}, \{\text{last}\}, \{\text{wash, wind}\}\}.$$

PARTIAL ORDERING

2.23. Let $\mathscr{C}$ be any collection of sets. Is the relation of set inclusion $\subseteq$ a partial order on $\mathscr{C}$?

Yes, since set inclusion is reflexive, antisymmetric, and transitive. That is, for any sets A, B, C in $\mathscr{C}$ we have: (i) $A \subseteq A$; (ii) if $A \subseteq B$ and $B \subseteq A$, then $A = B$; (iii) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

2.24. Consider the set $\mathbf{Z}$ of integers. Define $a\,R\,b$ by $b = a^r$ for some positive integer r. Show that R is a partial order on $\mathbf{Z}$, that is, show that R is: (*a*) reflexive; (*b*) antisymmetric, (*c*) transitive.

(*a*) R is reflexive since $a = a^1$.

(*b*) Suppose $a\,R\,b$ and $b\,R\,a$, say $b = a^r$ and $a = b^s$. Then $a = (a^r)^s = a^{rs}$. There are three possibilities: (i) $rs = 1$, (ii) $a = 1$, and (iii) $a = -1$. If $rs = 1$ then $r = 1$ and $s = 1$ and so $a = b$. If $a = 1$ then $b = 1^r = 1 = a$, and, similarly, if $b = 1$ then $a = 1$. Lastly, if $a = -1$ then $b = -1$ (since $b \neq 1$) and $a = b$. In all three cases, $a = b$. Thus R is antisymmetric.

(*c*) Suppose $a\,R\,b$ and $b\,R\,c$ are, say, $b = a^r$ and $c = b^s$. Then $c = (a^r)^s = a^{rs}$ and, therefore, $a\,R\,c$. Hence R is transitive.

Accordingly, R is a partial order on $\mathbf{Z}$.

Supplementary Problems

RELATIONS

2.25. Let $W = \{\text{Marc, Erik, Paul}\}$ and let $V = \{\text{Erik, David}\}$. Find: (*a*) $W \times V$; (*b*) $V \times W$; (*c*) $V \times V$.

2.26. Let $S = \{a, b, c\}$, $T = \{b, c, d\}$, and $W = \{a, d\}$. Construct the tree diagram of $S \times T \times W$ and then find $S \times T \times W$.

2.27. Find x and y if: (*a*) $(x + 2, 4) = (5, 2x + y)$; (*b*) $(y - 2, 2x + 1) = (x - 1, y + 2)$.

2.28. Prove: (*a*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$; (*b*) $A \times (B \cup C) = (A \times B) \cup (A \times B)$.

2.29. Let R be the following relation on $A = \{1, 2, 3, 4\}$:

$$R = \{(1,3), (1,4), (3,2), (3,3), (3,4)\}$$

(*a*) Find the matrix M_R of R.
(*b*) Find the domain and range of R.
(*c*) Find R^{-1}.
(*d*) Draw the directed graph of R.
(*e*) Find the composition relation $R \circ R$.

2.30. Let R and S be the following relations on $B = \{a, b, c, d\}$:

$$R = \{(a,a), (a,c), (c,b), (c,d), (d,b)\} \quad \text{and} \quad S = \{(b,a), (c,c), (c,d), (d,a)\}$$

Find the following composition relations: (*a*) $R \circ S$; (*b*) $S \circ R$; (*c*) $R \circ R$; (*d*) $S \circ S$.

2.31. Let R be the relation on the positive integers $\mathbf{N}$ defined by the equation $x + 3y = 12$; that is,

$$R = \{(x, y) : x + 3y = 12\}$$

(*a*) Write R as a set of ordered pairs.
(*b*) Find: (i) domain of R, (ii) range of R, and (iii) R^{-1}.
(*c*) Find the composition relation $R \circ R$.

PROPERTIES OF RELATIONS

2.32. Each of the following defines a relation on the positive integers $\mathbf{N}$:

(1) "x is greater than y".
(2) "xy is the square of an integer".
(3) $x + y = 10$.
(4) $x + 4y = 10$.

Determine which relations are (*a*) reflexive; (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

2.33. Let R and S be relations on a set A. Assuming A has at least three elements, state whether each of the following statements is true or false. If it is false, give a counterexample on the set $A = \{1, 2, 3\}$:

(*a*) If R and S are symmetric then $R \cap S$ is symmetric.
(*b*) If R and S are symmetric then $R \cup S$ is symmetric.
(*c*) If R and S are reflexive then $R \cap S$ is reflexive.
(*d*) If R and S are reflexive then $R \cup S$ is reflexive.
(*e*) If R and S are transitive then $R \cup S$ is transitive.
(*f*) If R and S are antisymmetric then $R \cup S$ is antisymmetric.
(*g*) If R is antisymmetric, then R^{-1} is antisymmetric.
(*h*) If R is reflexive then $R \cap R^{-1}$ is not empty.
(*i*) If R is symmetric then $R \cap R^{-1}$ is not empty.

2.34. Suppose R and S are relations on a set A and R is antisymmetric. Prove that $R \cap S$ is antisymmetric.

EQUIVALENCE RELATIONS

2.35. Prove that if R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

2.36. Let $S = \{1, 2, 3, \ldots, 19, 20\}$. Let R be the equivalence relation on S defined by $x \equiv y \pmod 5$, that is, $x - y$ is divisible by 5. Find the partition of S induced by R, i.e., the quotient set S/R.

2.37. Let $A = \{1, 2, 3, \ldots, 9\}$ and let $\sim$ be the relation on $A \times A$ defined by

$$(a, b) \sim (c, d) \quad \text{if} \quad a + d = b + c$$

(*a*) Prove that $\sim$ is an equivalence relation.
(*b*) Find $[(2, 5)]$, i.e., the equivalence class of $(2, 5)$.

Answers to Supplementary Problems

2.25. (*a*) $W \times V = \{(\text{Marc, Erik}), (\text{Marc, David}), (\text{Erik, Erik}), (\text{Erik, David}), (\text{Paul, Erik}), (\text{Paul, David})\}$.
(*b*) $V \times W = \{(\text{Erik, Marc}), (\text{David, Marc}), (\text{Erik, Erik}), (\text{David, Erik}), (\text{Erik, Paul}), (\text{David, Paul})\}$.
(*c*) $V \times V = \{(\text{Erik, Erik}), (\text{Erik, David}), (\text{David, Erik}), (\text{David, David})\}$.

2.26. The tree diagram of $S \times T \times W$ is shown in Fig. 2-11. The set $S \times T \times W$ is equal to

$$\{(a,b,a),(a,b,d),(a,c,a),(a,c,d),(a,d,a),(a,d,d),$$
$$(b,b,a),(b,b,d),(b,c,a),(b,c,d),(b,d,a),(b,d,d),$$
$$(c,b,a),(c,b,d),(c,c,a),(c,c,d),(c,d,a),(c,d,d)\}$$

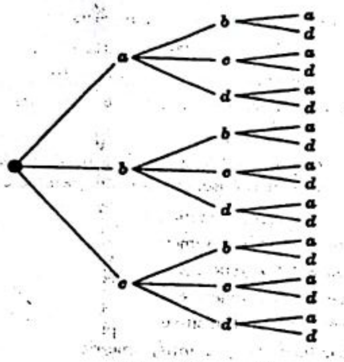


Fig. 2-11

2.27. (a) $x = 3; y = -2$; (b) $x = 2, y = 3$.

2.29. (a) $M_R = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) Domain $= \{1,3\}$, range $= \{2,3,4\}$.
(c) $R^{-1} = \{(3,1),(4,1),(2,3),(3,3),(4,3)\}$.
(d) See Fig. 2-12.
(e) $R \circ R = \{(1,2),(1,3),(1,4),(3,2),(3,3),(3,4)\}$.

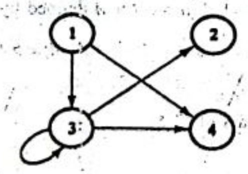


Fig. 2-12

2.30. (a) $R \circ S = \{(a,c),(a,d),(c,a),(d,a)\}$.
(b) $S \circ R = \{(b,a),(b,c),(c,b),(c,d),(d,a),(d,c)\}$.
(c) $R \circ R = \{(a,a),(a,b),(a,c),(a,d),(c,b)\}$.
(d) $S \circ S = \{(c,c),(c,a),(c,d)\}$.

2.31. (a) $\{(9,1),(6,2),(3,3)\}$
(b) (i) $\{9,6,3\}$, (ii) $\{1,2,3\}$, (iii) $\{(1,9),(2,6),(3,3)\}$
(c) $\{(3,3)\}$

2.32. (a) None; (b) (2) and (3); (c) (1) and (4); (d) all except (3).

2.33. All are true except (e) $R = \{(1,2)\}$, $S = \{(2,3)\}$ and (f) $R = \{(1,2)\}$, $S = \{(2,1)\}$.

2.36. $[\{1,6,11,16\}, \{2,7,12,17\}, \{3,8,13,18\}, \{4,9,14,19\}, \{5,10,15,20\}]$

2.37. (b) $\{(1,4),(2,5),(3,6),(4,7),(5,8),(6,9)\}$

Chapter 3

Functions and Algorithms

3.1 INTRODUCTION

One of the most important concepts in mathematics is that of a function. The terms "map", "mapping", "transformation", and many others mean the same thing; the choice of which word to use in a given situation is usually determined by tradition and the mathematical background of the person using the term.

Related to the notion of a function is that of an algorithm. The notation for presenting an algorithm and a discussion of its complexity is also covered in this chapter.

3.2 FUNCTIONS

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a *function* from A into B. The set A is called the *domain* of the function, and the set B is called the *codomain*.

Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f: a \to B$$

which is read: "f is a function from A into B", or "f takes (or; maps) A into B". If $a \in A$, then $f(a)$ (read: "f of a") denotes the unique element of B which f assigns to a; it is called the *image* of a under f, or the *value* of f at a. The set of all image values is called the *range* or *image* of f. The image of $f: A \to B$ is denoted by Ran (f), Im (f) or $f(A)$.

