Estimation and Comparison of Beta-Pricing Models

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Ninth PKU-NUS Annual International Conference on Quantitative Finance and Economics, May 2025

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Background: Beta Pricing Models

- Beta-pricing models, such as CAPM, provide a foundational framework for explaining cross-sectional expected returns via factor risk premia and asset-factor exposures (betas).

$$E(\boldsymbol{r}_t) = \boldsymbol{B\lambda}$$

- The two-pass regression is a standard estimation approach.
- Asset-specific Time series regression for betas, factor exposures

$$r_{t,i} = a_i + \boldsymbol{\beta}_i \boldsymbol{f}_t + \boldsymbol{e}_{t,i}, \quad i = 1, ..., N$$

- Cross-sectional regression for lambdas, risk premia

$$\bar{r}_{t,i} = \hat{\boldsymbol{\beta}}_i \boldsymbol{\lambda} + \alpha_i, \quad i = 1, ..., N$$

Background: Factor Zoo

- Traded Factors:
  - excess market return
  - small-minus-big (size), high-minus-low (value) (e.g., Fama and French, 1993, JFE)
  - up-minus-down (momentum) (e.g., Jegadeesh and Titman, 1993, JF)
  - ...
- Nontraded Factors:
  - consumption growth (e.g., Breeden et al., 1989, JF)
  - market liquidity (e.g., Pastor and Stambaugh, 2003, JPE)
  - intermediary capital ratio(e.g., He et al., 2017, JFE)
  - ...

Motivation: Model Comparison

- Comparing Traded Factor Models:
  - GRS test (Sharpe Ratio Increase) (e.g., Gibbons, Ross, and Shanken, 1989, ECTA)
  - Bayesian Marginal Likelihood (Bayesian GRS-type comparison) (e.g., Barillas and Shanken,2018, JF, Chib, Zeng, and Zhao,2020, JF)
  - Only applies to traded factors
- Comparing Models with Nontraded Factors:
  - Two-pass Cross-Sectional R^2 (e.g., Kan, Robotti, and Shanken, 2013, JF)
    - CSR^2 tends to increase with more factors.
  - Hansen-Jaganathan Distance (e.g., Kan and Robotti, 2009, RFS)
    - HJD tends to decrease with more factors.

Motivation: Weak Factors

- Most of the nontraded factors are weakly correlated with test assets, which will cause the problem of reduced-rank of loading matrix $\boldsymbol{B}$.
- Inference on risk premia becomes invalid with weak factors. (e.g., Kan and Zhang,1999, JF; Kleibergen, 2009, JoE).
- F-rank statistics to test the identification of risk premia. (e.g., Kleibergen and Zhan,2020, JF)
- Giglio, Xiu, and Zhang (2025, JF) perform test assets selection and remove test assets exposed to weak factors

We provide a unified framework for simultaneous beta-pricing model comparison and risk premia estimation, which can exclude weak factors.

Our Solution: Bayesian Marginal Likelihood Comparison

- We develop a Bayesian framework for estimating beta-pricing models with traded and nontraded factors.
- Crucially, the framework is designed to enable model comparisons via marginal likelihoods.
- Simulations confirm that this criterion avoids the overfitting bias of traditional metrics such as cross-sectional R^2 and H-J distance.
- Empirically, the optimal model selects 8 traded factors, excluding all nontraded ones, and achieves excellent out-of-sample performance.

Simulation Evidence: Model Comparison

- True model: MKTRF+SMB+HML+HKMcapital
- All combinations of models:
  - 6 traded factors and 4 nontraded factors
  - $(2^6 - 1) * (2^4) = 1008$ model specifications
  - Average across 100 simulations

