RSA Cryptography: Our World's Security

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Motivation

- During my freshman year, I heard a talk on Cryptography.
- The presentation was very interesting and blended the ideas of computer science with mathematics.
- The Imitation Game was also a part in my interest in encryption and decryption of codes.

History

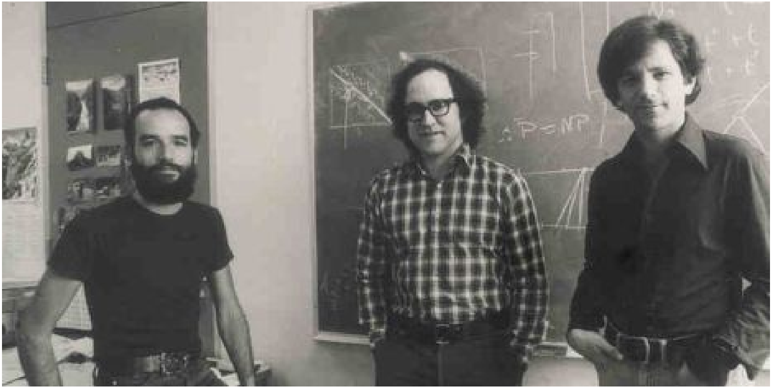
- Cryptography is the subject of transforming information, so it cannot be easily recovered without special knowledge.
- Julius Caesar is one of the first known people to use cryptography.
- He shifted the letters of the alphabet in order to encrypt messages he was sending.
- Originally, cryptography would be a system that could be represented as a receiver receiving many messages (locks) and having to keep track of many different keys that go to these locks in order to break them.

History

- James Ellis came up with an idea that would allow a public distribution of open locks all with the same key, but the locks could have a message kept inside then sent back to the person with the key so only he/she can open the lock and read the message.
- Sadly, Ellis was unable to come up with a mathematical solution to his idea.

History

- Ronald Rivest, Adi Shamir, and Leonard Adleman perfected public key cryptography or what we call RSA.



Receiver

- 1. First, the receiver picks 2 very large primes, p and q. Also, compute $n = pq$.
- 2. Compute $m = lcm(p-1, q-1)$.
- 3. Pick an e relatively prime to m.
- 4, Find d such that $ed \mod m = 1$.
- 5. Announce n and e publicly.
  NOTE: Each p and q are 100-200 digits each and n is anywhere between 200 and 400 digits.

Example

- As the receiver, choose two primes, say 37 and 73. Compute the product, which is 2701.
- Compute the lcm of 36 and 72, which is 72.
- Choose an e relatively prime to 72, say 7.
- Find a d such that $7d \mod 72 = 1$. Thus, $d = 31$.

Sender

- 1. Convert the message to a string of digits.
- 2. Break up the message into uniform blocks of digits; call them $M_1, M_2, ..., M_k$.
- 3. Calculate and send $R_i = M_i^e \mod n$.

Example

Suppose I want to send the message "YES"

- As the sender, convert the letters into a string of digits. Use a code for each of the 26 letters starting with a=01 through z=26 and blank with 00.
- We have 250519.
- Now, break these up into strings of length 4. This gives strings $M_1 = 2505, M_2 = 1900$.
- Compute $R_1 = 2505^7 \mod 2701 = 692$, $R_2 = 1900^7 \mod 2701 = 1734$.
- The sender will send 0692 and 1734.

Receiver

- 1. For each received message, R_i, calculate $R_i^d \mod n$.
- 2. Convert the string of digits back to a string of characters.

Example

- 0692 and 1734 will be received.
- As the receiver, now compute $692^{31} \mod 2701 = 2505$ and $1734^{31} \mod 2701 = 1900$.
- Now convert 25051900 back to letters to decrypt the message.
- We know each letter is encoded with a two-digit number.
- Split 25051900 into blocks of 2.
- This gives 25 (Y), 05 (E), 19 (S).

Mod Simplification

- Consider 692^{31}. If we were to compute this, we get a number 89 digits long.
- Take $692^{31} = (692^2)^8(692^3)^5$. The numbers are much smaller and easier to deal with.
- $692^{31} \mod 2701 \equiv ((692^2 \mod 2701)^8 \mod 2701) \cdot ((692^3 \mod 2701)^5 \mod 2701) \mod 2701$
- We only have 692^2 and 692^3, which are 6 and 9 digits, instead of 89.
- Now, to mod out by 2701, the problem is quicker by far.

Proof

Fermat's Little Theorem

If p is any prime number, and a is any integer such that $p \nmid a$, then $a^{p-1} \equiv 1 \mod p$.

Chinese Remainder Theorem

A system of linear congruences modulo pairwise relatively prime integers has a unique solution modulo the product of these moduli.

Proof

- Assume $de \equiv 1 \mod (p-1)(q-1)$. Then, there exists an integer, k, such that $de = 1 + k(p-1)(q-1)$.
- It follows that $R^d \equiv (M^e)^d = M^{de} = M^{1+k(p-1)(q-1)} \mod n$.
- $gcd(M,p) = gcd(M,q) = 1 \Rightarrow M^{p-1} \equiv 1 \mod p$ and $M^{q-1} \equiv 1 \mod q$.
- Therefore, $R^d \equiv M \cdot (M^{p-1})^{k(q-1)} \equiv M \cdot 1 = M \mod p$
- Also, $R^d \equiv M \cdot (M^{q-1})^{k(p-1)} \equiv M \cdot 1 = M \mod q$
- $gcd(p,q) = 1 \Rightarrow R^d \equiv M \mod pq$.

Future Work

- RSA can fuel more research around prime numbers.
- That being said, if someone figures out a way to factor large numbers into two primes rather quickly, security will be compromised.
- Some non-secret forms of communication, such as email, can be used between the sender and receiver to speed up the process since the RSA process is time consuming.
- Until 2007, RSA gave cash prizes for factoring large numbers into two primes.

References

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- For any further questions or comments, email me at seth.hamilton@valpo.edu.