Maths - Prime Numbers

Prime Number:

-) Any positive number which has exactly 2 factors.

-) It is divisible by 1 and itself.

Ex 1 => only 1 factor

-1 : {1, -1} => 2 factors

Composite Nos. => > 2 factors

~~All non-prime => composite~~

Is 1 composite? NO

Is 1 Prime? NO

1 is neither prime nor composite

Questions: Find no. of factors/divisors of N

N = 12 {1, 2, 3, 4, 6, 12} => 6

N = 16 {1, 2, 4, 8, 16} => 5

i = 2

[1, N]

Brute Force: count = 0

for(i = 1; i<=n; i++) {

if(n % i == 0) {

} count++;

T.C: $O(n)$

}

return count;

$a = N,\ b = 1$

$N = a \times b$

Better Approach:

$N = a \times b$ $\boxed{\{a \le b\}}$

=> Divisors always come in pairs

12 = 1 2 3 | 4 6 12

a | b

$a = 3$

16: 1 2 4 8 16

a	b
1	16
2	8
4	4

$N = a \times b$ $\{a \le b\}$

$a = b$

$N = a^2$

$\boxed{a = \sqrt{N}}$

$N = a \times b$

$\boxed{a : [1, \sqrt{N}]}$

$\boxed{b : [\sqrt{N}, N]}$

2

We always have a factor which is than $\sqrt{N}$.

less

$N = a \times b$

$a = i$

$b = \frac{N}{i}$

Compute root(N) $\Rightarrow O(\log n)$

$i \le root(N)$

$i^2 \le N$

```
count = 0;
for ( i = 1; i*i <= N ; i++) {
    if ( N % i == 0) {
        count ++;
        if(i != N/i) {
            count ++;
        }
    }
}
return count;
```

16

i = 1, {1, 16}

i = 2, {2, 8}

i = 3 ×

i = 4 {4, 4}

i and $\frac{N}{i}$ are equal

$T.C: [1, \sqrt{N}]$

$T.C: O(\sqrt{N})$

Question: Check if a number is prime?

N

```
if(N <= 1) return false;
for(i = 2; i*i <= N ; i++) {
    if( N % i == 0)
        return false;
}
return true;
```

25

i = 2 ×

i = 3 ×

i = 4 ×

i = 5 ✓

i = 6

i = 7

N: 1 and N

$[2, \sqrt{N}]$

Prime = 1 and N

If there is a factor in the $[2, \sqrt{N}]$, prime not a ...

$N = a \times b$

$[2 : \sqrt{N}]$ $[\sqrt{N}, N]$

If there is no factor in the range $[2, \sqrt{N}]$, will we have a factor in $[\sqrt{N}, N]$?

$[2, \sqrt{N}]$

$\boxed{T.C: O(\sqrt{N})}$

Question : Print all prime nums $\leq N$

$N = 20$ Arr $[2, 3, 5, 7, 11, 13, 17, 19]$

Brute force:

$\forall\, i \in [1, N]$

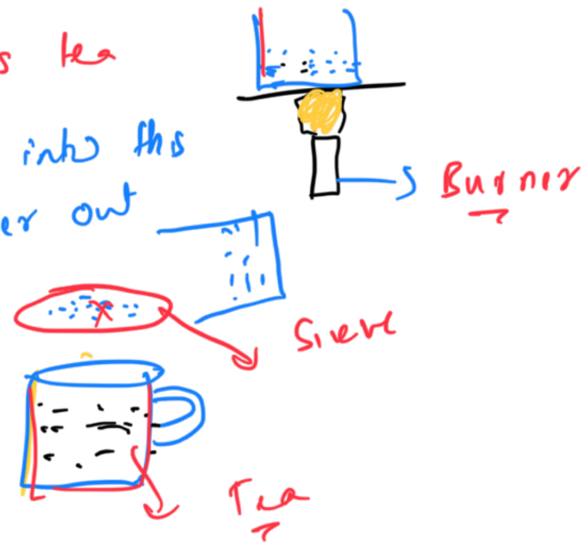
```
for(i = 1; i ≤ N; i++) {
    if( isPrime (i) )
        print(i);
}
```

T.C: $O(N \times \sqrt{N})$

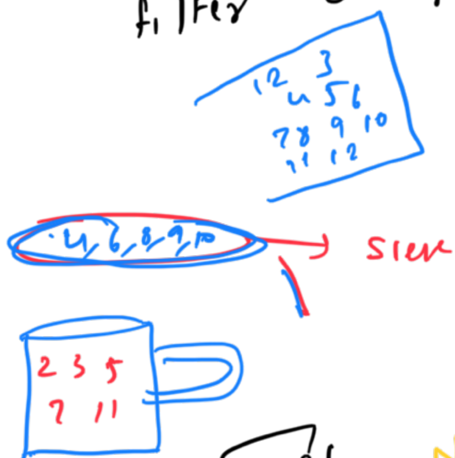
Sieve of Eratosthenes Algo

Sieve : Utensil to filter stuff.

