

Matrix Computations

Assignment1

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Q1

(a)

$$-\epsilon u'' + au'' = f(x), 0 < x < 1$$

$$u(0) = \alpha, u(1) = \beta.$$

For $x \in [0, 1]$, suppose that $x_j = jh, j = 0, 1, \dots, n$, then $h = \frac{1}{n}$.

Using discrete approximations:

$$u'(x_i) \approx \frac{u(x_i + h) - u(x_i - h)}{2h} = \frac{u(x_{i+1}) - u(x_{i-1}))}{2h}$$

$$u''(x_i) \approx \frac{u(x_i + h) - 2u(x_i) + u(x_i - h))}{h^2} = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1}))}{h^2}.$$

For every x_i , we have $-\epsilon u''(x_i) + au'(x_i) = f(x_i)$, for $i = 1, 2, \dots, n-1$.

Substituting these approximations for the derivatives into the differential equation, we obtain

$$-\epsilon \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1}))}{h^2} + a \frac{u(x_{i+1}) - u(x_{i-1}))}{2h} \approx f(x_i), \text{ for } i = 1, 2, \dots, n-1.$$

$$\text{so, } -\epsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + a \frac{u_{i+1} - u_{i-1}}{2h} \approx f_i, \text{ for } i = 1, 2, \dots, n-1.$$

$$\text{It means, } \begin{cases} -\epsilon \frac{u_2 - 2u_1 + u_0}{h^2} + a \frac{u_2 - u_0}{2h} = f_1 \\ -\epsilon \frac{u_3 - 2u_2 + u_1}{h^2} + a \frac{u_3 - u_1}{2h} = f_2 \\ \dots \\ -\epsilon \frac{u_n - 2u_{n-1} + u_{n-2}}{h^2} + a \frac{u_n - u_{n-2}}{2h} = f_{n-1} \end{cases}$$

$$\text{Also, } \left(\frac{a}{2h} - \frac{\epsilon}{h^2}\right)u_{i+1} + \left(\frac{2\epsilon}{h^2}\right)u_i + \left(-\frac{a}{2h} - \frac{\epsilon}{h^2}\right)u_{i-1} = f_i.$$

Since $u(0) = \alpha, u(1) = \beta$,

$$\begin{bmatrix} \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} & & & 0 \\ -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} & & \\ & -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} & \\ & & \ddots & \ddots & \ddots \\ & & & -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} \\ 0 & & & & -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} \end{bmatrix} * \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 - \left(-\frac{a}{2h} - \frac{\epsilon}{h^2}\right)\alpha \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} - \left(\frac{a}{2h} - \frac{\epsilon}{h^2}\right)\beta \end{bmatrix}$$

(b)

Since $f(x) = 1, a = 1, \alpha = \beta = 0$,

Function becomes $-\epsilon u'' + u'' = 1, 0 < x < 1, u(0) = 0, u(1) = 0$

Also, $\left(\frac{1}{2h} - \frac{\epsilon}{h^2}\right)u_{i+1} + \left(\frac{2\epsilon}{h^2}\right)u_i + \left(-\frac{1}{2h} - \frac{\epsilon}{h^2}\right)u_{i-1} = 1$

The exact solution is $U(x) = x - \frac{\exp\left(-\frac{1-x}{\epsilon}\right) - \exp\left(-\frac{1}{\epsilon}\right)}{1 - \exp\left(-\frac{1}{\epsilon}\right)}$.

Consider three different cases where the number of grid points are chosen as $n = 25, 50$ and 100 . For each subdivision points change the choice of the diffusion coefficient $\epsilon = 1, 0.1$ and 0.01

```
for n = [25 50 100]
    for epsilon = [1 0.1 0.01]
        % Use function from M-file
        u = ode_solver(n,epsilon);

        % Display the results
        fprintf('For n = %d and epsilon = %.2f, u = \n', n, epsilon);
        disp(u');
    end
end
```

For n = 25 and epsilon = 1.00, u =	0.0163	0.0315	0.0458	0.0590	0.0712	0.0821	0.0920	0.1005	0.1078	0.1138	0.1184	0.1220
For n = 25 and epsilon = 0.10, u =	0.0400	0.0800	0.1199	0.1598	0.1997	0.2396	0.2794	0.3190	0.3585	0.3978	0.4366	0.4750
For n = 25 and epsilon = 0.01, u =	0.0400	0.0800	0.1200	0.1600	0.2000	0.2400	0.2800	0.3200	0.3600	0.4000	0.4400	0.4800
For n = 50 and epsilon = 1.00, u =	0.0082	0.0162	0.0240	0.0315	0.0388	0.0458	0.0525	0.0590	0.0652	0.0712	0.0768	0.0820
For n = 50 and epsilon = 0.10, u =	0.0200	0.0400	0.0600	0.0799	0.0999	0.1199	0.1399	0.1598	0.1798	0.1997	0.2196	0.2396
For n = 50 and epsilon = 0.01, u =	0.0200	0.0400	0.0600	0.0800	0.1000	0.1200	0.1400	0.1600	0.1800	0.2000	0.2200	0.2400

