NS103 (a).

Quiz I → ODE → 15.
" II → PDE → 15.
Mid-Sem → ODE → 35/30.
End-Sem → PDE → 35/30.

(1) The flow of current in an electrical circuit
(2) The flow of fluids.
(3) Growth/Decay of Population.

ex 1:- $\frac{dy}{dx} + xy = x^2$

1st Order → 1st.
degree → 1

ex-2:- $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = \sin x$.

order = 3
degree = 1

ex:3 $\frac{du}{dt} + \left(\frac{\partial u}{\partial x}\right)^2 = 4$

ex-4 $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

more than one dependent variable,
dependent variable.

every dependent derivative/~~variable~~ is more than one time with itself or with each other.

ex5 $\left[\frac{dy}{dx}\right]^3 = \left[y^{1/3}\right]^3 = y \Rightarrow \left(\frac{dy}{dx}\right)^3 = y$

ex 6 $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0$ → 0 → Non-linear.

ex 7 :- $\frac{dy}{dx} + (y)^2 = 0$, → Non-linear.

ex.8 :- $\frac{d^2y}{dx^2} + (y)\left(\frac{dy}{dx}\right) = x^2$.

→ non-linear.

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(#) An ODE is linear If every dependent variable and derivative present in the ODE ~~across~~ occurs to first degree and product of derivative ~~or~~ and/or dependent variable don't occur.

⟹ **Solution of differential equations :-**

(#) General / Particular solution :- Particular solⁿ

unknown.

ODE : $\frac{dy}{dx} = 2x$ , $y = x^2$. , $y = x^2 + c$.

$c \in R$

algebric .

$x^2 - 1 = 0$ $x = \pm 1$ general soln. arbitary const.

$\log x = 0$ $\sin(hn) = 0$

---

e.x $\frac{dy}{dx} = x + 1$ (ODE) $y = \frac{x^2}{2} + x + 1$

$y = \frac{x^2}{2} + x + e.$

$y = \frac{x^2}{2} + x \mp \sqrt{3}$

$y = \frac{x^2}{2} + x + c.$

general solⁿ

ODE :-

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$$

$$y = -\frac{x^2}{4}.$$

$$\Rightarrow \frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{4} = 0.$$

Singular solⁿ

# First order, First degree ODE :-

General form :- $\frac{dy}{dx} = f(x, y)$ — (i)

→ Variable seperable :-

$$\int M(x)\,dx = \int N(y)\,dy + c$$

order = $n$ ⇒ $n$ no. of arbitary const.

**ex1** $\frac{dy}{dx}$ $\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + c.$

$\Rightarrow \quad \tan^{-1}(y) = \tan^{-1}(x) + c.$

**ex2:** $x \cdot \frac{dy}{dx} = \sqrt{1-y^2}.$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{x}.$$

$\Rightarrow \quad \sin^{-1}(y) = \ln x + c.$

**ex3** $x(1+y^2)\,dx = y(1+x^2)\,dy.$

$\Rightarrow \quad \frac{x\,\cancel{(1+y^2)}}{(1+x^2)\cancel{(1+y^2)}}\,dx = \frac{y\,\cancel{(1+x^2)}}{\cancel{(1+x^2)}(1+y^2)}\,dy.$

$\Rightarrow \quad \frac{dz}{\cancel{x}z} = \frac{du}{xu}.$

$\Rightarrow \quad \ln(1+x^2) = \ln(1+y^2) + c.$

# Homogeneous :-

$$\underline{M(x,y)}\,dx + \underline{N(x,y)}\,dy = 0$$

both homogeneous funct^n of same degree,

* **Homogeneous function :-**

f(x) is called homogeneous function of degree n if :-

$$\boxed{f(tx, ty) = t^n f(x,y)}$$

eg $x^2 + xy$.

$x \cot^{-1}\left(\frac{y}{x}\right) = t^{1}\, x \cot\left(\frac{y}{x}\right)$

$\sin\left(\frac{x}{y}\right) = t^0 \sin\left(\frac{x}{y}\right)$

$y \sin x =$

ex: $x^2 y\,dx - (x^3 + y^3)\,dy = 0$.

$\underset{M}{\downarrow}$ $\qquad$ $\underset{N}{\downarrow}$

homogeneous $f(x)$ of degree 3.

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{M(x,y)}{N(x,y)}$

$y = v \cdot x$.

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

$\Rightarrow v + x\dfrac{dv}{dx} = -\dfrac{M(x, vx)}{N(x, vx)} = f(v)$

$\Rightarrow \cancel{v +} \cdot x\dfrac{dv}{dx} = f(v) - v$.

$\Rightarrow \displaystyle\int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c \quad \Big| \quad v = \frac{y}{x}$.

$\Rightarrow v = \phi(x) + c$

$\Rightarrow \boxed{\dfrac{y}{x} = \phi(x) + c}$

$\cancel{y} =$

# Homogeneous ODE :-

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{--- (i)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

$$\Rightarrow y = v \cdot x$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{-M(x,vx)}{N(x,vx)} = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

$$c + \int \frac{dx}{x} = \int \frac{dv}{f(v)-v}$$

$\Rightarrow v = \frac{y}{x}$ put there for sol$^n$.

Ex-1: $x^2 y\,dx - (x^3+y^3)\,dy = 0$.

$M = x^2 y$.   $y = vx$.

$N = -(x^3+y^3)$

$\Rightarrow M = vx^3$.

$N = (v^3+1)x^3$.

$\Rightarrow v + x\dfrac{dy}{dx} = +\dfrac{M(x,y)}{N(x,y)}$.

$\Rightarrow v + x\dfrac{dy}{dx} = +\dfrac{v\cancel{x^3}}{(v^3+1)\cancel{x^3}}$

$\Rightarrow \cancel{x\,dx} = x\dfrac{dy}{dx} = \dfrac{+v}{(v^3+1)} \mp v$.

$=$

$\dfrac{dx}{x} = \dfrac{dy\,(v^3+1)}{-(v^4 \cancel{+\ldots})}$

$\dfrac{dx}{x} = -dy\left(\dfrac{1}{v} + \dfrac{1}{v^4}\right)$

$\Rightarrow -\ln x + c = \ln y + \left(-\dfrac{1}{3v^3}\right)$

$\Rightarrow \boxed{-\ln x + c = \ln\left(\dfrac{y}{x}\right) - \dfrac{x^3}{3y^3}}$.

$\Rightarrow$

Ans

# Special forms:-

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}, \quad \frac{a}{a'} \neq \frac{b}{b'}.$$

$$\left.\begin{matrix} a, b, c \\ a', b', c' \end{matrix}\right\} \in R$$

$ax+by+c=0 \Rightarrow y = -\frac{ax}{b} - \frac{c}{b}.$

$a'x+b'y+c'=0 \Rightarrow y = -\frac{a'x}{b'} - \frac{c'}{b'}$

$\Leftrightarrow -\frac{a}{b} \neq \frac{-a'}{b'}$

Let $P(h, k)$ be POI of their intersection

$x-h = X \Rightarrow x \to h \quad X \to 0$

$y-k = Y \Rightarrow y \to k \quad Y \to 0$

$x = h + X \Rightarrow \frac{dx}{dX} = 1$

$y = K + Y \Rightarrow \frac{dy}{dY} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dY} \cdot \frac{dY}{dX} \cdot \frac{dX}{dx}.$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dX}}$$

$$\frac{dY}{dX} = \frac{a(h+X) + b(k+Y) + c}{a'(h+X) + b'(K+Y) + c'}$$

$$= \frac{aX + bY + \cancel{c}(ah+bk+c)}{a'X+b'Y+(a'h+b'K+c)}$$

$$\Rightarrow \frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} \qquad \begin{bmatrix} ah+bK+c=0 \\ a'h+b'k+c=0 \end{bmatrix}$$

$\hookrightarrow$ homogeneous ODE of degree 1.

Let $\cancel{y}\, Y = v \cdot X$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + x\frac{dv}{dX} = \frac{aX + bv\cancel{X}}{a'X + b'v\cancel{X}} = \frac{a+bv}{a'+b'v}.$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\left(\frac{a+bv}{a'+b'v} - v\right)}$$

$$\Rightarrow \cancel{\frac{dx}{x} = \frac{dv(a'+b'v)}{a + a'v + bv - b'v^2}}.$$

$$= \cancel{\frac{dv(a'+b'v)}{}}$$

$$\Rightarrow \boxed{v = \frac{Y}{X} = \frac{y-k}{x-h}}$$

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Ex:- Solve the ODE.

$$\frac{dy}{dx} = \frac{x+y+4}{x-y-6} = \frac{ax+by+c}{a'x+b'y+c}$$

$a=1 \quad b=1 \quad c=4$

$a'=1 \quad b'=-1 \quad c'=-6$

$\frac{a}{a'} \neq \frac{b}{b'}$ → Special form.

$ax+by+c=0$ & $a'x+b'y+c'=0$ are not parallel.