Top 10 marginal likelihood models	logML	rank	CSR^2	H-J D
MKTRF+SMB+HML+HKMcapital	**55337.2**	**1.0**	62.6	0.0171
MKTRF+SMB+HML+CMA+HKMcapital	55284.7	3.7	63.4	0.0168
MKTRF+SMB+HML+RMW+HKMcapital	55284.5	3.8	63.3	0.0168
MKTRF+SMB+HML+UMD+HKMcapital	55284.1	3.9	63.5	0.0168
MKTRF+SMB+HML+PEAR+HKMcapital	55278.8	5.1	63.5	0.0168
MKTRF+SMB+HML+LIQ+HKMcapital	55278.8	5.1	63.6	0.0167
MKTRF+SMB+HML+PCEND+HKMcapital	55276.8	5.6	63.4	0.0169
MKTRF+SMB+HML	55243.6	11.5	48.0	0.0203
MKTRF+SMB+HML+RMW+CMA+HKMcapital	55231.2	13.0	64.1	0.0165
MKTRF+SMB+HML+CMA+UMD+HKMcapital	55231.1	13.1	64.2	0.0165

Empirical Highlight: Model Comarison

- 10 traded factors:
  - MKTRF, SMB, HML, CMA, RMW, UMD, BAB, QMJ, IA, ROE
- 8 Nontraded factors:
  - IndProd, Liq, LTY, M2_SA, PCEDG, PCEND, HKMcapital, PEAR

Rank	Top 5 Model	logML	CSR^2	H-J D	Prob
1	MKTRF + SMB + HML + RMW + CMA + UMD + BAB + QMJ	**322024**	49.03	0.0071	**1**
2	MKTRF + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE	322008	49.68	0.0071	0
3	MKTRF + SMB + HML + RMW + CMA + UMD + BAB + QMJ + HKMcapital	321905	49.10	0.0071	0
4	MKTRF + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE + HKMcapital	321895	49.74	0.0071	0
5	MKTRF + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PCEDG	321891	49.10	0.0071	0

- Model averaging and model selection are equivalent when the top-1 model has a probability of almost 1.

Model

Model

- Assume that the distributions for traded (T) (e.g., Fama French portfolio-based), nontraded factors (NT) (e.g., macroeconomic innovations) and returns are normal and take the stationary form

$$\begin{pmatrix} \boldsymbol{f}_t^T \\ \boldsymbol{f}_t^{NT} \\ \boldsymbol{r}_t \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \boldsymbol{\mu}^T \\ \boldsymbol{\mu}^{NT} \\ \boldsymbol{\mu}_r \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_T & \boldsymbol{\Sigma}_{T,NT} & \boldsymbol{\Sigma}_{T,r} \\ \boldsymbol{\Sigma}_{NT,T} & \boldsymbol{\Sigma}_{NT} & \boldsymbol{\Sigma}_{NT,r} \\ \boldsymbol{\Sigma}_{r,T} & \boldsymbol{\Sigma}_{r,NT} & \boldsymbol{\Sigma}_r \end{pmatrix}\right)$$

- Now assume that these factors are in the SDF M_t, and suppose, following Hansen and Jagannathan (1997), that M_t is given by

$$M_t = 1 - \boldsymbol{\lambda}' \boldsymbol{\Sigma}_f^{-1} (\boldsymbol{f}_t - \boldsymbol{\mu}_f), \quad \boldsymbol{\lambda} = (\boldsymbol{\lambda}^T, \boldsymbol{\lambda}^{NT})$$

Model (contd)

- Under the no-arbitrage condition, we must have the pricing restrictions

$$\mathbb{E}[M_t \boldsymbol{f}_t^{T'}] = 0, \quad \mathbb{E}[M_t \boldsymbol{r}_t'] = 0$$

- From the first of these pricing restrictions one can show that

$$\boldsymbol{\mu}^T = \boldsymbol{\lambda}^T$$

and from the second that

$$\boldsymbol{\mu}_r = \boldsymbol{B}\boldsymbol{\lambda} = \boldsymbol{B}^T\boldsymbol{\lambda}^T + \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT}$$

where $\boldsymbol{B} = \boldsymbol{\Sigma}_{f,r}\boldsymbol{\Sigma}_f^{-1}$ is the loading matrix.

Model (contd)

- Inserting these two pricing conditions together into the distribution of returns conditional on the factors we get

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^{T}\boldsymbol{f}_t^{T} + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t, \boldsymbol{e}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_e). \quad (1)$$

- With the distributional assumption of the factors,

$$\boldsymbol{f}_t = \boldsymbol{\mu} + \boldsymbol{u}_t, \quad \boldsymbol{u}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_u) \qquad (2)$$

we get a restricted TS model that can be used to estimate all the parameters in *one pass*.