Frequently, a function can be expressed by means of a mathematical formula. For example, consider the function which sends each real number into its square. We may describe this function by writing

$$f(x) = x^2 \quad \text{or} \quad x \mapsto x^2 \quad \text{or} \quad y = x^2$$

In the first notation, x is called a *variable* and the letter f denotes the function. In the second notation, the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value of x.

Remark: Whenever a function is given by a formula in terms of a variable x, we assume, unless it is otherwise stated, that the domain of the function is **R** (or the largest subset of **R** for which the formula has meaning) and the codomain is **R**.

EXAMPLE 3.1

Consider the function $f(x) = x^3$, i.e., f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$.

Let f assign to each country in the world its capital city. Here the domain of f is the set of countries in the world; the codomain is the list of cities of the world. The image of France is Paris; or, in other words, $f(\text{France}) = \text{Paris}$.

Figure 3-1 defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way. Here

$$f(a) = s, \quad f(b) = u, \quad f(c) = r, \quad f(d) = s$$

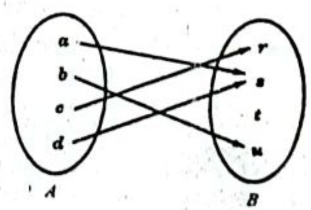


Fig. 3-1

The image of f is the set of image values, $\{r, s, u\}$. Note that t does not belong to the image of f because t is not the image of any element under f.

(*d*) Let A be any set. The function from A into A which assigns to each element that element itself is called the *identity function* on A and is usually denoted by 1_A or simply 1. In other words,

$$1_A(a) = a$$

for every element a in A.

(*e*) Suppose S is a subset of A, that is, suppose $S \subseteq A$. The *inclusion map* or *embedding* of S into A, denoted by $i: S \hookrightarrow A$, is the function defined by

$$i(x) = x$$

for every $x \in S$; and the *restriction* to S of any function $f: A \to B$, denoted by $f|_S$, is the function from S to B defined by

$$f|_A(x) = f(x)$$

for every $x \in S$.

Functions as Relations

There is another point of view from which functions may be considered. First of all, every function $f: A \to B$ gives rise to a relation from A to B called the *graph of* f and defined by

$$\text{Graph of } f = \{(a, b) : a \in A, b = f(a)\}$$

Two functions $f: A \to B$ and $g: A \to B$ are defined to be equal, written $f = g$, if $f(a) = g(a)$ for every $a \in A$; that is, if they have the same graph. Accordingly, we do not distinguish between a function and its graph. Now, such a graph relation has the property that each a in A belongs to a unique ordered pair (a, b) in the relation. On the other hand, any relation f from A to B that has this property gives rise to a function $f: A \to B$, where $f(a) = b$ for each (a, b) in f. Consequently, one may equivalently define a function as follows:

Definition: A function $f: A \to B$ is a relation from A to B (i.e., a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Although we do not distinguish between a function and its graph, we will still use the terminology "graph of f" when referring to f as a set of ordered pairs. Moreover, since the graph of f is a relation, we can draw its picture as was done for relations in general, and this pictorial representation is itself sometimes called the graph of f. Also, the defining condition of a function, that each $a \in A$ belongs to a unique pair (a, b) in f, is equivalent to the geometrical condition of each vertical line intersecting the graph in exactly one point.

EXAMPLE 3.2

(*a*) Let $f: A \to B$ be the function defined in Example 3.1(*c*). Then the graph of f is the following set of ordered pairs:

$$\{(a, s), (b, u), (c, r), (d, s)\}$$

(*b*) Consider the following relations on the set $A = \{1, 2, 3\}$:

$$f = \{(1, 3), (2, 3), (3, 1)\}$$
$$g = \{(1, 2), (3, 1)\}$$
$$h = \{(1, 3), (2, 1), (1, 2), (3, 1)\}$$

f is a function from A into A since each member of A appears as the first coordinate in exactly one ordered pair in f; here $f(1) = 3$, $f(2) = 3$ and $f(3) = 1$. g is not a function from A into A since $2 \in A$ is not the first coordinate of any pair in g and so g does not assign any image to 2. Also h is not a function from A into A

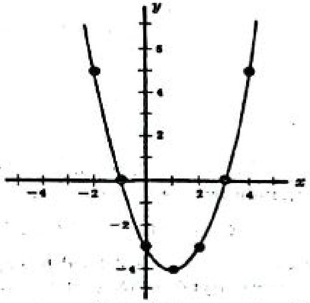
since $1 \in A$ appears as the first coordinate of two distinct ordered pairs in h, $(1,3)$ and $(1,2)$. If h is to be a function it cannot assign both 3 and 2 to the element $1 \in A$.

(c) By a real polynomial function, we mean a function $f: \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i are real numbers. Since $\mathbf{R}$ is an infinite set, it would be impossible to plot each point of the graph. However, the graph of such a function can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The points are usually obtained from a table where various values are assigned to x and the corresponding values of $f(x)$ are computed. Figure 3-2 illustrates this technique using the function $f(x) = x^2 - 2x - 3$.

x	$f(x)$
−2	5
−1	0
0	−3
1	−4
2	−3
3	0
4	5



Graph of $f(x) = x^2 - 2x - 3$

Fig. 3-2

Composition Function

Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Then we may define a new function from A to C, called the *composition* of f and g and written $g \circ f$, as follows:

$$(g \circ f)(a) \equiv g(f(a))$$

That is, we find the image of a under f and then find the image of $f(a)$ under g. This definition is not really new. If we view f and g as relations, then this function is the same as the composition of f and g as relations (see Section 2.6) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for relations.

Consider any function $f: A \to B$. Then

$$f \circ 1_A = f \quad \text{and} \quad 1_B \circ f = f$$

where 1_A and 1_B are the identity functions on A and B, respectively.

3.3 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A *onto* B or that f maps A *onto* B.

A function $f: A \to B$ is *invertible* if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function. The following theorem gives simple criteria which tells us when it is.

Theorem 3.1: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

If $f: A \to B$ is one-to-one and onto, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

Some texts use the terms *injective* for a one-to-one function, *surjective* for an onto function, and *bijective* for a one-to-one correspondence.

EXAMPLE 3.3 Consider the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined by the diagram of Fig. 3-3. Now f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$.

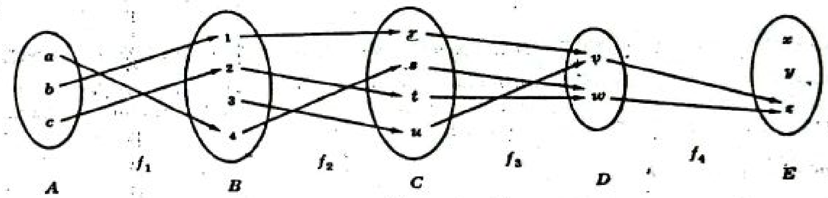


Fig. 3-3

As far as being onto is concerned, f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, i.e., $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of A, and f_4 is not onto since $x \in E$ is not the image under f_4 of any element of D.

Thus f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto. However, f_2 is both one-to-one and onto, i.e., is a one-to-one correspondence between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

Geometrical Characterization of One-to-One and Onto Functions

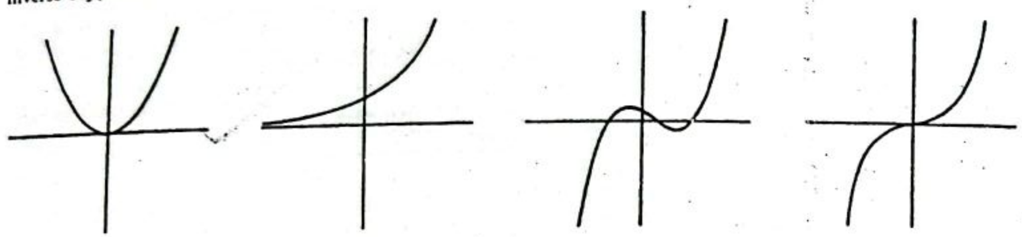
Since functions may be identified with their graphs, and since graphs may be plotted, we might wonder whether the concepts of being one-to-one and onto have geometrical meaning. We show that the answer is yes.

To say that a function $f: A \to B$ is one-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in the graph of f; hence each horizontal line can intersect the graph of f in at most one point. On the other hand, to say that f is an onto function means that for every $b \in B$ there must be at least one $a \in A$ such that (a, b) belongs to the graph of f; hence each horizontal line must intersect the graph of f at least once. Accordingly, if f is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph of f in exactly one point.

EXAMPLE 3.4 Consider the following four functions from **R** into **R**:

$$f_1(x) = x^2, \quad f_2(x) = 2^x, \quad f_3(x) = x^3 - 2x^2 - 5x + 6, \quad f_4(x) = x^3$$

The graphs of these functions appear in Fig. 3-4. Observe that there are horizontal lines which intersect the graph of f_1 twice and there are horizontal lines which do not intersect the graph of f_1 at all; hence f_1 is neither one-to-one nor onto. Similarly, f_2 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is both one-to-one and onto. The inverse of f_4 is the cube root function, i.e., $f_4^{-1}(x) = \sqrt[3]{x}$.



$f_1(x) = x^2$ $\qquad$ $f_2(x) = 2^x$ $\qquad$ $f_3(x) = x^3 - 2x^2 - 5x + 6$ $\qquad$ $f_4(x) = x^3$

Fig. 3-4

3.4 MATHEMATICAL FUNCTIONS, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This section presents various mathematical functions which appear often in the analysis of algorithms, and in computer science in general, together with their notation. We also discuss the exponential and logarithmic functions, and their relationship.

Floor and Ceiling Functions

Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically,

$\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer that does not exceed x.
$\lceil x \rceil$, called the *ceiling* of x, denotes the least integer that is not less than x.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$. For example,

$$\lfloor 3.14 \rfloor = 3, \quad \lfloor \sqrt{5} \rfloor = 2, \quad \lfloor -8.5 \rfloor = -9, \quad \lfloor 7 \rfloor = 7, \quad \lfloor -4 \rfloor = -4$$
$$\lceil 3.14 \rceil = 4, \quad \lceil \sqrt{5} \rceil = 3, \quad \lceil -8.5 \rceil = -8, \quad \lceil 7 \rceil = 7, \quad \lceil -4 \rceil = -4$$

Integer and Absolute Value Functions

Let x be any real number. The *integer value* of x, written INT(x), converts x into an integer by deleting (truncating) the fractional part of the number. Thus

$$\text{INT}(3.14) = 3, \quad \text{INT}(\sqrt{5}) = 2, \quad \text{INT}(-8.5) = -8, \quad \text{INT}(7) = 7$$

Observe that $\text{INT}(x) = \lfloor x \rfloor$ or $\text{INT}(x) = \lceil x \rceil$ according to whether x is positive or negative.