1) this mixture is tea
2) We pour tea into this sieve and filter out all the wa...



1) Initially, assume all numbers to be prime
2) We'll pass the numbers through a sieve and filter out prime nos.



$\boxed{3 \times 2 = 6}$
$3 \times 3 = 9$
$3 \times 4 = 12$
$3 \times 5 = 15$

if $(i \times i > N)$

$i = 3$

$\boxed{2 \times i}$

2×3, 6, 6+3, 9+3, 12+3, 15+3

$\boxed{5 \times 2}$

N=50

2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50

1st step: Removing all multiples of 2

1) Assume all numbers to be prime
2) Iterate over all prime nos, remove all of their multiple.

bool arr[N+1];

N = 10

0	1	2	3	4	5	6	7	8	9	10
T	T	T	T	F	T	T	T	T	T	T

arr[4] = F

```
{
prime[N+1] = {T};
prime[0] = F;
prime[1] = F;
                                  }
for( i = 2; i ≤ N; i++ ) {
            if(prime[i] = = true) {
                  for( j = 2·i;  j ≤ N;  j = j + i ) {
                        prime[j] = F;
                  }
            }
}
```

2i, 3i, 4i, 5i ...

Time Complexity

i=2 { 2, 4, 6, 8 . . . . . . n } $\frac{n}{2}$

N=20 i = { 2, 4, 6, 8, 10, 12, 14, 16, 18, 20 } $\frac{20}{2}$

Operations

operation

$i = 2,$ $\quad (2) \rightarrow$

$i = 3,$ $\quad \frac{n}{3}$ operations

$i = 4$ $\quad 0$ operation

$i = \{3, 6, 9, 12, 15, 18\}$ $\rightarrow \frac{N}{3}$

$i = 5,$ $\quad \frac{n}{5}$

$N = 20$

$i = 6,$ $\quad 0$

$i = 7,$ $\quad \frac{n}{7}$

operations = $\boxed{\frac{n}{2} + \frac{n}{3} + \frac{n}{5} + \frac{n}{7} + \frac{n}{11} + \frac{n}{13} + \frac{n}{17} + \frac{n}{19}}$

$\Rightarrow$ sum of reciprocals of prime numbers $\leq N$

$= n \boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots}$

AP ×

GP ×

HP ×

No pattern

$n \times \log N$

$< n\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots\right]$ HP

$= \boxed{\int_1^N \frac{1}{x} = \log x \Big|_1^N = \boxed{\log N}}$

Upper bound of our algo: $\Rightarrow \boxed{(N \log N)}$

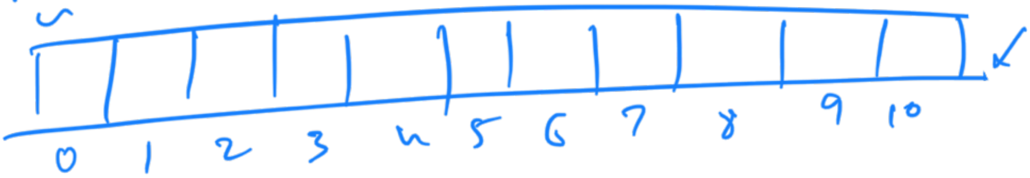
$\Rightarrow n\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \ldots\right]$

=) 1 L 2 $\boxed{\log(\log(n))}$

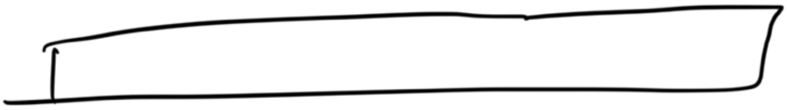
=) | T.C : $O(n \times \log(\log(n)))$

$\boxed{N+1}$

N = 10



S. C :



Extra array of size N;

$\boxed{S.C: O(N)}$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

=) $\left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \quad\right] =$

Pattern: $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} \ldots$

=) $\frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right]$

$\frac{1}{2} + \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots\right]$

$\boxed{\log n}$

S.C : $O(N)$

Prime Nos from $[1, 10^9]$

$\boxed{Arr[10^9]}$ { slap }

Segmented Sieve :