(#) $(h,k)$ be (POI)

$x=h+X \qquad y=k+Y$ ~~where~~

$$\frac{dy}{dx} = \frac{X+Y+(h+k+4)}{X-Y+(h-k-6)}$$

$\Rightarrow h+k+4=0 \quad | \quad h-k-6=0$

$\Rightarrow h+k=-4 \quad | \quad h-k=6$

$\Rightarrow 2h=2$

$\boxed{h=1}$, $\boxed{k=-5}$

$$\Rightarrow \frac{dY}{dX} = \frac{X+Y}{X-Y}$$

Let $Y = vX$.

$$v + X\frac{dv}{dY} = \frac{X+Y}{X-Y} = \frac{1+v}{1-v}.$$

$$X\frac{dv}{dY} = \frac{1+v}{1-v} - v.$$

$$X\frac{dv}{dY} = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v} = \frac{v^2+1}{1-v}.$$

$$\frac{dX}{X} = \frac{dv}{\cancel{1}+v^2+1}(1-v).$$

$$\ln X = \int\left(\frac{1}{v^2+1} - \frac{v}{v^2+1}\right)dv.$$

$$c + \ln X = \tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) \cancel{+c}.$$

$$\ln(x-1) + c = \tan^{-1}\left(\frac{Y}{X}\right) - \frac{1}{2}\ln\left(1+\frac{Y^2}{X^2}\right)$$

$$\Rightarrow \boxed{\ln(x-1) + c = \tan^{-1}\left(\frac{y+5}{x-1}\right) - \frac{1}{2}\ln\left(1+\frac{(y+5)^2}{(x-1)^2}\right)}$$

Ans

# Case 2 :-

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad , \frac{a}{a'} = \frac{b}{b'} = \frac{1}{\lambda}$$

$$\frac{dy}{dx} = \frac{ax+by+c}{a\lambda x+b\lambda y+c'} = \frac{ax+by+c}{\lambda(ax+by)+c'}$$

Let $z = ax+by$.

$$\Rightarrow \frac{dz}{dx} = a+b\frac{dy}{dx}.$$

$$\Rightarrow \frac{1}{b}\left(\frac{dz}{dx}-a\right) = \frac{dy}{dx} \longrightarrow (i)$$

$$\Rightarrow \frac{1}{b}\left(\frac{dz}{dx}-a\right) = \frac{z+c}{\lambda z+c'}$$

Replace $\boxed{z = ax+by.}$

ex:- Solve $\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$.

$\Rightarrow \quad z = x+y$. $\qquad \frac{a}{a'} = \frac{b}{b'} = \frac{1}{\lambda}$

$\Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx}$.

$$\frac{dy}{dx} = \left(\frac{dz}{dx} - 1\right)$$

$$\Rightarrow \left(\frac{dz}{dx} - 1\right) = \frac{z+4}{z-6}$$

$$\frac{dz}{dx} = \frac{z+4+z-6}{z-6}.$$

$$\frac{dz}{dx} = \frac{2(z-1)}{(z-6)}$$

$$\frac{(z-6)}{(z-1)} dz = 2\,dx.$$

$$\Rightarrow \frac{(z-1)-5}{(z-1)} dz = 2dx.$$

$$\Rightarrow dz - \frac{5}{z-1} dz = 2dx.$$

$$\Rightarrow z - 5\ln(z-1) = 2x + c$$

$$z = x+y.$$

$$\Rightarrow x+y - 5\ln(x+y-1) = 2x + c. \quad \text{Ans}$$

# Exact ODE :- An ordinary DE of form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$ is called exact if $\exists$ a funct$^n$ $U(x,y)$

s.t $\boxed{d(U(x,y)) = M(x,y)\,dx + N(x,y)\,dy}$

$$d(U(x,y)) = \frac{\partial U}{\partial x}\cdot dx + \frac{\partial U}{\partial y}\cdot dy$$

Total derivative of a function.

e.g $y\cdot dx + x\,dy = 0$.

$$y\cdot dx + x\,dy = d(xy) = \frac{\partial(xy)}{\partial x}\cdot dx + \frac{\partial(xy)}{\partial y}\cdot dy$$

$$= y\,dx + x\,dy.$$

$\therefore$ Exact ODE

$\therefore U = (x\cdot y)$

eg. $\dfrac{y\,dx - x\,dy}{y^2} = 0$

$\Rightarrow$ Total derivative of $d\left(\frac{x}{y}\right)$

$$U = \frac{x}{y}.$$

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## (#) How to check for exact ODE?

$\Rightarrow$ ODE: $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact iff and only if (iff) $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

e.g $y\,dx - x\,dy \Rightarrow \quad M = y \qquad N = -x.$

$$\Rightarrow \frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow \text{Not exact.}$$

## (#) How to solve?

Result - ODE: $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof: Given: $M(x,y)\,dx + N(x,y)\,dy = 0$. is exact

$\Rightarrow \exists$ a function $U(x,y)$ s.t Total derivative of $d(U(x,y)) = M\,dx + N\,dy$.

$$\Rightarrow \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = M\,dx + N\,dy.$$

$$= \frac{dM}{\partial y} = \frac{\partial^2 U}{\partial x \partial y} \qquad \frac{dN}{\partial y} = \frac{\partial^2 U}{\partial x \partial y}.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{(given)}$$

Let $V(x,y) = \int \overset{y = \text{const.}}{M(x,y)}\, dx$

$$\Rightarrow \frac{\partial V}{\partial x} = M(x,y)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial N}{\partial x}.$$

$$N(x,y) = \frac{\partial V}{\partial y} + \phi(y)$$

$$\Rightarrow M(x,y)\,dx + N(x,y)\,dy \cancel{= 0}.$$

$$\Rightarrow \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \phi(y) \cancel{= 0}\, dy.$$

$$\Rightarrow d(V(x,y)) + \frac{dF(y)}{dy} \cdot dy.$$

$$\frac{dF(y)}{dy} = \phi(y) \text{ say.}$$

$$\Rightarrow d[\underbrace{V(x,y) + F(y)}_{U}]$$

# Procedure :-

$$\int M(x,y)\,dx = V(x,y)$$

$\hookrightarrow$ y as const.

$$+ \int N(x,y)\,dy = c$$

$\hookrightarrow$ x as const.

$\downarrow$ choose those terms do not contain x.

**ex:-** Solve $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$.

**Soln:-** $M(x,y)\,dx + N(x,y)\,dy = 0$

$$M(x,y) = y\cos x + \sin y + y$$

$$N(x,y) = \sin x + x\cos y + x.$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1.$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{exact ODE.}$$

y constant                x as const

$$\Rightarrow \int (y\cos x + \sin y + y)\,dx + \int \cancel{x\cos y\, 0\, dy} = c$$

$$\Rightarrow \boxed{y\sin x + x\sin y + xy = c} \quad \text{Ans}$$

eg. Solve $e^y\,dx + (xe^y \cancel{dy} + 2y)\,dy = 0$.

$M = e^y$

$N = xe^y + 2y$.

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = e^y \\ \quad \parallel \\ \dfrac{\partial M}{\partial x} = e^y \end{array}\right\} \text{exact ODE}$$

$$\Rightarrow \int e^y\,dx + \cancel{x}\int 2y\,dy = c$$

$$\Rightarrow xe^y + \frac{2y^2}{2} = c$$

$$\Rightarrow \boxed{xe^y + y^2 = c} \quad \text{Ans}$$

eg:- Solve $x \cdot e^{x^2+y^2}\,dx + y(e^{x^2+y^2}+1)\,dy = 0$

$M = xe^{x^2+y^2}$ | $N = y(e^{x^2+y^2}+1)$

$\frac{\partial M}{\partial y} = xe^{x^2+y^2}\cdot 2y$ | $\frac{\partial M}{\partial x} = ye^{x^2+y^2}\cdot 2x + 0$

Exact ODE.

$\Rightarrow \int xe^{x^2+y^2}\,dx + \int y\,dy = C$

$\Rightarrow \cancel{e^{y^2}}\cdot \int xe^{x^2+y^2}\,dx + \frac{y^2}{2} = c$

~~$\Rightarrow e^{y^2}\cdot\left[e^{x^2}\frac{x^2}{2} - \int e^{x^2}\cdot 2x\right.$~~

$x^2+y^2 = t$

$\cancel{2}x\,dx = \cancel{2}\,dt$

$\int \frac{1}{2}e^{t}\,dt + \frac{y^2}{2} = c$

$\frac{1}{2}e^{t} + \frac{y^2}{2} = c$

$$\boxed{\frac{1}{2}e^{x^2+y^2} + \frac{y^2}{2} = c} \text{ Ans}$$

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# Integrating factor : (IF) : A function say $\mu(x,y)$ is called I.F if multiplication of it to the ODE $M(x,y)dx + N(x,y)dy = 0$ changes to exact ODE.

$$\Rightarrow \mu(x,y)\left[Mdx + Ndy\right] = d\left(v(x,y)\right)$$

$$\frac{1}{y^2}\left[y\,dx - x\,dy\right] = d\left(\frac{x}{y}\right) \Rightarrow \frac{1}{y^2} = \text{I.F.}$$

ex: $x\,dx + y\,dy + (x^2+y^2)x^2dx = 0.$

~~$\frac{\partial M}{\partial y} = 2yx^2$.~~

$$\frac{2}{2}\left(\frac{x}{x^2+y^2}dx + \frac{y\,dy}{x^2+y^2}\right) + x^2dx = 0.$$

$$\frac{1}{2}d\left[\ln(x^2+y^2)\right] + x^2dx = 0,$$

# How to find Integrating factor?

**Result** :- If $x.M + y.N \neq 0$ and

$M(x,y)\,dx + N(x,y)\,dy = 0$ is Homogeneous

Then $\dfrac{1}{xM + y.N}$ is the I.F.

