Equation of State and the Walecka Model in Neutron Star Physics

HARISH KUMAR S
Summer Internship in Comupational Nuclear Astrophysics
M.R.P.D Government College, Talwara, India

June, 2025

1. Introduction

Understanding matter under extreme conditions is a central goal of nuclear astrophysics. Particle physics offers insight into the fundamental constituents of matter and their interactions, particularly the strong interaction described by Quantum Chromodynamics (QCD). These concepts form the basis for modeling dense astrophysical environments such as neutron stars. The study of the equation of state (EoS) of nuclear matter is essential to understand the internal structure, stability, and evolution of neutron stars.

2. The Standard Model and QCD

The Standard Model organizes elementary particles into quarks, leptons, and gauge bosons. It describes three of the four known fundamental interactions: electromagnetic, weak, and strong. The strong interaction, governed by Quantum Chromodynamics (QCD), is responsible for binding quarks together to form hadrons, such as protons and neutrons.

QCD exhibits two important properties: confinement and asymptotic freedom. Confinement ensures quarks are never found in isolation, while asymptotic freedom implies that the interaction between quarks becomes weaker at high energies or short distances. These principles are foundational for understanding matter at extreme densities and temperatures, such as those present in the early universe and in the core of neutron stars.

3. QCD Phase Diagram and Effective Models

The QCD phase diagram maps different states of strongly interacting matter as functions of temperature (T) and baryon chemical potential (μ_B). At high temperatures and/or densities, hadronic matter is expected to undergo phase transitions into deconfined phases such as the quark-gluon plasma.

Since solving QCD directly at finite density is computationally challenging, effective models are employed. The Chiral Mean Field (CMF) model, for example, incorporates both hadronic and quark degrees of freedom and allows a smooth crossover between the two regimes. Frameworks like MUSES (Modular Unified Solver of Equations of State) are used to implement such models and generate thermodynamically consistent EoS tables for astrophysical applications.

4. The Walecka Model: Basics and Motivation

The Walecka model (also called Quantum Hadrodynamics-I) is a relativistic mean-field theory developed to describe nuclear matter. It models nucleons as Dirac fields interacting through the exchange of mesons. The key mediators are:

- The scalar meson (σ), which provides attractive interaction,
- The vector meson (ω^μ), which provides repulsion.

This model captures essential features of dense nuclear matter and is analytically tractable. It serves as a foundation for more sophisticated extensions used in astrophysical contexts.

5. Lagrangian Formulation and Saturation Properties

The Walecka model Lagrangian consists of kinetic terms for nucleons and mesons, along with interaction terms:

$$\mathcal{L} = \bar{\psi} i(i\gamma^\mu \partial\mu - m_N)\psi_i + g_\sigma \bar{\psi} i\sigma\psi_i - g\omega\bar{\psi} i\gamma^\mu \omega\mu\psi_i \quad +\frac{1}{2}(\partial^\mu \sigma \partial_\mu \sigma - m_\sigma^2 \sigma^2) - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2}m_\omega^2 \omega^\mu \omega_\mu$$

The model reproduces important saturation properties of nuclear matter:

- Saturation density $\rho_0 \approx 0.16$ fm^{-3},
- Binding energy per nucleon ≈ -16 MeV,
- Compressibility modulus K, symmetry energy S, and slope parameter L.

These empirical benchmarks are critical in validating theoretical models against observed properties of nuclei and neutron stars.

6. Mean-Field Approximation

In the mean-field approximation (MFA), the meson fields are treated as classical, spatially uniform fields. For infinite nuclear matter:

This leads to simplified equations:

$$[i\gamma^\mu \partial_\mu - M_i^* - g_\omega^i \gamma^0 \bar{\omega}0]\psi_i = 0 \ M_i^* \qquad = m_N - g\sigma^i \bar{\sigma} \ \bar{\sigma} = \frac{g_\sigma^i}{m_\sigma^2} n_{s_i}, \quad \bar{\omega}0 = \frac{g\omega^i}{m_\omega^2} n_{B_i}$$

This formalism allows derivation of thermodynamic quantities like energy density, pressure, and chemical potential.

7. Equation of State for Neutron Star Matter

To describe neutron stars, the model must include leptons and satisfy beta-equilibrium and charge neutrality conditions:

The energy density and pressure of the system include contributions from both nucleons and leptons:

These expressions form the basis for the equation of state, which is a necessary input for solving the Tolman-Oppenheimer-Volkoff (TOV) equations.

8. Mass-Radius Relation and TOV Equations

The TOV equations describe the structure of a static, spherically symmetric star in general relativity. They relate pressure and energy density to the mass and radius of a star:

$$\frac{dP(r)}{dr} = -\frac{G[\epsilon(r) + P(r)][m(r) + 4\pi r^3 P(r)]}{r^2[1 - 2Gm(r)/r]} \ \frac{dm(r)}{dr} \qquad = 4\pi r^2 \epsilon(r)$$

By integrating these equations with an appropriate EoS, one obtains the mass-radius relation of neutron stars. These theoretical predictions can then be compared with observational data from pulsars and gravitational wave detections.

9. Conclusion

The Walecka model, supported by mean-field approximations and extended through frameworks like the CMF model, enables accurate modeling of dense matter in neutron stars. The resulting equations of state and derived observables, such as mass-radius curves, provide a crucial link between fundamental nuclear theory and high-energy astrophysical phenomena.