Matrix Computations

CPSC 5006-EL

Assignment1

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Q1

(a)

$-\epsilon u^{''} + au^{''} = f(x), 0 < x < 1$
$u(0) = \alpha, u(1) = \beta.$

For $x \in [0, 1]$, suppose that $x_j = jh, j = 0, 1, ..., n$, then $h = \frac{1}{n}$.

Using discrete approximations:

$$u^{'}(x_i) \approx \frac{u(x_i + h) - u(x_i - h)}{2h} = \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$$
$$u^{''}(x_i) \approx \frac{u(x_i + h) - 2u(x_i) + u(x_i - h)}{h^2} = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}.$$

For every x_i, we have $-\epsilon u^{''}(x_i) + au^{'}(x_i) = f(x_i)$, for $i = 1, 2, ..., n - 1$.

Substituting these approximations for the derivatives into the differential equation, we obtain

$$-\epsilon \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + a\frac{u(x_{i+1}) - u(x_{i-1})}{2h} \approx f(x_i), \text{ for } i = 1, 2, ..., n - 1.$$

so, $-\epsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + a\frac{u_{i+1} - u_{i-1}}{2h} \approx f_i$, for $i = 1, 2, ..., n - 1$.

It means, $$\begin{cases} -\epsilon \frac{u_2 - 2u_1 + u_0}{h^2} + a\frac{u_2 - u_0}{2h} = f_1 \\ -\epsilon \frac{u_3 - 2u_2 + u_1}{h^2} + a\frac{u_3 - u_1}{2h} = f_2 \\ \dots \\ -\epsilon \frac{u_n - 2u_{n-1} + u_{n-2}}{h^2} + a\frac{u_n - u_{n-2}}{2h} = f_{n-1} \end{cases}$$

Also, $\left(\frac{a}{2h} - \frac{\epsilon}{h^2}\right)u_{i+1} + \left(\frac{2\epsilon}{h^2}\right)u_i + \left(-\frac{a}{2h} - \frac{\epsilon}{h^2}\right)u_{i-1} = f_i$.

Since $u(0) = \alpha, u(1) = \beta$,

$$\begin{bmatrix} \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} & & & & 0 \\ -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} & & & \\ & -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} & \frac{a}{2h} - \frac{\epsilon}{h^2} \\ 0 & & & & -\frac{a}{2h} - \frac{\epsilon}{h^2} & \frac{2\epsilon}{h^2} \end{bmatrix} * \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 - \left(-\frac{a}{2h} - \frac{\epsilon}{h^2}\right)\alpha \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} - \left(\frac{a}{2h} - \frac{\epsilon}{h^2}\right)\beta \end{bmatrix}$$

(b)

Since $f(x) = 1, a = 1, \alpha = \beta = 0$,

Function becomes $-\epsilon u'' + u'' = 1, 0 < x < 1, u(0) = 0, u(1) = 0$

Also, $\left(\frac{1}{2h} - \frac{\epsilon}{h^2}\right)u_{i+1} + \left(\frac{2\epsilon}{h^2}\right)u_i + \left(-\frac{1}{2h} - \frac{\epsilon}{h^2}\right)u_{i-1} = 1$

The exact solution is $U(x) = x - \dfrac{\exp\left(-\frac{1-x}{\epsilon}\right) - \exp\left(-\frac{1}{\epsilon}\right)}{1 - \exp\left(-\frac{1}{\epsilon}\right)}$.

Consider three different cases where the number of grid points are chosen as n = 25, 50 and 100. For each subdivision points change the choice of the diffusion coefficient ϵ = 1, 0.1 and 0.01