Find prime nos in this range

$[\quad a \quad b \leq 10^6]$

Filter [P_1, P_2] if $|P_2 - P_1| = \ldots$

2) $[10^9, 10^9 + 10^6]$

Prime Nos

$\boxed{T.C: O(n.\log(\log n))}$

$N = 2^{32}$

$\log_2 n = 32$

$\log \log n = \log 32 = \log 2^5 = \boxed{5}$

$\boxed{\text{Chandana}}$

$O(5 \times n)$

Almost => $\boxed{\text{Linear}}$

```
for(i=2; i*i <= N; i++) {
    if(prime[i] == true) {
        for(j = i*i ; j <= n; j = j+i)
            prime[j] = false;
```

$\boxed{T.C: N\log(\log(N))}$

$\sqrt{N} \times \log(N)$

$\boxed{T.C}$: $\boxed{\text{Slack Group}}$

i = 7

for(j = i*i

j = 2, 3, 4, 5, 6, 7

2) 14, 21, 28, 35, 42, 7x7

2, 3, 2, 5, 2, 7

j		
i = 5		
2	10	2
3	15	3
4	20	2
5	25	5

Question: Count no. of divisors / factors for all numbers [1, N]

N = 10

	1	2	3	4	5	6	7	8	9	10
factors:	1	2	2	3	2	4	2	4	3	4

→ Return the array.

Brute Force:

T.C: $\sqrt{n} \times n = O(n\sqrt{n})$

$N \leq 10^5$

Better:

Instead of marking false, increment the count

arr[N+1] = {0}

N = 15

→ 2, 4, 6, 8 output array

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
C =	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~
	1	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	1	~~1~~	~~1~~	~~1~~	1	~~1~~	1	~~1~~	~~1~~
		2	2	~~2~~	2	~~2~~		~~2~~	2	~~2~~		~~2~~		2	~~2~~
				3		3		3		3		~~3~~			3
												4			

3: 3, 6, 9, 12

4: 4, 8, 12, 16..

12 = 1, 2, 3, 4, 6, 12

12 = 1, 2, 3, 4, 6, 12 => 6

```
factor [N+1] = {0};
for (i = 1; i <= N; i++) {
    for (j = i; j <= n; j = j + i) {
        factors [j] ++;
    }
}
return d;
```

2, 4, 6, 8, 10....

$\frac{N}{2}$

i = 2 : $\frac{N}{2}$

i = 3 : $\frac{n}{3}$

i = 4 : $\frac{n}{4}$

$= n\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right)$

log n

$\boxed{\text{T.C: } O(n \log n)}$

S.C: O(n) ?

S.C: Extra space required to convert input to output

O(1)

S. C: O(1)

$$\log_b a = \log_c a \times \log_b c$$

2) $\log_2 N$ =) $\log_{10} N \times \log_2 10$

constant

S.C: O(n)

Question: Given an integer N, find the Smallest Prime Factor (SPF) for all number from 2 to N

N=10
for(i= 2 to 3)

4 = 2×2

N= 10

	2	3	4	5	6	7	8	9	10
				5	2	7	2	3	2
SPF :	2	3	2	-1	-1	-1	-1	-1	-1
SPF =	-1	-1	-1			7	2	3	2
SPF :	2	3	2	5	2	7			

5: 25

25, 50, 75, 100

36 : 2 × 2 × 3 × 3

{2, 2, 3, 3}

...(36) = 2

65
5×13

spf(36)

$\frac{36}{2} = 18$

$\frac{18}{2} = 9$

$\frac{9}{3} = 3$

$\frac{3}{3} = \boxed{1}$

N

$N \rightarrow \frac{N}{2} \rightarrow \frac{N}{4} \rightarrow \frac{N}{8} \rightarrow \frac{N}{16} \ldots$

$\frac{N}{3}$

N =

$$\underbrace{\frac{N}{2} \quad \frac{N}{2^2} \quad \frac{N}{2^3} \quad \ldots\ldots}_{K \text{ steps}} \quad \frac{N}{2^K}$$

$\frac{N}{2^K} = 1$

$N = 2^K$

$\boxed{K = \log N}$

$(i^* \le n)$

spf[N+1] = {-1}; $\boxed{i^* i \le N}$

i = 2, 3, 5, 7 ..

for (i = 2; i <= n; i++) {

if (spf[i] == -1) {

spf[i] = i

j = j + i*i

only for prime

for (j = i*i; j <= n; j += i) {

if (spf[j] == -1)

spf[j] = i;

}

}

j = j + i*i

}

return spf;

$$\left(\frac{N}{2} + \frac{N}{3} + \frac{N}{5} + \frac{N}{7} + \frac{N}{11} \ldots \right)$$

$\downarrow$

$\boxed{\text{T.C: } N \log(\log N)}$

S.C: O(1)

i = 7

7 => 7

14 => 2×7 => 2

21 => 3×7 => 3

28 => 4×7 => 2

35 => 5×7 => 5

42 => 6×7 => 2

49 => => $\boxed{i*i}$

→ Open doors

Question: No. of open doors

N doors numbered from 1 to N

N = 10

Person1. 1, 2, 3, 4, 5 . . . .

