

Propositional Probability IV: Binary Decision Diagrams¹

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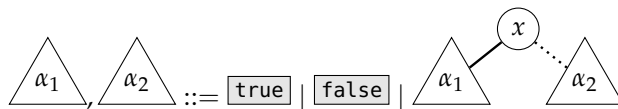
September 24, 2023

¹ CS7470 Fall 2023: Foundations of Probabilistic Programming.

- Note: not class next Tuesday, Sept 26 (Software Day)
- Join Slack, I make announcements there

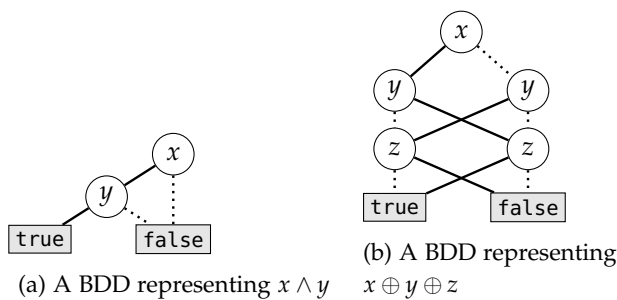
1 Binary decision diagrams (BDDs)

- **Problem:** PROP_S is not very expressive
- **Goal:** More expressive tractable languages
- What can we improve about PROP_S ? Observe that there may be *repetitious sub-syntactic terms*, and we'd like to exploit those, just like we observed in our memoized reduction scheme
- How can we represent repetitious subterms syntactically? By changing our syntax to permit *directed acyclic graphs*!
- Syntax of **binary decision diagrams** (BDDs):



I.e., each BDD is a rooted directed acyclic graph with terminal nodes labeled true and false, and internal *branch nodes* labeled by propositional variables. The solid edge of a branch node is called the *high edge*, and the dashed edge is called the *low edge*.

- Here are some example BDDs:



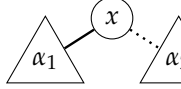
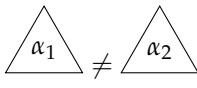
Solid edges are high edges, dotted edges are low edges. BDDs are read top-down, where the implied directionality of the arrow is top-down.

BDDs were introduced by Bryant [1992]. Meinel and Theobald [1998] is a good reference for learning how to implement and analyze BDDs. Knuth [2013] has a comprehensive discussion on BDDs as well.

Here we give a BNF grammar that permits directed acyclic graphs as terms. This is quite non-standard: we are implicitly permitting the structural re-use of terms in this grammar, and assuming an induction principle that “inducts on the structure of DAGs” in this definition of syntax. I don’t want to get into the weeds on this, so we will mostly see through examples how we write terms in this language and perform proofs by structural induction. See Braibant et al. [2014] for some discussion on this.

Figure 1: Example BDD programs.

- Observe that there can many equivalent ways of writing the same BDD for a particular denotation (show examples on board)
- A BDD is called *ordered* if each variable is used *at most once* and in a *fixed order*. We will consider only ordered BDDs.

- A BDD is called *reduced* if, for every node , it is the case that .

- We can enforce these constraints with a typing judgment that is reminiscent of *ordered affine linear logic* [Girard, 1987].

Let Γ be an *ordered* list of propositional variables. A term in ROBDD is a BDD term that satisfies the following typing judgment, written $\Gamma \vdash^r \triangle$:

$$\begin{array}{c}
 \Gamma \vdash^r \boxed{\text{true}} \quad \Gamma \vdash^r \boxed{\text{false}} \quad \frac{\Gamma \vdash^r \triangle \alpha}{x :: \Gamma \vdash^r \triangle \alpha} \text{ (WEAKEN)} \\
 \\
 \frac{\Gamma \vdash^r \triangle \alpha_1 \quad \Gamma \vdash^r \triangle \alpha_2 \quad \triangle \alpha_1 \neq \triangle \alpha_2}{x :: \Gamma \vdash^r \triangle \alpha_1 \text{---} x \text{---} \triangle \alpha_2}
 \end{array}$$

Theorem 1 (Canonicity of ROBDD). *For two well-typed ROBDD terms $\Gamma \vdash^r \alpha_1$ and $\Gamma \vdash^r \alpha_2$, it is the case that $\llbracket \Gamma \vdash^r \alpha_1 \rrbracket = \llbracket \Gamma \vdash^r \alpha_2 \rrbracket$ if and only if $\alpha_1 = \alpha_2$.*

Proof. It is straightforward to show that if $\alpha_1 = \alpha_2$ then $\llbracket \Gamma \vdash^r \alpha_1 \rrbracket = \llbracket \Gamma \vdash^r \alpha_2 \rrbracket$. The proof in the other direction follows by inducting on $|\Gamma|$.