The *absolute value* of the real number x, written ABS(x) or $|x|$, is defined as the greater of x or $-x$. Hence ABS(0) = 0, and, for $x \neq 0$, ABS(x) = x or ABS(x) = $-x$, depending on whether x is positive or negative. Thus

$$|-15| = 15, \quad |7| = 7, \quad |-3.33| = 3.33, \quad |4.44| = 4.44, \quad |-0.075| = 0.075$$

We note that $|x| = |-x|$ and, for $x \neq 0$, $|x|$ is positive.

Remainder Function; Modular Arithmetic

Let k be any integer and let M be a positive integer. Then

$$k \pmod{M}$$

(read k *modulo* M) will denote the integer remainder when k is divided by M. More exactly, $k \pmod{M}$ is the unique integer r such that

$$k = Mq + r \quad \text{where} \quad 0 \le r < M$$

When k is positive, simply divide k by M to obtain the remainder r. Thus

$$25 \ (\text{mod } 7) = 4, \quad 25 \ (\text{mod } 5) = 0, \quad 35 \ (\text{mod } 11) = 2, \quad 3 \ (\text{mod } 8) = 3$$

If k is negative, divide $|k|$ by M to obtain a remainder r'; then $k(\text{mod } M) = M - r'$ when $r' \neq 0$. Thus

$$-26 \ (\text{mod } 7) = 7 - 5 = 2, \quad -371 \ (\text{mod } 8) = 8 - 3 = 5, \quad -39 \ (\text{mod } 3) = 0$$

The term "mod" is also used for the mathematical congruence relation, which is denoted and defined as follows:

$$a \equiv b \pmod{M} \quad \text{if and only if} \quad M \text{ divides } b - a$$

M is called the *modulus*, and $a \equiv b \pmod{M}$ is read "a is congruent to b modulo M" The following aspects of the congruence relation are frequently useful:

$$0 \equiv M \pmod{M} \quad \text{and} \quad a \pm M \equiv a \pmod{M}$$

Arithmetic modulo M refers to the arithmetic operations of addition, multiplication, and subtraction where the arithmetic value is replaced by its equivalent value in the set

$$\{0, 1, 2, \ldots, M - 1\}$$

or in the set

$$\{1, 2, 3, \ldots, M\}$$

For example, in arithmetic modulo 12, sometimes called "clock" arithmetic,

$$6 + 9 \equiv 3, \quad 7 \times 5 \equiv 11, \quad 1 - 5 \equiv 8, \quad 2 + 10 \equiv 0 \equiv 12$$

(The use of 0 or M depends on the application.)

Exponential Functions

Recall the following definitions for integer exponents (where m is a positive integer):

$$a^m = a \cdot a \cdots a \ (m \text{ times}), \quad a^0 = 1, \quad a^{-m} = \frac{1}{a^m}$$

Exponents are extended to include all rational numbers by defining, for any rational number m/n,

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

For example,

$$2^4 = 16, \quad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \quad 125^{2/3} = 5^2 = 25$$

In fact, exponents are extended to include all real numbers by defining, for any real number x,

$$a^x = \lim_{r \to x} a^r, \quad \text{where } r \text{ is a rational number}$$

Accordingly, the exponential function $f(x) = a^x$ is defined for all real numbers.

Logarithmic Functions

Logarithms are related to exponents as follows. Let b be a positive number. The logarithm of any positive number x to be the base b, written

$$\log_b x$$

represents the exponent to which b must be raised to obtain x. That is,

$$y = \log_b x \quad \text{and} \quad b^y = x$$

are equivalent statements. Accordingly,

$$\log_2 8 = 3 \quad \text{since} \quad 2^3 = 8; \qquad \log_{10} 100 = 2 \quad \text{since} \quad 10^2 = 100$$
$$\log_2 64 = 6 \quad \text{since} \quad 2^6 = 64; \qquad \log_{10} 0.001 = -3 \quad \text{since} \quad 10^{-3} = 0.001$$

Furthermore, for any base b,

$$\log_b 1 = 0 \quad \text{since} \quad b^0 = 1$$
$$\log_b b = 1 \quad \text{since} \quad b^1 = b$$

The logarithm of a negative number and the logarithm of 0 are not defined.

Frequently, logarithms are expressed using approximate values. For example, using tables or calculators, one obtains

$$\log_{10} 300 = 2.4771 \quad \text{and} \quad \log_e 40 = 3.6889$$

as approximate answers. (Here $e = 2.718281\ldots$.)

Three classes of logarithms are of special importance: logarithms to base 10, called *common logarithms*; logarithms to base e, called *natural logarithms*; and logarithms to base 2, called *binary logarithms*. Some texts write

$$\ln x \quad \text{for} \quad \log_e x \quad \text{and} \quad \lg x \quad \text{or} \quad \operatorname{Log} x \quad \text{for} \quad \log_2 x$$

The term $\log x$, by itself, usually means $\log_{10} x$; but it is also used for $\log_e x$ in advanced mathematical texts and for $\log_2 x$ in computer science texts.

Frequently, we will require only the floor or the ceiling of a binary logarithm. This can be obtained by looking at the powers of 2. For example,

$$\lfloor \log_2 100 \rfloor = 6 \quad \text{since} \quad 2^6 = 64 \quad \text{and} \quad 2^7 = 128$$
$$\lceil \log_2 1000 \rceil = 9 \quad \text{since} \quad 2^8 = 512 \quad \text{and} \quad 2^9 = 1024$$

and so on.

Relationship between the Exponential and Logarithmic Functions

The basic relationship between the exponential and the logarithmic functions

$$f(x) = b^x \quad \text{and} \quad g(x) = \log_b x$$

is that they are inverses of each other; hence the graphs of these functions are related geometrically. This relationship is illustrated in Fig. 3-5 where the graphs of the exponential function $f(x) = 2^x$, the logarithmic function $g(x) = \log_2 x$, and the linear function $h(x) = x$ appear on the same coordinate axis. Since $f(x) = 2^x$ and $g(x) = \log_2 x$ are inverse functions, they are symmetric with respect to the linear function $h(x) = x$ or, in other words, the line $y = x$.

Figure 3-5 also indicates another important property of the exponential and logarithmic functions. Specifically, for any positive c, we have

$$g(c) < h(c) < f(c)$$

In fact, as c increases in value, the vertical distances $h(c) - g(c)$ and $f(c) - g(c)$ increase in value. Moreover, the logarithmic function $g(x)$ grows very slowly compared with the linear function $h(x)$, and the exponential function $f(x)$ grows very quickly compared with $h(x)$.

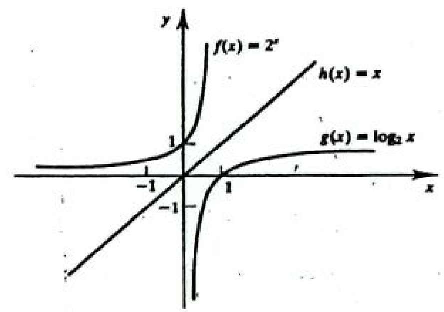


Fig. 3-5

3.5 SEQUENCES, INDEXED CLASSES OF SETS

Sequences and indexed classes of sets are special types of functions with their own notation. We discuss these objects in this section. We also discuss the summation notation here.

Sequences

A *sequence* is a function from the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ of positive integers into a set A. The notation a_n is used to denote the image of the integer n. Thus a sequence is usually denoted by

$$a_1, a_2, a_3, \ldots \quad \text{or} \quad \{a_n : n \in \mathbf{N}\} \quad \text{or simply} \quad \{a_n\}$$

Sometimes the domain of a sequence is the set $\{0, 1, 2, \ldots\}$ of nonnegative integers rather than $\mathbf{N}$. In such a case we say n begins with 0 rather than 1.

A *finite sequence* over a set A is a function from $\{1, 2, \ldots, m\}$ into A, and it is usually denoted by

$$a_1, a_2, \ldots, a_m$$

Such a finite sequence is sometimes called a *list* or an *m-tuple*.

EXAMPLE 3.5

(*a*) The familiar sequences

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots \quad \text{and} \quad 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots$$

may be formally defined, respectively, by

$$a_n = 1/n \quad \text{and} \quad b_n = 2^{-n}$$

where the first sequence begins with $n = 1$ and the second sequence begins with $n = 0$.

(*b*) The important sequence $1, -1, 1, -1, \ldots$ may be formally defined by

$$a_n = (-1)^{n+1} \quad \text{or, equivalently, by} \quad b_n = (-1)^n$$

where the first sequence begins with $n = 1$ and the second sequence begins with $n = 0$.

(c) (*Strings*) Suppose a set A is finite and A is viewed as a character set or an alphabet. Then a finite sequence over A is called a *string* or *word*, and it is usually written in the form $a_1 a_2 \dots a_m$, that is, without parenthesis. The number m of characters in the string is called its *length*. One also views the set with zero characters as a string; it is called the *empty string* or *null string*. Strings over an alphabet A and certain operations on these strings will be discussed in detail in Chapter 13.

Summation Symbol, Sums

Here we introduce the summation symbol $\sum$ (the Greek letter sigma). Consider a sequence $a_1, a_2, a_3, \dots$. Then the sums

$$a_1 + a_2 + \cdots + a_n \quad \text{and} \quad a_m + a_{m+1} + \cdots + a_n$$

will be denoted, respectively, by

$$\sum_{j=1}^{n} a_j \quad \text{and} \quad \sum_{j=m}^{n} a_j$$

The letter j in the above expressions is called a *dummy index* or *dummy variable*. Other letters frequently used as dummy variables are i, k, s and t.