Likelihood of the Model

- The factor's mean is estimated, rather than demeaned ex ante

$$\boldsymbol{f}_t = \boldsymbol{\mu} + \boldsymbol{u}_t, \boldsymbol{u}_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_u)$$

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^{T}\boldsymbol{f}_t^{T} + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t, \boldsymbol{e}_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_e)$$

- The likelihood of the model is given by

$$\mathcal{L} = P(\boldsymbol{R} \mid \boldsymbol{F}, \boldsymbol{\lambda}, \boldsymbol{B}, \Sigma_e, \boldsymbol{\mu})P(\boldsymbol{F} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}_u)$$
$$\propto -\frac{1}{2}\sum_{t=1}^{T} \boldsymbol{e}_t'\boldsymbol{\Sigma}_{\boldsymbol{e}}^{-1}\boldsymbol{e}_t - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{u}_t'\boldsymbol{\Sigma}_{\boldsymbol{u}}^{-1}\boldsymbol{u}_t.$$

- Likelihood integrates cross-sectional pricing constraints and time-series variations.

Bayesian Marginal Likelihood

- Model comparison based on the likelihood?
  - Likelihood increases when adding more factors.
- The marginal likelihood integrates over parameters.

$$ML = \int\int P(\boldsymbol{R} \mid \boldsymbol{F}, \boldsymbol{\lambda}, \boldsymbol{B}, \Sigma_e, \boldsymbol{\mu}) P(\boldsymbol{F} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}_u) P(\boldsymbol{\lambda} \mid \boldsymbol{B}, \Sigma_e)$$
$$\times P(\boldsymbol{B} \mid \Sigma_e, \boldsymbol{\mu}) P(\Sigma_e \mid \boldsymbol{\mu}) P(\boldsymbol{\mu} \mid \boldsymbol{\Sigma}_u) P(\boldsymbol{\Sigma}_u)\, d\boldsymbol{\lambda}\, d\boldsymbol{B}\, d\Sigma_e\, d\boldsymbol{\mu}\, d\boldsymbol{\Sigma}_u$$

- For weak factors, similar to the ridge penalty, increasing the dimension of $\boldsymbol{B}$ will decrease the $||\boldsymbol{R} - \boldsymbol{FB}||^2$ but increase $||\boldsymbol{B}||^2$.
- The prior of $\boldsymbol{B}$ is a normal density proportional to $exp(-||\boldsymbol{B}||^2)$

$$ML = \int p(\theta) lik(data|\theta) d\theta \propto \int exp(-||\boldsymbol{R} - \boldsymbol{FB}||^2 - ||\boldsymbol{B}||^2) d\theta$$

Bayesian Model Comparison for Beta-Pricing Model

- We have a full set of factors, which we define as $\boldsymbol{f}^*$. Different models choose different combinations within this full set
- The factor model is estimated based on this full set of factors.

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t, \quad \boldsymbol{f}_t \subset \boldsymbol{f}_t^*, \quad \boldsymbol{u}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_u)$$

- The return model is estimated based on the selected factors.

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT} \boldsymbol{\lambda}^{NT} + \boldsymbol{B}^T \boldsymbol{f}_t^T + \boldsymbol{B}^{NT} (\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t$$

$$\boldsymbol{e}_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_e)$$

- We must consider the full set of factors, $\boldsymbol{f}^*$, for model comparison, with the marginal likelihood comparable.

Prior Distribution

We rely on training sample prior distribution, to make valid model comparisons (e.g., Chib and Zeng,2020, JBES, Chib et al.,2024 MS)

- The model-specific priors must be proper for valid model comparison.
- To ensure that differences in marginal likelihood reflect genuine model fit rather than variations in prior specifications, the prior distributions must be comparable across models.
- The chosen priors should be minimally subjective, requiring little user input while maintaining robustness in inference.

MCMC

- The estimation is performed through the Gibbs Sampler.