For n = 100 and epsilon = 1.00, u =	0.0042	0.0082	0.0123	0.0162	0.0202	0.0240	0.0278	0.0315	0.0352	0.0388	0.0423	0
For n = 100 and epsilon = 0.10, u =	0.0100	0.0200	0.0300	0.0400	0.0500	0.0600	0.0700	0.0799	0.0899	0.0999	0.1099	0
For n = 100 and epsilon = 0.01, u =	0.0100	0.0200	0.0300	0.0400	0.0500	0.0600	0.0700	0.0800	0.0900	0.1000	0.1100	0

Plot the exact and approximate solutions on the same window for $n = 50$ and different values of $\epsilon = 1, 0.1$, and 1.

```
n = 50;
h = 1 / n; % Grid size

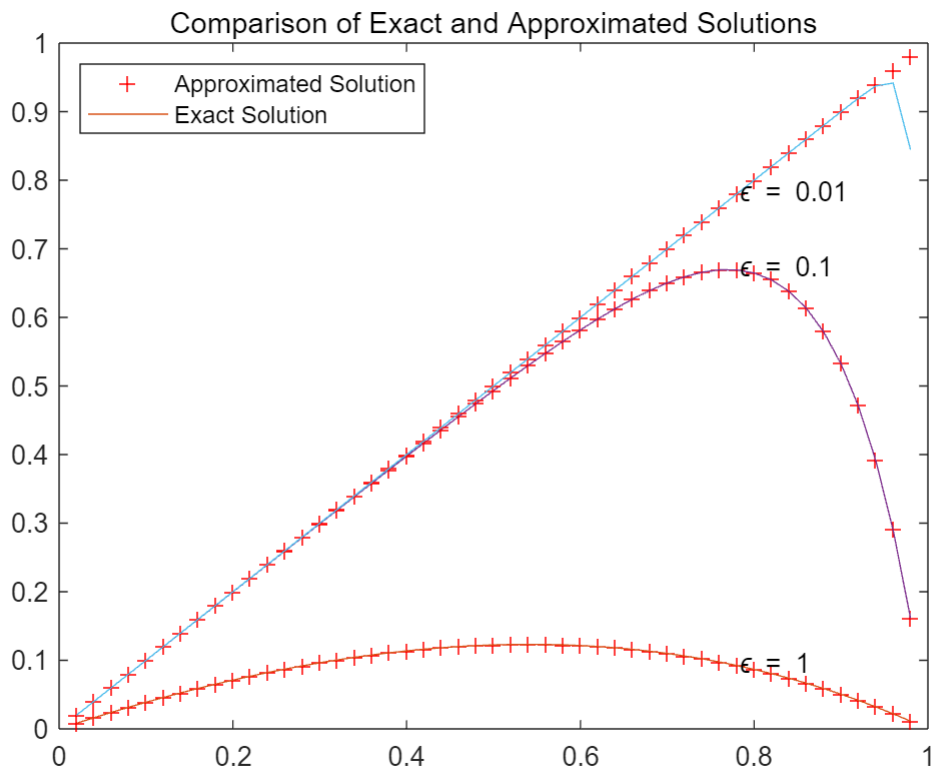
for epsilon = [1 0.1 0.01]
    % Use function from M-file
    u = ode_solver(n,epsilon);

    % Computing the exact solution
    t = linspace(h, 1-h, n-1);
    y = t - (exp(-(1-t)/epsilon) - exp(-1/epsilon)) / (1 - exp(-1/epsilon));

    % Plotting for comparison
    plot(t, u, 'r+'); hold on;
    plot(t, y, '-');

    % Adding epsilon value on the graph
    text(t(end-10), u(end-10), ['\epsilon = ', num2str(epsilon)]);
end

title('Comparison of Exact and Approximated Solutions');
legend('Approximated Solution', 'Exact Solution', 'Location', 'northwest');
```



When $\epsilon > 0.01$, the approximate solution is consistent with the exact solution; when $\epsilon = 0.01$, the approximate solution oscillates.

Q2

matvectime

```
matrixsize = 500
time = 0.0156
matrixsize = 1000
time = 0.1250
ratio = 8
matrixsize = 2000
time = 0.5469
ratio = 4.3750
matrixsize = 4000
time = 2.7812
ratio = 5.0857
```

Q3

Write a modified version of (1.3.5) for leading zeros for b .