EXAMPLE 3.6

$$\sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\sum_{j=2}^{5} j^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$$

$$\sum_{j=1}^{n} j = 1 + 2 + \cdots + n$$

The last sum in Example 3.6 appears often. It has the value $n(n+1)/2$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Thus, for example,

$$1 + 2 + \cdots + 50 = \frac{50(51)}{2} = 1275$$

Indexed Classes of Sets

Let I be any nonempty set, and let S be a collection of sets. An *indexing function* from I to S is a function $f: I \to S$. For any $i \in I$, we denote the image $f(i)$ by A_i. Thus the indexing function f is usually denoted by

$$\{A_i: i \in I\} \quad \text{or} \quad \{A_i\}_{i \in I} \quad \text{or simply} \quad \{A_i\}$$

The set I is called the *indexing set*, and the elements of I are called *indices*. If f is one-to-one and onto, we say that S is indexed by I.

The concepts of union and intersection are defined for indexed classes of sets by

$$\cup_{i \in I} A_i = \{x: x \in A_i \text{ for some } i \in I\} \quad \text{and} \quad \cap_{i \in I} A_i = \{x: x \in A_i \text{ for all } i \in I\}$$

In the case that I is a finite set, this is just the same as our previous definitions of union and intersection. If I is $\mathbf{N}$, we may denote the union and intersection by

$$A_1 \cup A_2 \cup \cdots \quad \text{and} \quad A_1 \cap A_2 \cap \cdots$$

respectively.

EXAMPLE 3.7 Let I be the set $\mathbf{Z}$ of integers. To each integer n we assign the following subset of $\mathbf{R}$:

$$A_n = \{x: x \leq n\}$$

In other words, A_n is the infinite interval $(-\infty, n]$.) For any real number a, there exist integers n_1 and n_2 such that $n_1 < a < n_2$; so $a \in A_{n_2}$ but $a \notin A_{n_1}$. Hence

$$a \in \cup_n A_n \quad \text{but} \quad a \notin \cap_n A_n$$

Accordingly,

$$\cup_n A_n = \mathbf{R} \quad \text{but} \quad \cap_n A_n = \varnothing$$

3.6 RECURSIVELY DEFINED FUNCTIONS

A function is said to be *recursively defined* if the function definition refers to itself. In order for the definition not to be circular, the function definition must have the following two properties:

(1) There must be certain arguments, called *base values*, for which the function does not refer to itself.
(2) Each time the function does refer to itself, the argument of the function must be closer to a base value.

A recursive function with these two properties is said to be *well-defined*.

The following examples should help clarify these ideas.

Factorial Function

The product of the positive integers from 1 to n, inclusive, is called "n factorial" and is usually denoted by $n!$; that is,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$$

It is also convenient to define $0! = 1$, so that the function is defined for all nonnegative integers. Thus we have

$$0! = 1, \quad 1! = 1, \quad 2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6, \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24,$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120, \quad 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

and so on. Observe that

$$5! = 5 \cdot 4! = 5 \cdot 24 = 120 \quad \text{and} \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720$$

This is true for every positive integer n; that is,

$$n! = n \cdot (n = 1)!$$

Accordingly, the factorial function may also be defined as follows:

Definition 3.1 **(Factorial Function):**

(a) If $n = 0$, then $n! = 1$.
(b) If $n > 0$, then $n! = n \cdot (n-1)!$

Observe that the above definition of n! is recursive, since it refers to itself when it uses $(n-1)!$. However:

(1) The value of $n!$ is explicitly given when $n = 0$ (thus 0 is a base value).
(2) The value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0.

Accordingly, the definition is not circular, or, in other words, the function is well-defined.

EXAMPLE 3.8 Let us calculate 4! using the recursive definitions. This calculation requires the following nine steps:

(1) $4! = 4 \cdot 3!$
(2) $\quad 3! = 3 \cdot 2!$
(3) $\quad\quad 2! = 2 \cdot 1!$
(4) $\quad\quad\quad 1! = 1 \cdot 0!$
(5) $\quad\quad\quad\quad 0! = 1$
(6) $\quad\quad\quad 1! = 1 \cdot 1 = 1$
(7) $\quad\quad 2! = 2 \cdot 1 = 2$
(8) $\quad 3! = 3 \cdot 2 = 6$
(9) $4! = 4 \cdot 6 = 24$

That is:

Step 1. This defines 4! in terms of 3!, so we must postpone evaluating 4! until we evaluate 3! This postponement is indicated by indenting the next step.

Step 2. Here 3! is defined in terms of 2!, so we must postpone evaluating 3! until we evaluate 2!

Step 3. This defines 2! in terms of 1!

Step 4. This defines 1! in terms of 0!

Step 5. This step can explicitly evaluate 0!, since 0 is the base value of the recursive definition.

Steps 6 to 9. We backtrack, using 0! to find 1!, using 1! to find 2!, using 2! to find 3!, and finally using 3! to find 4! This backtracking is indicated by the "reverse" indention.

Observe that we backtrack in the reverse order of the original postponed evaluations.

Level Numbers

Let P be a procedure or recursive formula which is used to evaluate $f(X)$ where f is a recursive function and X is the input. We associate a *level number* with each execution of P as follows. The original execution of P is assigned level 1; and each time P is executed because of a recursive call, its level is one more than the level of the execution that made the recursive call. The *depth* of recursion in evaluating $f(X)$ refers to the maximum level number of P during its execution.

Consider, for example, the evaluation of 4! Example 3.8, which uses the recursive formula $n! = n(n-1)!$. Step 1 belongs to level 1 since it is the first execution of the formula. Thus:

Step 2 belongs to level 2; Step 3 to level 3, ...; Step 5 to level 5.

On the other hand, Step 6 belongs to level 4 since it is the result of a return from level 5. In other words, Step 6 and Step 4 belong to the same level of execution. Similarly,

Step 7 belongs to level 3; Step 8 to level 2; and Step 9 to level 1.

Accordingly, in evaluating 4!, the depth of the recursion is 5.

Fibonacci Sequence

The celebrated Fibonacci sequence (usually denoted by $F_n, F_1, F_2, \ldots$) is as follows:

$$0, \; 1, \; 1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21, \; 34, \; 55, \ldots$$

That is, $F_0 = 0$ and $F_1 = 1$ and each succeeding term is the sum of the two preceding terms. For example, the next two terms of the sequence are

$$34 + 55 = 89 \quad \text{and} \quad 55 + 89 = 144$$

A formal definition of this function follows:

Definition 3.2 **(Fibonacci Sequence):**

(*a*) If $n = 0$ or $n = 1$, then $F_n = n$.
(*b*) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

This is another example of a recursive definition, since the definition refers to itself when it uses F_{n-2} and F_{n-1}. However:

(1) The base values are 0 and 1.
(2) The value of F_n is defined in terms of smaller values of n which are closer to the base values.

Accordingly, this function is well-defined.

Ackermann Function

The Ackermann function is a function with two arguments, each of which can be assigned any nonnegative interger, that is, 0, 1, 2, ... This function is defined as follows:

Definition 3.3 **(Ackermann Function):**

(*a*) If $m = 0$, then $A(m,n) = n + 1$.
(*b*) If $m \neq 0$ but $n = 0$, then $A(m,n) = A(m-1, 1)$.
(*c*) If $m \neq 0$ and $n \neq 0$, then $A(m,n) = A(m-1, A(m, n-1))$

Once more, we have a recursive definition, since the definition refers to itself in parts (*b*) and (*c*). Observe that $A(m,n)$ is explicitly given only when $m = 0$. The base criteria are the pairs

$$(0,0), \quad (0,1), \quad (0,2), \quad (0,3), \ldots, (0,n), \ldots$$

Although it is not obvious from the definition, the value of any $A(m,n)$ may eventually be expressed in terms of the value of the function on one or more of the base pairs.

The value of $A(1,3)$ is calculated in Problem 3.24. Even this simple case requires 15 steps. Generally speaking, the Ackermann function is too complex to evaluate on any but a trivial example. Its importance comes from its use in mathematical logic. The function is stated here mainly to give another example of a classical recursive function and to show that the recursion part of a definition may be complicated.

3.7 CARDINALITY

Two sets A and B are said to be ***equipotent***, or to have the *same number of elements* or the *same cardinality*, written $A \simeq B$, if there exists a one-to-one correspondence $f: A \to B$. A set A is *finite* if A is empty or if A has the same cardinality as the set $\{1, 2, \ldots, n\}$ for some positive integer n. A set is *infinite* if it is not finite. Familiar examples of infinite sets are the natural numbers **N**, the integers **Z**, the rational numbers **Q**, and the real numbers **R**.

We now introduce the idea of "cardinal numbers". We will consider cardinal numbers simply as symbols assigned to sets in such a way that two sets are assigned the same symbol if and only if they have the same cardinality. The cardinal number of a set A is commonly denoted by $|A|$, $n(A)$, or card (A). We will use $|A|$.

We use the obvious symbols for the cardinal numbers of finite sets. That is, 0 is assigned to the empty set $\varnothing$, and n is assigned to the set $\{1, 2, \ldots, n\}$. Thus $|A| = n$ if and only if A has the same cardinality as $\{1, 2, \ldots, n\}$, which implies that A has n elements.

The cardinal number of the infinite set **N** of positive integers is $\aleph_0$ ("aleph-naught"). This symbol was introduced by Cantor. Thus $|A| = \aleph_0$ if and only if A has the same cardinality as **N**.

EXAMPLE 3.9

(*a*) $|\{x, y, z\}| = 3$
(*b*) Let $E = \{2, 4$

EXAMPLE 3.9

(a) $|\{x, y, z\}| = 3$ and $|\{1, 3, 5, 7, 9\}| = 5$.

(b) Let $E = \{2, 4, 6, \ldots\}$, the set of even positive integers. The function $f: \mathbf{N} \to E$ defined by $f(n) = 2n$ is a one-to-one correspondence between the positive integers $\mathbf{N}$ and E. Thus E has the same cardinality as $\mathbf{N}$ and so we may write

$$|E| = \aleph_0$$

A set with cardinality $\aleph_0$ is said to be *denumerable* or *countably infinite*. A set which is finite or denumerable is said to be *countable*. One can show that the set $\mathbf{Q}$ of rational numbers is countable. In fact, we have the following theorem (proved in Problem 3.15) which we will use subsequently.

Theorem 3.2: A countable union of countable sets is countable.