Algorithm MCMC Sampling with Parameter Expressions

Initialize parameters: Set $\boldsymbol{\theta}^{(0)} = (\boldsymbol{B}^{(0)}, \boldsymbol{\Sigma}_e^{(0)}, \boldsymbol{\lambda}^{NT(0)}, \boldsymbol{\Sigma}_u^{(0)}, \boldsymbol{\mu}^{(0)})$
for $g = 1$ to $n_0 + M$ **do**
    Step 1: For $i = 1$ to $i = n$, sample $\boldsymbol{\beta}_i^{(g+1)} \mid \boldsymbol{\Sigma}_e^{(g)}, \boldsymbol{\lambda}^{NT(g)}, \boldsymbol{\Sigma}_u^{(g)}, \boldsymbol{\mu}^{(g)} \sim \mathcal{N}\left(\widehat{\boldsymbol{\beta}}_i^{(g)}, \mathcal{B}_i^{(g)}\right)$
    Step 2: For $i = 1$ to $i = n$, sample $\sigma_{ei}^2 \mid \boldsymbol{B}^{(g+1)}, \boldsymbol{\lambda}^{NT(g)}, \boldsymbol{\Sigma}_u^{(g)}, \boldsymbol{\mu}^{(g)} \sim \mathcal{IG}\left(\nu_{ei}^{(g)}, \delta_{ei}^{(g)}\right)$
    Step 3: Sample $\boldsymbol{\lambda}^{NT} \mid \boldsymbol{B}^{(g+1)}, \boldsymbol{\Sigma}_e^{(g+1)}, \boldsymbol{\Sigma}_u^{(g)}, \boldsymbol{\mu}^{(g)} \sim \mathcal{N}\left(\widehat{\boldsymbol{\lambda}}^{NT(g)}, \boldsymbol{L}^{NT(g)}\right)$
    Step 4: Sample $\boldsymbol{\mu}^{(g+1)} \mid \boldsymbol{B}^{(g+1)}, \boldsymbol{\Sigma}_e^{(g+1)}, \boldsymbol{\lambda}^{NT(g+1)}, \boldsymbol{\Sigma}_u^{(g)} \sim \mathcal{N}\left(\widehat{\boldsymbol{\mu}}^{(g)}, \boldsymbol{D}_1^{(g)}\right)$
    Step 5: Sample $\boldsymbol{\Sigma}_u^{(g+1)} \mid \boldsymbol{B}^{(g+1)}, \boldsymbol{\Sigma}_e^{(g+1)}, \boldsymbol{\lambda}^{NT(g+1)}, \boldsymbol{\mu}^{(g+1)} \sim \mathcal{IW}\left(\nu_u^{(g)}, \boldsymbol{S}_u^{(g)}\right)$
end for
After burn-in, use samples $\{\boldsymbol{\theta}^{(g)}\}_{g=1}^{N}$ to estimate the posterior distribution.

- Output from this sampling is used to estimated the marginal likelihood by Chib (1995)'s method.

Empirical

Data

- Simulation uses parameters calibrated from empirical data.
- Jan 1985 to Dec 2023.
- A large cross-section of test assets, including 302 equity portfolios downloaded from French's website.
- 10 Traded factors:
    - MKTRF, SMB, HML, CMA, RMW, UMD, BAB, QMJ, IA, ROE
- 8 Nontraded factors:
    - IndProd, Liq, LTY, M2_SA, PCEDG, PCEND, HKMcapital, PEAR