Suppose that $b_1 = b_2 = \dots = b_k = 0$

for $i = k + 1, k + 2, \dots, n$

for $i = k + 1, k + 2, \dots, i - 1$ (not executed when $i = k + 1$)

$b_i = b_i - g_{ij}b_j$

if $g_{ii} = 0$, set error flag, exit

$$b_i = b_i / g_{ii}$$

Q4

1.3.12

Write a nonrecursive algorithm in the spirit of (1.3.5).

```
G = [5 0 0; 2 -4 0; 1 2 3];  
b = [15; -2; 10];  
Y = forward_col(G, b)
```

```
Y = 3x1  
    3  
    2  
    1
```

1.3.14

(a) Count the operations in (1.3.13)

$$fbps = \sum_{j=1}^n \sum_{i=j+1}^n 2 = 2 \sum_{j=1}^n (n-j) = 2(n-1 + n-2 + \dots + 0) = n(n-1) \approx n^2$$

(b) Convince myself that the row- and column-oriented versions of forward substitution carry out exactly the same operations but not in the same order.

As the flops calculated before, $\text{flops_row} = \text{flops_column} = n^2$.

$$\text{flops_row} = 2 \sum_{i=1}^n (i-1) = 2(0 + 1 + \dots + n-1);$$

$$\text{flops_column} = 2 \sum_{j=1}^n (n-j) = 2(n-1 + n-2 + \dots + 0), \text{ they are not in the same order.}$$

Q5

Write an algorithm based on (1.4.13) and (1.14.14), and calculate R of Example 1.4.18 using the Cholesky's Algorithm.

```
A = [4 -2 4 2; -2 10 -2 -7; 4 -2 8 4; 2 -7 4 7];  
R = cholesky(A)
```

```
R = 4x4  
    2    -1     2     1  
    0     3     0    -2  
    0     0     2     1  
    0     0     0     1
```

Q6

1.4.21

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix} \text{ and } b = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}.$$

(a) Use the Cholesky method to show A is positive definite and compute its Cholesky factor

$$r_{11} = \sqrt{a_{11}} = \sqrt{16} = 4$$

$$r_{12} = \frac{a_{12}}{r_{11}} = \frac{4}{4} = 1, r_{13} = \frac{8}{4} = 2, r_{14} = \frac{4}{4} = 1$$

$$r_{22} = \sqrt{a_{22} - r_{12}^2} = \sqrt{10 - 1^2} = 3$$

$$r_{23} = \frac{a_{23} - r_{13}r_{12}}{r_{22}} = \frac{8 - 2}{3} = 2, r_{24} = \frac{4 - 1}{3} = 1$$

$$r_{33} = \sqrt{a_{33} - r_{13}^2 - r_{23}^2} = \sqrt{12 - 2^2 - 2^2} = 2$$

$$r_{34} = \frac{a_{34} - r_{14}r_{13} - r_{24}r_{23}}{r_{33}} = \frac{12 - 2^2 - 2}{2} = 3$$

$$r_{44} = \sqrt{a_{44} - r_{14}^2 - r_{24}^2 - r_{34}^2} = \sqrt{12 - 1 - 1 - 3^2} = 1$$

$$\text{so, } R = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

R can be computed without error, A is positive definite.

(b) Use forward and backward substitution to solve the linear system.

Since $Ax = R^T R x = b$, define that $Rx = y$, then $R^T y = b$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}$$

$$y_1 = \frac{b_1}{g_{11}} = \frac{32}{4} = 8$$

$$y_2 = \frac{(b_2 - g_{21}y_1)}{g_{22}} = \frac{(26 - 8)}{3} = 6$$

$$y_3 = \frac{(b_3 - g_{31}y_1 - g_{32}y_2)}{g_{33}} = \frac{(38 - 2 * 8 - 2 * 6)}{2} = 5$$

$$y_4 = \frac{(b_4 - g_{41}y_1 - g_{42}y_2 - g_{43}y_3)}{g_{44}} = \frac{(30 - 8 - 6 - 3 * 5)}{1} = 1$$

$$\text{so, } y = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix}, \text{ then } Rx=y$$

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix}$$

$$x_4 = \frac{y_4}{g_{44}} = \frac{1}{1} = 1$$

$$x_3 = \frac{(5-3)}{2} = 1$$

$$x_2 = \frac{(6-2-1)}{3} = 1$$

$$x_1 = \frac{(8-1-2-1)}{4} = 1$$

$$\text{so, } x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

1.4.22

Determine whether the matrices are positive definite.

```
A = [9 3 3;3 10 7;3 5 9];
try
    R = chol(A)
catch
    warning('The matrix is not positive definite.');
```

```
end

R = 3x3
     3     1     1
     0     3     2
     0     0     2
```

```
B = [4 2 6;2 2 5;6 5 29];
try
    R = chol(B)
catch
    warning('The matrix is not positive definite.');
```

```
end

R = 3x3
     2     1     3
     0     1     2
```

0 0 4

```
C = [4 4 8;4 -4 1;8 1 6];
try
    R = chol(C)
catch
    warning('The matrix is not positive definite.');
```

警告: The matrix is not positive definite.

```
D = [1 1 1;1 2 2;1 2 1];
try
    R = chol(D)
catch
    warning('The matrix is not positive definite.');
```

警告: The matrix is not positive definite.

