Cascade Algorithms in Wavelet Analysis †

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Abstract

In this paper we survey some recent results on cascade algorithms. Let a be a finitely supported sequence on $\mathbb{Z}$. The cascade operator Q_a is the the linear operator on $L_p(\mathbb{R})$ $(1 \le p \le \infty)$ given by

$$Q_a f := \sum_{j \in \mathbb{Z}} a(j) f(2 \cdot - j), \quad f \in L_p(\mathbb{R}).$$

The iteration scheme $Q_a^n f$ $(n = 1, 2, \ldots)$ is called the cascade algorithm associated with a. The L_p convergence of a cascade algorithm is characterized in terms of the p-norm joint spectral radius of two matrices associated with the corresponding mask. For the special case $p = 2$, convergence of a cascade algorithm is characterized in terms of the spectrum of the transition matrix associated with the mask. Then the basic theory on cascade algorithms is employed to give a unified treatment of orthogonal wavelets, biorthogonal wavelets, and fundamental refinable functions. Furthermore, we give a comprehensive review of biorthogonal wavelet bases. Our methods can be used to deal with more complicated problems such as biorthogonal wavelet bases on bounded domains. Finally, we extend our study of cascade algorithms to high dimensional spaces.

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§1. Introduction

In this paper we survey some recent results on cascade algorithms and their applications to wavelet analysis.

As usual, for $1 \le p \le \infty$, $L_p(\mathbb{R})$ denotes the Banach space of all (complex-valued) measurable functions f on $\mathbb{R}$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. The Fourier transform of a function $f \in L_1(\mathbb{R})$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}.$$

The domain of the Fourier transform can be naturally extended to compactly supported distributions.

Let a be a finitely supported sequence on $\mathbb{Z}$. The **cascade operator** Q_a is the linear operator on $L_p(\mathbb{R})$ $(1 \le p \le \infty)$ given by

$$Q_a f := \sum_{j \in \mathbb{Z}} a(j) f(2 \cdot - j), \quad f \in L_p(\mathbb{R}). \tag{1.1}$$

It is easily seen that Q_a is a bounded linear operator on $L_p(\mathbb{R})$. If ϕ is a fixed point of Q_a, *i.e.*, $Q_a \phi = \phi$, then ϕ satisfies the **refinement equation**

$$\phi = \sum_{j \in \mathbb{Z}} a(j) \phi(2 \cdot - j). \tag{1.2}$$

Correspondingly, the sequence a is called the **refinement mask**. Any function satisfying a refinement equation is called a **refinable function**.

If a is a mask with $\sum_{j \in \mathbb{Z}} a(j) = 2$, then it is known (see [1] and [5]) that there is a unique compactly supported distribution ϕ satisfying $\hat{\phi}(0) = 1$ and (1.2). This distribution is said to be **the normalized solution** to the refinement equation with mask a.

Taking the Fourier transform of both sides of (1.2), we obtain

$$\hat{\phi}(\xi) = H(\xi/2) \hat{\phi}(\xi/2), \quad \xi \in \mathbb{R},$$

where

$$H(\xi) := \frac{1}{2} \sum_{j \in \mathbb{Z}} a(j) e^{-ij\xi}, \quad \xi \in \mathbb{R}. \tag{1.3}$$

Clearly, H is 2π-periodic and $H(0) = 1$. Let f be a compactly supported function in $L_p(\mathbb{R})$. Then we also have

$$\widehat{Q_a f}(\xi) = H(\xi/2) \hat{f}(\xi/2), \quad \xi \in \mathbb{R}. \tag{1.4}$$

The iteration scheme $Q_a^n f$ $(n = 1, 2, \ldots)$ is called the **cascade algorithm** associated with a. Suppose there exists a function ϕ in $L_p(\mathbb{R})$ with $\hat{\phi}(0) = 1$ such that

$$\lim_{n\to\infty} \|Q_a^n f - \phi\|_p = 0.$$

Then ϕ is the normalized solution to the refinement equation (1.2). In what follows we will show that the initial function f must satisfy the Strang-Fix conditions of order 1 (see [21]), that is,

$$\hat{f}(0) = 1 \quad \text{and} \quad \hat{f}(2k\pi) = 0 \quad \forall\, k \in \mathbb{Z} \setminus \{0\}.$$

Let $f_n := Q_a^n f$, $n = 1, 2, \ldots$. It follows from (1.4) that

$$\hat{f}_n(2^{n+1}k\pi) = \hat{f}_{n-1}(2^n k\pi) = \cdots = \hat{f}(2k\pi), \quad k \in \mathbb{Z}.$$

Since the supports of f_n are uniformly bounded, we have $\lim_{n\to\infty} \|f_n - \phi\|_1 = 0$. Moreover,

$$|\hat{f}_n(\xi) - \hat{\phi}(\xi)| \le \|f_n - \phi\|_1 \quad \forall\, \xi \in \mathbb{R}.$$

Hence, for $k \in \mathbb{Z}$ we have

$$\hat{f}(2k\pi) = \lim_{n\to\infty} \hat{f}_n(2^{n+1}k\pi) = \lim_{n\to\infty} \hat{\phi}(2^{n+1}k\pi) = \begin{cases} 0 & \text{for } k \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{for } k = 0, \end{cases}$$

by the Riemann-Lebesgue lemma. Thus, f satisfies the Strang-Fix conditions of order 1.

We say that the cascade algorithm associated with a converges in the L_p norm if there exists a compactly supported function $\phi \in L_p(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|Q_a^n f - \phi\|_p = 0$$

for any compactly supported function f in $L_p(\mathbb{R})$ satisfying the Strang-Fix conditions of order 1. The L_∞ convergence is often referred to as uniform convergence.

Let u be a compactly supported function in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$. Then u satisfies the Strang-Fix conditions of order 1 if and only if

$$\sum_{j\in\mathbb{Z}} u(\cdot - j) = 1$$

(See [20]). Suppose g is a compactly supported function in $L_q(\mathbb{R})$, where $1/p + 1/q = 1$. For $h > 0$, let P_h be the linear operator defined by

$$P_h\phi := \sum_{j\in\mathbb{Z}} \langle \phi, g(\cdot/h - j)/h \rangle \phi(\cdot/h - j), \quad \phi \in L_p(\mathbb{R}),$$

where

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx, \quad \text{for } f \in L_p(\mathbb{R}) \text{ and } g \in L_q(\mathbb{R}).$$

Then we have the following result (see, *e.g.*, [12]):

$$\lim_{h\to 0} \|P_h\phi - \phi\|_p = 0, \quad \phi \in L_p(\mathbb{R}).$$

In particular, we may choose g to be the characteristic function of the interval $[0,1]$. In this case,

$$\langle \phi, g(\cdot/h - j)/h \rangle = \frac{1}{h}\int_{jh}^{(j+1)h} \phi(x)\, dx.$$

Let $\ell(\mathbb{Z})$ denote the linear space of all sequences on $\mathbb{Z}$, and let $\ell_0(\mathbb{Z})$ denote the linear space of all finitely supported sequences on $\mathbb{Z}$. For a bounded subset K of $\mathbb{R}$, let $\ell(K)$ denote the linear space of all sequences supported on $K \cap \mathbb{Z}$. For $j \in \mathbb{Z}$ we use δ_j to denote the sequence on $\mathbb{Z}$ given by $\delta_j(k) = \delta_{jk}$, $k \in \mathbb{Z}$, where δ_{jk} stands for the Kronecker sign, *i.e.*, $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. In particular, we write δ for δ_0. We use ∇ to denote the difference operator on $\ell(\mathbb{Z})$ given by $\nabla u = u - u(\cdot - 1)$ for $u \in \ell(\mathbb{Z})$. The convolution of two elements u and v in $\ell(\mathbb{Z})$ is defined by

$$u * v(j) := \sum_{k\in\mathbb{Z}} u(j-k)v(k), \quad j \in \mathbb{Z},$$

whenever the above sum makes sense. Moreover, $\ell_p(\mathbb{Z})$ denotes the Banach space of all (complex-valued) sequences b on $\mathbb{Z}$ such that $\|b\|_p < \infty$, where

$$\|b\|_p := \left(\sum_{j\in\mathbb{Z}} |b(j)|^p\right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and $\|b\|_\infty$ is the supremum of $|b|$ on $\mathbb{Z}$.

Suppose ϕ is a compactly supported function in $L_p(\mathbb{R})$. Then there exists a positive constant B (see [13]) such that

$$\left\| \sum_{j\in\mathbb{Z}} b(j)\phi(\cdot - j) \right\|_p \le B\|b\|_p \quad \forall\, b \in \ell_p(\mathbb{Z}).$$

We say that the shifts of a function ϕ in $L_p(\mathbb{R})$ are **stable** if there are two positive constants A and B such that

$$A\|b\|_p \le \left\| \sum_{j\in\mathbb{Z}} b(j)\phi(\cdot - j) \right\|_p \le B\|b\|_p \quad \forall\, b \in \ell_p(\mathbb{Z}).$$

It was proved in [14] that the shifts of a compactly supported function $\phi \in L_p(\mathbb{R})$ are stable if and only if, for any $\xi \in \mathbb{R}$, there exists an integer k such that

$$\hat{\phi}(\xi + 2k\pi) \neq 0.$$

The L_p convergence of the cascade algorithm associated with a will be characterized in terms of the p-norm joint spectral radius of two finite matrices derived from the mask a.

Let $\mathcal{A}$ be a finite collection of linear operators on a *finite dimensional* vector space V. A vector norm $\|\cdot\|$ on V induces a norm on the linear operators on V as follows. For a linear operator A on V, define

$$\|A\| := \max_{\|v\|=1} \big\{\|Av\|\big\}.$$

For a positive integer n we denote by $\mathcal{A}^n$ the Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \big\{(A_1, \dots, A_n) : A_1, \dots, A_n \in \mathcal{A}\big\}.$$

Let

$$\|\mathcal{A}^n\|_\infty := \max\big\{\|A_1 \cdots A_n\| : (A_1, \dots, A_n) \in \mathcal{A}^n\big\}.$$

Then the **uniform joint spectral radius** of $\mathcal{A}$ is defined to be

$$\rho_\infty(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_\infty^{1/n}.$$

The uniform joint spectral radius was introduced by Rota and Strang in [19]. Daubechies and Lagarias used the uniform joint spectral radius as a tool in their study of refinement equations (see [6]).

The p-norm joint spectral radius of a finite collection of linear operators was introduced by Jia in [11]. We define, for $1 \le p < \infty$,

$$\|\mathcal{A}^n\|_p := \left(\sum_{(A_1,\dots,A_n)\in\mathcal{A}^n} \|A_1 \cdots A_n\|^p \right)^{1/p}.$$

For $1 \le p \le \infty$, the p**-norm joint spectral radius** of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n}.$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on V.

We claim that

$$\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\ge1} \|\mathcal{A}^n\|_p^{1/n}.$$

Let $\rho := \inf_{n\ge1} \|\mathcal{A}^n\|_p^{1/n}$. To justify our claim we observe that, for positive integers r and s,

$$\|\mathcal{A}^{r+s}\|_p \le \|\mathcal{A}^r\|_p \|\mathcal{A}^s\|_p.$$

For $\varepsilon > 0$, there exists a positive integer s such that $\|\mathcal{A}^s\|_p^{1/s} \le \rho+\varepsilon$, *i.e.*, $\|\mathcal{A}^s\|_p \le (\rho+\varepsilon)^s$. Let n be a positive integer. Then n can be written as $n = ms + r$, where $m \ge 0$ is an integer and $0 \le r < s$. Consequently,

$$\|\mathcal{A}^n\|_p \le \|\mathcal{A}^{ms}\|_p \|\mathcal{A}^r\|_p \le \|\mathcal{A}^s\|_p^m \|\mathcal{A}^r\|_p \le (\rho+\varepsilon)^{sm} \|\mathcal{A}^r\|_p.$$

It follows that

$$\|\mathcal{A}^n\|_p^{1/n} \le (\rho+\varepsilon)^{1-r/n} \|\mathcal{A}^r\|_p^{1/n}.$$

Therefore, we have

$$\limsup_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} \le \rho+\varepsilon.$$

On the other hand,

$$\liminf_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} \ge \inf_{n\ge 1} \|\mathcal{A}^n\|_p^{1/n} = \rho.$$

But $\varepsilon > 0$ can be arbitrary. We conclude that $\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \rho$.

