Nim Game.

Game Theory.

Game being played by 2 players. Player 1 & 2. Game is such that it will end, and there has to be winner (no ties/draws)

Analysis of such games is game theory.

Q1 Number N. 2 Players. Each player plays turn by turn.
In one turn, player can replace current no x with any no $1 \le y < x$ $[1, x-1]$
Who will win the game.
Person who cannot make a move loses.

$50 \longrightarrow 37 \rightarrow 5 \rightarrow 1$ X

N

$1 \quad 2 \quad 3 \quad \cdots \quad N-1$

(Losing state)

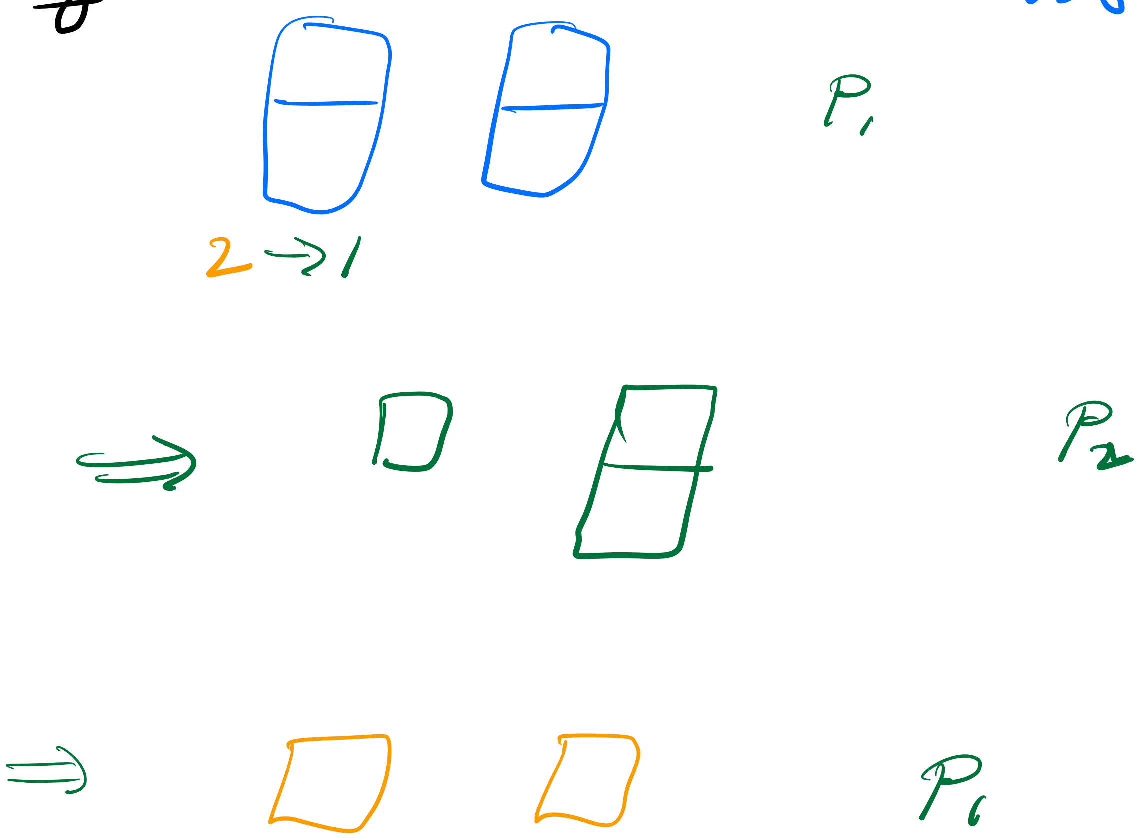
$N = 1 \qquad \text{win} = P_2$

$N \neq 1 \qquad \text{win} = P_1$

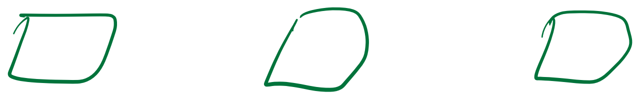
Q2 There are N piles, each with K stones. In a turn, you can convert exactly 1 pile of size x to y where $1 \le y < x$ and $\gcd(x, y) = 1$

Who will win?