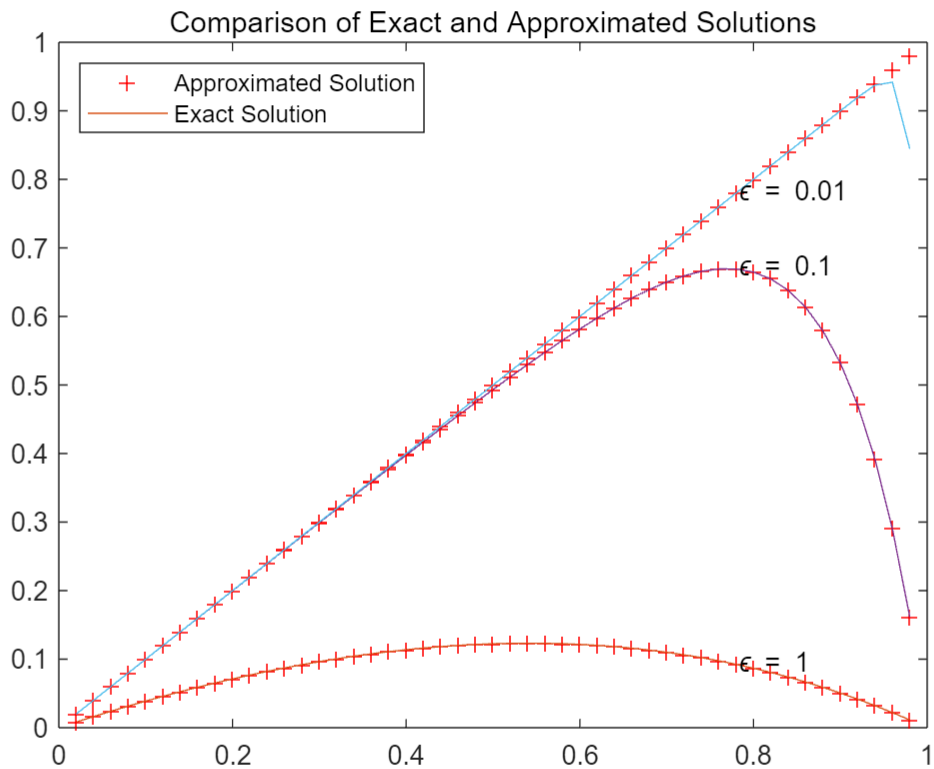
```
for n = [25 50 100]
    for epsilon = [1 0.1 0.01]
        % Use function from M-file
        u = ode_solver(n,epsilon);

        % Display the results
        fprintf('For n = %d and epsilon = %.2f, u = \n', n, epsilon);
        disp(u');
    end
end
```

```
For n = 25 and epsilon = 1.00, u =
    0.0163    0.0315    0.0458    0.0590    0.0712    0.0821    0.0920    0.1005    0.1078    0.1138    0.1184
For n = 25 and epsilon = 0.10, u =
    0.0400    0.0800    0.1199    0.1598    0.1997    0.2396    0.2794    0.3190    0.3585    0.3978    0.4366
For n = 25 and epsilon = 0.01, u =
    0.0400    0.0800    0.1200    0.1600    0.2000    0.2400    0.2800    0.3200    0.3600    0.4000    0.4400
For n = 50 and epsilon = 1.00, u =
    0.0082    0.0162    0.0240    0.0315    0.0388    0.0458    0.0525    0.0590    0.0652    0.0712    0.0768
For n = 50 and epsilon = 0.10, u =
    0.0200    0.0400    0.0600    0.0799    0.0999    0.1199    0.1399    0.1598    0.1798    0.1997    0.2196
```

```
For n = 50 and epsilon = 0.01, u =
    0.0200    0.0400    0.0600    0.0800    0.1000    0.1200    0.1400    0.1600    0.1800    0.2000    0.2200    0
For n = 100 and epsilon = 1.00, u =
    0.0042    0.0082    0.0123    0.0162    0.0202    0.0240    0.0278    0.0315    0.0352    0.0388    0.0423    0
For n = 100 and epsilon = 0.10, u =
    0.0100    0.0200    0.0300    0.0400    0.0500    0.0600    0.0700    0.0799    0.0899    0.0999    0.1099    0
For n = 100 and epsilon = 0.01, u =
    0.0100    0.0200    0.0300    0.0400    0.0500    0.0600    0.0700    0.0800    0.0900    0.1000    0.1100    0
```

Plot the exact and approximate solutions on the same window for n = 50 and different values of ϵ = 1, 0.1, and 1.

```
n = 50;
h = 1 / n; % Grid size

for epsilon = [1 0.1 0.01]
    % Use function from M-file
    u = ode_solver(n,epsilon);

    % Computing the exact solution
    t = linspace(h, 1-h, n-1);
    y = t - (exp(-(1-t)/epsilon) - exp(-1/epsilon)) / (1 - exp(-1/epsilon));

    % Plotting for comparison
    plot(t, u, 'r+'); hold on;
    plot(t, y, '-');

    % Adding epsilon value on the graph
    text(t(end-10), u(end-10), ['\epsilon = ', num2str(epsilon)]);
end

title('Comparison of Exact and Approximated Solutions');
legend('Approximated Solution', 'Exact Solution', 'Location', 'northwest');
```



When ϵ > 0.01, the approximate solution is consistent with the exact solution; when ϵ = 0.01, the approximate solution oscillates.