If $\mathcal{A}$ consists of a single linear operator A, then

$$\rho_p(\mathcal{A}) = \rho(A),$$

where $\rho(A)$ denotes the spectral radius of A, which is independent of p. If $\mathcal{A}$ consists of more than one element, then $\rho_p(\mathcal{A})$ depends on p in general. By some basic properties of ℓ_p spaces we have that, for $1 \le p \le r \le \infty$,

$$(\#\mathcal{A})^{1/r-1/p} \rho_p(\mathcal{A}) \le \rho_r(\mathcal{A}) \le \rho_p(\mathcal{A}),$$

where $\#\mathcal{A}$ denotes the number of elements in $\mathcal{A}$. Furthermore, it is easily seen from the definition of the joint spectral radius that $\rho(A) \le \rho_\infty(\mathcal{A})$ for any element A in $\mathcal{A}$.

Here is an outline of the paper. Section 2 is devoted to investigation of convergence of cascade algorithms. The L_p convergence of a cascade algorithm will be characterized in terms of the p-norm joint spectral radius of two matrices associated with the corresponding mask. For the special case $p = 2$, convergence of a cascade algorithm can be characterized in terms of the spectrum of the transition matrix associated with the mask. In Section 3, the basic theory on cascade algorithms developed in Section 2 will be employed to give a unified treatment of orthogonal wavelets, biorthogonal wavelets, and fundamental refinable functions. In Section 4, we will provide a comprehensive study of biorthogonal wavelet bases. The technique developed in this section can be used to deal with more complicated problems such as biorthogonal wavelet bases on bounded domains. Finally, in Section 5, we will extend our study of cascade algorithms to high dimensional spaces. In particular, we will give a characterization of uniform convergence of a cascade algorithm associated with a nonnegative mask in terms of stochastic matrices.

§2. Convergence of Cascade Algorithms

This section is devoted to a study of convergence of cascade algorithms. We mainly follow the lines of [11]. But some proofs are modified, so that the results in this section can be easily extended to more general situations.

Let a be a finitely supported sequence on $\mathbb{Z}$, and let Q_a be the cascade operator as defined in (1.1). We claim that

$$Q_a^n f = \sum_{j\in\mathbb{Z}} a_n(j) f(2^n \cdot - j), \tag{2.1}$$

where the sequences a_n are given by

$$a_1 = a \quad \text{and} \quad a_n(j) = \sum_{k\in\mathbb{Z}} a_{n-1}(k)a(j-2k), \quad j \in \mathbb{Z},\ n = 2, 3, \ldots. \tag{2.2}$$

This can be proved by induction on n. Indeed, (2.1) is valid for $n = 1$. Suppose (2.1) holds true for $n-1$. Then by the induction hypothesis we have

$$Q_a^n f = Q_a^{n-1}(Q_a f) = \sum_{k\in\mathbb{Z}} a_{n-1}(k)(Q_a f)(2^{n-1}\cdot - k) = \sum_{k\in\mathbb{Z}} a_{n-1}(k) \sum_{j\in\mathbb{Z}} a(j) f(2^n\cdot - 2k - j).$$

It follows that

$$Q_a^n f = \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{n-1}(k)a(j-2k) \right) f(2^n\cdot - j) = \sum_{j\in\mathbb{Z}} a_n(j) f(2^n\cdot - j).$$

This completes the induction procedure.

Theorem 2.1. *Let a be a finitely supported sequence on $\mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}} a(j) = 2$, and let $Q = Q_a$ be the cascade operator associated with a. Suppose u is a compactly supported function in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$, u satisfies the Strang-Fix conditions of order 1, and the shifts of u are stable. If there exists a function $\phi \in L_p(\mathbb{R})$ (a continuous function ϕ in the case $p = \infty$) such that*

$$\lim_{n\to\infty} \|Q^n u - \phi\|_p = 0,$$

then for any compactly supported function $v \in L_p(\mathbb{R})$ satisfying the Strang-Fix conditions of order 1 we also have

$$\lim_{n\to\infty} \|Q^n v - \phi\|_p = 0.$$

Proof. For $n = 0, 1, 2, \ldots$, let

$$f_n := \sum_{j\in\mathbb{Z}} b_n(j) u(2^n \cdot - j) \quad \text{and} \quad g_n := \sum_{j\in\mathbb{Z}} b_n(j) v(2^n \cdot - j),$$

where

$$b_n(j) := \frac{1}{h} \int_{jh}^{(j+1)h} \phi(x)\, dx \quad \text{with } h = \frac{1}{2^n}.$$

Since both u and v satisfy the Strang-Fix conditions of order 1, we have

$$\lim_{n\to\infty} \|\phi - f_n\|_p = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\phi - g_n\|_p = 0.$$

Observe that

$$Q^n u = \sum_{j\in\mathbb{Z}} a_n(j) u(2^n \cdot - j) \quad \text{and} \quad Q^n v = \sum_{j\in\mathbb{Z}} a_n(j) v(2^n \cdot - j),$$

where a_n $(n = 1, 2, \ldots)$ are the sequences given in (2.2). Thus, we obtain

$$Q^n u - f_n = \sum_{j\in\mathbb{Z}} \big[a_n(j) - b_n(j)\big] u(2^n \cdot - j).$$

Since the shifts of u are stable, there exists a constant $C_1 > 0$ such that

$$\|a_n - b_n\|_p \le C_1 \|(f_n - Q^n u)(2^{-n}\cdot)\|_p = 2^{n/p} C_1 \|f_n - Q^n u\|_p.$$

Furthermore, there exists a constant $C_2 > 0$ such that

$$\|(g_n - Q^n v)(2^{-n}\cdot)\|_p = \Big\|\sum_{j\in\mathbb{Z}} \big[a_n(j) - b_n(j)\big] v(\cdot - j)\Big\|_p \le C_2 \|a_n - b_n\|_p.$$

Combining the above two estimates together, we see that there exists a constant $C > 0$ such that

$$\|g_n - Q^n v\|_p \le C\,\|f_n - Q^n u\|_p.$$

Therefore we have

$$\begin{aligned} \|\phi - Q^n v\|_p &\le \|\phi - g_n\|_p + \|g_n - Q^n v\|_p \\ &\le \|\phi - g_n\|_p + C\|f_n - Q^n u\|_p \\ &\le \|\phi - g_n\|_p + C\big(\|\phi - f_n\|_p + \|\phi - Q^n u\|_p\big). \end{aligned}$$

But as $n \to \infty$, $\|\phi - g_n\|_p \to 0$, $\|\phi - f_n\|_p \to 0$, and $\|\phi - Q^n u\|_p \to 0$; hence we conclude that $\lim_{n\to\infty} \|Q^n v - \phi\|_p = 0$. $\square$

Let ϕ be the normalized solution to the refinement equation (1.2). Then $Q_a\phi = \phi$. Suppose $\phi \in L_p(\mathbb{R})$ (ϕ is continuous in the case $p = \infty$) and the shifts of ϕ are stable. In Theorem 2.1 we may choose u to be ϕ. Consequently, the cascade algorithm associated with mask a converges to ϕ in the L_p norm, provided the shifts of ϕ are stable.

For $\varepsilon \in \mathbb{Z}$, let A_ε be the linear operator on $\ell_0(\mathbb{Z})$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon + 2\alpha - \beta) v(\beta), \quad \alpha \in \mathbb{Z},\ v \in \ell_0(\mathbb{Z}). \tag{2.3}$$

Suppose $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \ldots, \varepsilon_n, \gamma \in \mathbb{Z}$. Then

$$(a_n * v)(\alpha) = (A_{\varepsilon_n} \cdots A_{\varepsilon_1} v)(\gamma). \tag{2.4}$$

This can be proved by induction on n. For $n = 1$, we have

$$(a_1 * v)(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\alpha - \beta) v(\beta) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon_1 + 2\gamma - \beta) v(\beta) = (A_{\varepsilon_1} v)(\gamma).$$

Hence, (2.4) is true for $n = 1$. Suppose $n > 1$ and (2.4) is valid for $n-1$. By (2.2) we have

$$(a_n * v)(\alpha) = \sum_{\beta\in\mathbb{Z}} v(\beta) a_n(\alpha-\beta) = \sum_{\beta\in\mathbb{Z}}\sum_{\eta\in\mathbb{Z}} v(\beta) a_{n-1}(\eta) a(\alpha-\beta-2\eta).$$

Write $\alpha = \varepsilon_1 + 2\alpha_1$, where $\alpha_1 := \varepsilon_2 + \cdots + 2^{n-2}\varepsilon_n + 2^{n-1}\gamma$. Consequently,

$$(a_n * v)(\alpha) = \sum_{\eta\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} a_{n-1}(\eta) v(\beta) a(\varepsilon_1 + 2\alpha_1 - \beta - 2\eta) = \sum_{\eta\in\mathbb{Z}} a_{n-1}(\eta)(A_{\varepsilon_1} v)(\alpha_1 - \eta).$$

By the induction hypothesis we obtain

$$(a_n * v)(\alpha) = \big(a_{n-1} * (A_{\varepsilon_1} v)\big)(\alpha_1) = (A_{\varepsilon_n}\cdots A_{\varepsilon_2})(A_{\varepsilon_1} v)(\gamma) = (A_{\varepsilon_n}\cdots A_{\varepsilon_2} A_{\varepsilon_1} v)(\gamma).$$

This completes the induction procedure.

As a consequence of (2.4), we have the following identity for $1 \le p \le \infty$:

$$\|a_n * v\|_p = \left(\sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \|A_{\varepsilon_n}\cdots A_{\varepsilon_1} v\|_p^p\right)^{1/p}, \quad v \in \ell_p(\mathbb{Z}). \tag{2.5}$$

Indeed, for $1 \le p < \infty$ we have

$$\|a_n * v\|_p^p = \sum_{\alpha\in\mathbb{Z}} |(a_n * v)(\alpha)|^p = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}}\sum_{\gamma\in\mathbb{Z}} \big|(a_n * v)(\varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma)\big|^p.$$

In light of (2.4), it follows that

$$\|a_n * v\|_p^p = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}}\sum_{\gamma\in\mathbb{Z}} \big|(A_{\varepsilon_n}\cdots A_{\varepsilon_1} v)(\gamma)\big|^p = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \|A_{\varepsilon_n}\cdots A_{\varepsilon_1} v\|_p^p.$$

This verifies (2.5) for $1 \le p < \infty$. For the case $p = \infty$ we have

$$\|a_n * v\|_\infty = \max_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \|A_{\varepsilon_n}\cdots A_{\varepsilon_1} v\|_\infty.$$

A necessary condition for the L_p convergence of the cascade algorithm associated with a is that a satisfies the **basic sum rule**:

$$\sum_{j\in\mathbb{Z}} a(2j) = \sum_{j\in\mathbb{Z}} a(2j+1) = 1. \tag{2.6}$$

In order to prove this statement we choose f to be the characteristic function of the interval $[0, 1)$ and consider the cascade algorithm $Q_a^n f$ $(n = 1, 2, \ldots)$. If $Q_a^n f$ converges to some ϕ in the L_p norm, then $Q_a^n(Q_a f)$ also converges to ϕ in the L_p norm. Hence, $Q_a f$ also satisfies the Strang-Fix conditions of order 1. Consequently,

$$0 = \widehat{Q_a f}(2\pi) = H(\pi)\hat{f}(\pi),$$

where H is the trigonometric polynomial given in (1.3). Since $\hat{f}(\pi) \neq 0$, it follows that $H(\pi) = 0$. Therefore,

$$\sum_{j\in\mathbb{Z}} a(j)(-1)^j = 0.$$

This in connection with $\sum_{j\in\mathbb{Z}} a(j) = 2$ gives (2.6), as desired.