Eg $N = 2$ $K = 2$ ans $= P_2$

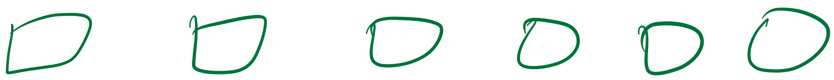


$N = 3$ $K = 1$



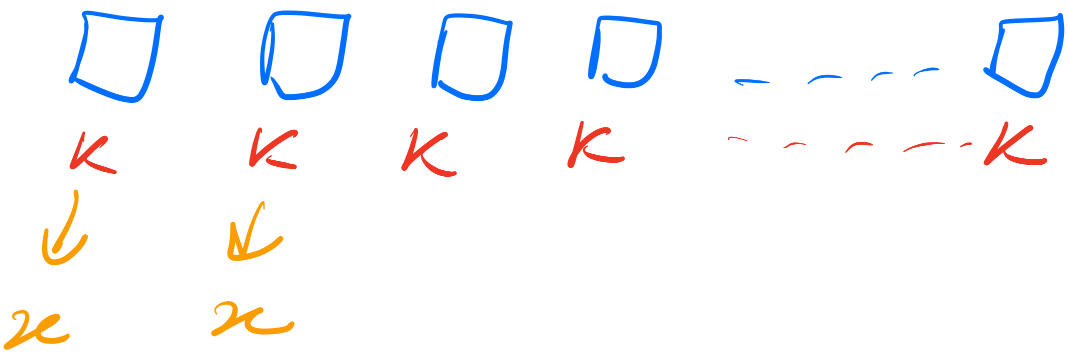
P_2

$N = 6$ $K = 1$ P_2



if $(K == 1)$
P_2 win

N is even



$N=4$ $\qquad$ $K=3$

	3	3	3	3
$\Rightarrow$	2	3	3	3
	2	2	3	3
	2	2	3	2
	2	2	2	2
	1	2	2	2
	1	1	2	2
	1	1	1	2
	1	1	1	1

if (N is even)
P_2 wins.

N=5 K=3

3 3 3 3 3

$1 \quad \underbrace{3 \quad 3 \quad 3 \quad 3}$ P_2

reduces to N= even case

hence P_1 wins,

if (K==1)
  P_2

if (N is even)
  p_2
else
  p_1

Q3 Make Palindrome.

string of lowercase alphabets.

In 1 move, a player can remove exactly 1 letter.

If the player, before his turn, can reorder s into a palindrome, that player wins.

Eg- aab P_1

Eg ab ⇒ b P_2

How to get a palindrome.

even a a b b c c a a

all chars have even freq

odd a a b b k c c a a

one char has odd freq.

if no of odd freq characters ≤ 1

you can directly convert to palin

what if odd freq chars = 2

P2 wins

abca

$a \Rightarrow 2$

$b \Rightarrow 1$

$c \Rightarrow 1$

$\Rightarrow$ aab P_2

$\Rightarrow$ abc P_2 $\Rightarrow$ bc $\Rightarrow P_1$

$\Rightarrow$ c P_2

- odd freq = 3 P_1 wins

abc $\rightarrow$ bc P_2 $\rightarrow$ c P_1

–

- odd freq = 4 P_2 wins.

abcd → bcd P_2 → cd P_1

→ d P_2

By obs, :

if odd freq = 0

P_1

if the odd freq is even

P_2

else

P_1

Eg – aabb abba

aabbc

Nim Game

N piles of stones, each with diff amount

$a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{N-1}$

In a move, a player can take any positive no of stones from any pile & throw them away

Person who cannot make a move, loses

Eg- $3 \rightarrow 0 \quad P_2 \qquad win = P_1$

$3, 3 \rightarrow 3\ 1 \rightarrow 1\ 1 \qquad win = P_2$

$\rightarrow 0\ 1 \rightarrow 0\ 0$

$3, 5 \rightarrow 3\ 3 \quad P_2 \qquad win = P_1$

Solution to nim game.

Current player wins if
$a_0 \wedge a_1 \wedge a_2 \text{----} \wedge a_{n-1}$ is non-zero

Proof

Obs: If xor is 0, then it is a losing state

$a_0 \quad a_1 \quad a_2 \quad \text{------} \quad a_{n-1}$ P_1

xor $= 0$ P_2

Let current xor $s = a_0 \wedge a_1 \wedge a_2 \wedge \text{.....} \wedge a_{n-1}$
s will have some largest set bit

Take number a_i where this largest bit is set

Convert $a_i \Rightarrow a_i \wedge s$

$a_0 \quad a_1 \quad a_2 \; --- \; a_i \; --- \; a_{n-1}$

$a_0 \quad a_1 \quad a_2 \; --- \; S \wedge a_i \; --- \; a_{n-1}$

$\underbrace{a_0 \wedge a_1 \wedge a_2 \; --- \wedge a_{n-1}}_{S} \wedge S$

$\wedge S = 0$

- Variations:
In a move, apart from removing stones, you can also add stones.

Ans: No change, same solution applies.

$a_0 \quad a_1 \quad a_2 \; ---- \; a_{n-1}$

Other player can reverse the move.

Q4 Given N piles. On one move you can remove only 1, 2 or 3 stones. The player who cannot move loses.

⇒ Try analysises for only pile

N:	1	2	3	4	5	6	7	8	9	10	11	12
Win	P_1	P_1	P_1	P_2	P_1	P_1	P_1	P_2	P_1	P_1	P_1	P_2

Obs: P_2 wins when

$N \% 4 == 0$

Thus we can reduce (N) any number to $N \% 4$

Piles ⇒ 10, 12, 7, 6, 16

2 0 3 2 0

Sprague Grundy Theorem

Once replaced with equivalent number this can be considered as the eq Nim Game.

If you want to explore more

Sprague Grundy Theorem

<done>