Second Derivative Test

Suppose $f(x)$ is a function of x that is twice differentiable at a stationary point x_0.

1. If $f''(x_0) > 0$, then f has a local minimum at x_0.

2. If $f''(x_0) < 0$, then f has a local maximum at x_0.

The extremum test gives slightly more general conditions under which a function with $f''(x_0) = 0$ is a maximum or minimum.

If $f(x, y)$ is a two-dimensional function that has a local extremum at a point (x_0, y_0) and has continuous partial derivatives at this point, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. The second partial derivatives test classifies the point as a local maximum or local minimum.

Define the second derivative test discriminant as

$$\begin{aligned} D &\equiv f_{xx} f_{yy} - f_{xy} f_{yx} \quad (1) \\ &= f_{xx} f_{yy} - f_{xy}^2. \quad (2) \end{aligned}$$

Then

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, the point is a local minimum.

2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, the point is a local maximum.

3. If $D < 0$, the point is a saddle point.

4. If $D = 0$, higher order tests must be used.