**Proof** :- ~~M.x + N.y~~

$$\Rightarrow \quad M\,dx + N\,dy = \frac{1}{2}\left[(M.x + N.y)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (M.x - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$\Rightarrow \quad \frac{M\,dx + N\,dy}{xM + yN} = \frac{1}{2}\left[\left(\frac{Mx+Ny}{Mx+Ny}\right)\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{Mx-Ny}{Mx+Ny}\right)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[d\left(\ln(x.y)\right) + \cancel{d}\, f\left(\frac{x}{y}\right) d\left[\ln\left(\frac{x}{y}\right)\right]\right]$$

$$\frac{x}{y} = e^{\ln(x/y)} = f\left(\ln\left(\frac{x}{y}\right)\right)$$

$$= \frac{1}{2}\left[d\left(\ln(xy)\right) + \overset{(dt)}{d\ln\left(\frac{x}{y}\right)} \cdot \overset{(t)}{f\ln\left(\frac{x}{y}\right)}\right]$$

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Solve:- $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$.

$M = x^2y - 2xy^2$ | $N = 3x^2y - x^3$.

$xM = x^3y - 2x^2y^2$ | $yN = 3x^2y^2 - x^3y$.

$\boxed{xM + yN = x^2y^2} \neq 0$,

$I.F = \dfrac{1}{x^2y^2}$

$\Rightarrow \dfrac{x^2y - 2xy^2}{x^2y^2}dx + \dfrac{3x^2y - x^3}{x^2y^2}dy = 0$.

$\Rightarrow \left(\dfrac{1}{y} - \dfrac{2}{x}\right)dx + \left(\dfrac{3}{y} - \dfrac{x}{y^2}\right)dy = 0$,

$M = \dfrac{1}{y} - \dfrac{2}{x}$.

$\dfrac{\partial M}{\partial y} = -\dfrac{1}{y^2}$.

$\dfrac{\partial N}{\partial x} = -\dfrac{1}{y^2}$.

$\rightarrow$ exact ODE,

$$\Rightarrow \int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y}\, dy = c.$$

$$\Rightarrow \boxed{\frac{x}{y} - 2\ln x + 3\ln y = c.} \quad \text{Ans}$$

(II) Result: If $(xM - yN) \neq 0$ And

$Mdx + Ndy = 0$ is of the form

$f_1(x \cdot y)\, y\, dx + f_2(x \cdot y)\, x\, dy = 0.$

Then I.F $= \dfrac{1}{xM - yN}$.

Proof: $$Mdx + Ndy = \frac{1}{2}\left[(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$\Rightarrow \frac{Mdx + Ndy}{xM - yN} = \frac{1}{2}\left[\left(\frac{Mx + Ny}{xM - yN}\right)\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{\cancel{Mx - Ny}}{\cancel{xM - Ny}}\right)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left(\frac{f_1(x \cdot y) y \cdot x + f_2(x \cdot y) \cdot x \cdot y}{x f_1(x \cdot y) \cdot y - y \cdot f_2(x \cdot y) \cdot x}\right) d(\ln(x \cdot y))$$

$$+ d\ln\left(\frac{x}{y}\right)$$

$$= \frac{1}{2}\left( f_3(x \cdot y)\, d\ln(x \cdot y) + d\ln\left(\frac{x}{y}\right)\right)$$

$$\because\ x \cdot y = e^{\ln(x \cdot y)}$$

$$= \frac{1}{2}\left( \underbrace{f_4(\ln(x \cdot y)) \cdot d\ln(x \cdot y)}_{\text{Total derivative of one}} + \underbrace{d\ln\left(\frac{x}{y}\right)}_{\text{Total derivative of another}} \right)$$

ex:- $(x^2 \cdot y^2 + x \cdot y + 1)\, y\, dx + (x^2 y^2 - x \cdot y + 1)\, x\, dy = 0.$

$$M = (x^2 \cdot y^2 + x \cdot y + 1)\, y$$

$$xM = xy(x^2y^2 + xy + 1)$$

$$yN = xy(x^2y^2 - x \cdot y + 1)$$

$$xM - yN \neq 0 \;=\; 2x^2y^2.$$

$$I.F = \frac{1}{xM - yN} = \frac{1}{2x^2y^2}.$$

$$\Rightarrow \frac{(x^2y^2 + xy + 1)\, y\,dx}{2x^2y^2} + \frac{(x^2y^2 - xy + 1)\, x\,dy}{2x^2y^2} = 0$$

$$\Rightarrow \left(\frac{y}{2} + \frac{\cancel{y}}{2x\cancel{y}} + \frac{y}{2x^2y^2}\right)dx + \left(\frac{x}{2} - \frac{\cancel{x}}{2\cancel{x}y} + \frac{x\,dy}{2x^2y^2}\right)$$

$$M = \frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}.$$

$$\frac{\partial M}{\partial y} = \frac{1}{2} - \frac{1}{2x^2y^2}.$$

$$\frac{\partial N}{\partial x} = \frac{1}{2} - \frac{1}{2x^2y^2}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

exact ODE.

$$\Rightarrow \int \frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\,dx + \int \frac{\cancel{x}}{\cancel{2}} - \frac{1}{2y} + \frac{\cancel{1}}{\cancel{2x}y^2}\,dy$$

$$\Rightarrow \frac{xy}{2} + \frac{\ln x}{2} - \frac{1}{2xy} + \cancel{\frac{xy}{2}} - \frac{\ln y}{2} - \cancel{\frac{1}{2xy}} = c$$

$$\Rightarrow \cancel{xy + \frac{1}{2}(\ln x - \ln y) - \frac{1}{xy} = c}.$$

$$\frac{xy}{2} + \frac{1}{2}\ln x - \frac{1}{2xy} - \frac{\ln y}{2} = c$$

Ans.

# Result: If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ only then.

$$I \cdot F = e^{\int f(x)\,dx}.$$

**Proof:** Let $\mu(x)$ is the I.F of

$$M\,dx + N\,dy = 0$$

$\Rightarrow$ $M\,\mu(x)\,dx + N\,\mu(x)\,dy = 0$ is exact.

$$\frac{\partial\, \mu(x) M(x,y)}{\partial y} = \frac{\partial}{\partial x}\left(\mu(x) N(x,y)\right)$$

$$\Rightarrow \mu(x)\frac{\partial M(x,y)}{\partial y} = \frac{d\mu(x)}{dx} \cdot N(x,y) + \mu(x)\frac{\partial N}{\partial x}.$$

$$\Rightarrow \mu(x)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\partial \mu(x)}{\partial x} \cdot N$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\partial \mu(x)}{\partial x} \cdot \frac{1}{\mu(x)}$$

$$\Rightarrow f(x) = \frac{d\mu(x)}{dx} \cdot \frac{1}{\mu x}.$$

$$\Rightarrow \int f(x)\,dx = \int \frac{\partial \mu(x)}{\mu(x)}$$

$$\Rightarrow \int f(x)\,dx = \ln \mu$$

$$\frac{\mu(x)}{c} = e^{\int f(x)dx}$$

$$\mu(x) = c \cdot \left(e^{\int f(x)dx}\right)$$

Let $c = 1$.

$$U(x) = e^{\int f(x)dx}.$$

Solve: $(x^2+y^2)dx - 2x \cdot y\,dy = 0$,

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y.$$

↳ Not exact.