Suppose a is supported on $[0, N]$, where N is a positive integer. It is easily seen that, for $j \leq 0$ and $k \geq N-1$, $\ell([j,k])$ is invariant under both A_0 and A_1. Let V be the linear space of those elements $v \in \ell([0,N])$ for which $\sum_{k\in\mathbb{Z}} v(k) = 0$. We claim that V is invariant under both A_0 and A_1 if and only if a satisfies the basic sum rule. Indeed, if a satisfies the basic sum rule, then for $\varepsilon = 0, 1$ and $v \in \ell_0(\mathbb{Z})$ we have

$$\sum_{j\in\mathbb{Z}} A_\varepsilon v(j) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a(\varepsilon + 2j - k)v(k) = \sum_{k\in\mathbb{Z}}\left[\sum_{j\in\mathbb{Z}} a(\varepsilon + 2j - k)\right]v(k) = \sum_{k\in\mathbb{Z}} v(k).$$

Hence, $v \in V$ implies $A_\varepsilon v \in V$ for $\varepsilon = 0, 1$. In other words, V is invariant under both A_0 and A_1. Conversely, suppose V is invariant under A_0. Since $\nabla\delta \in V$, we have $A_0(\nabla\delta) \in V$. Hence,

$$\sum_{j\in\mathbb{Z}} \big(A_0(\nabla\delta)\big)(j) = 0, \quad i.e., \quad \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a(2j-k)\nabla\delta(k) = 0.$$

It follows that

$$\sum_{j\in\mathbb{Z}} a(2j) - \sum_{j\in\mathbb{Z}} a(2j-1) = 0.$$

This show that a satisfies the basic sum rule.

Theorem 2.2. *The cascade algorithm associated with a converges in the L_p norm if and only if the following two conditions are satisfied:*
(a) *V is invariant under both A_0 and A_1;*
(b) *$\rho_p(A_0|_V, A_1|_V) < 2^{1/p}$.*

Proof. Suppose the cascade algorithm associated with a converges to ϕ in $L_p(\mathbb{R})$. Then a satisfies the basic sum rule. Hence, V is invariant under both A_0 and A_1.

Let f be the hat function given by $f(x) = \max\{1 - |x|, 0\}$, $x \in \mathbb{R}$. Let $f_n := Q_a^n f$, $n = 1, 2, \ldots$. Then we have $\|f_n - \phi\|_p \to 0$ as $n \to \infty$. By (2.1) we have

$$f_n = \sum_{j\in\mathbb{Z}} a_n(j) f(2^n \cdot - j), \quad n = 1, 2, \ldots,$$

where a_n $(n = 1, 2, \ldots)$ are the sequences given by (2.2). It follows that

$$f_n - f_n(\cdot - 1/2^n) = \sum_{j\in\mathbb{Z}} (\nabla a_n)(j) f(2^n \cdot - j).$$

Since f is the hat function, there exists a positive constant C such that

$$2^{-n/p}\|\nabla a_n\|_p \leq C\|f_n - f_n(\cdot - 1/2^n)\|_p.$$

We observe that

$$\|f_n - f_n(\cdot - 1/2^n)\|_p \le \|f_n - \phi\|_p + \|\phi - \phi(\cdot - 1/2^n)\|_p + \|\phi(\cdot - 1/2^n) - f_n(\cdot - 1/2^n)\|_p.$$

Consequently,

$$\lim_{n\to\infty} \|f_n - f_n(\cdot - 1/2^n)\|_p = 0.$$

Hence, we obtain

$$\lim_{n\to\infty} 2^{-n/p}\|\nabla a_n\|_p = 0. \tag{2.7}$$

Let $\mathcal{A} := \{A_0|_V, A_1|_V\}$. Fix a vector norm on V. Then there exist two positive constants C_1 and C_2 such that

$$C_1\|\nabla a_n\|_p \le \|\mathcal{A}^n\|_p \le C_2\|\nabla a_n\|_p, \quad 1 \le p \le \infty. \tag{2.8}$$

In order to prove (2.8) we observe that $\{\nabla\delta_j : j = 0, \ldots, N-1\}$ is a basis for V. Hence, there exist two positive constants C_3 and C_4 such that

$$C_3 \sum_{j=0}^{N-1} \|B(\nabla\delta_j)\|_p \le \|B\| \le C_4 \sum_{j=0}^{N-1} \|B(\nabla\delta_j)\|_p$$

for every linear operator B on V. Recall that

$$\|\mathcal{A}^n\|_p^p = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \|(A_{\varepsilon_n}\cdots A_{\varepsilon_1})|_V\|^p, \quad 1 \le p < \infty.$$

Moreover, by (2.5) we get

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \big\|(A_{\varepsilon_n}\cdots A_{\varepsilon_1})(\nabla\delta_j)\big\|_p^p = \|a_n*(\nabla\delta_j)\|_p^p = \|(\nabla a_n)*\delta_j\|_p^p = \|\nabla a_n\|_p^p \quad \forall j \in \mathbb{Z}.$$

For the case $p = \infty$, we have $\|\mathcal{A}^n\|_\infty = \max_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \|(A_{\varepsilon_n}\cdots A_{\varepsilon_1})|_V\|_\infty$ and

$$\max_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \big\|(A_{\varepsilon_n}\cdots A_{\varepsilon_1})(\nabla\delta_j)\big\|_\infty = \|a_n*(\nabla\delta_j)\|_\infty = \|\nabla a_n\|_\infty \quad \forall j \in \mathbb{Z}.$$

Hence, (2.8) is valid for $1 \le p \le \infty$.

Combining (2.8) and (2.7) together, we obtain

$$\lim_{n\to\infty} 2^{-n/p}\|\mathcal{A}^n\|_p = 0.$$

Recall that

$$\inf_{n\ge1} \|\mathcal{A}^n\|_p^{1/n} = \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \rho_p(A_0|_V, A_1|_V).$$

Consequently,

$$2^{-n/p}\|\mathcal{A}^n\|_p = \big(2^{-1/p}\|\mathcal{A}^n\|_p^{1/n}\big)^n \ge \big(2^{-1/p}\rho_p(A_0|_V, A_1|_V)\big)^n.$$

The above estimates tell us that

$$\lim_{n\to\infty}\left(2^{-1/p}\rho_p(A_0|_V, A_1|_V)\right)^n = 0.$$

Therefore, $2^{-1/p}\rho_p(A_0|_V, A_1|_V) < 1$. This completes the proof of the necessity part.

Let us establish the sufficiency part of the theorem. Suppose f is a compactly supported function in $L_p(\mathbb{R})$ (f is continuous in the case $p=\infty$) satisfying the Strang-Fix conditions of order 1. Observe that $Q_a^{n+1}f - Q_a^n f = Q_a^n g$, where $g := Q_a f - f$. For $x \in [0,1)$, let $v_x(j) := g(x+j)$, $j \in \mathbb{Z}$. We claim that $\sum_{j\in\mathbb{Z}} v_x(j) = 0$ for almost every $x \in [0,1)$. Since f satisfies the Strang-Fix conditions of order 1, we have $\sum_{j\in\mathbb{Z}} f(x+j) = 1$ for almost every $x \in \mathbb{Z}$. By condition (a), the mask a satisfies the basic sum rule. Hence, for almost every $x \in \mathbb{R}$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} Q_a f(x+j) &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a(k) f(2x+2j-k)\\ &= \sum_{k\in\mathbb{Z}}\left[\sum_{j\in\mathbb{Z}} a(k+2j)\right] f(2x-k) = \sum_{k\in\mathbb{Z}} f(2x-k) = 1.\end{aligned}$$

Consequently,

$$\sum_{j\in\mathbb{Z}} v_x(j) = \sum_{j\in\mathbb{Z}} Q_a f(x+j) - \sum_{j\in\mathbb{Z}} f(x+j) = 0 \quad \text{for } \ a.e.\ x \in \mathbb{R}.$$

This justifies our claim.

By (2.1) we have

$$Q_a^n g(2^{-n}x) = \sum_{j\in\mathbb{Z}} a_n(j) g(x-j), \quad x \in \mathbb{R}.$$

Hence, for $1 \le p < \infty$, we obtain

$$\begin{aligned}\|Q_a^n g\|_p^p &= \int_{\mathbb{R}} \left|Q_a^n g(y)\right|^p dy = 2^{-n}\int_{\mathbb{R}} \left|Q_a^n g(2^{-n}x)\right|^p dx\\ &= 2^{-n}\int_{\mathbb{R}}\left|\sum_{j\in\mathbb{Z}} a_n(j) g(x-j)\right|^p dx = 2^{-n}\sum_{k\in\mathbb{Z}}\int_{[0,1)+k}\left|\sum_{j\in\mathbb{Z}} a_n(j) g(x-j)\right|^p dx\\ &= 2^{-n}\int_{[0,1)}\sum_{k\in\mathbb{Z}}\left|\sum_{j\in\mathbb{Z}} a_n(j) g(x+k-j)\right|^p dx = 2^{-n}\int_{[0,1)} \|a_n * v_x\|_p^p\, dx.\end{aligned}$$

It follows from (2.5) that

$$\|a_n * v\|_p \le \|\mathcal{A}^n\|_p \|v\|_p \quad \forall\, v \in \ell_p(\mathbb{Z}).$$

By condition (b), $\rho := \rho_p(A_0|_V, A_1|_V) < 2^{1/p}$. Hence, $\rho < \eta 2^{1/p}$ for some η, $0 < \eta < 1$. Thus, there exists a positive constant C such that $\|\mathcal{A}^n\|_p \le C(\eta 2^{1/p})^n$ for all $n \in \mathbb{N}$. Consequently,

$$\|a_n * v_x\|_p \le C(\eta 2^{1/p})^n \|v_x\|_p.$$

Note that $\int_{[0,1)} \|v_x\|_p^p \, dx = \|g\|_p^p$. Therefore,

$$\|Q_a^n g\|_p^p \le 2^{-n} C^p (\eta 2^{1/p})^{pn} \|g\|_p^p = C^p \eta^{np} \|g\|_p^p.$$

It follows that

$$\|Q_a^{n+1} f - Q_a^n f\|_p = \|Q_a^n g\|_p \le C\eta^n \|g\|_p.$$

Similarly, for the case $p = \infty$, we have

$$\|Q_a^{n+1} f - Q_a^n f\|_\infty = \|Q_a^n g\|_\infty \le C\eta^n \|g\|_\infty.$$

Since $0 < \eta < 1$, the above estimates tell us that $(Q_a^n f)_{n=1,2,\ldots}$ converges in $L_p(\mathbb{R})$. In other words, the cascade algorithm associated with a converges in the L_p norm. $\square$

In what follows, we use $\omega(f,h)_p$ to denote the L_p modulus of continuity of f:

$$\omega(f,h)_p := \sup_{|t|\le h} \|f - f(\cdot - t)\|_p, \qquad h > 0.$$

For $1 \le p \le \infty$ and $0 < \mu \le 1$, we use $\mathrm{Lip}(\mu, L_p(\mathbb{R}))$ to denote the linear space of all functions f in $L_p(\mathbb{R})$ for which

$$\sup_{t>0} \omega(f,t)_p / t^\mu < \infty.$$

In particular, if $p = \infty$, then $\mathrm{Lip}(\mu, L_\infty(\mathbb{R}))$ is the Lipschitz space $\mathrm{Lip}\,\mu$ (see [8, p. 51]).

Suppose the cascade algorithm associated with mask a converges to ϕ in $L_p(\mathbb{R})$. By Theorem 2.2 we have $\rho_p(A_0|_V, A_1|_V) < 2^{1/p}$. Hence, there exists some η, $0 < \eta < 1$, such that $\rho_p(A_0|_V, A_1|_V) < \eta 2^{1/p}$. Consequently, there exists a positive constant C_1 such that $\|\mathcal{A}^n\|_p \le C_1(\eta 2^{1/p})^n \ \forall\, n \in \mathbb{N}$. This in connection with (2.8) gives $\|\nabla a_n\|_p \le C_2(\eta 2^{1/p})^n$, where C_2 is a constant independent of n. Note that

$$\phi = Q_a^n \phi = \sum_{j\in\mathbb{Z}} a_n(j)\phi(2^n \cdot - j).$$

It follows that

$$\phi - \phi(\cdot - 1/2^n) = \sum_{j\in\mathbb{Z}} (\nabla a_n)(j)\phi(2^n \cdot - j).$$

Therefore, we obtain

$$\|\phi - \phi(\cdot - 1/2^n)\|_p \le C_3 2^{-n/p} \|\nabla a_n\|_p \le C_2 C_3 \eta^n \quad \forall\, n \in \mathbb{N},$$

where C_3 is a constant independent of n. Let $\mu := -\log_2 \eta > 0$. Then $\eta = 2^{-\mu}$. Suppose

$$h = \frac{1}{2^n} + \frac{d_1}{2^{n+1}} + \frac{d_2}{2^{n+2}} + \cdots,$$

where $d_1, d_2, \ldots \in \{0,1\}$. By what has been proved, we have

$$\|\phi - \phi(\cdot - h)\|_p \le C_2 C_3 \sum_{m=n}^{\infty} 2^{-m\mu} \le C h^{\mu},$$

where C is a positive constant independent of h. This shows that $\phi \in \mathrm{Lip}(\mu, L_p(\mathbb{R}))$ for some $\mu > 0$, whenever the corresponding cascade algorithm converges in $L_p(\mathbb{R})$.

