Combinatorics

Definition: Deals with the arrangement of object according to some pattern & counting no. of ways it can be done.

n(Task) = 1+1+1 = 3 (Addition Principle)

No. of ways of attempting the test.

→ 3 T/F Questions

Task 1 ⇒ Q1   T/F/_ (2) (3)

Task 2 ⇒ Q2   T/F/_ (2) (3)

Task 3 ⇒ Q3   T/F/_ (2) (3)

$\boxed{2^3 = 8}$ $3 \times 3 \times 3 = \boxed{27}$ $3!$ 2^3 2^3 $3 \times 3 \times 3 = \boxed{27}$

→ When to multiply / add?

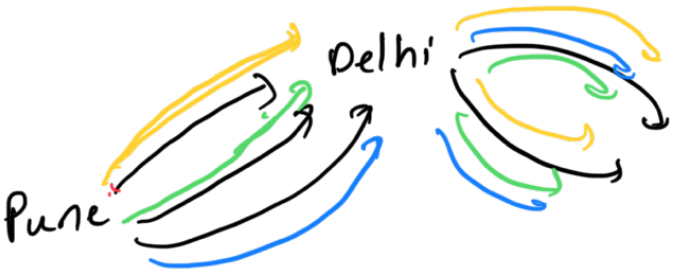
Multiplicative Principle

Task A ⇒ n ways

Task B ⇒ m ways

This principle says that no of ways of doing both the tasks = $n \times m$

$2 + 2 + 2 = 6$



$= \boxed{3 \times 2}$

Task 1   Task 2

3 ⇐ Task A ⇒ Travel from Pune to Delhi

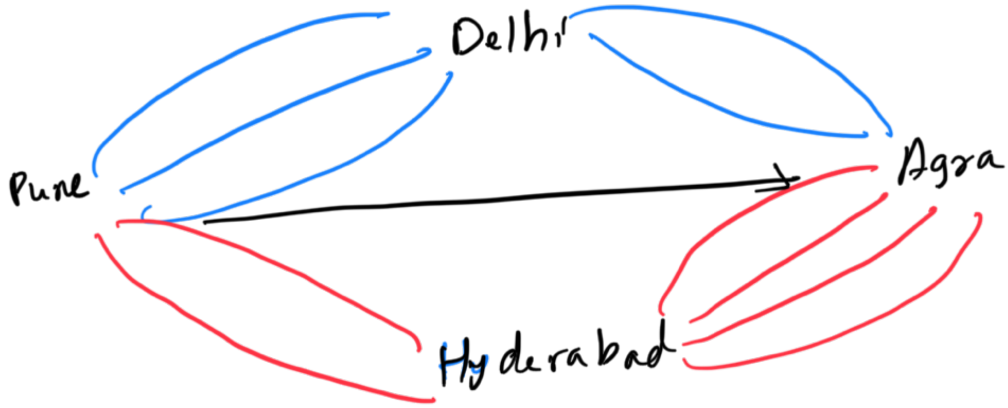
Task B ⇒ Travel from Delhi to Agra

2 ⇒ Task B

Task A & Task B ⇒ Travel from Pune to Agra (via Delhi)

Addition Principle:

No of ways of doing Task A or Task B = n + m



Task A : Pune to Agra (via Delhi) $3 \times 2 = 6$

Task B : Pune to Agra (via Hyderabad) $2 \times 4 = 8$

Task A or Task B = Pune to Agra (via Delhi or Hyderabad)

$= 6 + 8 = 14$

Task C: Direct from Pune to Agra (1)

⇒ (Task A) + (task B) + (task C)

$= 6 + 8 + 1 = \boxed{15}$

AND ⇒ Multiplication

OR ⇒ Addition

Permutations

ordered arrangements of objects

abc ≠ acb

abc ⇒

a b c
a c b
b a c
b c a
c a b
c b a

$\}$ 6 Permutations

S = abc (distinct)

$\underline{3}$ $\underline{2}$ $\underline{1}$ = $3 \times 2 \times 1 = 3!$

TASK A, TASK B, TASK C

abc

$\underline{a}$ $\underline{b}$ $\underline{c}$

⇒ $\boxed{N! \text{ Permutations}}$

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$$\boxed{N! = N \times (N-1)!}$$