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-(-2y)}{-2xy} = -\frac{2}{x} = f(x)$$

only

$$I.F = e^{\int -\frac{2}{x}dx}$$

$$= e^{-2\ln x}$$

$$\Rightarrow (x)^{-2} = \frac{1}{x^2}.$$

$$\Rightarrow \frac{x^2+y^2}{x^2}dx - \frac{2xy}{x^2}dy = 0.$$

$$\Rightarrow \left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2}.$$

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$$\int\left(1+\frac{y^2}{x^2}\right)dx + \int 0 \cdot dy = c.$$

$$\Rightarrow \boxed{x - \frac{y^2}{x} = c.}$$ Ans

(#) Result: If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = f(y)$ only.

$$I.F = e^{-\int f(y)\,dy}$$

Sol$^n$ Let $U(y) = I.F$.

$$U(y)\left[M\,dx + N\,dy\right] = 0 \text{ is exact.}$$

$$\frac{\partial}{\partial y}\left(U(y)M\right) = \frac{\partial}{\partial x} U(y) N.$$

$$\Rightarrow \frac{\partial U(y)}{\partial y} \cdot M + U(y)\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \cdot U(y)$$

$$\Rightarrow \frac{\partial U(y)}{dy} M = \cancel{\partial N} -\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \cdot U(y)$$

$$\Rightarrow \frac{\partial U(y)}{U(y)} = -f(y)\,dy.$$

$$\Rightarrow \ln\left(\frac{U}{c}\right) = -\int f(y)\,dy.$$

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$$\boxed{u(y) = e^{-\int f(y)\,dy}}$$

(4) Solve:- $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dx = 0.$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x.$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x.$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy)}$$

$$= \cancel{\frac{2y}{2y}\left(6x^2y^3 + 4\right)}$$

$$= \frac{2}{y}\left(\frac{y}{2}\left(\frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy}\right)\right) = \frac{2}{y}.$$

$$\boxed{\cancel{I.F.} = \frac{2}{y}}$$

$\Rightarrow$ $2(6x^2y^3 + 4x)dx + (4x^3y^2 - \frac{2x^2}{y})dy = 0$

$\Rightarrow$ $\frac{\partial M}{\partial y} = 18x^2y^2$

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$$I.F = e^{\int -\frac{2}{y} dy} = e^{\ln y^{-2}} = \frac{1}{y^2}.$$

$$\Rightarrow \left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 6x^2y - \frac{2x}{y^2},$$

exact ODE.

$$\frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}.$$

$$\Rightarrow \int \left(3x^2y^2 + \frac{2x}{y}\right)dx + 0 = c$$

$$\Rightarrow \boxed{x^3y^2 + \frac{x^2}{y} = c}$$ Ans.

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## * Linear ~~ord~~ ODE :-

$$\frac{dy}{dx} + P(x)\cdot y = Q(y)$$

$$\Rightarrow dy + P(x)\cdot y\,dx = Q(y)\,dx$$

$$\Rightarrow \underbrace{(P(x)\cdot y - Q(x))}_{M}\,dx + \underbrace{1}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = P(x) \qquad \frac{\partial N}{\partial x} = 0$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{P(x)}{N} = P(x) \rightarrow \text{only funct}^{n} \text{ of } x.$$

$$\boxed{I.F \Rightarrow e^{\int P(x)dx}}$$

$$\Rightarrow e^{\int P(x)dx}\left[\frac{dy}{dx} + P(x)\cdot y\right] = Q(x)e^{\int P dx}$$

$$\Rightarrow e^{\int P(x)dx}\frac{dy}{dx} + P(x)\cdot y\, e^{\int P dx} = Q(x)e^{\int P dx}$$

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$$\frac{d}{dx} e^{\int P dx} = e^{\int P dx} \cdot P$$

$$\Rightarrow \frac{dy}{dx} \cdot e^{\int P dx} + y \frac{d}{dx}\left(e^{\int P dx}\right) = Q\, e^{\int P dx}$$

$$\Rightarrow \int \frac{d}{dx}\left(y \cdot e^{\int P dx}\right) = \int Q\, e^{\int P dx} dx + c$$

$$\Rightarrow y\, e^{\int P dx} = \int Q(x)\, e^{\int P dx} \cdot dx + c$$

$$\Rightarrow y = e^{-\int P dx}\left[\int Q(x) e^{\int P dx} dx + c\right]$$

first $\rightarrow$ $e^{\int P dx}$

$\downarrow$

$$\frac{d}{dx}\left(y \cdot e^{\int P dx}\right) = \cancel{\int} Q(x)\, e^{\int P dx}$$

then integrate

## (#) Linear ODE :-

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$I.F = e^{\int p\,dx}$$

$$\frac{d}{dx}\left(y\,e^{\int p\,dx}\right) = Q(x)\,e^{\int p\,dx}.$$

$$\Rightarrow \quad y\cdot e^{\int p\,dx} = \int Q(x)\,e^{\int p\,dx}\,dx + c$$

$$y = e^{-\int p\,dx}\left[\int Q(x)e^{\int p\,dx}\,dx + c\right]$$

ex:- $$x\frac{dy}{dx} - (x+1)y = x^2 - x^3.$$

$$e^{\int p\,dx}.$$

$$\frac{dy}{dx} - \left(1+\frac{1}{x}\right)y = x - x^2.$$

$$P = -1 - 1/x \quad \Big| \quad Q = x - x^2.$$

$$e^{\int p\,dx} = e^{\int -\left(1+\frac{1}{x}\right)dx}.$$

$$= e^{-x-\ln x}$$

$$= (e^{-x})\,e^{-\ln x},$$

$$= \frac{1}{e^x}\cdot\frac{1}{x} = \frac{1}{xe^x}.$$

$$\cancel{y =}$$

$$\frac{d}{dx}\left(y\cdot \cancel{e}\,\frac{1}{e^{x}\cdot x}\right) = \frac{(x-x^2)\cancel{e}\,e^{-x}}{x}$$

$$= (1-x)e^{-x}$$

$$y\cdot\frac{e^{-x}}{x} = \int (1-x)\,e^{-x}$$

$$= \int (e^{-x} - xe^{-x})\,dx + c$$

$$y\cdot\frac{e^{-x}}{x} = -e^{-x} - \left[x\int e^{-x}\,dx - \int \frac{e^{-x}}{-1}\,dx + c\right.$$

$$= -e^{-x} - \left[\frac{xe^{-x}}{-1} + \frac{e^{-x}}{-1}\right]$$

$$y\frac{e^{-x}}{x} = xe^{-x} + c$$

$$\boxed{y = xe^{x}\left[xe^{-x}\right] + c\cdot e^{x}\cdot x.}$$

Ans

$\log(mn) = \log(m+n)$

Solve: $\frac{dy}{dx} + y\tan x = \sin(2x)$ ; $y(0) = 1$
where $x = 0$
$y = 1$

$\Rightarrow e^{\int P dx} = e^{\int \tan x \, dx} = e^{\int \frac{\sin x}{\cos x} dx}$

$= e^{\sec^2 x} = e^{-\ln(\pm\cos x)} = \frac{1}{\cos x}$.

$\Rightarrow \frac{d}{dx}\left(y \cdot e^{\sec^2 x}\right) = \sin 2x$ .

$\ln(\pm\cos x)$

$\frac{d}{dx} y \cdot \frac{1}{\cos x} = \sin 2x \cdot \sec x$.

$= 2\sin x$

$y \sec x = \int 2\sin x \, dx + c$

$= 2(-\cos x) + c$

$y = -2\cos^2 x + c(\cos x)$ .

at $x = 0$ . $y = 1$

$1 = -2 + c$ .

$c = 3$

$y = -2\cos^2 x + 3\cos x$ Ans

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## ODE which can be converted into linear DE :-

⟹ **Bernoulli equation :-**

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n \quad , \quad n \neq 1 .$$

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = Q(x)$$

$$\text{Let } y^{1-n} = z .$$

$$(1-n)\, y^{1-n-1}\frac{dy}{dx} = \frac{dz}{dx} .$$

$$\Rightarrow (1-n)\, y^{-n}\frac{dy}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \left(\frac{1}{1-n}\right)\frac{dz}{dx}$$

Now;

$$\Rightarrow \left(\frac{1}{1-n}\right)\frac{dz}{dx} + P(x)\, z = Q(x) .$$

$$\Rightarrow \frac{dz}{dx} + (1-n)\, P(x) . z = Q(x)\,(1-n)$$

$$I.F = e^{\int (1-n)\, P(x)\, dx} .$$

$$\Rightarrow \frac{d}{dx}\left(z \cdot e^{\int (1-n)\, P(x)\, dx}\right) = Q(x)\,(1-n) \cdot e^{\int (1-n)\, P(x)\, dx} .$$

then $\boxed{z = y^{1-n}}$

Solve:- $\frac{dy}{dx} + \frac{y}{x} = y^2 \frac{\log x}{x}$.

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{yx} = \frac{\log x}{x}.$$

$$\frac{1}{y} = z.$$

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}.$$

$$\Rightarrow -\frac{dz}{dx} + \frac{z}{x}. = \frac{\log x}{x}.$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}.$$

$$e^{\int -\frac{1}{x}dx.} = e^{-\ln x.} = \frac{1}{x}.$$

$$\Rightarrow \frac{d}{dx}\left(z \cdot \frac{1}{x}\right) = -\frac{\log x}{x} \cdot \frac{1}{x}.$$

$$\Rightarrow z \cdot \frac{1}{x} = \int -\frac{\log x}{x^2}\, dx + c.$$

$$= -\left[\log x\left(\frac{-1}{x}\right) - \int \frac{1}{x}\left(\frac{-1}{x}\right) dx\right]$$

$$= -\left[-\frac{\log x}{x} + \left(-\frac{1}{x}\right)\right]$$

$$z = \log x + 1 + cx.$$ Ans

$$\boxed{\frac{1}{y} = \log x + 1 + cx}$$ Ans

# Ricatti equation :

$$\boxed{\frac{dy}{dx} + P(x)\,y + Q(x)\,y^2 = R(x).}$$

Let $y_1(x)$ be a particular solution.