In the rest of this section we will discuss the L_2 convergence of cascade algorithms. This problem was investigated in terms of subdivision operators by Villemoes [23], and by Goodman, Micchelli, and Ward [9]. More generally, Zhou [24] showed that the 2-norm joint spectral radius of a finite collection of square matrices is equal to the spectral radius of a certain finite matrix derived from the given matrices.

Suppose $a \in \ell_0(\mathbb{Z})$. For $n = 1, 2, \ldots$, let a_n be the sequences given in (2.2). Let T_a be the **transition operator** on $\ell_0(\mathbb{Z})$ given by

$$T_a v(j) := \sum_{k\in\mathbb{Z}} a(2j-k)v(k), \quad j \in \mathbb{Z},\ v \in \ell_0(\mathbb{Z}).$$

Clearly, T_a is the same as the linear operator A_0 defined in (2.3). Hence, by (2.4) we have the following formula:

$$T_a^n v(j) = (a_n * v)(2^n j) = \sum_{k\in\mathbb{Z}} a_n(2^n j - k)v(k), \quad j \in \mathbb{Z},\ n \in \mathbb{N}. \tag{2.9}$$

For two functions f and g in $L_2(\mathbb{R})$, we define

$$f \odot g(x) := \int_{\mathbb{R}} f(x+y)\overline{g(y)}\, dy, \quad x \in \mathbb{R}.$$

Evidently, $f \odot g$ is continuous and $\|f \odot g\|_\infty \le \|f\|_2 \|g\|_2$. Similarly, for two sequences u and v in $\ell_2(\mathbb{Z})$, we define

$$u \odot v(j) := \sum_{k\in\mathbb{Z}} u(j+k)\overline{v(k)}, \quad j \in \mathbb{Z}.$$

Also, $\|u \odot v\|_\infty \le \|u\|_2 \|v\|_2$.

Let $b := a \odot a/2$. We claim that

$$\lim_{n\to\infty} \|a_n * v\|_2^{1/n} = \sqrt{2\rho(T_b|_W)}, \quad v \in \ell_0(\mathbb{Z}), \tag{2.10}$$

where W is the minimal invariant subspace of the transition operator T_b generated by $w := v \odot v$.

For $n = 1, 2, \ldots$, let b_n be the sequences given by

$$b_1 = b \quad \text{and} \quad b_n(j) = \sum_{k\in\mathbb{Z}} b_{n-1}(k) b(2j-k), \quad j \in \mathbb{Z}.$$

Let $v_n := a_n * v$ and $w_n := v_n \odot v_n$. It can be easily proved that $w_n = 2^n b_n * w$. Hence, it follows from (2.9) that

$$2^n T_b^n w(j) = w_n(2^n j), \quad j \in \mathbb{Z}.$$

Since $w_n = v_n \odot v_n$, we have

$$2^n \|T_b^n w\|_\infty \le \|w_n\|_\infty \le \|v_n\|_2^2.$$

On the other hand,

$$2^n T_b^n w(0) = w_n(0) = \sum_{k \in \mathbb{Z}} v_n(k)\overline{v_n(k)} = \|v_n\|_2^2.$$

Consequently,

$$\|v_n\|_2^2 = 2^n \|T_b^n w\|_\infty.$$

Therefore,

$$\lim_{n\to\infty} \|a_n * v\|_2^{2/n} = \lim_{n\to\infty} \|v_n\|_2^{2/n} = \lim_{n\to\infty} \left(2^n \|T_b^n w\|_\infty\right)^{1/n} = 2\rho(T_b|_W).$$

This verifies (2.10).

Theorem 2.3. *Let a be a sequence supported on $[0, N]$ and satisfying $\sum_{j\in\mathbb{Z}} a(j) = 2$. Then the cascade algorithm associated with a converges in the L_2 norm if and only if the following two conditions are satisfied:*

(a) *a satisfies the basic sum rule;*

(b) *$\rho(T_b|_W) < 1$, where $W := \{w \in \ell([-N, N]) : \sum_{j\in\mathbb{Z}} w(j) = 0\}$.*

Proof. Suppose conditions (a) and (b) are satisfied. Let $v := \nabla\delta$. Then $w := v * v$ belongs to W. In light of (2.10) we have

$$\lim_{n\to\infty} \|\nabla a_n\|_2^{1/n} = \lim_{n\to\infty} \|a_n * (\nabla\delta)\|_2^{1/n} \le \sqrt{2\rho(T_b|_W)} < \sqrt{2}.$$

Let $\mathcal{A} := \{A_0|_V, A_1|_V\}$, where $V := \{v \in \ell([0, N]) : \sum_{j\in\mathbb{Z}} v(j) = 0\}$. It follows from (2.8) that

$$\rho_2(A_0|_V, A_1|_V) = \lim_{n\to\infty} \|\mathcal{A}^n\|_2^{1/n} = \lim_{n\to\infty} \|\nabla a_n\|_2^{1/n}.$$

Hence, $\rho_2(A_0|_V, A_1|_V) < 2^{1/2}$. Therefore, the cascade algorithm associated with a converges in $L_2(\mathbb{R})$, by Theorem 2.2.

Conversely, suppose the cascade algorithm associated with a converges to ϕ in $L_2(\mathbb{R})$. Then a satisfies the basic sum rule. Consequently, b satisfies the basic sum rule. Let ϕ_0 be the characteristic function of the interval $[0, 1)$, and let $\phi_n := Q_a^n \phi_0$, $n = 1, 2, \ldots$. Then $\|\phi_n - \phi\|_2 \to 0$ as $n \to \infty$. Moreover, let $g_n := \phi_n \odot \phi_n$ $(n = 0, 1, \ldots)$ and $g := \phi \odot \phi$. It is easily seen that $g_n = Q_b^n g_0$, $n = 1, 2, \ldots$. We have

$$\begin{aligned}\|g_n - g\|_\infty &= \|\phi_n \odot \phi_n - \phi \odot \phi\|_\infty \\ &\le \|\phi_n \odot (\phi_n - \phi)\|_\infty + \|(\phi_n - \phi) \odot \phi\|_\infty \\ &\le (\|\phi_n\|_2 + \|\phi\|_2)\|\phi_n - \phi\|_2.\end{aligned}$$

Hence, $Q_b^n g_0$ converges to g in $L_\infty(\mathbb{R})$. Therefore, we must have $\rho(T_b|_W) < 1$. $\square$

We remark that condition (b) holds true if and only if 1 is a simple eigenvalue of the transition matrix $(b(2j - k))_{-N\le j,k\le N}$ and all the other eigenvalues are less than 1 in modulus.

§3. Applications to Wavelet Analysis

In this section, the basic theory on cascade algorithms developed in the last section will be used to give a unified treatment of orthogonal wavelets, biorthogonal wavelets, and fundamental refinable functions.

To begin with, we discuss orthogonal refinable functions. It was Daubechies who first constructed smooth orthogonal wavelets with compact support (see [4]). The following theorem gives a characterization for a refinable function to be orthogonal.

Theorem 3.1. *Let ϕ be the normalized solution of the refinement equation (1.2), and let H be the trigonometric polynomial given in (1.3). Then $\{\phi(\cdot - j) : j \in \mathbb{Z}\}$ forms an orthonormal system in $L_2(\mathbb{R})$ if and only if the following two conditions are satisfied:*
(a) $|H(\xi)|^2 + |H(\xi+\pi)|^2 = 1$ *for all* $\xi \in \mathbb{R}$;
(b) *The cascade algorithm associated with a converges in the L_2 norm.*

Proof. Let us first establish the necessity part of the theorem. Suppose the shifts of ϕ are orthonormal. Then $\sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2\pi k)|^2 = 1$ for all $\xi \in \mathbb{R}$. Since $\hat{\phi}(\xi) = H(\xi/2)\hat{\phi}(\xi/2)$, $\xi \in \mathbb{R}$, we deduce that

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2\pi k)|^2 = \sum_{k\in\mathbb{Z}} \bigl|H(\xi/2+\pi k)\hat{\phi}(\xi/2+\pi k)\bigr|^2 \\ &= |H(\xi/2)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi/2+2\pi k)|^2 + |H(\xi/2+\pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi/2+\pi+2\pi k)|^2 \\ &= |H(\xi/2)|^2 + |H(\xi/2+\pi)|^2. \end{aligned}$$

This verifies condition (a). Moreover, if the shifts of ϕ are orthonormal, then the shifts of ϕ are stable. Hence, the cascade algorithm associated with a converges in the L_2 norm, by Theorem 2.1.

For the sufficiency part we observe that condition (a) implies invariance of orthonormality under the cascade operator Q_a. Suppose f is a compactly supported function in $L_2(\mathbb{R})$ and the shifts of f are orthornormal. We wish to show that the shifts of $Q_a f$ are also orthonormal. For this purpose it suffices to show that $\sum_{k\in\mathbb{Z}} |(Q_a f)\hat{\ }(\xi+2\pi k)|^2 = 1$ for all $\xi \in \mathbb{R}$. Since $(Q_a f)\hat{\ }(\xi) = H(\xi/2)\hat{f}(\xi/2)$, $\xi \in \mathbb{R}$, we have

$$\sum_{k\in\mathbb{Z}} |(Q_a f)\hat{\ }(\xi+2\pi k)|^2 = \sum_{k\in\mathbb{Z}} \bigl|H(\xi/2+\pi k)\hat{f}(\xi/2+\pi k)\bigr|^2.$$

The right of the above equation is equal to

$$|H(\xi/2)|^2 \sum_{k\in\mathbb{Z}} |\hat{f}(\xi/2+2\pi k)|^2 + |H(\xi/2+\pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{f}(\xi/2+\pi+2\pi k)|^2.$$

But $\sum_{k\in\mathbb{Z}} |\hat{f}(\xi/2+2\pi k)|^2 = \sum_{k\in\mathbb{Z}} |\hat{f}(\xi/2+\pi+2\pi k)|^2 = 1$. Hence,

$$\sum_{k\in\mathbb{Z}} \bigl|H(\xi/2+\pi k)\hat{f}(\xi/2+\pi k)\bigr|^2 = |H(\xi/2)|^2 + |H(\xi/2+\pi)|^2 = 1,$$

where condition (a) has been used to derive the last equality. This shows that the shifts of $Q_a f$ are orthonormal.

Let ϕ_0 be the characteristic function of the interval $[0,1)$, and let $\phi_n := Q_a^n \phi_0$ for $n = 1, 2, \ldots$. Clearly, the shifts of ϕ_0 are orthonormal. By what has been proved, the shifts of ϕ_n are also orthonormal. In other words,

$$\langle \phi_n, \phi_n(\cdot - j)\rangle = \delta_{0j}, \quad j \in \mathbb{Z}.$$

Since $\|\phi_n - \phi\|_2 \to 0$ as $n \to \infty$, we obtain

$$\langle \phi, \phi(\cdot - j)\rangle = \lim_{n\to\infty} \langle \phi_n, \phi_n(\cdot - j)\rangle = \delta_{0j}, \quad j \in \mathbb{Z}.$$

Consequently, the shifts of ϕ are orthonormal. $\square$

Suppose the mask a is supported on $[0, N]$. Then the sequence $b := a \odot a/2$ is supported on $[-N, N]$. As we mentioned at the end of Section 2, the cascade algorithm associated with a converges in the L_2 norm if and only if 1 is a simple eigenvalue of the transition matrix $(b(2j-k))_{-N\le j,k\le N}$ and all the other eigenvalues are less than 1 in modulus. Thus, Theorem 3.1 recovers a result of Lawton [16] on orthogonality of a refinable function in terms of the eigenvalue condition of the corresponding transition matrix.

