Lecture 2: Propositional Probability[1]

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September 14, 2023

[1] CS7470 Fall 2023: Foundations of Probabilistic Programming.

Today we will see an example of our first "probabilistic programming language". I put that in quotes because it is not going to look like a typical programming language like C or Java. We will build our way up to programming languages that look more familiar to you, but today we will encounter a "baby" PPL that is nonetheless surprisingly useful. That baby PPL will be based on *propositional logic*. Along the way, we will see the basics of (discrete) probability that we will need for the first part of the course.

"Programming languages are languages, a means of expressing computations in a form comprehensible to both people and machines." Harper [2016]

1 Probability

We begin with some formal definitions from *probability*, which will give us a formal basis for describing relative degrees of certainty we give to facts about the world:

"Whereas truth values in logic characterize the formulas under discussion, uncertainty characterizes *invisible* facts." Pearl [1988]

The formal notion of probability used here was introduced by Kolmogorov; see Kolmogorov and Bharucha-Reid [2018]. There have been many introductions and interesting discussions of formal notions of probability over the years; see Jaynes [2003] for an interesting discussion. I also like the discussion in Graham et al. [1989, Chapter 8]

Definition 1 *Let Ω be a set called the **sample space**; assume that $|\Omega|$ is at most countably-infinite; we will deal with larger classes of sample spaces later on. A function $\Pr : \Omega \to [0, 1]$ is called a **probability measure on** Ω if it satisfies that $\sum_{\omega \in \Omega} \Pr(\omega) = 1$. The pair $(\Omega, \Pr)$ is called a **discrete probability space**.*

For example, a sample space could be the set of all pairs of six-sided dice rolls:

$$\Omega_{dice} = \{(1\ 1), (1\ 2), (1\ 3), \cdots, (6\ 6)\}$$

Then, we would assign a *uniform probability distribution* over all dice rolls:

$$\text{For all } \omega \in \Omega_{dice}, \Pr_{dice}(\omega) = 1/36.$$

Then, we have a probability space $(\Omega_{dice}, \Pr_{dice})$. Or, continuing with our propositional example, Table 1 gives an example of a probability table that describes a probability distribution over the sample space of instances for a 2-variable propositional sentence involving variables x and y.

ω	$\Pr(\omega)$
(`true`, `true`)	0.1
(`true`, `false`)	0.2
(`false`, `true`)	0.3
(`false`, `false`)	0.4

Table 1: A simple probability table.

There is a substantial amount of standard notation for working with probability spaces, which we briefly review here; a nice summary resource is Darwiche [2009, Chapter 3]. An **event** is a subset of the sample space $E \subseteq \Omega$. We define the probability of an event to be the sum total of all the elements of Ω that are contained in the event:

$$\Pr(E) \triangleq \sum_{\omega \in E} \Pr(\omega). \tag{1}$$

Continuing with our above example, $\Pr(\{(d_1, d_2) \mid d_1 = 1\})$ is $1/6$.

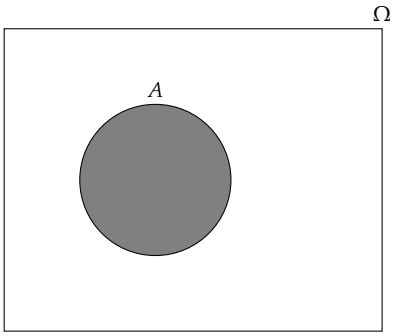


Figure 1: An event is a subset of the sample space Ω. Here, the event A is visualized using a Venn-diagram as a subset of the sample space Ω.

Typically in probability we work with quantities that are derived from the sample space; these are called random variables. Formally, a **random variable** is a map out of the sample space. For instance, we can define a random variable $S : \Omega_{dice} \to \mathbb{N}$ to be to sum two dice rolls π_1 to be the first die's value, and π_2 to be the second,where $\mathbb{N}$ is the set of natural numbers:

$$S((d_1, d_2)) = d_1 + d_2, \quad \pi_1((d_1, d_2)) = d_1, \quad \pi_2((d_1, d_2)) = d_2. \tag{2}$$

A random variable induces a probability distribution on its co-domain. For a random variable $X : \Omega \to D$ acting on a probability space $(\Omega, \Pr)$, we can compute the probability of a particular element $d \in D$ by computing the total probability mass in Ω that is mapped to d, i.e.:

$$\Pr(X = x) \triangleq \Pr(X^{-1}(x)) = \sum_{\{\omega \in \Omega \mid X(\omega) = x\}} \Pr(\omega). \tag{3}$$

Another way of interpreting this notation is that the event $X = x$ is the set of elements of Ω such that $X(\omega) = x$.