$$\frac{dy_1}{dx} + P(x)\cdot y_1 + Q(x)y_1^2 - R(x) = 0 .$$

Let $\boxed{y = y_1 + z}$ → unknown. be general sol$^n$

⇓ Particular sol$^n$

$$\frac{d}{dx}(y_1+z) + P(x)(y_1+z) + Q(x)(y_1+z)^2 \cancel{- R(x) = 0} = R(x)$$

$$\Rightarrow \frac{dy_1}{dx} + \cancel{\frac{dz}{dx}} P(x)\cdot y_1 + Q(x)y_1^2 + 2y_1 z\,Q(x) + z^2 Q(x) + P(x)\cdot z + \frac{dz}{dx} = R(x)$$

$$\Rightarrow \left(\frac{dy_1}{dx} + P(x)y_1 + Q(x)\,y_1^2 - R(x)\right) + 2y_1 z\,Q(x) + z^2 Q(x) + P(x)z + \frac{dz}{dx} = 0$$

→ 0

$$\frac{dz}{dx} + \left[2y_1 Q(x) + P(x)\right] z + Q(x) z^2 = 0.$$

$$\frac{dz}{dx} + \left[2y_1 Q(x) + P(x)\right] z = -Q(x) z^2.$$

$$\frac{1}{z^2}\frac{dz}{dx} + \left[2y_1 Q(x) + P(x)\right] \times \frac{1}{z} = -Q(x)$$

$$\frac{1}{z} = t.$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{dt}{dx}.$$

$$-\frac{dt}{dx} + \left[2y_1 Q(x) + P(x)\right] t = -Q(x)$$

$$\frac{dt}{dx} - \left[2y_1 Q(x) + P(x)\right] t = Q(x).$$

$$\Rightarrow \boxed{I.F = e^{\int -\left(2y_1 Q(x) + P(x)\right) dx}}.$$

Solve:
$$\frac{dy}{dx} = y^2 - \frac{2}{x^2} \quad ; \quad y_1 = \frac{1}{x} \; (P.S.)$$

$$\Rightarrow \boxed{\frac{dy}{dx} + \underbrace{P(x)}_{0} \cdot y + (-1)\,y^2 = \underbrace{-\frac{2}{x^2}}_{R(x)}}$$

Bernaulli

Let:
$$y = y_1 + z$$

$$\frac{d}{dx}(y_1 + z) = (y_1 + z)^2 - \frac{2}{x^2}.$$

$$\Rightarrow \frac{dy_1}{dx} + \frac{dz}{dx} = y_1^2 + z^2 + 2y_1 z - \frac{2}{x^2}$$

$$\Rightarrow \underbrace{\cancel{\left(\frac{dy_1}{dx} - y_1^2 + \frac{2}{x^2}\right)}}_{0} + \left(\frac{dz}{dx} - 2y_1 z - z^2\right) = 0.$$

$$\Rightarrow \frac{dz}{dx} - 2y_1 z = z^2.$$

$$\Rightarrow \frac{1}{z^2}\frac{dz}{dx} - \frac{2y_1}{z} = 1$$

$$\frac{1}{z} = t.$$

$$\Rightarrow -\frac{1}{z^2}\frac{dz}{dx} = \frac{dt}{dx}.$$

$$\Rightarrow -\frac{dt}{dx} - 2y_1 t = 1.$$

$$\frac{dt}{dx} + \underbrace{2y_1}_{P(x)} t = \underbrace{-1}_{Q(x)}.$$

$$e^{\int P(x)dx} \Rightarrow e^{\int 2y_1 dx}.$$

$$= e^{\int \frac{2}{x} dx} = e^{2\ln x}.$$

$$\boxed{I.F = x^2}$$

$$\frac{d}{dx}(t \cdot x^2) = -1 x^2.$$

$$t \cdot x^2 = \int -1 \cdot x^2 dx + c.$$

$$t \cdot x^2 = -\frac{1}{3} \cdot x^3 + c.$$

$$t = -\frac{1x}{3} + \frac{cx^{-2}}{\cancel{x^2}}.$$

$$\Rightarrow \frac{1}{z} = \frac{-x}{3} + \frac{c}{x^2}.$$

$$\Rightarrow \frac{1}{z} = \frac{-x^3 + 3c}{3x^2}.$$

$$\boxed{z = \frac{3x^2}{3c - x^3}.}$$

$$\boxed{y = \frac{1}{x} + \frac{3x^2}{3c - x^3}}$$ Ans

# First order higher degree :-

$$\left(\frac{dy}{dx}\right)^n + Q_1(x,y)\left(\frac{dy}{dx}\right)^{n-1} + \cdots + Q_{n-1}(x,y)\frac{dy}{dx}$$

$$+ Q_n(x,y)\,y = 0 .$$

$\Rightarrow$ Let $P = \frac{dy}{dx} \Rightarrow P^n + Q_1(x,y)P^{n-1} + \cdots$

$\cdots Q_{n-1}(x,y)P + Q_n(x,y)y = 0$

**Solvable for P :-**

(1) Can be written as.

$(P-\alpha_1)(P-\alpha_2)\cdots(P-\alpha_n) = 0 \quad \alpha_i \in R/\mathbb{C}$

$\Rightarrow (P-\alpha_i) = 0$ for $i = 1, 2, \ldots, n$

$\Rightarrow \left(\frac{dy}{dx}\right) = \alpha_i(x,y)$ for all $i$

$\Rightarrow f_i(x, y, c_i)$ for $i = 1, 2 \ldots n$

$\hookrightarrow n$ solⁿ.

$c_i = c$ for all $i = 1, 2, \ldots n.$

$\Rightarrow f_i(x, y, c) = 0$ for $i = 1, 2 \ldots n.$

$\Rightarrow f_1(x,y,c)\cdot f_2(x,y,c) \cdot f \cdots f_n(x,y,c) = 0$

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(11) Solve $x^2\left(\frac{dy}{dx}\right)^2 + (x \cdot y)\left(\frac{dy}{dx}\right) - 6y^2 = 0$.

Let $\frac{dy}{dx} = p$.

$\Rightarrow x^2 p^2 + xyp - 6y^2 = 0$

$x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$.

$\Rightarrow xp(xp+3y) - 2y(xp+3y) = 0$.

$\Rightarrow (xp-2y)(xp+3y) = 0$.

$\left(p - \frac{2y}{x}\right)\left(p + \frac{3y}{x}\right) = 0$.

| | |
|---|---|
| $p = \frac{2y}{x}$ | $p = -\frac{3y}{x}$. |
| $\frac{dy}{dx} = \frac{2y}{x}$ | $\frac{dy}{dx} = -\frac{3y}{x}$. |
| $\Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$. | $\frac{dy}{y} = -3\frac{dx}{x}$. |
| $\Rightarrow \ln y = 2\ln x + \ln c$ | $\ln y = -3\ln x + \ln c$ |
| $y = x^2c$ | $y = \frac{c}{x^3}$. |
| $y - x^2c = 0$ (3) | $y - \frac{c}{x^3} = 0$. (4). |

$$\Rightarrow (y - x^2 c)\left(y - \frac{c}{x^3}\right) = 0 \rightarrow \text{sol}^n$$

# Soluable for y :-

$\Rightarrow$ Solve: $y = F(x, p)$ — (i)

$$\Rightarrow \frac{dy}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial p}\cdot\frac{dp}{dx}.$$

$$\Rightarrow \boxed{p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\left(\frac{dp}{dx}\right)}$$ first order. first degree. unknown P.

$\Rightarrow$

$$\Psi(x, p, c) = 0 \quad \cdot \text{ eq } (2)$$

$\Rightarrow$ eliminate P between (1) and (2)

# Solve

$$y = 2\left(\frac{dy}{dx}\right)\cdot x - x\left(\frac{dy}{dx}\right)^2$$

Let $\frac{dy}{dx} = p.$

$$\Rightarrow y = 2p\cdot x - xp^2. \quad — (1)$$

$$= F(x, p)$$

$$\Rightarrow \frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - p^2 - x\cdot 2p\frac{dp}{dx}.$$

$$\boxed{p = 2x\frac{dp}{dx} + 2p - p^2 - 2px\frac{dp}{dx}} \rightarrow (2)$$

$$p^2 - p = (2x - 2px)\frac{dp}{dx}.$$

$$\frac{p(p-1)}{2x(1-p)} = \frac{dp}{dx}.$$

$$\cancel{\frac{dp}{dx} = \frac{p}{2x}}.$$

$$(p-1)\left[p + 2x\frac{dp}{dx}\right] = 0.$$

$p = 1$

$$\frac{dp}{dx} = -\frac{p}{2x}.$$

$$2\ln p = -\ln x + \ln c.$$

$$\Rightarrow \cancel{p^2 = \frac{c}{x}}.$$

$$\Rightarrow \cancel{x}\cdot\frac{1}{x} = p^2 \cdot c.$$

$$[\psi(x, p, c) = 0]$$

$$p^2 = \frac{1}{xc}. \Rightarrow p = \frac{\pm 1}{(xc)^{1/2}}.$$

$$\Rightarrow y = 2x\left(\frac{\pm 1}{\sqrt{xc}}\right) - \cancel{x}\cdot\frac{1}{xc}$$

$$y = 2\sqrt{\frac{x}{c}} - \frac{1}{c}, \quad y = -2\sqrt{\frac{x}{c}} - \frac{1}{c}.$$

General sol$^n$

$$p - 1 = 0$$

$$p = 1 \Rightarrow y = 2x - x.$$

$$\Rightarrow y = x \Rightarrow \text{singular soln.}$$

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(7) Solve :- $x - y\frac{dy}{dx} = a\left(\frac{dy}{dx}\right)^2$.