Base case: Assume $|\Gamma| = 0$. Assume $\llbracket \emptyset \vdash^r \alpha_1 \rrbracket = \llbracket \emptyset \vdash^r \alpha_2 \rrbracket$. By our typing judgment for ROBDD, α_1 and α_2 must each be either `true` or `false`; these two cases have different semantics (either \emptyset or $\{\top\}$), so we can conclude that this implies that $\alpha_1 = \alpha_2$.

Inductive argument: (come back if time, if no time, see Bryant [1992] and Knuth [2013] for an extended discussion)

□

- We can enforce this ordering constraint with a typing judgment that is reminiscent of *ordered affine linear logic* [Girard, 1987].

Let Γ be an *ordered* list of propositional variables. A term in **ROBDD** is a BDD term that satisfies the following typing judgment, written $\Gamma \vdash \triangle$:

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 \Gamma \vdash \boxed{\text{true}} \quad \Gamma \vdash \boxed{\text{false}} \quad \frac{\Gamma \vdash \triangle}{x :: \Gamma \vdash \triangle} \text{ (WEAKEN)} \\
 \\
 \frac{\Gamma \vdash \triangle_{\alpha_1} \quad \Gamma \vdash \triangle_{\alpha_2} \quad \triangle_{\alpha_1} \neq \triangle_{\alpha_2}}{x :: \Gamma \vdash^r \triangle_{\alpha_1} \overset{x}{\bullet} \triangle_{\alpha_2}}
 \end{array}$$

- Semantics of BDDs: Let Γ be an ordered list of propositional variables. Then:

$$\begin{aligned}
 \llbracket \Gamma \vdash \boxed{\text{true}} \rrbracket &= \llbracket \Gamma \rrbracket \\
 \llbracket \Gamma \vdash \boxed{\text{false}} \rrbracket &= \emptyset \\
 \llbracket \Gamma \vdash \triangle_{\alpha_1} \overset{x}{\bullet} \triangle_{\alpha_2} \rrbracket &= \{I \in \llbracket \Gamma \vdash \triangle_{\alpha_1} \rrbracket \mid I(x) = \text{true}\} \cup \{I \in \llbracket \Gamma \vdash \triangle_{\alpha_2} \rrbracket \mid I(x) = \text{false}\}
 \end{aligned}$$

We define $\llbracket \Gamma \rrbracket$ inductively:

$$\begin{aligned}
 \llbracket [] \rrbracket &= \{\top\} \\
 \llbracket x :: \Gamma \rrbracket &= \{[x \mapsto \text{true}] \cup I \mid I \in \llbracket \Gamma \rrbracket\} \cup \{[x \mapsto \text{false}] \cup I \mid I \in \llbracket \Gamma \rrbracket\},
 \end{aligned}$$

where \top is the empty map.

Probabilistic BDD

- Just like how we made propositional logic, we can define a probabilistic version of BDDs and give them a probabilistic semantics. We call this language **BDD**.
- Syntax of **BDD**: pairs (α, w) , where α is a BDD and w is a map from propositional variables to probabilities
- The denotational semantics are nearly identical to **PROB_S**, so we

skip then here. The operational semantic are quite similar as well:

$$\begin{array}{c}
 (\boxed{\text{true}}, w) \Downarrow 1 \qquad (\boxed{\text{false}}, w) \Downarrow 0 \\
 \\
 \begin{array}{ccc}
 \triangle \alpha_1 & & \triangle \alpha_2 \\
 \Downarrow v_1 & & \Downarrow v_2
 \end{array} \\
 \hline
 \begin{array}{c}
 \begin{array}{ccc}
 \triangle \alpha_1 & \circ x & \triangle \alpha_2 \\
 & \swarrow \quad \searrow & \\
 & \text{---} &
 \end{array} \\
 (\begin{array}{ccc} \triangle \alpha_1 & & \triangle \alpha_2 \end{array}, w) \Downarrow w(x) + v_1 + w(\bar{x})v_2
 \end{array}
 \end{array}$$

- Is the above a a tractable runtime for BDD? This question is a bit subtle and depends on how we define structural induction on BDDs. For now, let's assume that we are caching as traverse (just like the memoization routine from last time); this memoization routine is a surely a tractable runtime for BDD.