Next, we investigate biorthogonal refinable functions. Let ϕ and $\tilde\phi$ be two compactly supported functions in $L_2(\mathbb{R})$. We say that the shifts of ϕ and $\tilde\phi$ are **biorthogonal** if

$$\langle \phi(\cdot - j), \tilde\phi(\cdot - k)\rangle = \delta_{jk}, \quad j, k \in \mathbb{Z}.$$

By the Poisson summation formula, the shifts of ϕ and $\tilde\phi$ are biorthogonal if and only if

$$\sum_{k\in\mathbb{Z}} \hat\phi(\xi + 2k\pi)\overline{\hat{\tilde\phi}(\xi + 2k\pi)} = 1 \quad \forall\, \xi \in \mathbb{R}.$$

Suppose ϕ and $\tilde\phi$ are refinable:

$$\phi = \sum_{j\in\mathbb{Z}} a(j)\phi(2\cdot - j) \quad \text{and} \quad \tilde\phi = \sum_{j\in\mathbb{Z}} b(j)\tilde\phi(2\cdot - j), \tag{3.1}$$

where a and b are finitely supported sequences on $\mathbb{Z}$. By taking the Fourier transform of both sides of the refinement equations in (3.1), we obtain

$$\hat\phi(\xi) = H(\xi/2)\hat\phi(\xi/2) \quad \text{and} \quad \hat{\tilde\phi}(\xi) = G(\xi/2)\hat{\tilde\phi}(\xi/2), \quad \xi \in \mathbb{R},$$

where

$$H(\xi) := \frac{1}{2}\sum_{j\in\mathbb{Z}} a(j)e^{-ij\xi} \quad \text{and} \quad G(\xi) := \frac{1}{2}\sum_{j\in\mathbb{Z}} b(j)e^{-ij\xi}, \quad \xi \in \mathbb{R}. \tag{3.2}$$

If the shifts of ϕ and $\tilde{\phi}$ are biorthogonal, then for $\xi \in \mathbb{R}$,

$$\begin{aligned}
1 &= \sum_{k\in\mathbb{Z}} \hat{\phi}(2\xi + 2k\pi)\, \overline{\hat{\tilde{\phi}}(2\xi + 2k\pi)} \\
&= \sum_{k\in\mathbb{Z}} H(\xi + k\pi)\hat{\phi}(\xi + k\pi)\, \overline{G(\xi + k\pi)}\, \overline{\hat{\tilde{\phi}}(\xi + k\pi)} \\
&= \sum_{\varepsilon=0,1} H(\xi + \varepsilon\pi)\overline{G(\xi + \varepsilon\pi)} \sum_{k\in\mathbb{Z}} \hat{\phi}(\xi + \varepsilon\pi + 2k\pi)\, \overline{\hat{\tilde{\phi}}(\xi + \varepsilon\pi + 2k\pi)} \\
&= H(\xi)\overline{G(\xi)} + H(\xi + \pi)\overline{G(\xi + \pi)}.
\end{aligned}$$

Theorem 3.2. *The shifts of ϕ and $\tilde{\phi}$ are biorthogonal if and only if the following two conditions are satisfied:*
(a) $H(\xi)\overline{G(\xi)} + H(\xi + \pi)\overline{G(\xi + \pi)} = 1$ *for* $\xi \in \mathbb{R}$*;*
(b) *The cascade algorithms associated with a and b converge in $L_2(\mathbb{R})$.*

Proof. We have proved necessity of condition (a). If the shifts of ϕ and $\tilde{\phi}$ are biorthogonal, then they are stable; hence, the cascade algorithms associated with a and b converge in $L_2(\mathbb{R})$, by Theorem 2.1.

In order to establish sufficiency of conditions (a) and (b), we consider the cascade operators Q_a and Q_b given by

$$Q_a f := \sum_{j\in\mathbb{Z}} a(j) f(2\cdot - j) \quad \text{and} \quad Q_b g := \sum_{j\in\mathbb{Z}} b(j) g(2\cdot - j),$$

where f and g are compactly supported functions in $L_2(\mathbb{R})$. If condition (a) holds true, and if the shifts of f and g are biorthogonal, then the shifts of $Q_a f$ and $Q_b g$ are also biorthogonal. Indeed, for $\xi \in \mathbb{R}$, the expression

$$\sum_{k\in\mathbb{Z}} \widehat{Q_a f}(\xi + 2k\pi)\, \overline{\widehat{Q_b g}(\xi + 2k\pi)}$$

is equal to

$$\sum_{\varepsilon=0,1} H(\xi + \varepsilon\pi)\, \overline{G(\xi + \varepsilon\pi)} \sum_{k\in\mathbb{Z}} \hat{f}(\xi + \varepsilon\pi + 2k\pi)\, \overline{\hat{g}(\xi + \varepsilon\pi + 2k\pi)}.$$

Using biorthgonality of the shifts of f and g, we obtain

$$\sum_{k\in\mathbb{Z}} \widehat{Q_a f}(\xi + 2k\pi)\, \overline{\widehat{Q_b g}(\xi + 2k\pi)} = H(\xi)\overline{G(\xi)} + H(\xi + \pi)\overline{G(\xi + \pi)} = 1.$$

This justifies our claim.

Choose f_0 and g_0 to be the characteristic function of the interval $[0,1)$. Clearly, the shifts of f_0 and g_0 are biorthogonal. For $n = 1, 2, \ldots$, let $f_n := Q_a^n f_0$ and $g_n := Q_b^n g_0$. By what has been proved, the shifts of f_n and g_n are biorthogonal. In other words,

$$\langle f_n(\cdot - j), g_n(\cdot - k)\rangle = \delta_{jk}, \quad j, k \in \mathbb{Z}.$$

By condition (b), $\|f_n - \phi\|_2 \to 0$ and $\|g_n - \tilde{\phi}\|_2 \to 0$ as $n \to \infty$. Hence, we obtain

$$\langle \phi(\cdot - j), \tilde{\phi}(\cdot - k)\rangle = \lim_{n\to\infty} \langle f_n(\cdot - j), g_n(\cdot - k)\rangle = \delta_{jk}, \quad j, k \in \mathbb{Z}.$$

This shows that the shifts of ϕ and $\tilde{\phi}$ are biorthogonal. $\square$

In a slightly different form, Theorem 3.2 was established by Cohen and Daubechies [2]. But their proof relies on the fact that a trigonometric polynomial of one variable has only a finite number of zeros in the interval $[0, 2\pi]$. By contrast, the proof given here can be easily extended to the multivariate case (see Section 5).

Finally, let us study refinable fundamental functions. A continuous function on $\mathbb{R}$ is said to be **fundamental** if

$$\phi(0) = 1 \quad \text{and} \quad \phi(j) = 0 \quad \forall\, j \in \mathbb{Z} \setminus \{0\}.$$

Suppose in addition that ϕ is compactly supported and satisfies the refinement equation

$$\phi(x) = \sum_{j\in\mathbb{Z}} a(j)\phi(2x - j), \quad x \in \mathbb{R}.$$

Setting $x = k$ in the above equation, we obtain $\phi(k) = a(2k)$. Hence,

$$a(0) = 1 \quad \text{and} \quad a(2k) = 0 \quad \forall\, k \in \mathbb{Z} \setminus \{0\}.$$

If a finitely supported sequence a satisfies the above conditions and $\sum_{j\in\mathbb{Z}} a(2j+1) = 1$, then we say that a is **interpolatory**. Thus, in order for a continuous refinable function with compact support to be fundamental, the corresponding mask must be interpolatory. In general, this condition is not sufficient. For example, let $\phi(x) := \max\{1 - |x|/3, 0\}$, $x \in \mathbb{R}$. Then

$$\phi(x) = \frac{1}{2}\phi(2x+3) + \phi(2x) + \frac{1}{2}\phi(2x-3), \quad x \in \mathbb{R}.$$

Hence, ϕ is refinable and the corresponding mask a is interpolatory. But ϕ is not fundamental.

Theorem 3.3. *If a is an interpolatory mask and the cascade algorithm associated with a converges uniformly, then the limit function ϕ is fundamental. Conversely, if ϕ is a continuous fundamental function on $\mathbb{R}$ with compact support and ϕ satisfies the refinement* *equation with mask a, then a is interpolatory and the cascade algorithm associated with a converges uniformly.*

Proof. If a is interpolatory and f is a continuous fundamental function on $\mathbb{R}$, then $Q_a f$ is also fundamental. Indeed, for $k \in \mathbb{Z}$ we have

$$(Q_a f)(k) = \sum_{j\in\mathbb{Z}} a(j) f(2k-j) = a(2k) = \delta(k), \quad k \in \mathbb{Z}.$$

Let $\phi_n := Q_a^n \phi_0$, where ϕ_0 is the hat function given by $\phi_0(x) := \max\{1-|x|, 0\}$, $x \in \mathbb{R}$. Note that ϕ_0 is fundamental. Hence, for each $n = 1, 2, \ldots$, ϕ_n is fundamental. If the cascade algorithm associated with a converges uniformly, then there exists a continuous function ϕ such that $\|\phi_n - \phi\|_\infty \to 0$ as $n \to \infty$. It follows that

$$\phi(k) = \lim_{n\to\infty} \phi_n(k) = \delta(k) \quad \forall\, k \in \mathbb{Z}.$$

This shows that ϕ is fundamental.

Conversely, suppose that ϕ is a continuous fundamental function on $\mathbb{R}$ with compact support and ϕ satisfies the refinement equation with mask a. Then a is interpolatory. Moreover, the shifts of ϕ are stable. Consequently, the cascade algorithm associated with a converges uniformly, by Theorem 2.1. $\square$

In [7] Deslauriers and Dubuc constructed a family of refinable fundamental functions with increasing smoothness, which are related to the orthogonal refinable functions constructed by Daubechies [4].

§4. Biorthogonal Wavelet Bases

Let ϕ and $\tilde{\phi}$ be two compactly supported functions in $L_2(\mathbb{R})$. Suppose ϕ and $\tilde{\phi}$ satisfy the refinement equations in (3.1) with masks a and b, respectively. Let H and G be the trigonometric polynomials given in (3.2). Suppose the shifts of ϕ and $\tilde{\phi}$ are biorthogonal. By Theorem 3.2, this is the case if and only if the cascade algorithms associated with a and b converge in the L_2 norm and $H(\xi)\overline{G(\xi)} + H(\xi+\pi)\overline{G(\xi+\pi)} = 1$ for all $\xi \in \mathbb{R}$. It is easily seen that the latter is equivalent to the following relation:

$$\sum_{j\in\mathbb{Z}} a(2k+j)\overline{b(j)} = 2\delta_{0k} \quad \forall\, k \in \mathbb{Z}. \tag{4.1}$$

Since the cascade algorithms associated with a and b converge in the L_2 norm, the masks a and b satisfy the basic sum rule. Moreover, there exists some $\mu > 0$ such that ϕ and $\tilde{\phi}$ belong to $\mathrm{Lip}\,(\mu, L_2(\mathbb{R}))$.

For $\mu > 0$, we denote by $H^\mu(\mathbb{R})$ the Sobolev space of all functions $f \in L_2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1+|\xi|^\mu)^2 \, d\xi < \infty.$$

For $0 < \mu \le 1$, it is easily seen that $H^\mu(\mathbb{R}) \subseteq \mathrm{Lip}\,(\mu, L_2(\mathbb{R}))$.