Given two random variables X and Y defined on the same sample space Ω, we quite often want to characterize their joint behavior. The **joint probability distribution** of two random variables relates them:

$$\Pr(X = x, Y = y) \triangleq \Pr(X^{-1}(x) \cap Y^{-1}(y)). \tag{4}$$

For example, returning to the dice situation above, we can compute the joint probability for any particular pair of dice rolls:

$$\Pr(\pi_1 = 1, \pi_2 = 1) = \sum_{\{(d_1, d_2) \in \Omega_{dice} \mid d_1 = 1, d_2 = 1\}} \Pr((d_1, d_2)) = 1/36.$$

Given a joint probability distribution $\Pr(X, Y)$, it is often useful to "forget" one of the random variables and consider only the distribution over the remaining variables. This is called the **marginal**

probability, and it is computed by summing over all possible values the random variable can take on. Concretely, the marginal probability of X is given by "summing out" Y:

$$\Pr(X = x) \triangleq \sum_y \Pr(X = x, Y = y) \tag{5}$$

One of the most common operations one performs in probability is *conditioning*: computing the probability that some event happens *given an observation* that some other event has occurred. For instance, one might ask, "what is the probability of that the sum of a pair of dice rolls is even given that one of them is 1"? Formally, these questions are instances of *conditional probability*; notationally, the above question would be written as $\Pr(S \text{ is even} \mid \pi_1 = 1)$; the bar separates two events, and is read "given".

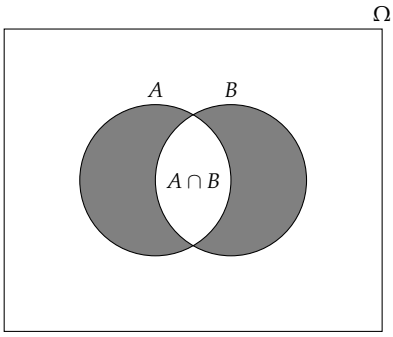


Figure 2: It is useful to visualize set intersection when understanding conditioning. By definition, $\Pr(A \mid B) = \Pr(A \cap B) / \Pr(B)$. Intuitively, this corresponds to considering the event at the intersection of these two events, and *re-normalizing* that probability based on the viewpoint of the restricted new sample space B.

Given a joint probability distribution $\Pr(X, Y)$, we define the **conditional probability** of X given Y is:

$$\Pr(X = x \mid Y = y) \triangleq \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)}. \tag{6}$$

There are a number of relationships between these quantities that are commonly used. One is the law of total probability, which can be derived by combining equations (6) and (5):

$$\Pr(X = x) = \sum_y \Pr(X = x \mid Y = y) \Pr(y). \tag{7}$$

It is sometimes the case that conditioning on one random variable has no effect: for instance, in our dice-roll example, observing the value of the first die tells you nothing about the value of the second die. Formally, we say two random variables X and Y are **independent**, written $X \perp\!\!\!\perp Y$, if the following holds:

$X \perp\!\!\!\perp Y$ if for all x, y it holds that $\Pr(X = x \mid Y = y) = \Pr(X = x)$.

Exercise 1 (⋆) *See Equation 2. What is* $\Pr(S = 3 \mid \pi_1 = 1)$*?*

Exercise 2 (⋆) *See Equation 2. Show that* $S \not\perp\!\!\!\perp \pi_1$ *and* $\pi_1 \perp\!\!\!\perp \pi_2$*.*

Exercise 3 (⋆) *Let* X *and* Y *be random variables defined on the same sample space. Show that, if* $X \perp\!\!\!\perp Y$*, then* $\Pr(X = x, Y = y) = \Pr(X = x) \times \Pr(Y = y)$*.*

2 *Probabilistic models*

One of our main goals in this class is to automate probabilistic reasoning on a computer. For this purpose, our discussion so far of

probability has been too abstract: we are told nothing about *how* these various probabilistic quantities can be computed, or how we can represent probability distributions to the computer. The means of describing a distribution to a computer is called a **probabilistic model**; it is a data-structure. The simplest kind of probabilistic model is a look-up table (or dictionary):

Definition 2 *A **probability lookup table** $\mathcal{T}$ is a lookup-table that maps elements of the sample space $\omega \in \Omega$ to the probability of that element* $\Pr(\omega)$*. Lookup tables support constant-time lookups.*

For instance, if we are given a discrete sample space with 4 elements $\Omega = \{1, 2, 3, 4\}$, then we can represent this as a dictionary or look-up table:

$$\mathcal{T} = [1 \mapsto 0.1, 2 \mapsto 0.2, 3 \mapsto 0.3, 4 \mapsto 0.4], \tag{8}$$

The **size** is the amount of memory required to represent the model to the computer; it is denoted $|\cdot|$. In the case of a look-up table, the size is equal to $|\Omega|$.