Q7

Prove that $n \times n$ matrix has $n^3/3$ flops.

for $i = 1, \dots, n$

for $k = 1, \dots, i - 1$ (not executed when $i=1$)

$$a_{ii} = a_{ii} - a_{ki}^2$$

if $a_{ii} \leq 0$, set error flag

$$a_{ii} = \sqrt{a_{ii}} \text{ (this is } r_{ii})$$

for $j = i + 1, \dots, n$ (not executed when $i=n$)

$$a_{ij} = a_{ij} - a_{ki}a_{kj}$$

$$a_{ij} = a_{ij}/a_{ii} \text{ (this is } r_{ij})$$

We have $\sum_{i=1}^n i^2 = \frac{n^3}{3} + O(n^2)$.

$$\begin{aligned}
fbps &= \sum_{i=1}^n \sum_{k=1}^{i-1} 2 + \sum_{i=1}^n \sum_{j=i+1}^n \sum_{k=1}^{i-1} 2 \\
&= 2 \sum_{i=1}^n (i-1) + 2 \sum_{i=1}^n (n-i)(i-1) \\
&= n(n-1) + 2n \sum_{i=1}^n (i-1) - 2 \sum_{i=1}^n i^2 + 2 \sum_{i=1}^n i \\
&= n^2 - n + n^3 - 2 \frac{n^3}{3} + O(n^2) \\
&\approx \frac{n^3}{3}
\end{aligned}$$

Q8

Use outer-product formulation of Cholesky's method to calculate the Cholesky factor

$$B = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 \\ s & \hat{R}^T \end{bmatrix} \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{R} \end{bmatrix}$$

$$r_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$s^T = r_{11}^{-1} b^T = \frac{1}{2} [2 \ 6] = [1 \ 3]$$

$$\tilde{A} = \hat{A} - ss^T = \begin{bmatrix} 2 & 5 \\ 5 & 29 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} [1 \ 3] = \begin{bmatrix} 1 & 2 \\ 2 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 20 \end{bmatrix} = \begin{bmatrix} r_{22} & 0 \\ r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{22} & r_{23} \\ 0 & r_{33} \end{bmatrix}$$

$$r_{22} = \sqrt{1} = 1$$

$$r_{23} = \frac{2}{1} = 2$$

$$r_{33} = \sqrt{20 - 2^2} = 4$$

$$\text{so, } R = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

Q9

Write an outer-product formulation of Cholesky algorithm, and impliment it in 1.4.22.

```
A = [9 3 3;3 10 7;3 5 9];
try
```

```

R = chol_cholesky_outerproduct(A)
catch
    warning('The matrix is not positive definite.');
```

```

end

R = 3x3
    3    1    1
    0    3    2
    0    0    2
```

```

B = [4 2 6;2 2 5;6 5 29];
try
    R = chol_cholesky_outerproduct(B)
catch
    warning('The matrix is not positive definite.');
```

```

end

R = 3x3
    2    1    3
    0    1    2
    0    0    4
```

```

C = [4 4 8;4 -4 1;8 1 6];
try
    R = chol_cholesky_outerproduct(C)
catch
    warning('The matrix is not positive definite.');
```

警告: The matrix is not positive definite.

```

D = [1 1 1;1 2 2;1 2 1];
try
    R = chol_cholesky_outerproduct(D)
catch
    warning('The matrix is not positive definite.');
```

警告: The matrix is not positive definite.

Q10

```
matcholtime
```

```

matrixsize = 500
time = 0.0156
matrixsize = 1000
time = 0.1250
ratio = 8
matrixsize = 2000
time = 0.5469
ratio = 4.3750
matrixsize = 4000
time = 4.2188
ratio = 7.7143
```

This matlab code uses data that obeys a normal distribution to generate 500*500, 1000*1000, 2000*2000, 3000*3000 matrices, and testing the time they spent on the cholesky decomposition.

The ratio should be near to 8, because $\text{ratio} = \frac{n_2^3}{n_1^3} = \left(\frac{n_2}{n_1}\right)^3 = 2^3 = 8$.