$N! = N \times (N-1) \times (N-2)!$

Question: – No. of ways to arrange any R characters among N characters (Permutations)

S = $\boxed{abcde}$

$\boxed{\begin{matrix} N = 5 \\ R = 2 \end{matrix}}$ ⇒ $\dfrac{5!}{(5-2)!} = \dfrac{5!}{3!}$ = $5 \times 4 = 20$

(a/b/c/d/e)

$\underset{1^{st}}{\underline{5}}$ $\underset{2^{nd}}{\underline{4}}$

$\neq 5!$

abcde

$= \underline{5}\ \underline{4}\ \underline{3} = 5 \times 4 \times 3 = \boxed{160 \text{ ways}}$

$3!$

$= 5 \times 4 \times \cancel{3 \times 2 \times 1}$

$\cancel{3 \times 2 \times 1}$

$= \boxed{5 \times 4}$

$$\frac{5!}{(5-3)!} = \frac{5}{2!} \leftarrow \frac{5!}{2!} = \boxed{\frac{N!}{(N-R)!}}$$

$$= \boxed{\frac{N!}{(N-R)!} = {}^{N}P_{R}}$$

Permutation

How many chars do you want to arrange

$$^{N}P_{R} = \frac{N!}{(N-R)!}$$

Permutations with Repetition

str = "aabbbc"

$N = 6$

ways = 6!

$$\frac{6!}{2!\ 3!}$$

frequency of a

frequency of b

$$\frac{N!}{r_1! \times r_2! \times r_3! \times \dots r_k!}$$

where r_i = frequency of i^{th} character

str = abbacccca deefff

15!

= 3

freq(a)

freq(b) = 2

freq(c) = 4

freq(d) = 1

freq(e) = 2

freq(f) = 3

(N!)

$3! \times 2! \times 4! \times 1! \times 2! \times 3!$

$$\frac{N!}{r_1! \times r_2! \times r_3! \ldots}$$

S = " a a b b b c "

b_1 b_2 b_3

consider a permutation, a b c a b b

freq(b) = 3

a	b_1	c	a	b_2	b_3
a	b_2	c	a	b_1	b_3
a	b_1	c	a	b_3	b_2
a	b_2	c	a	b_3	b_1
a	b_3	c	a	b_1	b_2
a	b_3	c	a	b_2	b_1

Duplicate

5 4 3 2 1

$\boxed{abcabb}$

N!

Permutation:

b_1	a	b_2	a	b_3	c
b_1	a	b_3	a	b_2	c
b_2	a	b_1	a	b_3	c
b_2	a	b_3	a	b_1	c
b_3	a	b_1	a	b_2	c
b_3	a	b_2	a	b_1	c

6 Permutation

permutation is occurring 6 times

N!

Every

$$\Rightarrow \boxed{\frac{N!}{6}}$$

S = a a b b b c

$\underline{6}$ $\underline{5}$ $\underline{4}$ $\underline{3}$ $\underline{2}$ $\underline{1}$

Why divide?

$\Rightarrow$ $\boxed{a\ b\ b\ b\ a\ c}$ $\boxed{6!}$

$a\ b_1\ b_2\ b_3\ a\ c.$

(Backtracking)

$\boxed{a\ b\ b\ b\ a\ c}$

$a\ b_1\ b_2\ b_3\ a\ c$
$a\ b_1\ b_3\ b_2\ a\ c$
$a\ b_2\ b_1\ b_3\ a\ c$
$a\ b_2\ b_3\ b_1\ a\ c$
$a\ b_3\ b_1\ b_2\ a\ c$
$a\ b_3\ b_2\ b_1\ a\ c$

6 duplicate

$\boxed{a\ a\ c\ b\ b\ b}$ $\Rightarrow$

6 duplicates

$$\frac{N!}{6} = \frac{N!}{3!}$$

$\#(a_s) = 2$

$\boxed{a\ b\ b\ b\ a\ c}$

$a_1\ b\ b\ b\ a_2\ c$
$a_2\ b\ b\ b\ a_1\ c$

2 duplicate

$$= \frac{\left(\frac{N!}{3!}\right)}{2!}$$

s = 'aaa'

$$\frac{N!}{} = \frac{3!}{} = \boxed{1}$$

$= \boxed{aaa}$

$3!$ $3!$

Merge Sort

Combinations

selection of objects

S = a b c d e

$N = 5, R = 4$

$\boxed{\{abcd\}}, \{abce\}, \{abde\}, \{acde\}, \{bcde\}$ — 5 ways

$${}^5C_4 = \frac{5!}{(5-4)! \times 4!} = \frac{5!}{1 \times 4!} = \boxed{5}$$

abcd
abdc
acbd

abce
abec
acbe
aceb

→ 52 cards, No. of ways of choosing 3 cards.