Querying

Now that we have a concrete description of a probability distribution, we can describe algorithms for automatically *querying* for properties about these representations. Different kinds of probabilistic models will support different kinds of queries, and have different complexity-theoretic properties of those queries. cWe have already seen the most basic kind of query: given some event $E \subseteq \Omega$, compute the probability of that event $\Pr(E)$. This immediately raises the question: how do we represent events? The most basic approach is a **set-event**, which supports constant-time membership checks (i.e., given some event $\omega \in \Omega$, a set-event E can answer the question of whether or not $\omega \in E$ in constant time). Now, we can give an algorithm for computing the probability of a set-event for a particular probability look-up table:

Algorithm 1: $\text{QueryEvent}_1(\Pr, E)$

Data: A probability look-up table Pr; an set-event E
Result: The probability of the event $\Pr(E)$

```
a ← 0;
for each ω ∈ Ω do
    if ω ∈ E then a ← a + Pr(ω);
end
return a
```

What is the time-complexity of running $\text{QueryEvent}_1(\Pr, E)$? Since evaluating $\Pr(\omega)$ and checking membership $\omega \in E$ are both constant time, the overall time complexity is $|\Omega|$: the cost of iterating the outer loop from Lines 2–4. Is this an acceptable cost? Typically, *no*! The size of the sample space Ω is typically prohibitively large: it represents the set of *all* possibilities.

This leads us to *the fundamental design challenge of probabilistic modeling*. We desire a *concise* (i.e., small in representation size, for instance polynomial in $\log |\Omega|$) language for describing probability distributions and queries; ideally, this language is usable and interpretable by a user. Additionally, we desire we want our queries to be *tractable*: when we compute the probability of an event (or perhaps some other kind of query), we want it to be as efficient as possible in the representation size. We will see that these two attributes, tractability and conciseness, are in tension. In the case of lookup-table distributions and set-events, queries are tractable – they are typically linear $|\mathcal{T}|$, but these representations are not concise: they are as large as the sample space.

3 *Propositional logic*

What is a good language for describing probability distributions that is an alternative to lookup-table distributions and set-events? Throughout the course we will slowly grow our vocabulary for describing these things, but for now we begin with a simple language for describing sets of possibilities: *propositional logic*.

Propositional logic is a formal language and has two components: syntax and semantics. The syntax tells us how to write valid statements in propositional logic, and the semantics tell us how to interpret those statements. For example, we may wish to express the logical statement "if it rains, then it is wet". To do this, we will introduce two **propositional variables** r and w: r is true if and only if it is raining, and w is true if and only if it is wet. We can relate these two variables by using **propositional connectives**. In the above example, we will use the **implication connective** $r \Rightarrow w$, which states that if r is true then so is w.

We can summarize these connectives using the following grammar, which tells us how to form any **propositional formula** φ, α, or β in propositional logic by combining propositional variables (lower-case letters) with any of the standard propositional connectives (conjunction *land*, disjunction $\vee$, unary negation $\neg$, and bi-implication $\Leftrightarrow$):[2]

$$\varphi, \alpha, \beta ::= V \mid \neg\varphi \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \Rightarrow \beta \mid \alpha \Leftrightarrow \beta \mid (\alpha) \tag{9}$$

[2] This style of grammar is often called Backus-Naur style.

Using the above grammar, we can write down some more formu-

lae, like $(a \wedge b) \vee (c \wedge \neg d)$ and $(a \Leftrightarrow b) \wedge w$.

Semantics of propositional logic

We now know how to write down valid syntactic formulae in propositional logic, but we have not yet described what they *mean*: how these formulae relate to the original quantities in the real world, and how to unambiguously interpret these various connectives? Importantly, we want to be able to interpret propositional formulae in terms of sets: this will let us connect back to our earlier goals for efficiently representing events and distributions.

One way to interpret a formula is by asking if, for a particular state of the world, is the formula true? To do this, we can fix an **interpretation** $I : V \to \{\texttt{true}, \texttt{false}\}$, which is a map that maps any variable $v \in V$ to a truth assignment; intuitively, this interpretation will capture some state of the world. For example, in our running example, an interpretation that states that it is currently raining and not wet would be:[3]

$$I_{ex} \triangleq [r \mapsto \texttt{true}, w \mapsto \texttt{false}] \triangleq [r, \overline{w}]. \tag{10}$$

[3] As notation, we will sometimes abbreviate $r \mapsto \texttt{true}$ as an unadorned r, and $r \mapsto \texttt{false}$ as $\overline{r}$. Additionally, we use the notation $\triangleq$ to denote definitional equality.