In other words, if $A_1, A_2, \ldots$ are each countable sets then the union

$$A_1 \cup A_2 \cup A_3 \cup \cdots$$

is also a countable set.

An important example of an infinite set which is uncountable, i.e., not countable, is given by the following theorem which is proved in Problem 3.16.

Theorem 3.3: The set I of all real numbers between 0 and 1 is uncountable.

Inequalities and Cardinal Numbers

One also wants to compare the size of two sets. This is done by means of an inequality relation which is defined for cardinal numbers as follows. For any sets A and B, we define $|A| \le |B|$ if there exists a function $f: A \to B$ which is one-to-one. We also write

$$|A| < |B| \quad \text{if} \quad |A| \le |B| \quad \text{but} \quad |A| \ne |B|$$

For example, $|\mathbf{N}| < |I|$, where $I = \{x: 0 \le x \le 1\}$, since the function $f: \mathbf{N} \to I$ defined by $f(n) = 1/n$ is one-to-one, but $|\mathbf{N}| \ne |I|$ by Theorem 3.3.

Cantor's theorem, which follows and which we prove in Problem 3.28, tells us that the cardinal numbers are unbounded.

Theorem 3.4 (Cantor): For any set A, we have $|A| < |\text{Power}(A)|$ (where Power(A) is the power set of A, i.e., the collection of all subsets of A).

The next theorem tells us that the inequality relation for cardinal numbers is symmetric.

Theorem 3.5 (Schroeder-Bernstein): Suppose A and B are sets such that

$$|A| \le |B| \quad \text{and} \quad |B| \le |A|$$

Then $|A| = |B|$.

We prove an equivalent formulation of this theorem in Problem 3.29.

3.8 ALGORITHMS AND FUNCTIONS

An algorithm M is a finite step-by-step list of well-defined instructions for solving a particular problem, say, to find the output $f(X)$ for a given function f with input X. (Here X may be a list or set of values.) Frequently, there may be more than one way to obtain $f(X)$, as illustrated by the following examples. The particular choice of the algorithm M to obtain $f(X)$ may depend on the "efficiency" or "complexity" of the algorithm; this question of the complexity of an algorithm M is formally discussed in the next section.

EXAMPLE 3.10 (Polynomial Evaluation) Suppose, for a given polynomial $f(x)$ and value $x = a$, we want to find $f(a)$, say,

$$f(x) = 2x^3 - 7x^2 + 4x - 15 \quad \text{and} \quad a = 5$$

This can be done in the following two ways.

(a) (*Direct Method*): Here we substitute $a = 5$ directly in the polynomial to obtain

$$f(5) = 2(125) - 7(25) + 4(5) - 7 = 250 - 175 + 20 - 15 = 80$$

Observe that there are $4 + 3 + 1 = 8$ multiplications and 3 additions. In general, evaluating a polynomial of degree n directly would require approximately.

$$n + (n-1) + \cdots + 1 = \frac{n(n-1)}{2} \text{ multiplications} \quad \text{and} \quad n \text{ additions}$$

(b) (*Horner's Method or Synthetic Division*): Here we rewrite the polynomial by successively factoring out x (on the right) as follows:

$$f(x) = (2x^2 - 7x + 4)x - 15 = (((2x - 7)x + 4)x - 15$$

Then

$$f(5) = ((3)5 + 4)5 - 15 = (19)5 - 15 = 95 - 15 = 80$$

For those familiar with synthetic division, the above arithmetic is equivalent to the following synthetic division:

$$\begin{array}{r|rrrrrrr} 5 & 2 & - & 7 & + & 4 & - & 15 \\ & & & 10 & + & 15 & + & 95 \\ \hline & 2 & + & 3 & + & 19 & + & 80 \end{array}$$

Observe that here there are 3 multiplications and 3 additions. In general, evaluating a polynomial of degree n by Horner's method would require approximately

$$n \text{ multiplications} \quad \text{and} \quad n \text{ additions}$$

Clearly Horner's method (b) is more efficient than the direct method (a).

EXAMPLE 3.11 (Greatest Common Divisor) Let a and b be positive integers with, say, $b < a$; and suppose we want to find $d = \text{GCD}(a, b)$, the greatest common divisor of a and b. This can be done in the following two ways.

(a) (*Direct Method*): Here we find all the divisors of a, say by testing all the numbers from 2 to $a/2$, and all the divisors of b. Then we pick the largest common divisor. For example, suppose $a = 258$ and $b = 60$. The divisors of a and b follow:

$$\begin{aligned} a &= 258; \quad \text{divisors:} \quad 1, \ 2, \ 3, \ 6, \ 86, \ 129, \ 258 \\ b &= 60; \quad \text{divisors;} \quad 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 10, \ 12, \ 15, \ 20, \ 30, \ 60 \end{aligned}$$

Accordingly, $d = \text{GCD}(258, 60) = 6$.

(b) (*Euclidean Algorithm*): Here we divide a by b to obtain a remainder r_1. (Note $r_1 < b$.) Then we divide b by the remainder r_1 to obtain a second remainder r_2. (Note $r_2 < r_1$.) Next we divide r_1 by r_2 to obtain a third remainder r_3. (Note $r_3 < r_2$.) We continue dividing r_k by r_{k+1} to obtain a remainder r_{k+2}. Since

$$a > b > r_1 > r_2 > r_3 \ldots \tag{*}$$

eventually we obtain a remainder $r_m = 0$. Then $r_{m-1} = \text{GCD}(a, b)$. For example, suppose $a = 258$ and $b = 60$. Then:

(1) Dividing $a = 258$ by $b = 60$ yields the remainder $r_1 = 18$.
(2) Dividing $b = 60$ by $r_1 = 18$ yields the remainder $r_2 = 6$.
(3) Dividing $r_1 = 18$ by $r_2 = 6$ yields the remainder $r_3 = 0$.

Thus $r_2 = 6 = \text{GCD}(258, 60)$.

The Euclidean algorithm is a very efficient way to find the greatest common divisor of two positive integers a and b. The fact that the algorithm ends follows from (*). The fact that the algorithm yields $d = \mathrm{GCD}(a, b)$ is not obvious; it is discussed in Section 11.6.

3.9 COMPLEXITY OF ALGORITHMS

The analysis of algorithms is a major task in computer science. In order to compare algorithms, we must have some criteria to measure the efficiency of our algorithms. This section discusses this important topic.

Suppose M is an algorithm, and suppose n is the size of the input data. The time and space used by the algorithm are the two main measures for the efficiency of M. The time is measured by counting the number of "key operations"; for example:

(a) In sorting and searching, one counts the number of comparisons.
(b) In arithmetic, one counts multiplications and neglects additions.

Key operations are so defined when the time for the other operations is much less than or at most proportional to the time for the key operations. The space is measured by counting the maximum of memory needed by the algorithm.

The *complexity* of an algorithm M is the function $f(n)$ which gives the running time and/or storage space requirement of the algorithm in terms of the size n of the input data. Frequently, the storage space required by an algorithm is simply a multiple of the data size. Accordingly, unless otherwise stated or implied, the term "complexity" shall refer to the running time of the algorithm.

The complexity function $f(n)$, which we assume gives the running time of an algorithm, usually depends not only on the size n of the input data but also on the particular data. For example, suppose we want to search through an English short story TEXT for the first occurrence of a given 3-letter word W. Clearly, if W is the 3-letter word "the", then W likely occurs near the beginning of TEXT, so $f(n)$ will be small. On the other hand, if W is the 3-letter word "zoo", then W may not appear in TEXT at all, so $f(n)$ will be large.

The above discussion leads us to the question of finding the complexity function $f(n)$ for certain cases. The two cases one usually investigates in complexity theory are as follows:

(1) *Worst case*: The maximum value of $f(n)$ for any possible input.
(2) *Average case*: The expected value of $f(n)$.

The analysis of the average case assumes a certain probabilistic distribution for the input data; one possible assumption might be that the possible permutations of a data set are equally likely. The average case also uses the following concept in probability theory. Suppose the numbers $n_1, n_2, \ldots, n_k$ occur with respective probabilities $p_1, p_2, \ldots, p_k$. Then the *expectation* or *average value* E is given by

$$E = n_1p_1 + n_2p_2 + \ldots + n_kp_k$$

These ideas are illustrated below.

Linear Search

Suppose a linear array DATA contains n elements, and suppose a specific ITEM of information is given. We want either to find the location LOC of ITEM in the array DATA, or to send some message, such as LOC = 0, to indicate that ITEM does not appear in DATA. The linear search algorithm solves this problem by comparing ITEM, one by one, with each element in DATA. That is, we compare ITEM with DATA[1], then DATA[2], and so on, until we find LOC such that ITEM = DATA[LOC].

The complexity of the search algorithm is given by the number C of comparisons between ITEM and DATA[K]. We seek $C(n)$ for the worst case and the average case.

(1) ***Worst Case***: Clearly the worst case occurs when ITEM is the last element in the array DATA or is not there at all. In either situation, we have

$$C(n) = n$$

Accordingly, $C(n) = n$ is the worst-case complexity of the linear search algorithm.

(2) ***Average Case***: Here we assume that ITEM does appear in DATA, and that is is equally likely to occur at any position in the array. Accordingly, the number of comparisons can be any of the numbers $1, 2, 3, \ldots, n$, and each number occurs with probability $p = 1/n$. Then

$$\begin{aligned} C(n) &= 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + \cdots + n \cdot \frac{1}{n} \\ &= (1 + 2 + \cdots + n) \cdot \frac{1}{n} \\ &= \frac{n(n+1)}{2} \cdot \frac{1}{n} = \frac{n+1}{2} \end{aligned}$$

This agrees with our intuitive feeling that the average number of comparisons needed to find the location of ITEM is approximately equal to half the number of elements in the DATA list.

Remark: The complexity of the average case of an algorithm is usually much more complicated to analyze than that of the worst case. Moreover, the probabilistic distribution that one assumes for the average case may not actually apply to real situations. Accordingly, unless otherwise stated or implied, the complexity of an algorithm shall mean the function which gives the running time of the worst case in terms of the input size. This is not too strong an assumption, since the complexity of the average case for many algorithms is proportional to the worst case.