$\Rightarrow \frac{dy}{dx} = p$.

$\Rightarrow x - yp = ap^2$.

$\Rightarrow x - ap^2 = yp$.

$\Rightarrow \frac{x - ap^2}{p} = y = f(x,p)$,

$y = \frac{x}{p} - a \cdot p$.

$\frac{dy}{dx} = \frac{1}{p} + x\left(\frac{-1}{p^2}\right)\frac{dp}{dx} - a \cdot \frac{dp}{dx}$.

$p = \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx} - a \cdot \frac{dp}{dx}$.

$\cancel{p - \frac{1}{p}}\ \frac{1}{p} - p = \left(\frac{x}{p^2} + ap^2\right)\frac{dp}{dx}$.

$\Rightarrow \frac{p^2 - 1}{p} = \frac{x + ap^2}{p^2}\left(-\frac{dp}{dx}\right)$

$\Rightarrow \frac{p^2(p^2-1)}{p(x+ap^2)} = -\frac{dp}{dx}$.

$\Rightarrow \frac{p(p^2-1)}{(x+ap^2} = -\frac{dp}{dx}$.

$$\frac{x + ap^2}{p(p^2-1)} = -\frac{dx}{dp}$$

$$\Rightarrow \boxed{\frac{dx}{dp} + \left[\frac{1}{p(p^2-1)}\right] x = \frac{ap^2}{p(1-p^2)}}$$

$$I \cdot F = e^{\int \frac{1}{p(p^2-1)} dp}$$

$$\frac{1}{p(p^2-1)} = \frac{A}{p} + \frac{Bp + C}{p^2-1}$$

$$\cancel{\frac{1}{p(p-1)(p+1)} = \frac{A}{}}$$

$$\frac{1}{p(p^2-1)} = \frac{A(p^2-1) + (Bp+C)p}{p(p^2-1)}$$

$$\Rightarrow e^{\int -\frac{1}{p} dp} \circ e^{\int + \frac{p\,dp}{(p^2-1)}}$$

$$\Rightarrow e^{-\ln p} \circ e^{+\frac{1}{2}\ln(p^2-1)}$$

$$\Rightarrow \frac{1}{p} \circ \frac{1}{\sqrt{p^2-1}} = \frac{\sqrt{p^2-1}}{p}$$

$$\Rightarrow \frac{d}{dp}\left(\frac{\sqrt{p^2-1}}{p}\cdot x\right) = \frac{\sqrt{p^2-1}}{\cancel{p}}\cdot\frac{ap^{\cancel{2}}}{\cancel{p}(1-p^2)}$$

$$= \frac{\sqrt{p^2-1}\ a}{-(p^2-1)}$$

$$\int \frac{d}{dp}\left(\frac{\sqrt{p^2-1}\cdot x}{p}\right) = \frac{-a}{\sqrt{p^2-1}}$$

$$\frac{x\cdot\sqrt{p^2-1}}{p} =$$

$$x = \frac{p}{\sqrt{p^2-1}}\left[\int \frac{-a\,dp}{\sqrt{p^2-1}} + c\right)$$

$$y = \frac{x}{p} - ap$$

$$\boxed{y = f_1(p)}$$

# Tutorial 1:

1. (a) $\left(\frac{d^2x}{dt^2}\right)^3 + 7\left(\frac{d^2x}{dt^2}\right)^2$.

(c) $y = \frac{dy}{du} + \frac{5u}{\frac{dy}{du}}$.

$y\frac{dy}{du} = \left(\frac{dy}{du}\right)^2 + 5u$.

↳ ORDER 1 degree 2

(d) $\frac{5d^2x}{dt^2} + 2\left[1-\left(\frac{dx}{dt}\right)^3\right]^{1/2} - x = 0$.

(g) $(1+x^2)dy - x(1+y^2)dx = 0$.

$\frac{dy}{1+y^2} = \frac{x}{1+x^2}dx$.

$\Rightarrow \tan(y) = \ln(1+x^2)^{1/2} + c$.

(3) $\frac{dy}{dx} - \frac{y^2}{x} = \frac{9}{x}$.

$x\frac{dy}{dx} - y^2 = 9$.

$\frac{dy}{y^2+(3)^2} = \frac{dx}{x}$.

$\Rightarrow \frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right) = \ln x + c$.

$\cos 2\theta = 2\cos^2\theta - 1$

(c) $\dfrac{dy}{\cos^2 ax} - \dfrac{dx}{y} = 0.$

$\Rightarrow \quad y\,dy = \cos^2 ax\,dx.$

$y\,dy = \dfrac{1+\cos 2ax}{2}\,dx.$

$\dfrac{y^2}{2} = \dfrac{x}{2} + \dfrac{\sin 2ax}{4a} + c.$

(d) $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} = 0.$

$\dfrac{dx}{\sqrt{1-x^2}} = \dfrac{-1}{\sqrt{1-y^2}}\,dy,$

$\Rightarrow \quad \sin^{-1}(x) = -\sin^{-1}(y) + c.$

(e) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$

$\Rightarrow \quad \dfrac{\sec^2 x}{\tan x}\,dx = -\dfrac{\sec^2 y}{\tan y}\,dy.$

$\Rightarrow \quad \ln(\tan x) = -\ln(\tan y) + c$

f. $x(y^2-1)\,dx - y(x^2-1)\,dy = 0.$

$\Rightarrow \quad \dfrac{x}{x^2-1}\,dx = \dfrac{y}{y^2-1}\,dx,$

$\Rightarrow \quad \dfrac{1}{2}\ln(x^2-1) = \dfrac{1}{2}\ln(y^2-1) + c$

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(9) $(x+y)\,dx + dy = 0$,

$\Rightarrow$ ~~$\left(\frac{x}{y}+1\right)dx + \frac{dy}{y} = 0$~~,

$$\frac{dy}{dx} = -x - y,$$

$$\frac{dy}{dx} + y = -x.$$

$$e^{\int 1\,dx} \Rightarrow e^{x}.$$

$$y \cdot e^{x} = \int -x e^{x}\,dx.$$

$$y \cdot e^{x} = -\left[x e^{x} - \int 1 \cdot e^{x}\,dx\right.$$

$$= -\left[x e^{x} - e^{x} + c\right].$$

$$y = \frac{1}{e^{x}}\left[e^{x} - x e^{x} + c\right]$$ Ans

Q.) $\dfrac{dy}{dx} = \dfrac{2x}{\cos y}$

$\Rightarrow \cos y \, dy = 2x \, dx$

$\Rightarrow \sin y = x^2 + c$

Q.) $\dfrac{dy}{3x^2} = \dfrac{dx}{1-e^{-y}}$

$(1-e^{-y}) \, dy = 3x^2 \, dx$

$\Rightarrow y + e^{-y} = x^3 + c$ Ans

Q.) $xy \, dx + (x+1) \, dy = 0$

$\Rightarrow \dfrac{x}{1+x} \, dx = -\dfrac{dy}{y}$

$\Rightarrow \dfrac{x+1-1}{x+1} \, dx = -\dfrac{dy}{y}$

$\Rightarrow \left(1 - \dfrac{1}{x+1}\right) dx = -\dfrac{dy}{y}$

$\Rightarrow x - \ln(x+1) = -\ln y$ Ans

$$\boxed{(x+y)dx + dy = 0, \quad \boxed{x+y=t}}$$

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$$x\,dy = 2\sqrt{y-1}\,dx$$

$$\Rightarrow \frac{dy}{(\sqrt{y-1})^{2}} = \frac{2\,dx}{x}$$

$$\Rightarrow \cancel{2}\sqrt{y-1} = \cancel{2}\ln x + c'$$

Q.4 $(x^2-2y^2)\,dx + xy\,dy = 0$

$$\Rightarrow y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow (x^2 - 2v^2x^2)\,dx + x^2v\,dy = 0$$

$$\Rightarrow (1-2v^2)\,dx + v\,dy = 0$$

$$\frac{2v^2-1}{v} = \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{2v^2-1-v^2}{v} = x\frac{dv}{dx}$$

$$\frac{v^2-1}{v} = x\frac{dv}{dx}$$

$$\frac{dx}{x} = \frac{v}{v^2-1}\,dv$$

$$\Rightarrow \ln x = \frac{1}{2}\ln|v^2-1| + c \text{ Ans}$$

$$\Rightarrow \ln x = \frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2-1\right) + c \text{ Ans}$$

Q.4 $2xy\,dy - (y^2 - x^2)\,dx = 0$.