Now we can interpret propositional formulae for a particular interpretation I. This will be our formal notion of semantics:

Definition 3 (Semantics of propositional formulae) *An instance I is called a **model** for a formula φ, written $I \models \varphi$, if the following inductive description holds:*

$$\begin{aligned}
I &\models a && \text{iff } I(a) = \texttt{true} \\
I &\models \neg\varphi && \text{iff } I \not\models \varphi, \text{ i.e. it is not the case that } I \models \varphi \\
I &\models \alpha \wedge \beta && \text{iff } I \models \alpha \text{ and } I \models \beta \\
I &\models \alpha \vee \beta && \text{iff } I \models \alpha \text{ or } I \models \beta \\
I &\models \alpha \Rightarrow \beta && \text{iff } I \not\models \alpha \text{ or } I \models \beta \\
I &\models \alpha \Leftrightarrow \beta && \text{iff } I \models \alpha, I \models \beta \text{ or } I \not\models \alpha, I \not\models \beta.
\end{aligned}$$

Now we have a nice formal description of when a formula holds for a given interpretation I: this is our first example in the course of a formal language semantics. Here is an example:

$$[x, \overline{y}] \models x \vee y, \text{ since } [x, \overline{y}](x) = \texttt{true}.$$

Given a logical formula, it is often useful to determine *the set of all models* for that formula. In some sense, this tells us everything there is to know about that formula. A common way to do this is via

a **truth table**, which lists (1) all possible interpretations for a fixed set of propositional variables, and (2) whether or not each interpretation models a particular formula. For example, we can give the truth table for the formula $x \vee \overline{y}$ as:

Truth tables as a way of reasoning about propositional formulae were introduced by Wittgenstein and Monk [2013].

I	$I \models x \vee y$?
$x\ y$	`true`
$x\ \overline{y}$	`true`
$\overline{x}\ y$	`true`
$\overline{x}\ \overline{y}$	`false`

Table 2: A simple truth table.

Propositional probability

Now we are ready to return to our earlier discussion and ask: *can we use propositional logic as the basis for an effective probabilistic model*? The first question is: what is our sample space? It's clear so far that a logical choice of sample space is the set of all possibly instances for a fixed set of Boolean formulae; this sample space will have size 2^n, where n is the number of propositional variables. Now, if we want to efficiently evaluate queries over this sample space, we need need two more components:

1. A way to efficiently represent a probability distribution on the sample space;
2. A way to efficiently represent queries over that sample space.

Let's tackle (1) first. What is an alternative representation of a distribution that avoids the space-explosion we saw in the lookup-table representation? We will need to be more clever about how we represent probabilities. One useful choice is to assume that all random variables are *independent from each other*, and therefore we can concisely describe a joint distribution over all of them:

Definition 4 (Fully factorized probabilistic model) *Let $X_1, X_2, \cdots, X_n$ be jointly independent random variables (i.e., for any pair X_i, X_j, it is the case that $X_i \perp\!\!\!\perp X_j$). Then, a fully-factorized probabilistic model is a collection of n probability lookup tables $\Pr(X_i)$, one for each i. The joint probability is computed as $\Pr(X_1, X_2, \cdots, X_n) \triangleq \prod_i^n \Pr(X_i)$.*

Observe that a fully-factorized model only requires $O(\sum_i |X_i|)$ space, which is significantly smaller than $|\Omega|$.[4] This is a big improvement over plain lookup tables, but it comes at a cost of expressivity: there are some distributions that cannot be represented in a fully-factorized way.

[4] The notation $|X|$ refers to the size of a random variable's co-domain.

In the context of propositional logic, there is a natural notion of a fully-factorized model. Let Ω be the collection of n-element instances. Then, each propositional variable corresponds to a projection out of that sample space. Let's call this the **fully factorized propositional distribution**: each variable in the theory will be associated with an independent probability of being `true` or `false`. For example, we may have to propositional variables x and y; the fully factorized distribution could be given by two independent lookup tables:

$$[x \mapsto 0.1, \overline{x} \mapsto 0.9], \quad [y \mapsto 0.2, \overline{y} \mapsto 0.8]. \tag{11}$$

Then, the joint probability $\Pr(x, \overline{y})$ is 0.1×0.8.