Rate of Growth; Big O Notation

Suppose M is an algorithm, and suppose n is the size of the input data. Clearly the complexity $f(n)$ of M increases as n increases. It is usually the rate of increase of $f(n)$ that we want to examine. This is usually done by comparing $f(n)$ with some standard function, such as

$$\log_2 n, \quad n, \quad n \log_2 n, \quad n^2, \quad n^3, \quad 2^n$$

The rates of growth for these standard functions are indicated in Fig. 3-6, which gives their approximate values for certain values of n. Observe that the functions are listed in the order of their rates of growth: the logarithmic function $\log_2 n$ grows most slowly, the exponential function 2^n grows most rapidly, and the polynomial functions n^c grow according to the exponent c.

n \ $g(n)$	$\log n$	n	$n \log n$	n^2	n^3	2^n
5	3	5	15	25	125	32
10	4	10	40	100	10^3	10^3
100	7	100	700	10^4	10^6	10^{30}
1000	10	10^3	10^4	10^6	10^9	10^{300}

Fig. 3-6 Rate of growth of standard functions.

The way we compare our complexity function $f(n)$ with one of the standard functions is to use the functional "big O" notation which we formally define below.

efinition: Let $f(x)$
order $g(x$

Definition: Let $f(x)$ and $g(x)$ be arbitrary functions defined on **R** or a subset of **R**. We say "$f(x)$ is of order $g(x)$", written

$$f(x) = O(g(x))$$

if there exists a real number k and a positive constant C such that, for all $x > k$, we have

$$|f(x)| \leq C|g(x)|$$

We also write

$$f(x) = h(x) + O(g(x)) \qquad \text{when} \qquad f(x) - h(x) = O(g(x))$$

(The above is called the "big O" notation since $f(x) = o(g(x))$ has an entirely different meaning.)

Consider now a polynomial $P(x)$ of degree m. Then we show in Problem 3.27 that $P(x) = O(x^m)$. Thus, for example,

$$7x^2 - 9x + 4 = O(x^2) \qquad \text{and} \qquad 8x^3 - 576x^2 + 832x - 248 = O(x^3)$$

Complexity of Well-known Algorithms

Assuming $f(n)$ and $g(n)$ are functions defined on the positive integers, then

$$f(n) = O(g(n))$$

means that $f(n)$ is bounded by a constant multiple of $g(n)$ for almost all n.

To indicate the convenience of this notation, we give the complexity of certain well-known searching and sorting algorithms in computer science:

(*a*) Linear search: $O(n)$
(*b*) Binary search: $O(\log n)$
(*c*) Bubble sort: $O(n^2)$
(*d*) Merge-sort: $O(n \log n)$

Solved Problems

FUNCTIONS

3.1. State whether or not each diagram in Fig. 3-7 defines a function from $A = \{a, b, c\}$ into $B = \{x, y, z\}$.

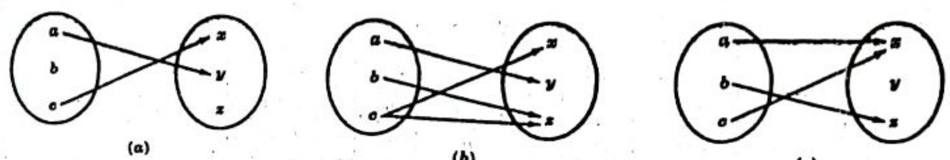


Fig. 3-7

(*a*) No. There is nothing assigned to the element $b \in A$.
(*b*) No. Two elements, x and z, are assigned to $c \in A$.
(*c*) Yes.

3.2. Let $X = \{1, 2, 3, 4\}$. Determine whether or not each relation below is a function from X into X.

(*a*) $f = \{(2,3), (1,4), (2,1)(3,2), (4,4)\}$.
(*b*) $g = \{(3,1), (4,2), (1,1)\}$.
(*c*) $h = \{(2,1), (3,4), (1,4), (2,1), (4,4)\}$.

Recall that a subset f of $X \times X$ is a function $f: X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.

(*a*) No. Two different ordered pairs $(2, 3)$ and $(2, 1)$ in f have the same number 2 as their first coordinate.
(*b*) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.
(*c*) Yes. Although $2 \in X$ appears as the first coordinate in two ordered pairs in h, these two ordered pairs are equal.

3.3. Let A be the set of students in a school. Determine which of the following assignments defines a function on A.

(*a*) To each student assign his age.
(*b*) To each student assign his teacher.
(*c*) To each student assign his sex.
(*d*) To each student assign his spouse.

A collection of assignments is a function on A if and only if each element a in A is assigned exactly in one element. Thus:

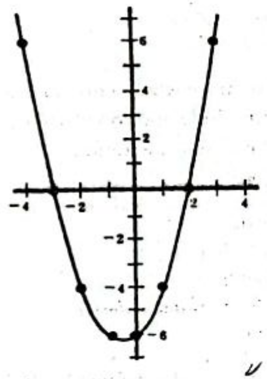
(*a*) Yes, because each student has one and only one age.
(*b*) Yes, if each student has only one teacher; no, if any student has more than one teacher.
(*c*) Yes.
(*d*) No, if any student is not married; yes otherwise.

3.4. Sketch the graph of:

(*a*) $f(x) = x^2 + x - 6$
(*b*) $g(x) = x^3 - 3x^2 - x + 3$

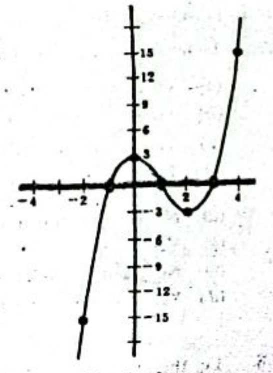
Set up a table of values for x and then find the corresponding values of the function. Since the functions are polynomials, plot the points in a coordinate diagram and then draw a smooth continuous curve through the points. See Fig. 3-8.

x	$f(x)$
-4	6
-3	0
-2	-4
-1	-6
0	-6
1	-4
2	0
3	6



Graph of f

x	$g(x)$
-2	-15
-1	0
0	3
1	0
2	-3
3	0
4	15



Graph of g

Fig. 3-8

3.5. Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3-9. Find the composition function $g \circ f: A \to C$.

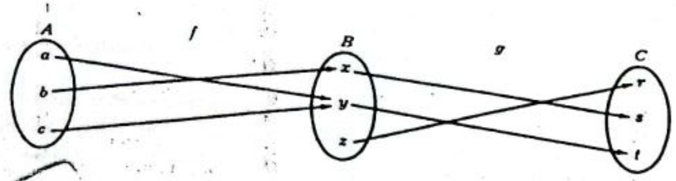


Fig. 3-9

We use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \to y \to t, \qquad b \to x \to s, \qquad c \to y \to t$$

3.6. Let the functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula defining the composition function $g \circ f$.

Compute $g \circ f$ as follows: $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$.

Observe that the same answer can be found by writing

$$y = f(x) = 2x + 1 \quad \text{and} \quad z = g(y) = y^2 - 2$$

and then eliminating y from both equations:

$$z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

3.7. Determine if each function is one-to-one.

(*a*) To each person on the earth assign the number which corresponds to his age.
(*b*) To each country in the world assign the latitude and longitude of its capital.
(*c*) To each book written by only one author assign the author.
(*d*) To each country in the world which has a prime minister assign its prime minister.

(*a*) No. Many people in the world have the same age.
(*b*) Yes.
(*c*) No. There are different books with the same author.
(*d*) Yes. Different countries in the world have different prime ministers.

3.8. Let the functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-10.

(*a*) Determine if each function is onto.
(*b*) Find the composition function $h \circ g \circ f$.

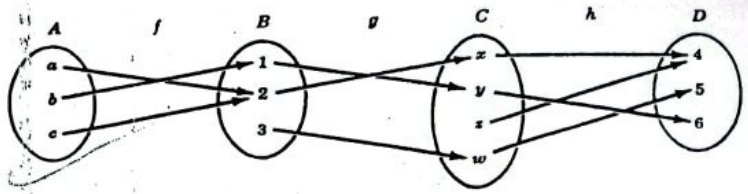


Fig. 3-10

(*a*) The function $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A.
The function $g: B \to C$ is not onto since $z \in C$ is not the image of any element in B.
The function $h: C \to D$ is onto since each element in D is the image of some element of C.

(*b*) Now $a \to 2 \to x \to 4$, $b \to 1 \to y \to 6$, $c \to 2 \to x \to 4$. Hence $h \circ g \circ f = \{(a, 4), (b, 6), (c, 4)\}$.

3.9. Consider functions $f: A \to B$ and $g: B \to C$. Prove the following:

(*a*) If f and g are one-to-one, then the composition function $g \circ f$ is one-to-one.
(*b*) If f and g are onto functions, then $g \circ f$ is an onto function.

(*a*) Suppose $(g \circ f)(x) = (g \circ f)(y)$; then $g(f(x)) = g(f(y))$. Hence $f(x) = f(y)$ because g is one-to-one. Furthermore, $x = y$ since f is one-to-one. Accordingly, $g \circ f$ is one-to-one.

(*b*) Let c be any arbitrary element of C. Since g is onto, there exists a $b \in B$ such that $g(b) = c$. Since f is onto, there exists an $a \in A$ such that $f(a) = b$. But then

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

Hence each $c \in C$ is the image of some element $a \in A$. Accordingly, $g \circ f$ is an onto function.

3.10. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse function f^{-1}. Find a formula for f^{-1}.

Let y be the image of x under the function f:

$$y = f(x) = 2x - 3$$

Consequently, x will be the image of y under the inverse function f^{-1}. Solve for x in terms of y in the above equation:

$$x = (y + 3)/2$$

Then $f^{-1}(y) = (y + 3)/2$. Replace y by x to obtain

$$f^{-1}(x) = \frac{x + 3}{2}$$

which is the formula for f^{-1} using the usual independent variable x.

3.11. Prove the following generalization of DeMorgan's law: For any class of sets $\{A_i\}$, we have

$$(\cup_i A_i)^c = \cap_i A_i^c$$

We have:

$$x \in (\cup_i A_i)^c \quad \text{iff } x \notin \cup_i A_i, \quad \text{iff } \forall_i \in I, x \notin A_i, \quad \text{iff } \forall_i \in I, x \in A_i^c, \quad \text{iff } x \in \cap_i A_i^c$$

Therefore, $(\cup_i A_i)^c = \cap_i A_i^c$. (Here we have used the logical notations iff for "if and only if" and $\forall$ for "for all".)