Q2

```
matvectime
```

```
matrixsize = 500
time = 0.0156
matrixsize = 1000
time = 0.1250
ratio = 8
matrixsize = 2000
time = 0.5469
ratio = 4.3750
matrixsize = 4000
time = 2.7812
ratio = 5.0857
```

The ratio should be near to 4, because $\text{ratio} = \frac{n_2^2}{n_1^2} = \left(\frac{n_2}{n_1}\right)^2 = 2^2 = 4.$

Q3

Write a modified version of (1.3.5) for leading zeros for b.

Suppose that $b_1 = b_2 = ... = b_k = 0$

for $i = k+1, k+2, ..., n$

for $i = k+1, k+2, ..., i-1$ (not executed when i = k+1)

$b_i = b_i - g_{ij} b_j$

if $g_{ii} = 0$, set error flag, exit

$b_i = b_i / g_{ii}$

Q4

1.3.12

Write a nonrecursive algorithm in the spirit of (1.3.5).

```
G = [5 0 0;2 -4 0; 1 2 3];
b = [15;-2;10];
Y = forward_col(G, b)
```

```
Y = 3×1
     3
     2
     1
```

1.3.14

(a) Count the operations in (1.3.13)

$$f\,bps = \sum_{j=1}^{n} \sum_{i=j+1}^{n} 2 = 2 \sum_{j=1}^{n} (n-j) = 2(n-1+n-2+...+0) = n(n-1) \approx n^2$$

(b) Convince myself that the row- and column-oriented versions of forward substitution carry out exactly the same operations but not in the same order.

As the flops calculated before, flops_row = flops_column = n^2.

flops_row = $2\sum_{i=1}^{n}(i-1) = 2(0+1+...+n-1);$

flops_column = $2\sum_{j=1}^{n}(n-j) = 2(n-1+n-2+...+0)$, they are not in the same order.

Q5

Write an algorithm based on (1.4.13) and (1.14.14) , and calculate R of Example 1.4.18 using the Cholesky's Algorithm.

```
A = [4 -2 4 2;-2 10 -2 -7;4 -2 8 4;2 -7 4 7];
R = cholesky(A)
```

```
R = 4×4
     2    -1     2     1
     0     3     0    -2
     0     0     2     1
     0     0     0     1
```

Q6

1.4.21

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix} \text{ and } b = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}.$$

(a) Use the Cholesky method to show A is positive definite and compute its Cholesky factor

$$r_{11} = \sqrt{a_{11}} = \sqrt{16} = 4$$

$$r_{12} = \frac{a_{12}}{r_{11}} = \frac{4}{4} = 1, r_{13} = \frac{8}{4} = 2, r_{14} = \frac{4}{4} = 1$$

$$r_{22} = \sqrt{a_{22} - r_{12}^2} = \sqrt{10 - 1^2} = 3$$

$$r_{23} = \frac{a_{23} - r_{13}r_{12}}{r_{22}} = \frac{8-2}{3} = 2, r_{24} = \frac{4-1}{3} = 1$$

$$r_{33} = \sqrt{a_{33} - r_{13}^2 - r_{23}^2} = \sqrt{12 - 2^2 - 2^2} = 2$$

$$r_{34} = \frac{a_{34} - r_{14}r_{13} - r_{24}r_{23}}{r_{33}} = \frac{12 - 2^2 - 2}{2} = 3$$

$$r_{44} = \sqrt{a_{44} - r_{14}^2 - r_{24}^2 - r_{34}^2} = \sqrt{12 - 1 - 1 - 3^2} = 1$$

so, $R = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

R can be computed without error, A is positive definite.

(b) Use forward and backward substitution to solve the linear system.

Since $\mathrm{Ax} = R^T R\, x = b$, define that Rx=y, then $R^T y = b$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}$$

$$y_1 = \frac{b_1}{g_{11}} = \frac{32}{4} = 8$$

$$y_2 = \frac{(b_2 - g_{21}y_1)}{g_{22}} = \frac{(26-8)}{3} = 6$$

$$y_3 = \frac{(b_3 - g_{31}y_1 - g_{32}y_2)}{g_{33}} = \frac{(38 - 2*8 - 2*6)}{2} = 5$$

$$y_4 = \frac{(b_4 - g_{41}y_1 - g_{42}y_2 - g_{43}y_3)}{g_{44}} = (30 - 8 - 6 - 3*5) = 1$$