NOW WE RETURN TO AN EARLIER QUESTION: how do we efficiently represent events in the propositional setting? Events are a subset of the sample space. The sample space in propositional probability is the set of all instances. Then, we can represent events by *propositional formulae*:

$$\Pr(\varphi) \triangleq \sum_{I \models \varphi} \Pr(\varphi). \tag{12}$$

For instance, for the fully-factorized distribution in Eq. 11, we can compute the probability of a propositional sentence $x \vee y$ by summing the probability mass over each instance that models this formula:

$$\Pr(x \vee y) = \Pr([x, y]) + \Pr([x, \overline{y}]) + \Pr(\overline{x}, y). \tag{13}$$

This is implemented by the following algorithm:

Algorithm 2: $\text{QueryPropositionalEvent}_1(\Pr, \varphi)$

Data: A fully-factorized distribution Pr; a propositional formula φ
Result: The probability of the event $\Pr(\varphi)$

```
1 a ← 0;
2 for each I ∈ Ω do
3 |   if I ⊨ φ then a ← a + Pr(ω);
4 end
5 return a
```

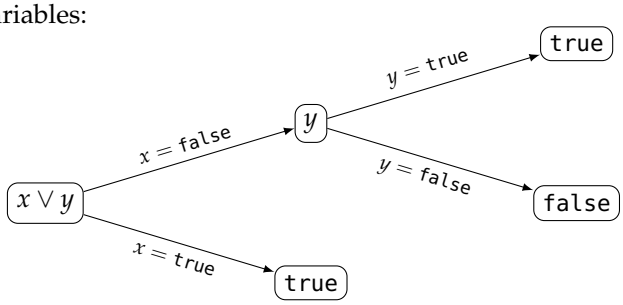
Unfortunately, QueryPropositionalEvent is *still linear* in the size of $|\Omega|$ due to the outer loop on Lines 2–4. In all cases, regardless of the structure of φ, we have to consider all all the instances, which will not scale.

Efficiently evaluating queries via search

How do we avoid scaling in $|\Omega|$ during querying? In other words: is it possible to design a scalable automated reasoning strategy for our tiny probabilistic programming language that we've made so far, with fully-factorized propositional distributions and queries given as propositional formulae? In the worst case, it may not be possible to avoid considering all possible instances, but we can try to avoid this worst-case explosion by delicately *exploiting the problem structure*: in particular, we want to simultaneously exploit the fully-factorized structure of the distribution along with the fact that all events are written as propositional formulae.

One approach is to *exploit the structure of models of the query* φ: sometimes, a query may only have a single model, or perhaps most instances are models. In these cases, we waste a lot of effort enumerating all models; it is much more efficient to *search* for the few satisfying instances.

For example, consider the formula $x \vee y$. We can recursively decompose this formula by branching on assignments to propositional variables:



The root of this binary tree is the original formula $x \vee y$. Each branch assigns a propositional variable to a Boolean value. The children are the resulting formula evaluated on the *partial instance* given by the path up until that point: for instance, if we assign $x = \texttt{false}$, we are left with the formula $\texttt{false} \vee y = y$. Leaves of the tree denote whether or not a particular assignment is a model.[5]

You should convince yourself that the above algorithm is correct by trying it on a few examples. It is a good exercise to prove it correct inductively. What is the runtime of Algorithm 7? In the worst-case, it is still $|\Omega|$. But, sometimes – depending on the structure of φ – it can do better by terminating the recursion earlier in a base-case on Lines 1 and 2, rather than exhaustively exploring all instances like Algorithm 5.

[5] (under construction)

Exercise 4 (⋆) *Give the truth table for* $(a \Leftrightarrow b) \wedge (\neg c)$.

Exercise 5 (⋆) *Give an example of a joint probability distribution that cannot be represented in a fully-factorized way.*

Algorithm 3: PrEvent(Pr, φ)

Data: A fully-factorized distribution Pr; a propositional formula φ; a decision order σ.
Result: The probability $\Pr(\varphi)$.

```
1 if φ = true then return 1;
2 if φ = false then return 0;
3 v ← head(σ);
4 σ′ ← tail(σ);
5 φ_v ← fix v = true in φ;
6 φ_v̄ ← fix v = false in φ;
7 return Pr(v)×PrEvent(Pr, φ_v) + Pr(v̄)×PrEvent(Pr, φ_v̄)
```

Exercise 6 ($\star$) *Assume the following fully-factorized distribution on propositional variables:*

$$[x \mapsto 0.1, \overline{x} \mapsto 0.9], \quad [y \mapsto 0.3, \overline{y} \mapsto 0.7].$$

Use an exhaustive search tree to compute $\Pr(x \Leftrightarrow y)$.

Exercise 7 ($\star\star$) *Prove Algorithm 7 correct.*

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