RDINALITY
3.12. Find the c
(*a*) $A = \{$
(*b*) $B = \{$

CARDINALITY

3.12. Find the cardinal number of each set.

(a) $A = \{a, b, c, \ldots, y, z\}$ (d) $D = \{10, 20, 30, 40, \ldots\}$
(b) $B = \{1, -3, 5, 11, -28\}$ (e) $E = \{6, 7, 8, 9, \ldots\}$
(c) $C = \{x: x \in \mathbf{N}, x^2 = 5\}$

(a) $|A| = 26$ since there are 26 letters in the English alphabet.
(b) $|B| = 5$.
(c) $|C| = 0$ since there is no positive integer whose square is 5, i.e., since C is empty.
(d) $|D| = \aleph_0$ because $f: \mathbf{N} \to D$, defined by $f(n) = 10n$, is a one-to-one correspondence between $\mathbf{N}$ and D.
(e) $|E| = \aleph_0$ because $g: \mathbf{N} \to E$, defined by $g(n) = n + 5$ is a one-to-one correspondence between $\mathbf{N}$ and E.

3.13. Show that the set $\mathbf{Z}$ of integers has cardinality $\aleph_0$.

The following diagram shows a one-to-one correspondence between $\mathbf{N}$ and $\mathbf{Z}$:

$$\begin{array}{ccccccccccc} \mathbf{N} = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\ \mathbf{Z} = & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & \ldots \end{array}$$

That is, the following function $f: \mathbf{N} \to \mathbf{Z}$ is one-to-one and onto:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (1-n)/2 & \text{if } n \text{ is odd} \end{cases}$$

Accordingly, $|\mathbf{Z}| = |\mathbf{N}| = \aleph_0$.

3.14. Let $A_1, A_2, \ldots$ be a countable number of finite sets. Prove that the union $S = \cup_i A_i$ is countable.

Essentially, we list the elements of A_1, then we list the elements of A_2 which do not belong to A_1, then we list the elements of A_3 which do not belong to A_1 or A_2, i.e., which have not already been listed, and so on. Since the A_i are finite, we can always list the elements of each set. This process is done formally as follows.

First we define sets $B_1, B_2, \ldots$ where B_i contains the elements of A_i which do not belong to preceding sets, i.e., we define

$$B_1 = A_1 \quad \text{and} \quad B_k = A_k \backslash (A_1 \cup A_2 \cup \cdots \cup A_{k-1})$$

Then the B_i are disjoint and $S = \cup_i B_i$. Let $b_{i1}, b_{i2}, \ldots, b_{im_i}$ be the elements of B_i. Then $S = \{b_{ij}\}$. Let $f: S \to \mathbf{N}$ be defined as follows:

$$f(b_{ij}) = m_1 + m_2 + \cdots + m_{i-1} + j$$

If S is finite, then S is countable. If S is infinite then f is a one-to-one correspondence between S and $\mathbf{N}$. Thus S is countable.

3.15. Prove Theorem 3.2: A countable union of countable sets is countable.

Suppose $A_1, A_2, A_3, \ldots$ are a countable number of countable sets. In particular, suppose $a_{i1}, a_{i2}, a_{i3}, \ldots$ are the elements of A_i. Define sets $B_2, B_3, B_4, \ldots$ as follows:

$$B_k = \{a_{ij}: i + j = k\}$$

For example, $B_6 = \{a_{15}, a_{24}, a_{33}, a_{42}, a_{51}\}$. Observe that each B_k is finite and

$$S = \cup_i A_i = \cup_k B_k$$

By the preceding problem $\cup_k B_k$ is countable. Hence $S = \cup_i A_i$ is countable and the theorem is proved.

3.16. Prove Theorem 3.3: The set I of all real numbers between 0 and 1 inclusive is uncountable.

The set I is clearly infinite, since it contains $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Suppose I is denumberable. Then there exists a one-to-one correspondence $f: \mathbf{N} \to I$. Let $f(1) = a_1, f(2) = a_2, \ldots$; that is, $I = \{a_1, a_2, a_3, \ldots\}$. We list the elements $a_1, a_2, \ldots$ in a column and express each in its decimal expansion:

$$\begin{aligned} a_1 &= 0.x_{11}x_{12}x_{13}x_{14}\ldots \\ a_2 &= 0.x_{21}x_{22}x_{23}x_{24}\ldots \\ a_3 &= 0.x_{31}x_{32}x_{33}x_{34}\ldots \\ a_4 &= 0.x_{41}x_{42}x_{43}x_{44}\ldots \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where $x_{ij} \in \{0, 1, 2, \ldots, 9\}$. (For those numbers which can be expressed in two different decimal expansions, e.g., $0.200\,0000\ldots = 0.199\,9999\ldots$, we choose the expansion which ends with nines.)

Let $b = 0.y_1y_2y_3y_4\ldots$ be the real number obtained as follows:

$$y_i = \begin{cases} 1 & \text{if } x_{ii} \neq 1 \\ 2 & \text{if } x_{ii} = 1 \end{cases}$$

Now $b \in I$. But

$$\begin{aligned} b &\neq a_1 \text{ because } y_1 \neq x_{11} \\ b &\neq a_2 \text{ because } y_2 \neq x_{22} \\ b &\neq a_3 \text{ because } y_3 \neq x_{33} \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Therefore b does not belong to $I = \{a_1, a_2, \ldots\}$. This contradicts the fact that $b \in I$. Hence the assumption that I is denumerable must be false, so I is uncountable.

SPECIAL MATHEMATICAL FUNCTIONS

3.17. Find: (a) $\lfloor 7.5 \rfloor$; $\lfloor -7.5 \rfloor, \lfloor -18 \rfloor$; (b) $\lceil 7.5 \rceil, \lceil -7.5 \rceil, \lceil -18 \rceil$.

(a) By definition, $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x, hence $\lfloor 7.5 \rfloor = 7, \lfloor -7.5 \rfloor = -8$, $\lfloor -18 \rfloor = -18$.
(b) By definition, $\lceil x \rceil$ denotes the least integer that is not less than x, hence $\lceil 7.5 \rceil = 8, \lceil -7.5 \rceil = -7$, $\lceil -18 \rceil = -18$.

3.18. Find: (a) $25 \pmod 7$; (b) $25 \pmod 5$; (c) $-35 \pmod{11}$; (d) $-3 \pmod 8$.

When k is positive, simply divide k by the modulus M to obtain the remainder r. Then $r = k \pmod M$. If k is negative, divide $|k|$ by M to obtain the remainder r'. Then $k \pmod M = M - r'$ (when $r' \neq 0$). Thus:

(a) $25 \pmod 7 = 4$, (c) $-35 \pmod{11} = 11 - 2 = 9$,
(b) $25 \pmod 5 = 0$, (d) $-3 \pmod 8 = 8 - 3 = 5$.

3.19. Using arithmetic modulo $M = 15$, evaluate: (a) $9 + 13$; (b) $7 + 11$; (c) $4 - 9$; (d) $2 - 10$.

Use $a + M \equiv a \pmod M$:

(a) $9 + 13 = 22 \equiv 22 - 15 = 7$. (c) $4 - 9 = -5 \equiv -5 + 15 = 10$.
(b) $7 + 11 = 18 \equiv 18 - 15 = 3$. (d) $2 - 10 = -8 \equiv -8 + 15 = 7$.

3.20. Simplify: (a) $\dfrac{n!}{(n-1)!}$; (b) $\dfrac{(n+2)!}{n!}$.

(a) $\dfrac{n!}{(n-1)!} = \dfrac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = n$ or, simply, $\dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$.

(*b*) $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = (n+2)(n+1) = n^2+3n+2$

or simply, $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1) = n^2+3n+2$.

3.21. Evaluate: (*a*) $\log_2 8$; (*b*) $\log_2 64$; (*c*) $\log_{10} 100$; (*d*) $\log_{10} 0.001$.

(*a*) $\log_2 8 = 3$ since $2^3 = 8$. (*c*) $\log_{10} 100 = 2$ since $10^2 = 100$.

(*b*) $\log_2 64 = 6$ since $2^6 = 64$. (*d*) $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$.

Remark: Frequently, logarithms are expressed using approximate values. For example, using tables or calculators, we obtain

$$\log_{10} 300 = 2.4771 \quad \text{and} \quad \log_e 40 = 3.6889$$

as approximate answers. (Here $e = 2.718281\ldots$.)

RECURSIVE FUNCTIONS

3.22. Let a and b be positive integers, and suppose Q is defined recursively as follows:

$$Q(a,b) = \begin{cases} 0 & \text{if } a < b \\ Q(a-b,b)+1 & \text{if } b \le a \end{cases}$$

(*a*) Find: (i) $Q(2,5)$, (ii) $Q(12,5)$.
(*b*) What does this function Q do? Find $Q(5861,7)$.

(*a*) (i) $Q(2,5) = 0$ since $2 < 5$.

(ii) $Q(12,5) = Q(7,5)+1$
$= [Q(2,5)+1]+1 = Q(2,5)+2$
$= 0+2 = 2$.

(*b*) Each time b is subtracted from a, the value of Q is increased by 1. Hence $Q(a,b)$ finds the quotient when a is divided by b. Thus $Q(5861,7) = 837$.

3.23. Let n denote a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L(\lfloor n/2 \rfloor)+1 & \text{if } n > 1 \end{cases}$$

Find $L(25)$ and describe what this function does.