$\Rightarrow$ $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$\Rightarrow$ $2v^2\,dy - (v^2 - 1)\,dx = 0$.

$$\Rightarrow \cancel{2v^2}\ \frac{dy}{dx} = \frac{v^2 - 1}{2v^{\cancel{2}}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v^{\cancel{2}}}.$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^{\cancel{3}}}{2v}.$$

$$x\frac{du}{dx} = \frac{-1 - v^2}{2v}.$$

$$\Rightarrow \frac{2v\,dv}{1+v^2} = -\frac{dx}{x}.$$

$\Rightarrow$ $\ln(1+v^2) = -\ln x + c$ Ans

Q.5 ~~$(x^2 - 2y^2)\,dx$~~

$$x^2 \frac{dy}{dx} - 3xy - 2y^2 = 0.$$

$$y = vx.$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$\Rightarrow \frac{dy}{dx} - 3v - 2v^2 = 0.$$

$$v + x\frac{dv}{dx} = 3v + 2v^2.$$

$$x\frac{dv}{dx} = 2v + 2v^2.$$

$$\frac{\cancel{2v}\,dv}{v(v+1)} = 2\frac{dx}{x}.$$

$$\left(\frac{1}{v} - \frac{1}{v+1}\right)dv = \frac{2\,dx}{x}.$$

$$\ln v - \ln(v+1) = 2\ln x + C \quad \text{Ans}$$

$$\ln\left(\frac{y}{x}\right) - \ln\left(\frac{y}{x} + 1\right) = 2\ln x + C \quad \text{Ans}$$

$$\frac{dy}{dx} = \frac{y}{x} + \tan yx.$$

## (iii) Solvable for $x$ :-

* $x = f(y, p)$ where $p = \frac{dy}{dx}$ — (i)

Differentiate (i) w.r.t $y$,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial f}{\partial y} \cdot \frac{dy}{dy} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dy}.$$

$$\Rightarrow \frac{1}{p} = \lambda\left(y, p, \frac{dp}{dy}\right) \quad \text{— (2)}.$$

First order

(2) ~~E~~. is 1st order 1st degree.

ODE. solving (1), we get $\psi(y, p, c) = 0$ — (3)

Eliminate $P$ between (1) & (3)

Solve:- $y\left(\frac{dy}{dx}\right)^2 - 2x\frac{dy}{dx} + y = 0$.

$$\frac{y\left(\frac{dy}{dx}\right)^2 + y}{2\left(\frac{dy}{dx}\right)} = x.$$

$p = \frac{dy}{dx}$.

$$\Rightarrow x = \frac{yp^2 + y}{2p}$$

$$\Rightarrow x = \cancel{\frac{p^2}{2} + \frac{1}{2}}. \quad \frac{yp}{2} + \frac{y}{2p}.$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2}p + \frac{y}{2}\frac{dp}{dy} + \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy}.$$

$$\Rightarrow \frac{2}{p} = p + \frac{1}{p} + y\frac{dp}{dy} - \frac{y}{p^2}\frac{dp}{dy}.$$

$$\Rightarrow 2 = p^2 + 1 + yp\frac{dp}{dy} - \frac{y}{p^2}\cdot\frac{dp}{dy}.$$

$$\Rightarrow \cancel{yp\frac{dp}{dy} - \frac{1}{p}\cdot y}.$$

$$\Rightarrow \frac{1}{p} - p = y\frac{dp}{dy}\left[1 - \frac{1}{p^2}\right].$$

$$\Rightarrow 1 - p^2 = y\frac{dp}{dy}\left(\frac{p^2-1}{p}\right)$$

$$\Rightarrow (1-p^2)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0.$$

$$(1-p^2)\left(\frac{p + y\frac{dp}{dy}}{p}\right) = 0$$

$$\left(\frac{1-p^2}{p}\right)\left(p + y\frac{dp}{dy}\right) = 0.$$

general sol$^n$

Singular sol$^n$

$$\Rightarrow p = -y\frac{dp}{dy}.$$

$$\frac{dy}{y} = -\frac{dp}{p}$$

$$\Rightarrow -\ln y = +\ln p + \ln c. \Rightarrow \frac{1}{y} = p\cdot c. \Rightarrow 1(ypA$$

$y \cdot p^2 - 2xp + y = 0$

$$p = \frac{1}{y-c}$$

$\Rightarrow y \cdot \frac{1}{y^2c^2} - 2x\frac{1}{yc} + y = 0$. G.S

$\Rightarrow$

$$\frac{1-p^2}{p} = 0.$$

$$p^2 = 1$$

$$\Rightarrow p = \pm 1.$$

$\Rightarrow y \cdot 1 - 2x(\pm 1) + y = 0.$

$\Rightarrow y = \pm x.$

---

**One Variable missing :-**

| | |
|---|---|
| $f(x, p) = 0$ (Solvable for p.) | or $f(y, p) = 0$. |
| $\Rightarrow p = \psi(x)$ | $p = \psi(y)$ |
| $\frac{dy}{dx} = \psi(x)$ | $\frac{dy}{dx} = \psi(y)$. |
| $\phi_1(x, y, c)$ | $\phi_2(x, y, c) = 0$. |

# Solvable for

$$f(x, p) = 0$$

$$\Rightarrow x = \psi(p) \quad \text{--- (1)}$$

differentiate w.r.t P.

$$\Rightarrow \frac{dx}{dp} = \frac{d\psi}{dp}$$

$$\Rightarrow dx = \left(\frac{d\psi}{dp}\right) dp$$

$$\frac{dy}{dx} = p$$

$$p\,dx = dy$$

$$\Rightarrow p\,dx = \frac{d\psi}{dp} \cdot p\,dp$$

$$\Rightarrow dy = \frac{d\psi}{dp} \cdot p\,dp$$

$$\Rightarrow \left(\frac{dy}{dp}\right) = p \cdot \frac{d\psi}{dp}$$

Solving it we get

$$\psi_3(y, p, c) = 0 \quad \text{--- (2)}$$

eliminate P b/w (1) & (2)

$$f(y, p) = 0$$

$$\Rightarrow y = \psi(p) \quad \text{--- (3)}$$

$$\Rightarrow \frac{dy}{dp} = \frac{d\psi}{dp}$$

$$\Rightarrow dy = \frac{d\psi}{dp} \cdot dp$$

$$\Rightarrow \frac{dy}{dx} = p$$

$$\Rightarrow dy = p\,dx$$

$$\Rightarrow p\,dx = \frac{d\psi}{dp} \cdot dp$$

$$\Rightarrow dx = \frac{1}{p} \frac{d\psi}{dp} \cdot dp$$

$$\left(\frac{dx}{dp}\right) = \frac{1}{p} \frac{d\psi}{dp}$$

Solving it we get

$$\psi_4(x, p, c) = 0 \rightarrow (4)$$

eliminate P between (3) & (4)

Solve $x^2 = \left(\frac{dy}{dx}\right)^2 (a^2 - x^2)$. $a \in R$

$x^2 = p^2 (a^2 - x^2)$

$\Rightarrow \quad p^2 = \frac{x^2}{a^2 - x^2}$.

$\Rightarrow \quad p = \frac{\pm x}{(a^2 - x^2)^{1/2}}$

$\frac{dy}{dx} = \frac{\pm x}{(a^2 - x^2)^{1/2}}$,

$dy = \frac{\pm x \, dx}{(a^2 - x^2)^{1/2}}$.

~~y~~

$a^2 - x^2 = t$.

$y = \pm \frac{1}{2} \int t^{-1/2} dt + c$,

$y = \pm \frac{\cancel{2}}{\cancel{2}} t^{1/2} + c$.

$$\boxed{y = \pm (a^2 - x^2)^{1/2} + c}$$ Ans

$(y - c)^2 = (a^2 - x^2)$ Ans

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$\Rightarrow$ Solve : $x^2 = \left(\frac{dy}{dx}\right)^2 (a^2 - x^2)$.

$$x^2 = p^2(a^2 - x^2) \quad = p = \pm \frac{x}{\sqrt{a^2 - x^2}}.$$

$$x^2(1+p^2) = p^2 a^2.$$

$$x^2 = \frac{p^2 a^2}{(1+p^2)}$$

$$x = \frac{pa}{\sqrt{1+p^2}}.$$

$$\Rightarrow \frac{dx}{dp} = \frac{a}{\sqrt{1+p^2}} \pm pa\left(\frac{-1}{\cancel{2}}\right) \cdot \frac{1}{(1+p^2)^{3/2}} \cdot \cancel{2}p.$$

$$\cancel{= \frac{\pm a}{\sqrt{1+p^2}} + \frac{p^2 a}{(1+p^2)^{3/2}}}$$

$$\frac{dx}{dp} = \frac{a(1+p^2) - p^2 a}{(1+p^2)^{3/2}}.$$

$$\frac{1}{a}\frac{dx}{dp} = \frac{1}{(1+p^2)^{3/2}}.$$

$\frac{dy}{dx} = p$.