Let

$$\psi := \sum_{j\in\mathbb{Z}} (-1)^j\, \overline{b(1-j)}\, \phi(2\cdot - j) \quad \text{and} \quad \tilde{\psi} := \sum_{j\in\mathbb{Z}} (-1)^j\, \overline{a(1-j)}\, \tilde{\phi}(2\cdot - j). \tag{4.2}$$

For $j, k \in \mathbb{Z}$, let

$$\psi_{jk} := \sqrt{2}^j \psi(2^j \cdot - k) \quad \text{and} \quad \tilde{\psi}_{jk} := \sqrt{2}^j \tilde{\psi}(2^j \cdot - k). \tag{4.3}$$

It was proved by Cohen and Daubechies [2] (also see [3]) that $(\psi_{jk})_{j,k\in\mathbb{Z}}$ and $(\tilde{\psi}_{jk})_{j,k\in\mathbb{Z}}$ are Riesz bases for $L_2(\mathbb{R})$. In this section we shall give a self-contained proof for this important result. The technique developed in this section can be used to deal with more complicated problems such as biorthogonal wavelet bases on bounded domains.

A sequence $(f_n)_{n=1,2,\ldots}$ in a Hilbert space H is called a **Riesz sequence** if there exist two positive constants A and B such that

$$A\left(\sum_{n=1}^{\infty} |c_n|^2\right)^{1/2} \le \left\| \sum_{n=1}^{\infty} c_n f_n \right\| \le B\left(\sum_{n=1}^{\infty} |c_n|^2\right)^{1/2}$$

for all sequences $(c_n)_{n=1,2,\ldots}$ in ℓ_2. A Riesz sequence $(f_n)_{n=1,2,\ldots}$ is called a **Riesz basis** if additionally the linear span of $\{f_n : n = 1, 2, \ldots\}$ is dense in H.

The inner product of two elements f and g in H is denoted by $\langle f, g\rangle$. A sequence $(g_n)_{n=1,2,\ldots}$ in H is called a **Bessel sequence** if there exists a positive constant B such that

$$\sum_{n=1}^{\infty} \big|\langle f, g_n\rangle\big|^2 \le B\|f\|^2 \quad \forall\, f \in H.$$

We observe that ψ lies in $H^\mu(\mathbb{R})$ for some $\mu > 0$. Moreover, since b satisfies the basic sum rule, we have $\int_{\mathbb{R}} \psi(x)\, dx = 0$. The following theorem tells us that $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Bessel sequence in $L_2(\mathbb{R})$.

Theorem 4.1. *Assume $\psi \in H^\mu(\mathbb{R})$ for some $\mu > 0$. If ψ furthermore is compactly supported and $\int_{\mathbb{R}} \psi(x)\, dx = 0$, then there is a positive constant B such that*

$$\sum_{j,k\in\mathbb{Z}} \big|\langle f, \psi_{jk}\rangle\big|^2 \le B\|f\|_2^2 \quad \forall\, f \in L_2(\mathbb{R}),$$

where $\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k)$, $x \in \mathbb{R}$. In other words, $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Bessel sequence in $L_2(\mathbb{R})$.

This theorem was established by Villemoes [22]. Its proof will be given at the end of this section.

We claim that the sequences $(\psi_{jk})_{j,k\in\mathbb{Z}}$ and $(\tilde{\psi}_{jk})_{j,k\in\mathbb{Z}}$ are biorthogonal, that is,

$$\langle \psi_{jk}, \tilde{\psi}_{rs}\rangle = \delta_{jr}\delta_{ks} \quad \forall\, j, k, r, s \in \mathbb{Z}. \tag{4.4}$$

To justify our claim we first verify (4.4) for the case $j = r = 0$. In this case we have

$$\langle \psi_{0k}, \tilde{\psi}_{0s} \rangle = \Big\langle \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha \, \overline{b(1-\alpha)} \, \phi(2\cdot - 2k - \alpha), \sum_{\beta \in \mathbb{Z}} (-1)^\beta \, \overline{a(1-\beta)} \, \tilde{\phi}(2\cdot - 2s - \beta) \Big\rangle.$$

Note that

$$\langle \phi(2\cdot - 2k - \alpha), \tilde{\phi}(2\cdot - 2s - \beta) \rangle = \begin{cases} 1/2 & \text{if } 2k + \alpha = 2s + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\langle \psi_{0k}, \tilde{\psi}_{0s} \rangle = \frac{1}{2} \sum_{\alpha \in \mathbb{Z}} a\big(2(s-k) + 1 - \alpha\big) \, \overline{b(1-\alpha)} = \delta_{ks},$$

where (4.1) has been used to derive the second equality. For $j, k \in \mathbb{Z}$, let

$$\phi_{jk} := \sqrt{2}^j \phi(2^j \cdot - k) \quad \text{and} \quad \tilde{\phi}_{jk} := \sqrt{2}^j \tilde{\phi}(2^j \cdot - k).$$

Then we have

$$\begin{aligned} \langle \psi_{0k}, \tilde{\phi}_{0s} \rangle &= \Big\langle \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha \overline{b(1-\alpha)} \, \phi(2\cdot - 2k - \alpha), \sum_{\beta \in \mathbb{Z}} b(\beta) \, \tilde{\phi}(2\cdot - 2s - \beta) \Big\rangle \\ &= \frac{1}{2} \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha \overline{b(1-\alpha)} \, \overline{b(2k - 2s + \alpha)} = 0. \end{aligned}$$

Similarly, $\langle \phi_{0k}, \tilde{\psi}_{0s} \rangle = 0$. By using a simple change of variables we obtain

$$\langle \psi_{jk}, \tilde{\psi}_{js} \rangle = \delta_{ks}, \quad \langle \psi_{jk}, \tilde{\phi}_{js} \rangle = 0, \quad \text{and} \quad \langle \phi_{jk}, \tilde{\psi}_{js} \rangle = 0.$$

In particular, (4.4) is valid for $j = r$. For $j > r$, $\tilde{\psi}_{rs}$ is a linear combination of $\tilde{\phi}_{jn}$ ($n \in \mathbb{Z}$). Hence, $\langle \psi_{jk}, \tilde{\psi}_{rs} \rangle = 0$ for $j > r$. Similarly, $\langle \psi_{jk}, \tilde{\psi}_{rs} \rangle = 0$ for $j < r$. This verifies (4.4) for the case $j \neq r$.

Thus, $(\psi_{jk})_{j,k \in \mathbb{Z}}$ and $(\tilde{\psi}_{jk})_{j,k \in \mathbb{Z}}$ are biorthogonal Bessel sequences in $L_2(\mathbb{R})$. The following theorem tells us that they are Riesz sequences.

Theorem 4.2. *Let $(f_n)_{n=1,2,\ldots}$ and $(g_n)_{n=1,2,\ldots}$ be two Bessel sequences in a Hilbert space H. If, in addition, they are biorthogonal, i.e., $\langle f_m, g_n \rangle = \delta_{mn}$, then both $(f_n)_{n=1,2,\ldots}$ and $(g_n)_{n=1,2,\ldots}$ are Riesz sequences.*

Proof. Let $(c_n)_{n=1,2,\ldots}$ be a sequence in ℓ_2. For $N \in \mathbb{N}$, let $u_N := \sum_{n=1}^N c_n f_n$. Then we have

$$\langle u_N, u_N \rangle = \sum_{n=1}^N c_n \langle f_n, u_N \rangle.$$

By the Schwarz inequality we deduce from the above equation that

$$\|u_N\|^4 \le \left(\sum_{n=1}^N |c_n|^2\right)\left(\sum_{n=1}^N \bigl|\langle f_n, u_N\rangle\bigr|^2\right). \tag{4.5}$$

But the sequence $(f_n)_{n=1,2,\ldots}$ is a Bessel sequence. Hence, there exists a constant A such that

$$\sum_{n=1}^\infty \bigl|\langle f_n, u_N\rangle\bigr|^2 \le A\|u_N\|^2, \tag{4.6}$$

where A is independent of N. Combining (4.5) and (4.6) together, we obtain

$$\|u_N\|^4 \le A\|u_N\|^2 \sum_{n=1}^N |c_n|^2.$$

It follows that

$$\|u_N\|^2 \le A \sum_{n=1}^N |c_n|^2. \tag{4.7}$$

Since $(c_n)_{n=1,2,\ldots}$ is a sequence in ℓ_2, this shows that $(u_N)_{N=1,2,\ldots}$ converges to some element u in H. Letting $N \to \infty$ in (4.7), we obtain

$$\|u\|^2 \le A \sum_{n=1}^\infty |c_n|^2. \tag{4.8}$$

We may write u as $\sum_{n=1}^\infty c_n f_n$. Clearly, biorthogonality of $(f_n)_{n=1,2,\ldots}$ and $(g_n)_{n=1,2,\ldots}$ implies $c_n = \langle u, g_n\rangle$. Since $(g_n)_{n=1,2,\ldots}$ is a Bessel sequence, there exists a constant B such that

$$\sum_{n=1}^\infty |c_n|^2 = \sum_{n=1}^\infty \bigl|\langle u, g_n\rangle\bigr|^2 \le B\|u\|^2. \tag{4.9}$$

By (4.8) and (4.9) we conclude that $(f_n)_{n=1,2,\ldots}$ is a Riesz sequence. Similarly, $(g_n)_{n=1,2,\ldots}$ is also a Riesz sequence. □

We are in a position to establish the main result of this section.

Theorem 4.3. *Let ϕ and $\tilde\phi$ be two compactly supported functions in $L_2(\mathbb{R})$ such that the shifts of ϕ and $\tilde\phi$ are biorthogonal. Suppose ϕ and $\tilde\phi$ are the normalized solutions to the refinement equations in (3.1) with masks a and b, respectively. For $j,k \in \mathbb{Z}$, let $(\psi_{jk})_{j,k\in\mathbb{Z}}$ and $(\tilde\psi_{jk})_{j,k\in\mathbb{Z}}$ be defined as in (4.2) and (4.3). Then $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Riesz basis for $L_2(\mathbb{R})$. Moreover,*

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \tilde\psi_{jk}\rangle \psi_{jk}, \quad f \in L_2(\mathbb{R}),$$

where the series converges in the L_2 norm.

Proof. For $j \in \mathbb{Z}$, let P_j be the linear operator on $L_2(\mathbb{R})$ given by

$$P_j f := \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{jk} \rangle \phi_{jk}, \quad f \in L_2(\mathbb{R}),$$

where $\phi_{jk} := \sqrt{2}^j \phi(2^j \cdot - k)$ and $\tilde{\phi}_{jk} := \sqrt{2}^j \tilde{\phi}(2^j \cdot - k)$. Note that $\|P_j\| = \|P_0\|$ for all $j \in \mathbb{Z}$. By the hypothesis of the theorem, both ϕ and $\tilde{\phi}$ satisfy the Strang-Fix conditions of order 1. Hence, $\sum_{k \in \mathbb{Z}} \phi(\cdot - k) = 1$ and $\int_{\mathbb{R}} \tilde{\phi}(x)\, dx = 1$. Consequently, P_j preserves constants, *i.e.*, $P_j 1 = 1$. Therefore, $\|P_n f - f\|_2 \to 0$ as $n \to \infty$. Moreover, $\|P_n f\|_2 \to 0$ as $n \to -\infty$. For this, one can first prove that $\lim_{n \to -\infty} \|P_n g\|_2 = 0$ for any continuous function g with compact support. For general $f \in L_2(\mathbb{R})$ we argue as follows. Given $\varepsilon > 0$, there exists a compactly supported continuous function g such that $\|f - g\|_2 \le \varepsilon / \|P_0\|$. It follows that $\|P_n(f - g)\|_2 \le \varepsilon$ for all n. There exists a positive integer N such that $\|P_n g\|_2 \le \varepsilon$ for all $n \le -N$. Consequently, $\|P_n f\|_2 \le 2\varepsilon$ for all $n \le -N$. This shows that $\|P_n f\|_2 \to 0$ as $n \to -\infty$.