$${}^{52}C_3 = \frac{52!}{(52-3)! \times 3!}$$

⇒ No. of ways of choosing R objects from objects is denoted by ${}^nC_R$

$${}^NC_R = \frac{N!}{(N-R)! \times R!}$$

A = a b c d

... d, bc, bd, cd} = 6

$N = 4$ { ab, ac, ad, ... }

$R = 2$

$^4C_2 = \frac{4!}{2!\ 2!} = \frac{4\times3\times2\times1}{2\times2} = \boxed{6}$

! => factorial

5

$$^NC_R = \boxed{\frac{N!}{(N-R)!\ R!}} = \frac{\left(\frac{N!}{(N-R)!}\right)}{R!} = \boxed{\frac{^NP_R}{R!}}$$

$S =$ a b c d e

$N = 5$

$R = 3$

$^5P_3 = \frac{5!}{2!} = 5\times4\times3 = \boxed{60}$

$\boxed{\text{Total permutations} = 60}$

No. of ways of arranging 3(R) characters among N characters

60

→ a b c
a c b
b a c
b c a
c a b
c b a

$\boxed{\text{only 1}}$

3!

→ b c d
b c d
b d c
c b d
c d b
d b c
d c b

6

2) if we have to chose R characters, divide by $\boxed{R!}$

$$^NC_R = \#\,\text{combinations} = \boxed{\frac{^NP_R}{R!}}$$

$\Big[\, N_{\cdot}$ ... $\frac{N!}{\ \ }$

$R \qquad (N-R)! \cdot R!$

$S = \{a_1 a_2 c\}$

$\{a_1 c, \; a_2 c, \; a_1 a_2\}$

3 combination

$\{a_1 a_2 c\}$

if $a_1 c$ and $a_2 c$ as considered

Properties :

1) $N_{C_0} = 1$

$$\frac{N!}{(N-0)! \times 0!} = \frac{N!}{N!} = \boxed{1}$$

2) $N_{C_1} = N$

$a_1 \; a_2 \; a_3 \; a_4 \; a_5 \ldots a_n$

3) N balls

→ place r balls in bag 1 and remaining balls in Bag 2

$\boxed{n_{C_r}}$

r balls — Bag 1

$N-r$ — Bag 2

ways of choosing r balls from N balls $= N_{C_r}$

2) Place $N-r$ balls in Bag 2

$\Rightarrow \boxed{N_{C_{N-r}}}$

$$^nC_r = {}^nC_{n-r}$$

$$\frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-(n-r))!\,(n-r)!}$$

$$\frac{n!}{r!\,(n-r)!}$$

10 balls, $R = 4$

4	6 Balls
B1	B2

1) $\boxed{^{10}C_4}$

2)

→ Placing 4 balls in Bag 1

→ $\boxed{^{10}C_6}$

$\boxed{^{10}C_4 = {}^{10}C_6}$

$$^NC_R = \frac{N!}{(N-R)!\,R!}$$

$R = R+1$

* Important

4) $^NC_R + \boxed{^NC_{R+1}} =$

⇒ Maths Proof

$$\frac{N!}{(N-R)!\,R!} + \frac{N!}{(N-R-1)!\,(R+1)!}$$

$$N!\left[\frac{1}{(N-R)!\,R!} + \frac{1}{(N-R-1)!\,(R+1)!}\right]$$

$$\frac{N!}{(N-R-1)!\,R!}\left[\frac{1}{(N-R)}+\frac{1}{R+1}\right]$$

$$=\frac{N!}{(N-R-1)!\,R!}\left[\frac{\cancel{R}+1+N-\cancel{R}}{(N-R)(R+1)}\right]$$

$$=\frac{N!\,(N+1)}{(N-R-1)!\,R!\,(N-R)\,(R+1)}=\frac{(N+1)!}{(N-R)!\,(R+1)!}$$

$$\Downarrow$$

$$\boxed{{}^{N+1}C_{R+1}}$$

$(N-R-1)! \times (N-R) = (N-R)!$

N.