$\frac{dy}{p} = dx$

$$dx = (1+p^2)^{-3/2} \cdot a \cdot dp.$$

$$\frac{dy}{p} = (1+p^2)^{-3/2}\, a\, dp.$$

$1 + p^2 = t$

$2p\,dp = dt$

$$\Rightarrow dy = \frac{ap}{(1+p^2)^{3/2}} dp.$$

$$\Rightarrow y = \frac{a}{2}\int \frac{dt}{t^{3/2}} = -\frac{2a}{2} \cdot \frac{1}{t^{1/2}}.$$

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$$y = \frac{-a}{t^{1/2}}.$$

$$y = \frac{-a}{\sqrt{1+p^2}}.$$

$$(y-c) = \frac{-a}{\sqrt{1+p^2}}.$$

$$(y-c)^2 = \frac{a^2}{1+p^2}$$

$$= \frac{a^2}{1 + \frac{x^2}{a^2-x^2}}.$$

$$\Rightarrow \frac{(a^2-x^2)\,\cancel{a^2}}{\cancel{(a^2-x^2)+x^2}}$$

$$\boxed{(y-c)^2 = (a^2-x^2)}$$ Ans

#

# One Variable missing :

$f(x, p) = 0$ or $f(y, p) = 0$

Let $x = \phi(t)$ and $p = \psi(t)$

$$\frac{dy}{dx} = p \Rightarrow p\,dx = dy$$

$$\frac{dx}{dt} = \frac{d\,\phi(t)}{dt} \Rightarrow dx = \frac{d\,\phi(t)}{dt} \cdot dt .$$

$$\Rightarrow p\,dx = p\,\phi'(t)\,dt .$$

$$\Rightarrow \int dy = \int \psi(t)\,\phi'(t)\,dt + c$$

$$\Rightarrow y(t) = \int \psi(t)\,\phi'(t)\,dt + c .$$

$x, y$ are parametric form of eq$^n$.

$y = \phi(t)$ and $p\ \psi(t)$

$$\frac{dy}{dt} = \frac{d\,\phi(t)}{dt}$$

$$\Rightarrow dy = \phi'(t)\,dt .$$

$$\Rightarrow \frac{dy}{p} = \frac{1}{p}\,\phi'(t)\,dt .$$

$$\Rightarrow \int dx = \int \frac{1}{p}\frac{\phi'(t)}{\psi(t)}\,dt + c .$$

$$\Rightarrow x(t) = \int \frac{\phi'(t)}{\psi(t)}\,dt + c$$

Parametric form .

$x = p^3 - p - \cancel{\phi}\ 1$

Let $p = t$.

$x = t^3 - t - 1$.

$\Rightarrow \frac{dx}{dt} = 3t^2 - 1$.

$dx = (3t^2 - 1)\,dt$.

$p\,dx = t(3t^2 - 1)\,dt$

$dy = (3t^3 - t)\,dt$

$y = \frac{3t^4}{4} - \frac{t^2}{2} + c$.

eliminate $p$ get solⁿ,

$y^2 + p^2 = a^2$, $a \in R$

$y = a\cos t$. $\qquad p = a\sin t$

OR

$y = a\sin t$. $\qquad p = a\cos t$.

$\frac{dy}{dt} = a(-\sin t)$

$\frac{dy}{p} = \frac{-a\sin t\,dt}{a\sin t}$.

$\int dx = -\int dt + c$

$\Rightarrow x = -t + c$

$$t = \cos^{-1}\left(\frac{y}{a}\right)$$

$$x = -\cos^{-1}\left(\frac{y}{a}\right) + c$$

$$\cos^{-1}\left(\frac{y}{a}\right) = c - x.$$

$$\frac{y}{a} = \cos(c-x)$$

$$\boxed{y = a\cos(c-x)}$$ Ans

# The Clairaut's form :-

$$y = x \cdot p + f(p)$$

diff w.r.t $x$:

$$\frac{dy}{dx} = \cancel{p + \frac{d f(p)}{dx}}\ p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

$$\Rightarrow \cancel{p} = \cancel{p} + \frac{dp}{dx}\left(x + f'(p)\right)$$

$$\Rightarrow \frac{dp}{dx}\left[x + f'(p)\right] = 0$$

Either $\frac{dp}{dx} = 0$ or $x + f'(p) = 0$.

"Singular solⁿ".

$$dp = 0 \cdot dx$$

$$p = 0 + c$$

$$\boxed{p = c}$$

$$\Rightarrow y = x \cdot \cancel{p}c + f(c)$$

↳ general solⁿ.

just replace p by c you will get general solⁿ

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Solve $y = \frac{dy}{dx} \cdot x + \frac{a}{\frac{dy}{dx}}$.

$$\Rightarrow y = p \cdot x + \frac{a}{p}.$$

$$\Rightarrow y = cx + \frac{a}{c} \rightarrow \text{general sol}^n$$

$$\Rightarrow \frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}.$$

$$\cancel{p} = \cancel{p} + \frac{dp}{dx}\left(x - \frac{a}{p^2}\right)$$

$$x = \frac{a}{p^2}.$$

$$p = \pm\sqrt{\frac{a}{x}}.$$

$$\Rightarrow y = \pm\sqrt{\frac{a}{x}} \cdot x + \frac{a}{\pm\sqrt{\frac{a}{x}}}$$

Singular sol$^n$

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Soln:- $\left(x\frac{dy}{dx}-y\right)\left(x-\frac{dy}{dx}y\right) = 2\frac{dy}{dx}.$

$\Rightarrow (xp-y)(x-py) = 2p.$

$\Rightarrow x^2p - xpy - yx + p^2y^2 = 2p.$

$\Rightarrow$ ~~$p(x^2+y^2)$~~ Let $x^2 = u$ . & $y^2 = v$.

$2x\,dx = du \quad | \quad 2y\,dy = dv.$

$$\frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}\frac{dy}{dx}.$$

$$\boxed{\frac{dv}{du} = \frac{y}{x}\cdot p}$$

$$\Rightarrow \boxed{p = \frac{x}{y}\frac{dv}{du}}$$

$$\Rightarrow \left(\frac{x^2}{y}\frac{dv}{du} - y\right)\left(x - x\frac{dv}{du}\right) = \frac{2x}{y}\frac{dv}{du}$$

$$\Rightarrow \left(\frac{x^2\frac{dv}{du} - y^2}{y}\right)x\left(1-\frac{dv}{du}\right) = \frac{2x}{y}\frac{dv}{du}$$

$$\Rightarrow \left(u\frac{dv}{du} - v\right)\left(1-\frac{dv}{du}\right) = 2\frac{dv}{du}.$$

~~Let~~

$$u\frac{dv}{du} - v = \frac{2\frac{dv}{du}}{\left(1-\frac{dv}{du}\right)}$$

$$V = u\frac{dv}{du} - \frac{2\frac{dv}{du}}{1 - \frac{dv}{du}}$$

Let $\boxed{P_1 = \frac{dv}{du}.}$

$$V = u \cdot P_1 - \frac{2P_1}{1 - P_1}.$$

general sol$^n$

$$V = u \cdot c - \frac{2c}{1-c} \rightarrow \text{general sol}^n$$

$$\cancel{\pm\sqrt{y} = \pm\sqrt{x}c - \frac{2c}{1-c}}$$

$$y^2 = x^2 \cdot c - \frac{2c}{1-c}. \quad G.S$$

The Lagrange form:-

$$y = (f(p)) \cdot x + g(p)$$

If $f(p) = p \Rightarrow$ Clairout's form.

diff w.r.t to $x$.

$$\Rightarrow \frac{dy}{dx} = f(p) + x f'(p) \cdot \frac{dp}{dx} + g'(p) \frac{dp}{dx}.$$

$$\frac{dy}{dx} = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx}.$$

$$p - f(p) = \frac{dp}{dx}\left[xf'(p) + g'(p)\right].$$

$$\Rightarrow \quad \frac{dx}{dp} = \left[\frac{f'(p)}{p-f(p)}\right]x + \frac{g'(p)}{p-f(p)}$$

$$\Rightarrow \quad \frac{dx}{dp} + \frac{f'(p)}{f(p)-p}\,x = \frac{g'(p)}{p-f(p)}.$$

$$If = e^{\int \frac{f'(p)}{f(p)-p}\,dp}.$$

$$\frac{d}{dp}(x \cdot I.f) = I.f \times \left(\frac{g'(p)}{p-f(p)}\right)$$

$$\Rightarrow \quad \boxed{x = \psi(p, c)}$$

eliminate $p$

---

(#) solve $y = (2p)x - p^3$.

$$\Rightarrow \frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 3p^2\frac{dp}{dx}.$$

$$\Rightarrow p = 2p + (2x - 3p^2)\frac{dp}{dx}.$$

$$\Rightarrow -p = 2x - 3p^2\frac{dp}{dx}.$$

$$\Rightarrow p = (3p^2 - 2x)\frac{dp}{dx}$$

$$\frac{dx}{dp} = 3p - \frac{2x}{p}.$$

$$\frac{dx}{dp} + \frac{2x}{p} = 3p.$$

$$IF = e^{\int \frac{2}{p} dp}.$$

$$= e^{2\ln p}. = p^2.$$

$$x \cdot p^2 = \int 3p \cdot p^2 \, dp.$$

$$x \cdot p^2 = \frac{3p^4}{4} + c.$$

$$x = \frac{3p^2}{4} + \frac{c}{p^2}$$

$$y = 2px - p^3$$

} General sol$^n$

Elimination of $p$ from (i) & (ii) gives the sol$^n$

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