Find $L(25)$ recursively as follows:

$$\begin{aligned} L(25) &= L(12)+1 \\ &= [L(6)+1]+1 = L(6)+2 \\ &= [L(3)+1]+2 = L(3)+3 \\ &= [L(1)+1]+3 = L(1)+4 = 0+4 = 4 \end{aligned}$$

Each time n is divided by 2, then the value of L is increased by 1. Hence L is the greatest integer such that

$$2^L \le n$$

Accordingly,
$$L(n) = \lfloor \log_2 n \rfloor$$

3.24. Use the definition of the Ackermann function to find $A(1,3)$.

We have the following 15 steps:

(1) $A(1,3) = A(0, A(1,2))$
(2) $A(1,2) = A(0, A(1,1))$
(3) $A(1,1) = A(0, A(1,0))$
(4) $A(1,0) = A(0,1)$
(5) $A(0,1) = 1+1 = 2$
(6) $A(1,0) = 2$
(7) $A(1,1) = A(0,2)$
(8) $A(0,2) = 2+1 = 3$
(9) $A(1,1) = 3$
(10) $A(1,2) = A(0,3)$
(11) $A(0,3) = 3+1 = 4$
(12) $A(1,2) = 4$
(13) $A(1,3) = A(0,4)$
(14) $A(0,4) = 4+1 = 5$
(15) $A(1,3) = 5$

The forward indention indicates that we are postponing an evaluation and are recalling the definition, and the backward indention indicates that we are backtracking. Observe that (*a*) of the definition is used in Steps 5, 8, 11 and 14; (*b*) in Step 4; and (*c*) in Steps 1, 2 and 3. In the other steps we are backtracking with substitutions.

MISCELLANEOUS PROBLEMS

3.25. Find the domain D of each of the following real-valued functions of a real variable:

(*a*) $f(x) = \dfrac{1}{x-2}$. (*c*) $f(x) = \sqrt{25-x^2}$.
(*b*) $f(x) = x^2 - 3x - 4$. (*d*) $f(x) = x^2$ where $0 \le x \le 2$.

When a real-valued function of a real variable is given by a formula $f(x)$, then the domain D consists of the largest subset of $\mathbf{R}$ for which $f(x)$ has meaning and is real, unless otherwise specified.

(*a*) f is not defined for $x-2=0$, i.e., for $x=2$; hence $D = \mathbf{R}\setminus\{2\}$.
(*b*) f is defined for every real number; hence $D = \mathbf{R}$.
(*c*) f is not defined when $25-x^2$ is negative; hence $D = [-5,5] = \{x: -5 \le x \le 5\}$.
(*d*) Here, the domain of f is explicitly given as $D = \{x: 0 \le x \le 2\}$.

3.26. For any $n \in \mathbf{N}$, let $D_n = (0, 1/n)$, the open interval from 0 to $1/n$. Find:

(*a*) $D_3 \cup D_7$; (*b*) $D_3 \cap D_{20}$; (*c*) $D_s \cup D_t$; (*d*) $D_s \cap D_t$.

(*a*) Since $(0,1/3)$ is a superset of $(0,1/7)$, $D_3 \cup D_7 = D_3$.
(*b*) Since $(0,1/20)$ is a subset of $(0,1/3)$, $D_3 \cap D_{20} = D_{20}$.
(*c*) Let $m = \min(s,t)$, that is, the smaller of the two numbers s and t; then D_m is equal to D_s or D_t and contains the other as a subset. Hence $D_s \cup D_t = D_m$.
(*d*) Let $M = \max(s,t)$, that is, the larger of the two numbers s and t; then $D_s \cap D_t = D_M$.

3.27. Suppose $P(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_m n^m$ has degree m. Prove $P(n) = O(n^m)$.

Let $b_0 = |a_0|, b_1 = |a_1|, \ldots, b_m = |a_m|$. Then, for $n \ge 1$,

$$P(n) \le b_0 + b_1 n + b_2 n^2 + \cdots + b_m n^m = \left(\frac{b_0}{n^m} + \frac{b_1}{n^{m-1}} + \cdots + b_m\right) n^m$$
$$\le (b_0 + b_1 + \cdots + b_m) n^m = M n^m$$

where $M = |a_0| + |a_1| + \cdots + |a_m|$. Hence $P(n) = O(n^m)$.

For example, $5x^3 + 3x = O(x^3)$ and $x^5 - 4\,000\,000 x^2 = O(x^5)$.

3.28. Prove

3.28. Prove Theorem 3.4 (Cantor): $|A| < |\text{Power}(A)|$ (where Power (A) is the power set of A).

The function $g: A \to \text{Power}(A)$ defined by $g(a) = \{a\}$ is clearly one-to-one; hence $|A| \le |\text{Power}(A)|$. If we show that $|A| \ne |\text{Power}(A)|$, then the theorem will follow. Suppose the contrary, that is, suppose $|A| = |\text{Power}(A)|$ and that $f: A \to \text{Power}(A)$ is a function which is both one-to-one and onto. Let $a \in A$ be called a "bad" element if $a \notin f(a)$, and let B be the set of bad elements. In other words,

$$B = \{x: x \in A,\ x \notin f(x)\}$$

Now B is a subset of A. Since $f: A \to \text{Power}(A)$ is onto, there exists $b \in A$ such that $f(b) = B$. Is b a "bad" element or a "good" element? If $b \in B$ then, by definition of B, $b \notin (b) = B$, which is impossible. Likewise, if $b \notin B$ then $b \in f(b) = B$, which is also impossible. Thus the original assumption that $|A| = |\text{Power}(A)|$ has led to a contradiction. Hence the assumption is false, and so the theorem is true.

3.29. Prove the following equivalent formulation of the Schroeder-Bernstein Theorem 3.5: Suppose $X \supseteq Y \supseteq X_1$ and $X \simeq X_1$. Then $X \simeq Y$.

Since $X \simeq X_1$, there exists a one-to-one correspondence (bijection) $f: X \to X_1$. Since $X \supseteq Y$, the restriction of f to Y, which we also denote by f, is also one-to-one. Let $f(Y) = Y_1$. Then Y and Y_1 are equipotent,

$$X \supseteq Y \supseteq X_1 \supseteq Y_1$$

and $f: Y \to Y_1$ is bijective. But now $Y \supseteq X_1 \supseteq Y_1$ and $Y \simeq Y_1$. For similar reasons, X_1 and $f(X_1) = X_2$ are equipotent,

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2$$

and $f: X_1 \to X_2$ is bijective. Accordingly, there exists equipotent sets $X, X_1, X_2, \ldots$ and equipotent sets $Y, Y_1, Y_2, \ldots$ such that

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2 \supseteq Y_2 \supseteq X_3 \supseteq Y_3 \supseteq \cdots$$

and $f: X_k \to X_{k+1}$ and $f: Y_k \to Y_{k+1}$ are bijective.

Let

$$B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \cap \cdots$$

Then

$$X = (X\setminus Y) \cup (Y\setminus X_1) \cup (X_1\setminus Y_1) \cup \cdots \cup B$$
$$Y = (Y\setminus X_1) \cup (X_1\setminus Y_1) \cup (Y_1\setminus X_2) \cup \cdots \cup B$$

Furthermore, $X\setminus Y,\ X_1\setminus Y_1,\ X_2\setminus Y_2, \ldots$ are equipotent. In fact, the function

$$f: (X_k\setminus Y_k) \to (X_{k+1}\setminus Y_{k+1})$$

is one-to-one and onto.

Consider the function $g: X \to Y$ defined by the diagram in Fig. 3-11. That is,

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_k\setminus Y_k \text{ or } x \in X\setminus Y \\ x & \text{if } x \in Y_k\setminus X_k \text{ or } x \in B \end{cases}$$

Then g is one-to-one and onto. Therefore $X \simeq Y$.

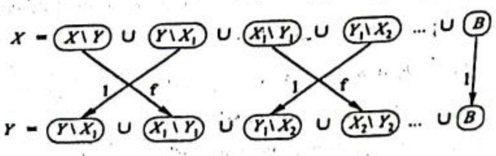


Fig. 3-11

Supplementary Problems

FUNCTIONS

3.30. Let $W = \{a, b, c, d\}$. Determine whether each set of ordered pairs is a function from W into W.

(a) $\{(b,a),(c,d),(d,a),(c,d),(a,d)\}$ (c) $\{(a,b),(b,b),(c,b),(d,b)\}$
(b) $\{(d,d),(c,a),(a,b),(d,b)\}$ (d) $\{(a,a),(b,a),(a,b),(c,d)\}$

3.31. Let the function g assign to each name in the set {Britt, Martin, David, Alan, Rebecca} the number of different letters needed to spell the name. Write g as a set of ordered pairs.

3.32. Let $W = \{1,2,3,4\}$ and let $g: W \to W$ be defined by Fig. 3-12. (a) Write g as a set of ordered pairs. (b) Find the image of g. (c) Write the composition function $g \circ g$ as a set of ordered pairs.

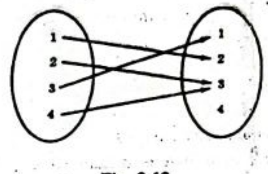


Fig. 3-12

3.33. Let $V = \{1,2,3,4\}$ and let

$$f = \{(1,3),(2,1),(3,4),(4,3)\} \quad \text{and} \quad g = \{(1,2),(2,3),(3,1),(4,1)\}$$

Find: (a) $f \circ g$; (b) $g \circ f$; (c) $f \circ f$.

3.34. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x - 7$. Find a formula for the inverse function $f^{-1}: \mathbf{R} \to \mathbf{R}$.

PROPERTIES OF FUNCTIONS

3.35. Prove: If $f: A \to B$ and $g: B \to A$ satisfy $g \circ f = 1_A$, then f is one-to-one and g is onto.

3.36. Prove Theorem 3.1: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

3.37. Prove: If $f: A \to B$ is invertible with inverse function $f^{-1}: B \to A$, then $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

3.38. For each positive integer n in N, let A_n be the following subset of the real numbers R:

$$A_n = (0, 1/n) = \{x: 0 < x < 1/n\}$$

Find: (a) $A_c \cup A_8$ (c) $\cup(A_i: i \in J)$ (e) $\cup(A_i: i \in K)$
(b) $A_3 \cap A_7$ (d) $\cap(A_i: i \in J)$ (f) $\cap(A_i: i \in K)$

where J is a finite subset of N and K is an infinite subset of N.

3.39. Consider an indexed class of sets $\{A_i: i \in I\}$, a set B and an index i_0 in I. Prove:

(a) $B \cap (\cup_i A_i) = \cup_i (B \cap A_i)$.
(b) $\cap(A_i: i \in I) \subseteq A_{i_0} \subseteq \cup(A_i: i \in I)$.