$(R+1)\,R!$

$$\frac{(N+1)!}{(N-R)!\,(R+1)!} \Leftarrow \frac{(N+1)!}{(N+1-R-1)!\,(R+1)!}$$

$$\boxed{{}^{N}C_{R}+{}^{N}C_{R+1}={}^{N+1}C_{R+1}}$$

Rel

$${}^{N+1}C_{R+1}={}^{N}C_{R}+{}^{N}C_{R+1}$$

$N+1 \longleftrightarrow N$

$R+1 \longleftrightarrow R$

$$\boxed{{}^{N}C_{R}={}^{N-1}C_{R}+{}^{N-1}C_{R-1}} \;\star$$

$\boxed{R}$

$R \Rightarrow [0, N]$

$\boxed{0 \le R \le N}$

Question:

Given, n, r, M $\qquad N \le 10^3$

$0 \le r \le n$

compute $\quad {}^{N}C_{r} \,\%\, M$

$N = 5, \quad r = 2, \quad M = \boxed{10}, \qquad {}^{5}C_{2} = 10 \,\%\, 10 = \boxed{0}$

$\qquad {}^{4}C_{2} = 6 \,\%\, 10 = \boxed{6}$

$N = 100$
$R = 50$

$N = 4, \quad r = 2 \qquad M = 10,$

$\frac{100!}{50!\, 50!}$ $\qquad {}^{4}C_{2} = \frac{4 \times 3}{2} = 6$

$$\left(\frac{N!}{R!\,(N-R)!}\right) \% \, M \Rightarrow (N!) \% M \times (R!)^{-1} \% M \times (N-R!)^{-1} \% M$$