Theorem 2. *BDD is more expressive than PROP_S.*

Proof. To show this, we must show two things: (1) that there is an efficient compilation from PROP_S to BDD, and (2) that there is no efficient compilation from BDD to PROP_S. Showing (1) is straightforward. Showing (2) is quite tricky.

In general, there are the following strategies we can use to establish that there is no efficient semantics-preserving translation:

- *Denotational lower-bounds:* Show that there is a particular denotation that separates the two languages
- *Reduction:* Show that the existence of a efficient semantics-preserving map can be used to give an efficient algorithm for something we know is hard (i.e., solving SAT)

In this case it is easier to give a denotational lower-bound². We will have the following two lemmas that establish separation between PROP_S and BDD:

Lemma 1. *For any integer N, There exists a BDD of size less than 2n whose denotation is equal to $\llbracket \bigoplus_i^N x_i \rrbracket$.*

Lemma 2. *For any N there does not exist a PROP_S formula represents the denotation given by $\llbracket \bigoplus_i^N x_i \rrbracket$ whose size is less than 2^N.*

□

Compiling PROP_S to BDD

- **Goal:** Build a compiler from PROP to BDD. The weight function translation is simple. We focus on translating formulae, which now

² ...but, if you can find a reduction, I'd be interested in seeing it!

In the literature this process is typically called *top-down knowledge compilation*. See [Darwiche and Marquis, 2002, Oztok and Darwiche, 2015, Darwiche, 2004].

has a contextual compilation relation $(\pi, \varphi) \rightsquigarrow \text{BDD}$, where now ρ is a map from PROP to BDD :

Diagram illustrating the semantic difference between the two lambda terms:

- Top row: $(\pi, \text{true}) \rightsquigarrow \boxed{\text{true}}$ and $(\pi, \text{false}) \rightsquigarrow \boxed{\text{false}}$
- Middle row: $(\pi, \varphi[\text{true}/x]) \rightsquigarrow \triangle \alpha_1$ and $\varphi[\text{false}/x] \rightsquigarrow \triangle \alpha_2$. Below these, a diagram shows a circle labeled x connected by a solid line to $\triangle \alpha_1$ and by a dotted line to $\triangle \alpha_2$, with $\varphi \rightsquigarrow$ to the left.
- Bottom row: $\varphi[\text{true}/x] \rightsquigarrow \triangle \alpha_1$ and $(\pi, \varphi[\text{false}/x]) \rightsquigarrow \triangle \alpha_2$. Below these, a diagram shows $(x :: \pi, \varphi) \rightsquigarrow \triangle \alpha_1$.

●

Theorem 3. *The above compilation rule produces well-typed BDDs.*

- **Observe:** This looks almost identical to our memoized semantics \Downarrow^m . In some sense, the memoized semantics are efficient if and only if there exists a compilation to BDD. This means that the syntax of BDD perfectly captures the subset of PROP that can be executed efficiently by \Downarrow^m