We claim that

$$P_1 f - P_0 f = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{0k} \rangle \psi_{0k}. \tag{4.10}$$

In order to verify (4.10) we express ψ_{0k} and $\tilde{\psi}_{0k}$ as follows:

$$\psi(x - k) = \sum_{r \in \mathbb{Z}} (-1)^r \overline{b(1 - r)} \phi(2x - 2k - r) = \sum_{r \in \mathbb{Z}} (-1)^r \overline{b(1 - r + 2k)} \phi(2x - r),$$

$$\tilde{\psi}(x - k) = \sum_{s \in \mathbb{Z}} (-1)^s \overline{a(1 - s)} \tilde{\phi}(2x - 2k - s) = \sum_{s \in \mathbb{Z}} (-1)^s \overline{a(1 - s + 2k)} \tilde{\phi}(2x - s).$$

Hence,

$$\sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{0k} \rangle \psi_{0k} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^{r+s} a(1 - s + 2k) \overline{b(1 - r + 2k)} \langle f, \tilde{\phi}(2 \cdot - s) \rangle \phi(2 \cdot - r).$$

Moreover,

$$\phi(x - k) = \sum_{r \in \mathbb{Z}} a(r) \phi(2x - 2k - r) = \sum_{r \in \mathbb{Z}} a(r - 2k) \phi(2x - r),$$

$$\tilde{\phi}(x - k) = \sum_{s \in \mathbb{Z}} b(s) \tilde{\phi}(2x - 2k - s) = \sum_{s \in \mathbb{Z}} b(s - 2k) \tilde{\phi}(2x - s).$$

Hence,

$$\sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{0k} \rangle \phi_{0k} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} a(r - 2k) \overline{b(s - 2k)} \langle f, \tilde{\phi}(2 \cdot - s) \rangle \phi(2 \cdot - r).$$

Thus, (4.10) will be established if we can prove

$$\sum_{k \in \mathbb{Z}} (-1)^{r+s} a(1 - s + 2k) \overline{b(1 - r + 2k)} + \sum_{k \in \mathbb{Z}} a(r - 2k) \overline{b(s - 2k)} = 2\delta_{rs}. \tag{4.11}$$

The proof of (4.11) will be divided into two cases: $r+s$ is odd and $r+s$ is even. If $r+s$ is odd, letting $k \to (r+s-1)/2-k$ in the second sum of (4.11), we obtain

$$\sum_{k\in\mathbb{Z}} a(r-2k)\overline{b(s-2k)} = \sum_{k\in\mathbb{Z}} a(1-s+2k)\overline{b(1-r+2k)}.$$

This verifies (4.11) for the case when $r+s$ is odd. If $r+s$ is even, letting $k \to (r+s)/2-k$ in the first sum of (4.11), we obtain

$$\sum_{k\in\mathbb{Z}} a(1-s+2k)\overline{b(1-r+2k)} = \sum_{k\in\mathbb{Z}} a(r+1-2k)\overline{b(s+1-2k)}.$$

Consequently, the left-hand side of (4.11) is equal to

$$\sum_{k\in\mathbb{Z}} a(r+1-2k)\overline{b(s+1-2k)} + \sum_{k\in\mathbb{Z}} a(r-2k)\overline{b(s-2k)} = \sum_{j\in\mathbb{Z}} a(r+j)\overline{b(s+j)} = 2\delta_{rs},$$

where (4.1) has been used to derive the last equality. This verifies (4.11) for the case when $r+s$ is even.

Let n be a positive integer. It follows from (4.10) that

$$P_{n+1}f - P_{-n}f = \sum_{j=-n}^{n} \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_{jk}\rangle \psi_{jk}, \quad f \in L_2(\mathbb{R}).$$

But $\|(P_{n+1}f - P_{-n}f) - f\|_2 \to 0$ as $n \to \infty$. Hence, the linear span of $\{\psi_{jk} : j,k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{R})$. By Theorems 4.1 and 4.2, $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Riesz sequence. Therefore, $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Riesz basis for $L_2(\mathbb{R})$. Moreover, by what has been proved, we have

$$f = \lim_{n\to\infty} (P_{n+1}f - P_{-n}f) = \lim_{n\to\infty} \sum_{j=-n}^{n} \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_{jk}\rangle \psi_{jk} = \sum_{j,k\in\mathbb{Z}} \langle f, \tilde{\psi}_{jk}\rangle \psi_{jk}, \quad f \in L_2(\mathbb{R}),$$

where the series converges in the L_2 norm. This completes the proof of the theorem. □

It remains to prove Theorem 4.1. Its proof requires two auxiliary lemmas. The following lemma extends [22, Lemma 3.1] from the case $p=2$ to the general case $1 \le p \le \infty$.

Lemma 4.1. *Suppose $f \in Lip\,(\mu, L_p(\mathbb{R}))$ for some $\mu > 0$ and $g \in L_q(\mathbb{R})$, $1/p + 1/q = 1$. Assume that g has compact support and $\int_{\mathbb{R}} g(x)\,dx = 0$. Let*

$$b_n(k) := \left\langle 2^{-n/p} f(2^{-n}\cdot), g(\cdot - k)\right\rangle, \quad k \in \mathbb{Z},\ n = 0, 1, 2, \ldots.$$

Then $\|b_n\|_p \le C2^{-n\mu}$, where C is a constant depending only on g, f and μ.

Proof. Let $f_n(x) := 2^{-n/p} f(2^{-n}x)$, $x \in \mathbb{R}$. By Theorem 2.4 in [8, Chap. 6] there exists a constant C_1 and a function $u_n \in L_p(\mathbb{R})$ such that $u_n' \in L_p(\mathbb{R})$,

$$\|f_n - u_n\|_p \le C_1\omega(f_n, 1)_p \quad \text{and} \quad \|u_n'\|_p \le C_1\omega(f_n, 1)_p, \tag{4.12}$$

where the constant C_1 is independent of n. For $t \in \mathbb{R}$ we have

$$\|f_n - f_n(\cdot - t)\|_p = \|2^{-n/p} f(2^{-n}\cdot) - 2^{-n/p} f(2^{-n}\cdot - 2^{-n}t)\|_p = \|f(\cdot) - f(\cdot - 2^{-n}t)\|_p.$$

Since f lies in $\mathrm{Lip}\,(\mu, L_p(\mathbb{R}))$, there exists a constant C_2 such that

$$\|f(\cdot) - f(\cdot - 2^{-n}t)\|_p \le C_2(2^{-n}|t|)^\mu.$$

Hence,

$$\omega(f_n, 1)_p = \sup_{|t|\le 1} \|f_n - f_n(\cdot - t)\|_p \le C_2 2^{-n\mu}. \tag{4.13}$$

We may write $b_n(k) = c_n(k) + d_n(k)$, where

$$c_n(k) := \langle f_n - u_n, g(\cdot - k)\rangle \quad \text{and} \quad d_n(k) := \langle u_n, g(\cdot - k)\rangle, \quad k \in \mathbb{Z}.$$

Since $g \in L_q(\mathbb{R})$ is compactly supported, by [13, Theorem 3.1], there exists a constant C_3 (independent of n) such that

$$\|c_n\|_p \le C_3\|f_n - u_n\|_p. \tag{4.14}$$

Let $h(x) := \int_{-\infty}^x g(y)\,dy$. Since $\int_{\mathbb{R}} g(y)\,dy = 0$, h is compactly supported. Moreover, $h' = g$. Consequently,

$$d_n(k) = \langle u_n, h'(\cdot - k)\rangle = -\langle u_n', h(\cdot - k)\rangle.$$

Hence, there exists a constant C_4 (independent of n) such that

$$\|d_n\|_p \le C_4\|u_n'\|_p. \qquad (4.15).$$

Combining the estimates (4.12)–(4.15) together, we see that there exists a constant C such that $\|b_n\|_p \le C2^{-n\mu}$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. This completes the proof of the lemma. □

Lemma 4.2. *Suppose $a \in \ell_2(\mathbb{Z})$ and w is a sequence supported on $[-K2^n, K2^n]$, where $K \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Let*

$$u(k) := \sum_{j\in\mathbb{Z}} a(2^n k - j)w(j) \quad \textit{and} \quad v(k) := \sum_{j\in\mathbb{Z}} w(k - 2^n j)a(j), \quad k \in \mathbb{Z}.$$

Then

$$\|u\|_2 \le \sqrt{2K+1}\,\|a\|_2\|w\|_2 \quad \textit{and} \quad \|v\|_2 \le \sqrt{2K+1}\,\|a\|_2\|w\|_2.$$

Proof. Since w is supported on $[-K2^n, K2^n]$, by the Schwarz inequality we have

$$|u(k)|^2 \le \sum_{j=-2^nK}^{2^nK} \bigl|a(2^n k - j)\bigr|^2 \sum_{j=-2^nK}^{2^nK} |w(j)|^2, \quad k \in \mathbb{Z}.$$

It follows that

$$\sum_{k\in\mathbb{Z}} |u(k)|^2 \le \sum_{k\in\mathbb{Z}} \sum_{j=-2^nK}^{2^nK} |a(2^nk-j)|^2 \sum_{j\in\mathbb{Z}} |w(j)|^2.$$

But

$$\sum_{k\in\mathbb{Z}} \sum_{j=-2^nK}^{2^nK} |a(2^nk-j)|^2 \le (2K+1) \sum_{k\in\mathbb{Z}} |a(k)|^2.$$

Hence, $\|u\|_2 \le \sqrt{2K+1}\|a\|_2\|w\|_2$.

In order to prove the second inequality we write $k = 2^nr+s$, where $0 \le s < 2^n$ and $r \in \mathbb{Z}$. Fix s for the time being. Since w is supported on $[-K2^n, K2^n]$, $w(2^nr+s-2^nj) \ne 0$ only if $-K2^n \le 2^nr+s-2^nj \le K2^n$, or $-K+r+2^{-n}s \le j \le K+r+2^{-n}s$. By the Schwarz inequality we have

$$|v(2^nr+s)|^2 \le \sum_{-K+r+2^{-n}s\le j\le K+r+2^{-n}s} |w(2^nr+s-2^nj)|^2 \sum_{-K+r+2^{-n}s\le j\le K+r+2^{-n}s} |a(j)|^2.$$

It follows that

$$\begin{aligned}\sum_{r\in\mathbb{Z}} |v(2^nr+s)|^2 &\le \sum_{j\in\mathbb{Z}} |w(s-2^nj)|^2 \left[\sum_{r\in\mathbb{Z}} \sum_{-K+r+2^{-n}s\le j\le K+r+2^{-n}s} |a(j)|^2\right].\\ &\le \sum_{j\in\mathbb{Z}} |w(s-2^nj)|^2\,(2K+1) \sum_{j\in\mathbb{Z}} |a(j)|^2.\end{aligned}$$

Hence,

$$\|v\|_2^2 = \sum_{s=0}^{2^n-1} \sum_{r\in\mathbb{Z}} |v(2^nr+s)|^2 \le (2K+1)\|a\|_2^2 \sum_{s=0}^{2^n-1} \sum_{j\in\mathbb{Z}} |w(s-2^nj)|^2.$$

This shows that $\|v\|_2 \le \sqrt{2K+1}\,\|a\|_2\|w\|_2$. $\square$

Proof of Theorem 4.1 Let h be the orthonormal Haar wavelet, defined by $h(x) = 1$ for $0 \le x < 1/2$, $h(x) = -1$ for $1/2 \le x < 1$, and $h(x) = 0$ elsewhere. Defining $h_{jk}(x) = 2^{j/2}h(2^jx-k)$, $j,k \in \mathbb{Z}$, we see that $(h_{jk})_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L_2(\mathbb{R})$. Decompose $f = \sum_{j,k\in\mathbb{Z}} a_j(k)h_{jk}$ and then compute $b_j(k) := \langle f, \psi_{jk}\rangle$, $j,k \in \mathbb{Z}$. We have

$$\langle h_{rs}, \psi_{jk}\rangle = \int_{\mathbb{R}} 2^{r/2}h(2^rx-s)\,2^{j/2}\overline{\psi(2^jx-k)}\,dx = \int_{\mathbb{R}} 2^{-n/2}h(2^{-n}x-s)\,\overline{\psi(x-k)}\,dx,$$

where $n := j-r$. For $n \in \mathbb{N}_0$, define

$$w_{n1}(k) := \int_{\mathbb{R}} 2^{-n/2}h(2^{-n}x)\,\overline{\psi(x-k)}\,dx, \quad k \in \mathbb{Z}.$$

Then for $j \geq r$ we have

$$\langle h_{rs}, \psi_{jk} \rangle = w_{n1}(k - 2^n s).$$

Similarly, for $n \in \mathbb{N}_0$ define

$$w_{n2}(k) := \int_{\mathbb{R}} h(x+k)\, 2^{-n/2} \overline{\psi(2^{-n}x)}\, dx, \quad k \in \mathbb{Z}.$$