$b^{-1} \bmod M$ can be computed only if b and M are coprime. $\gcd(b, M) = 1$

${}^{N}C_{0} = 1, \quad {}^{N}C_{1} = N \qquad {}^{N}C_{N} = 1$

$${}^{N}C_{R} = {}^{N-1}C_{R} + {}^{N-1}C_{R-1}$$

$N = 5, \quad R = 2$

${}^{5}C_{2} = {}^{4}C_{2} + {}^{4}C_{1}$

$^4C_2 = {}^3C_2 + {}^3C_1$

$^3C_2 = {}^2C_2 + {}^2C_1$

^NC_R

$R : [$

$R = [0, N]$

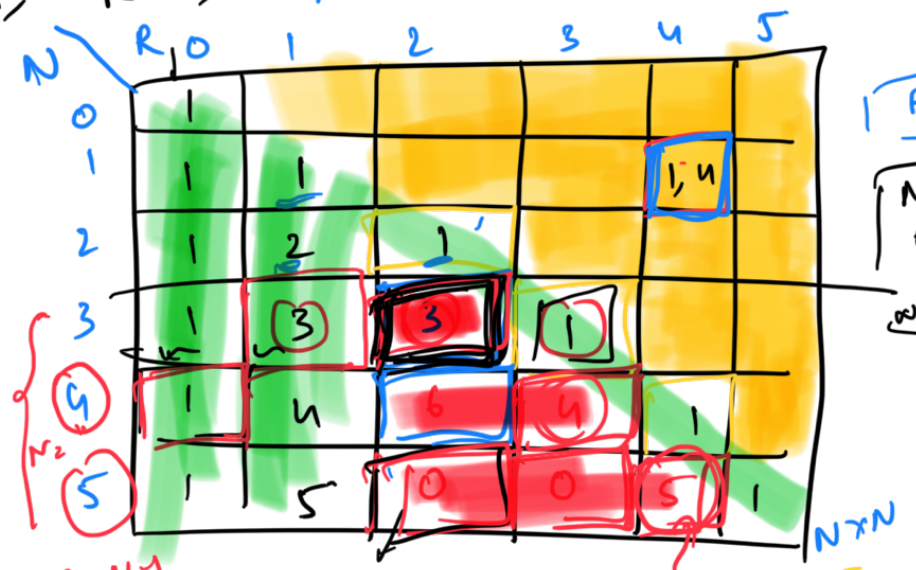
$N = 3, \; R = 0, 1, 2, 3$

$^NC_1 = N$

$N, \; R \in [0, N]$

$N = 5, \; R = 2, \; M = 10$

$R = 0, 1, 2, 3$



$N = 3$

$R = 2$

matrix =

$^4C_2 = {}^3C_2 + {}^3C_1$

$^4C_0$

$r = 3$

$c = 2$

$R > N$

$N = 1$

$R = 4$

$N = R$

$(^5C_2) \% 10$

$4 \% 10 = 4$

$N \times N$

5×5

For N, $R = N - 1$

$R = 0$

$^nC_R = 1$

$^3C_1 \% 10$

$6 + 4 = (10) \% 10$

$(6 + 4) \% 10$

$^NC_1 = N$

$^nC_n = 1$

$^3C_2 = {}^2C_2 + {}^2C_1 = (2 + 1) \% 10$

$^nC_R \quad r \le N$

PASCAL'S TRIANGLE

$$\boxed{mat[i][j] = mat[i-1][j] + mat[i-1][j-1]}$$

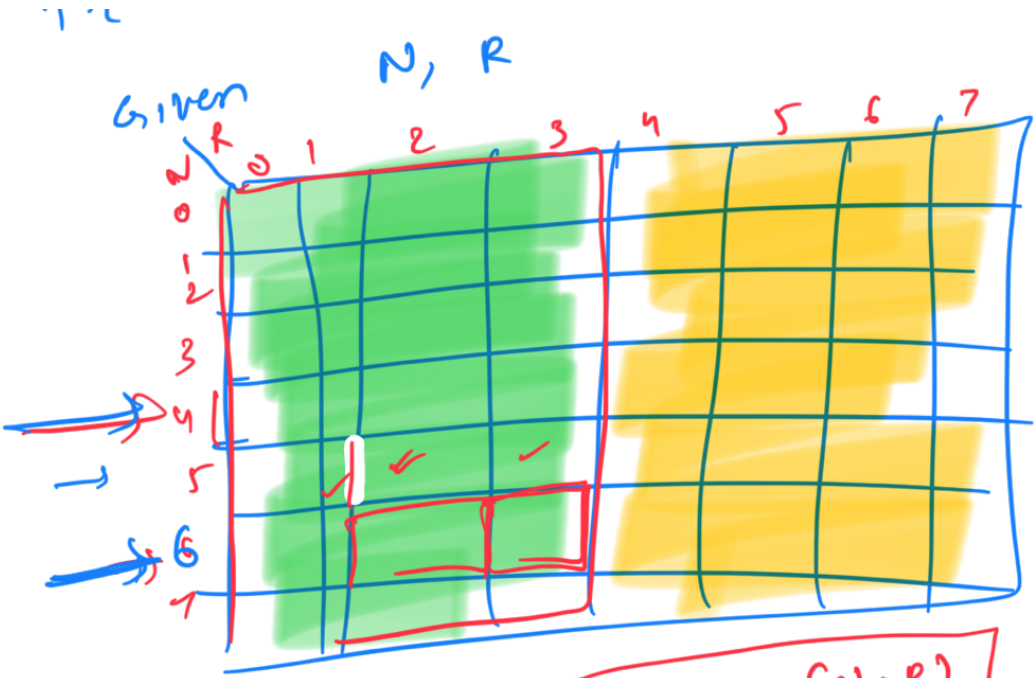
Dynamic Programming (Tabulation)

T.C: $O(n^2)$

Given N, R

$R=3$

$^7C_3$

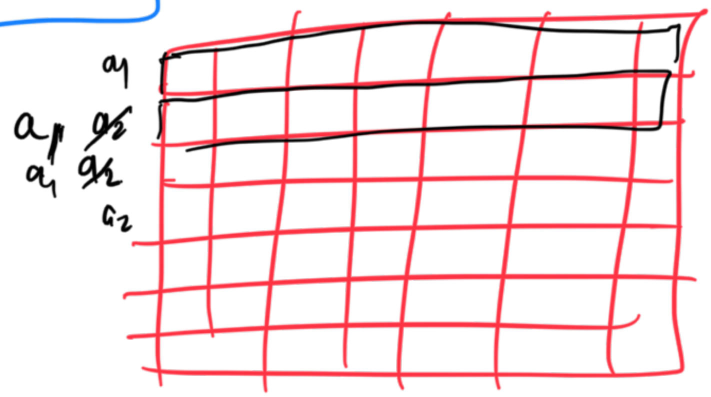


Maintain 2 array of size r.