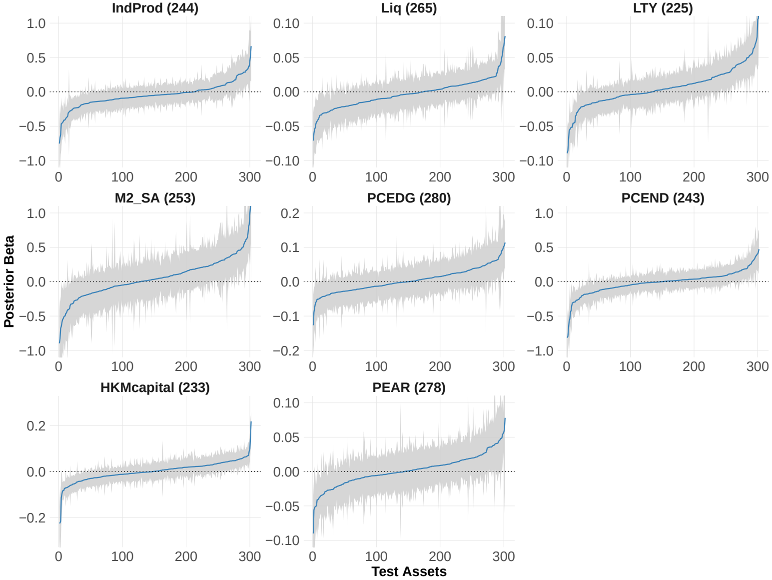
Absolute Test of Single Nontraded Factor

- Third row: log *BF* (Bayes Factor), the difference in log Marginal Likelihood between models with and without the nontraded factor.

	IndProd	LIQ	LTY	M2_SA	PCEDG	PCEND	HKMcapital	PEAR
	-0.68	4.57	1.36	-0.38	0.82	-0.70	1.78	-3.68
CAPM	[-0.83,-0.53]	[3.56,5.56]	[0.74,1.96]	[-0.45,-0.32]	[0.50,1.13]	[-0.84,-0.56]	[1.13,2.36]	[-4.24,-3.09]
	-424	-308	405	-312	340	-300	1291	-22
	0.07	-1.43	0.46	0.16	-1.10	-0.07	0.67	0.40
FF5	[-0.06,0.20]	[-2.31,-0.52]	[-0.43,1.34]	[0.09,0.24]	[-1.63,-0.55]	[-0.21,0.07]	[-0.03,1.35]	[-1.05,1.69]
	-332	-251	-351	-427	-122	-184	1	-239
	-0.04	-2.48	2.98	0.15	1.31	-0.25	1.50	2.52
All	[-0.17,0.09]	[-3.51,-1.45]	[2.15,3.75]	[0.08,0.22]	[0.45,1.99]	[-0.40,-0.11]	[0.76,2.24]	[1.55,3.40]
	-257	-261	-174	-350	-130	-162	-111	-182

- LTY and PCEDG show marginal contribution over CAPM, and HKMcapital even improves over FF5.
- No nontraded factor enhances the All benchmark.

Posterior Beta of Nontraded Factors (mostly zero)



Empirical Highlight: Model Comarison

- Top 5 models in 131,072 model specifications

Rank	Top 5 Model	logML	CSR^2	H-J D	Prob
1	MKTRF + SMB + HML + RMW + CMA + UMD + BAB + QMJ	**322024**	49.03	0.0071	**1**
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- Model averaging and model selection are equivalent when the top-1 model has a probability of almost 1.

Out-of-sample Model Performance

- Estimate factor model implied tangency portfolio weights (1985-2003).
- Fix weights for out-of-sample evaluation (2004-2023).

Panel A : Model-implied				
	Bayes SR	WLS SR	Bayes MDD	WLS MDD
Rank 1	0.983	0.600	0.231	0.428
Rank 2	0.975	0.613	0.246	0.407
Rank 3	0.878	0.588	0.307	0.445
CAPM	0.568	0.568	0.482	0.482
FF3	0.650	0.559	0.428	0.437
FF5	0.883	0.796	0.256	0.246
ALL	0.749	0.813	0.560	0.384
Panel B: EW and MVE				
EW SR	MVE SR		EW MDD	MVE MDD
0.526	0.566		0.491	0.472

Summary

Summary

- Our Bayesian framework is designed to enable beta-pricing model estimation and comparisons via marginal likelihoods, balancing in-sample fit against model complexity and penalizing specifications that include weak factors.
- Empirically, we evaluate each nontraded factor for its incremental contribution beyond benchmark traded factor models and find all tested nontraded factors to be weak.
- Our analysis shows that the proposed framework is an effective screening tool that favors models that satisfy cross-sectional pricing constraints, capture time-series dynamics, and exclude weak factors.

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