Then for $j < r$ we have

$$\langle h_{rs}, \psi_{jk} \rangle = w_{n2}(2^n k - s),$$

where $n := r - j$. Hence, for $j, k \in \mathbb{Z}$ we have

$$\begin{aligned} b_j(k) = \langle f, \psi_{jk} \rangle &= \sum_{r,s \in \mathbb{Z}} a_r(s) \langle h_{rs}, \psi_{jk} \rangle \\ &= \sum_{n=0}^{\infty} \sum_{s \in \mathbb{Z}} a_{j-n}(s) w_{n1}(k - 2^n s) + \sum_{n=1}^{\infty} \sum_{s \in \mathbb{Z}} a_{j+n}(s) w_{n2}(2^n k - s). \end{aligned}$$

Note that there exists a positive integer K such that both w_{n1} and w_{n2} are supported on $[-K2^n, K2^n]$. By Lemma 4.2, there exists a positive constant C_1 such that

$$\|b_j\|_2 \leq C_1 \left[\sum_{n=0}^{\infty} \|a_{j-n}\|_2 \|w_{n1}\|_2 + \sum_{n=1}^{\infty} \|a_{j+n}\|_2 \|w_{n2}\|_2 \right].$$

Note that $\psi \in H^\mu(\mathbb{R}) \subseteq \mathrm{Lip}(\mu, L_2(\mathbb{R}))$, where $\mu > 0$. Hence, by Lemma 4.1 we obtain

$$\|w_{n1}\|_2 \leq C_2 2^{-n\mu} \quad \text{and} \quad \|w_{n2}\|_2 \leq C_2 2^{-n\mu},$$

where C_2 is a constant independent of n. Consequently,

$$\|b_j\|_2 \leq C_1 C_2 \left[\sum_{n=0}^{\infty} \|a_{j-n}\|_2 2^{-n\mu} + \sum_{n=1}^{\infty} \|a_{j+n}\|_2 2^{-n\mu} \right].$$

It follows that

$$\sum_{j \in \mathbb{Z}} \|b_j\|_2^2 \leq (C_1 C_2)^2 \sigma^2 \sum_{j \in \mathbb{Z}} \|a_j\|_2^2,$$

where $\sigma := \sum_{n \in \mathbb{Z}} 2^{-\mu|n|} < \infty$. But $\sum_{j \in \mathbb{Z}} \|a_j\|_2^2 = \|f\|_2^2$. Therefore, there exists a positive constant B such that

$$\sum_{j,k \in \mathbb{Z}} \left| \langle f, \psi_{jk} \rangle \right|^2 = \sum_{j \in \mathbb{Z}} \|b_j\|_2^2 \leq B \|f\|_2^2 \quad \forall f \in L_2(\mathbb{R}).$$

This completes the proof of Theorem 4.1. $\square$

§5. Multivariate Refinable Functions

In this section we extend our study of cascade algorithms to high dimensional spaces. Several results on characterization of convergence of cascade algorithms are given. In particular, uniform convergence of a cascade algorithm associated with a nonnegative mask is characterized in terms of stochastic matrices. Finally, cascade algorithms are used to investigate multivariate biorthogonal and fundamental refinable functions. These results are stated without proof. For detailed proofs the reader is referred to relevant papers.

Let us consider the following refinement equation

$$\phi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(M\cdot - \alpha),$$

where a is a finitely supported sequence on $\mathbb{Z}^s$, called the refinement mask, and M is an $s\times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$, called a **dilation matrix**.

If $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m := |\det M|$, then there exists a unique compactly supported distribution ϕ with $\hat{\phi}(0) = 1$ satisfying the above refinement equation. We say that ϕ is the normalized solution of the refinement equation.

Let Q_a be the cascade operator defined by

$$Q_a f := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) f(M\cdot - \alpha), \quad f \in L_p(\mathbb{R}^s).$$

Suppose the normalized solution ϕ lies in $L_p(\mathbb{R}^s)$. If $\lim_{n\to\infty} \|Q_a^n f - \phi\|_p = 0$, then f must satisfy the Strang-Fix conditions of order 1. In other words,

$$\sum_{\alpha\in\mathbb{Z}^s} f(\cdot - \alpha) = 1.$$

We say that the cascade algorithm associated with a converges in the L_p norm if there exists a compactly supported function ϕ in $L_p(\mathbb{R}^s)$ such that

$$\lim_{n\to\infty} \|Q_a^n f - \phi\|_p = 0$$

for any compactly supported function $f \in L_p(\mathbb{R}^s)$ satisfying the Strang-Fix conditions of order 1.

Theorems 5.1 and 5.2, established by Han and Jia [10], are generalizations of Theorems 2.1 and 2.2.

Theorem 5.1. *Suppose u is a compactly supported function in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$), u satisfies the Strang-Fix conditions of order 1, and the shifts of u are stable. If there exists a function $\phi \in L_p(\mathbb{R}^s)$ (a continuous function in the case $p = \infty$) such that*

$$\lim_{n\to\infty} \|Q_a^n u - \phi\|_p = 0,$$

then for any compactly supported function $v \in L_p(\mathbb{R}^s)$ satisfying the Strang-Fix conditions of order 1 we also have

$$\lim_{n\to\infty} \|Q_a^n v - \phi\|_p = 0.$$

Consequently, if the normalized solution ϕ of the refinement equation lies in $L_p(\mathbb{R}^s)$ (ϕ is a continuous function in the case $p=\infty$), and if the shifts of ϕ are stable, then the cascade algorithm associated with mask a converges in the L_p norm.

If the cascade algorithm associated with a converges in the L_p norm ($1 \le p \le \infty$), then a satisfies the **basic sum rule**:

$$\sum_{\alpha\in\mathbb{Z}^s} a(\gamma + M\alpha) = 1 \quad \forall\, \gamma \in \mathbb{Z}^s.$$

Let E be a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^s/M\mathbb{Z}^s$. We assume that E contains 0. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\varepsilon + M\gamma$, where $\varepsilon \in E$ and $\gamma \in \mathbb{Z}^s$.

By $\ell_0(\mathbb{Z}^s)$ we denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$. The support of an element $a \in \ell_0(\mathbb{Z}^s)$ is defined by $\operatorname{supp} a := \{\alpha \in \mathbb{Z}^s : a(\alpha) \ne 0\}$. For a finite subset K of $\mathbb{Z}^s$, we denote by $\ell(K)$ the linear subspace of $\ell_0(\mathbb{Z}^s)$ consisting of all sequences supported on K.

For $\varepsilon \in E$, let A_ε be the linear operator on $\ell_0(\mathbb{Z}^s)$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon + M\alpha - \beta)v(\beta), \quad v \in \ell_0(\mathbb{Z}^s),\ \alpha \in \mathbb{Z}^s.$$

Let $K := \mathbb{Z}^s \cap (\sum_{n=1}^{\infty} M^{-n}H)$, where $H := (\operatorname{supp} a - E) \cup M([0,1]^s)$. Then $\ell(K)$ is invariant under A_ε for each $\varepsilon \in E$. Moreover, if a satisfies the basic sum rule, then

$$V := \Big\{ v \in \ell(K) : \sum_{\alpha\in\mathbb{Z}^s} v(\alpha) = 0 \Big\}$$

is invariant under A_ε for each $\varepsilon \in E$.

Theorem 5.2. *The cascade algorithm associated with mask a and dilation matrix M converges in the L_p norm ($1 \le p \le \infty$) if and only if a satisfies the basic sum rule and $\rho_p\big(\{A_\varepsilon|_V : \varepsilon \in E\}\big) < m^{1/p}$, where $m := |\det M|$.*

The following theorem on L_2 convergence of cascade algorithms was obtained independently by Lawton, Lee, and Shen [17], and by Han and Jia [10].

Theorem 5.3. *For a given mask a and a dilation matrix M, let b be the sequence given by $b(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha+\beta)\overline{a(\beta)}/m$, $\alpha \in \mathbb{Z}^s$, and let $K := \mathbb{Z}^s \cap (\sum_{n=1}^{\infty} M^{-n}(\operatorname{supp} b))$. Then the cascade algorithm associated with mask a and dilation matrix M converges in the L_2 norm if and only if the following two conditions are satisfied:*

(a) *a satisfies the basic sum rule;*

(b) *The matrix $B := (b(M\alpha - \beta))_{\alpha,\beta\in K}$ has 1 as its simple eigenvalue and all the other eigenvalues are less than 1 in modulus.*

The uniform convergence of a cascade algorithm associated with a nonnegative mask can be characterized in terms of stochastic matrices. A square matrix $(a_{ij})_{1\le i,j\le r}$ is called a **column-stochastic** matrix if $a_{ij} \ge 0$ for all $i,j = 1,\dots,r$ and $\sum_{i=1}^{r} a_{ij} = 1$ for all $j = 1,\dots,r$. A column-stochastic matrix A is called **scrambling** if each pair of columns of A have positive entries in some common row. The following theorem was established by Jia and Zhou [15].

Theorem 5.4. *Let a be a nonnegative mask. Then the cascade algorithm associated with mask a converges uniformly if and only if*

(a) *a satisfies the basic sum rule, and*

(b) *there is a positive integer k such that for every k-tuple $(\varepsilon_1, \ldots, \varepsilon_k) \in E^k$, the finite matrix $\big((A_{\varepsilon_k} \cdots A_{\varepsilon_1})(\alpha, \beta)\big)_{\alpha,\beta \in K}$ is scrambling.*

Let us discuss applications of cascade algorithms to biorthogonal refinable functions and wavelets. Given two compactly supported functions ϕ and $\tilde{\phi}$ in $L_2(\mathbb{R}^s)$, we say that the shifts of ϕ and $\tilde{\phi}$ are biorthogonal if

$$\langle \phi(\cdot - \alpha), \tilde{\phi}(\cdot - \beta) \rangle = \delta_{\alpha\beta} \quad \forall\, \alpha, \beta \in \mathbb{Z}^s.$$

Suppose ϕ and $\tilde{\phi}$ are refinable:

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(M \cdot - \alpha) \quad \text{and} \quad \tilde{\phi} = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) \tilde{\phi}(M \cdot - \alpha),$$

where a and b are finitely supported sequences on $\mathbb{Z}^s$ such that

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) = m = |\det M|.$$

Let

$$H(\xi) := \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha \cdot \xi} \quad \text{and} \quad G(\xi) := \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) e^{-i\alpha \cdot \xi}, \quad \xi \in \mathbb{R}^s.$$

Let Ω be a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^s / M^T \mathbb{Z}^s$. We assume that Ω contains 0.

Theorem 5.5. *The shifts of ϕ and $\tilde{\phi}$ are biorthogonal if and only if the following two conditions are satisfied:*

(a) $\sum_{\omega \in \Omega} H\big(\xi + (M^T)^{-1} 2\pi\omega\big) \overline{G\big(\xi + (M^T)^{-1} 2\pi\omega\big)} = 1$ *for all $\xi \in \mathbb{R}$;*

(b) *The cascade algorithms associated with a and b converge in $L_2(\mathbb{R}^s)$.*

By Theorem 5.3, the L_2 convergence of the cascade algorithms associated with a and b can be characterized in terms of the eigenvalue condition on the corresponding transition matrices. Thus, Theorem 5.5 recovers the results of Long and Chen [18] on biorthogonal wavelet bases.

Finally, we mention some useful facts about multivariate fundamental refinable functions. A continuous function ϕ on $\mathbb{R}^s$ is said to be fundamental if $\phi(0) = 1$ and $\phi(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \setminus \{0\}$. A mask a is called interpolatory (with respect to the lattice $M\mathbb{Z}^s$) if

$$a(0) = 1 \quad \text{and} \quad a(M\alpha) = 0 \quad \forall\, \alpha \in \mathbb{Z}^s \setminus \{0\}.$$

Theorem 5.6. *If a is an interpolatory mask and the cascade algorithm associated with a converges uniformly, then the limit function is fundamental. Conversely, if ϕ is a continuous fundamental function on $\mathbb{R}^s$ with compact support and ϕ satisfies the refinement equation with mask a, then the cascade algorithm associated with a converges uniformly.*

By using Theorem 5.4, it can be proved that for any dilation matrix M, there exists an interpolatory nonnegative mask a such that the normalized solution of the refinement equation is fundamental.

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