$\boxed{S.C: O(r)}$

$\boxed{\begin{array}{l} T.C: O(N*R) \\ S.C: O(N*R) \end{array}}$

$a_1 = a_2$



Question: M is a prime number

$0 \le R \le N < P$

$a^P \% mod$

$\left({}^NC_R\right) \% M$

$\left(\frac{N!}{(N-R)!\ R!}\right) \% M$

(Fast Exponent

$\log(M)$

$((N-R)!)^{M-2} \% M$

$\log(M)$

$(R!)^{M-2} \% M$

$O(R) \to O(R \log M)$

$= \boxed{(N!)\%M \times ((N-R)!)\%M \times (R!)\%M}$

$O(N)$ → $(N-R)!$ & M should be co-prime

→ $R!$ & M should be co-prime

$N = 13,\ r = 11,\ M = 10$

$R! = \{(11 \times 10 \times 9 \ldots) \ldots\}$

$M = 7$

$\gcd(R!, M) = 1$?

if $(R < M)$

$R!$ and M will be co-prime

$M = 19$ $r = 17!$

$(17!)$

T.C : $O(N + \log M)$

→ precompute $N!$, $(N-R)!$, $(R)!$

$\underbrace{(N-R)!,\ (R)!}_{O(N)}$

$(N-R)! \to P$, $(R)! \to Q$

$= (P)^{M-2}\%M$ $(Q)^{M-2}\%M$

```
int mat[N+1][N+1] = {0};

for(row= 0; row<= N; row++){
    mat[row][0] = 1;
```

mat[row][1] = row;
mat[row][row] = 1;

}

for(row=3; row <= N; row++){
for(col=2; col < row ; col++)
mat[row][col] = mat[row-1][col] +
mat[row-1][col-1];

}

$(N-R) < M$

M is a Prime

$^5C_7 = \quad ^5C_r$

$r = 0, 1, 2, 3, 4, 5$

$r : [0, N]$

$r : [0, 5]$

$\boxed{\begin{array}{l} gcd((N-R)!, M) = 1 \\ gcd(R!, M) = 1 \end{array}}$

$(N-R) \% M$

$0 \le r \le N < M$

$$\Rightarrow \boxed{(N!)\%M \times [(N-R)!]^{M-2}\%M \times [(R!)]^{M-2}\%M}$$

$$[(N-R)!]^{-1}\%M = ((N-R)!)^{M-2}\%M$$

$a^{P-2}\%P$

Fast-Exponent

$$\left(\frac{N!}{r!(N-R)!}\right)\%M \quad \text{Prm}$$

$$\left(\frac{a}{b}\right)\%M = ((a\%M) \times (b^{-1})\%M)\%M$$

Fermat's thm

$$(b^{M-2})\%M$$

$\boxed{(b, M) = 1}$

gcd(b,

$(a)^b \% m = (a\%m)^b \% m$

$\Rightarrow$ $\boxed{gcd((N-R)!, M) = 1}$ $\Leftarrow$ if $N-R < M$ & $R < M$

$gcd(R!, M) = 1$

2) M is a prime, $gcd(x, M) > 1$?

$gcd(1, M) = 1$

R! should not be multiple of M

$M = 7$

$gcd(x, 7) =$

$gcd(14, 7) = 7$

$M = 11$

$R = 12 \Rightarrow R! = 12 \times 11 \times 10 \cdots$

$gcd(R!, M) =$

$R \geq 10$

$R < 10 ?$

^nC_r

$n = 100$

$r = 50$

$M = 101$

$\Rightarrow \left(\boxed{(100!)\%101} \times (50!)^{99}\%101 \times (50!)^{99}\%101\right)\%101$

```
ans = 1;
for ( i = 1; i <= 100; i++ ) {
    ans = (ans * i) % 101
}
```

$^7C_3 = \frac{7 \times 6 \times 5}{1 \times 2 \times 3}$

$$^{100}C_{50} = \left(\frac{(100 \times 99 \times 98 \cdots\cdots 50)}{1 \times 2 \times 3 \cdots\cdots 50} \right) \% M$$