so, $y = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix}$, then Rx=y

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix}$$

$$x_4 = \frac{y_4}{g_{44}} = \frac{1}{1} = 1$$

$$x_3 = \frac{(5-3)}{2} = 1$$

$$x_2 = \frac{(6-2-1)}{3} = 1$$

$$x_1 = \frac{(8-1-2-1)}{4} = 1$$

so, $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

1.4.22

Determine whether the matrices are positive definite.

```
A = [9 3 3;3 10 7;3 5 9];
try
    R = chol(A)
catch
    warning('The matrix is not positive definite.');
end
```

```
R = 3×3
     3     1     1
     0     3     2
```

```
     0     0     2
```

```
B = [4 2 6;2 2 5;6 5 29];
try
    R = chol(B)
catch
    warning('The matrix is not positive definite.');
end
```

```
R = 3×3
     2     1     3
     0     1     2
     0     0     4
```

```
C = [4 4 8;4 -4 1;8 1 6];
try
    R = chol(C)
catch
    warning('The matrix is not positive definite.');
end
```

```
警告: The matrix is not positive definite.
```

```
D = [1 1 1;1 2 2;1 2 1];
try
    R = chol(D)
catch
    warning('The matrix is not positive definite.');
end
```

```
警告: The matrix is not positive definite.
```

Q7

Prove that n*n matrix has $n^3/3$ flops.

for $i = 1, ..., n$

for $k = 1, ..., i - 1$ (not executed when i=1)

$a_{ii} = a_{ii} - a_{ki}^2$

if $a_{ii} \leq 0$, set error flag

$a_{ii} = \sqrt{a_{ii}}$ (this is r_{ii})

for $j = i + 1, ..., n$ (not executed when i=n)

$a_{ij} = a_{ij} = a_{ki}a_{kj}$

$a_{ij} = a_{ij}/a_{ii}$ (this is r_{ij})

We have $\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + O(n^2)$.

$$f\,bps = \sum_{i=1}^{n}\sum_{k=1}^{i-1} 2 + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=1}^{i-1} 2$$

$$= 2\sum_{i=1}^{n}(i-1) + 2\sum_{i=1}^{n}(n-i)(i-1)$$

$$= n(n-1) + 2n\sum_{i=1}^{n}(i-1) - 2\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i$$

$$= n^2 - n + n^3 - 2\frac{n^3}{3} + O(n^2)$$

$$\approx \frac{n^3}{3}$$

Q8

Use outer-product formulation of Cholesky's method to calculate the Cholesky factor

$$B = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 \\ s & \hat{R}^T \end{bmatrix} \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{R} \end{bmatrix}$$

$$r_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$s^T = r_{11}^{-1} b^T = \frac{1}{2}[2 \quad 6] = [1 \quad 3]$$

$$\tilde{A} = \hat{A} - ss^T = \begin{bmatrix} 2 & 5 \\ 5 & 29 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}[1 \quad 3] = \begin{bmatrix} 1 & 2 \\ 2 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 20 \end{bmatrix} = \begin{bmatrix} r_{22} & 0 \\ r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{22} & r_{23} \\ 0 & r_{33} \end{bmatrix}$$

$$r_{22} = \sqrt{1} = 1$$

$$r_{23} = \frac{2}{1} = 2$$

$$r_{33} = \sqrt{20 - 2^2} = 4$$

so, $R = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

Q9

Write an outer-product formulation of Cholesky algorithm, and impliment it in 1.4.22.

```
A = [9 3 3;3 10 7;3 5 9];
try
    R = cholesky_outerproduct(A)
catch
    warning('The matrix is not positive definite.');
end
```

```
R = 3×3
     3     1     1
     0     3     2
     0     0     2
```

```
B = [4 2 6;2 2 5;6 5 29];
try
    R = cholesky_outerproduct(B)
catch
    warning('The matrix is not positive definite.');
end
```

```
R = 3×3
     2     1     3
     0     1     2
     0     0     4
```

```
C = [4 4 8;4 -4 1;8 1 6];
try
    R = cholesky_outerproduct(C)
catch
    warning('The matrix is not positive definite.');
end
```

```
警告: The matrix is not positive definite.
```

```
D = [1 1 1;1 2 2;1 2 1];
try
    R = cholesky_outerproduct(D)
catch
    warning('The matrix is not positive definite.');
end
```

```
警告: The matrix is not positive definite.
```

Q10

```
matcholtime
```

```
matrixsize = 500
time = 0.0156
matrixsize = 1000
time = 0.1250
ratio = 8
matrixsize = 2000
time = 0.5469
ratio = 4.3750
matrixsize = 4000
time = 4.2188
```

```
ratio = 7.7143
```

This matlab code uses data that that obeys a normal distribution to generate 500*500, 1000*1000, 2000*2000, 3000*3000 matrices, and testing the time they spent on the cholesky decomposition.

The ratio should be near to 8, because $\text{ratio} = \frac{n_2^3}{n_1^3} = \left(\frac{n_2}{n_1}\right)^3 = 2^3 = 8$.