2 Compositional compilation

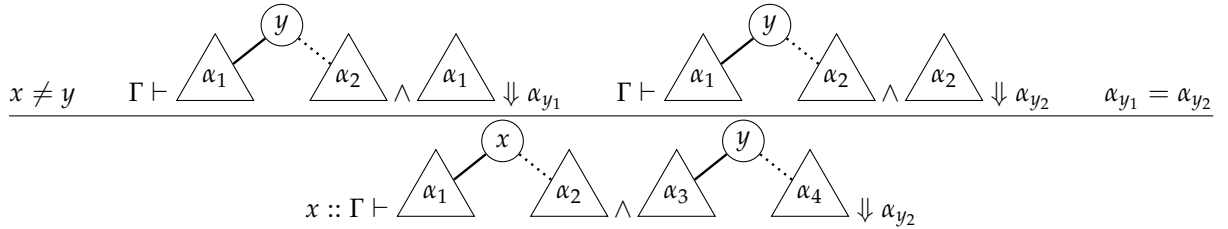
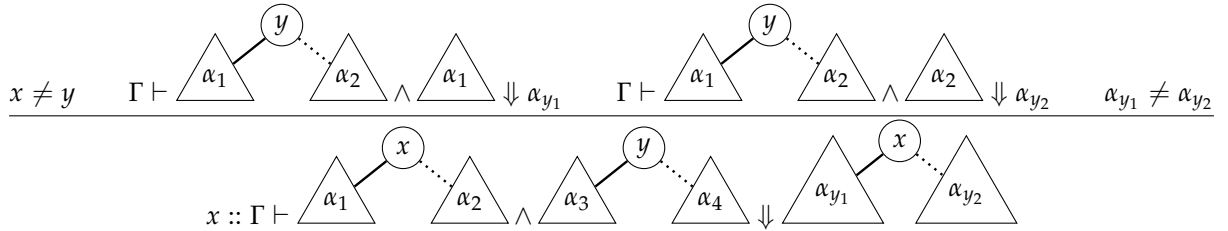
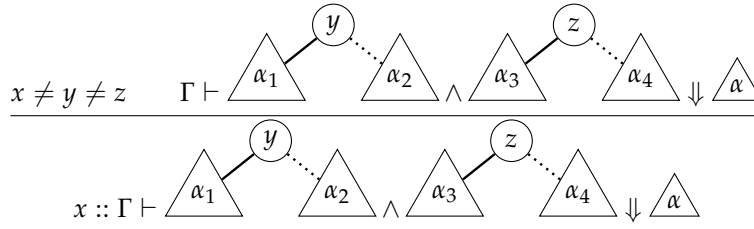
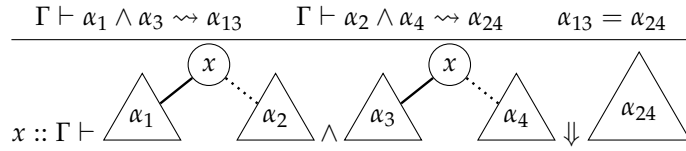
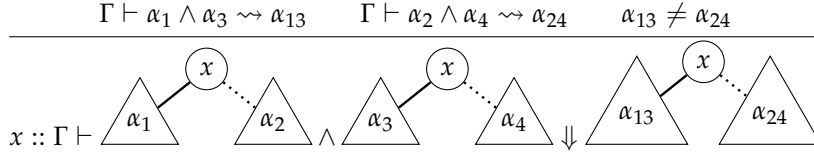
- So far we have compiled by inducting on variables
- **Problem:** this is not *compositional*! A compilation process is compositional if it works by compiling big programs out of smaller sub-programs. I.e., it would have a rule that looks something like:

$$\frac{\alpha \rightsquigarrow \alpha' \quad \beta \rightsquigarrow \beta'}{\alpha \wedge \beta \rightsquigarrow \alpha' \times \beta'}$$

- Compositional compilation is great: gives us modular reasoning about performance, ... (other reasons?)
- **Goal:** Design a compositional process for compiling PROP_S to BDD
- How can we build big BDDs out of smaller ones? Define a way to compose together BDDs, again using a type-directed step-relation

$\Gamma \vdash \alpha_1 \wedge \alpha_2 \Downarrow \beta$:

$\Gamma \vdash \boxed{\text{true}} \wedge \alpha \Downarrow \alpha$ $\Gamma \vdash \boxed{\text{false}} \wedge \text{false} \Downarrow \boxed{\text{false}}$...



Theorem 4 (Correctness). *If $\Gamma \vdash \alpha_1$, $\Gamma \vdash \alpha_2$, and $\Gamma \vdash \alpha_1 \wedge \alpha_2 \Downarrow \beta$, then $\llbracket \alpha_1 \rrbracket \cap \llbracket \alpha_2 \rrbracket = \llbracket \beta \rrbracket$.*

Theorem 5 (Type preservation). *If $\Gamma \vdash \alpha_1$, $\Gamma \vdash \alpha_2$, and $\Gamma \vdash \alpha_1 \wedge \alpha_2 \Downarrow \alpha$, then $\Gamma \vdash \alpha$.*

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