Toggle: open => close

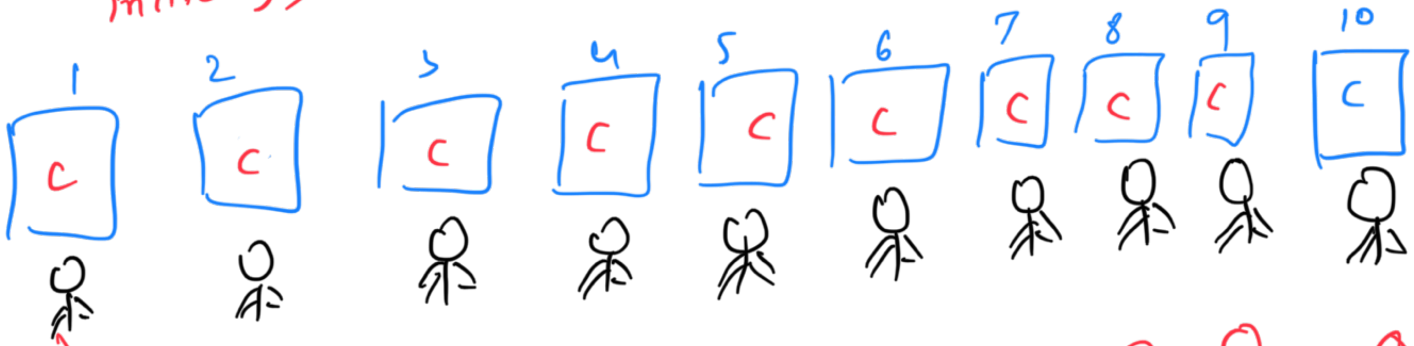
close => open

Person 2: 2, 4, 6, 8, 10 . . .

Person 3: 3, 6, 9, 12 . . .

Find no. of open doors

Initially, all doors are closed



p=1	O	O	O	O	O	O	O	O	O	O
p=2	O	C	O	C	O	C	O	C	O	C
p=3	O	C	C	C	O	O	O	C	C	C
p=4	O	C	C	O	O	O	O	O	C	C
p=5	O	C	C	O	C	O	O	O	C	O
p=10	O	C	C	O	C	C	C	C	O	C

3 doors are open

No. of toggles of a door N

$\boxed{2}$

N = 18 : 1, 2, 3, 6, 9, 18

Even factors: closed ✓

Odd factors : open ✓

2 4 6 8 10 12 16 18 2

Factors come in pairs

Even Factors

18 (1, 18)

(2, 9)

(3, 6)

16 : (1, 16)

(2, 8)

(4, 4)

Perfect square

(4×4)

1, 2, 4, 8, 16 = 5

odd no of Factors

$2 \times 2 \times 1 \times 1$

N = 1 2 3 4 5 6 7 8 9 10

O C C O C C C C O C

Ans = 3

Perfect Squares ≤ N

```
count = 0;
for (i = 1; i*i <= N; i++) {
    count++;
}
return count;
```

T.C : $O(\sqrt{n})$

i = 1 to $\sqrt{N}$

$\sqrt{101} = 10.02$

$1^2, 2^2, \ldots 10^2$

$\sqrt{N} = \lfloor 10.01 \rfloor$

= N = 101

Doors = $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2$

floor (sqrt(N))

T.C to compute sqrt(N) = O(logn)

Binary Search

Sum of divisors of all nos

spf :

S.C :

S.C is the extra space that we use apart from input and output

S.C: O(1)

input: N O(1)

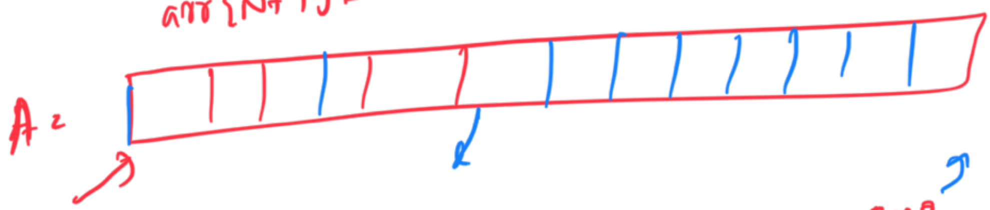
output: S pf !

$\boxed{S.C: O(1)}$

---

arr[N+1] =

A =



3x2

10 = 2x5

SPF	2	3	4	5	6	7	8	9	10
spf:	~~-1~~ 2	~~-1~~ 3	~~-1~~ 2	~~-1~~ 5	~~-1~~ 2	~~-1~~ 7	~~-1~~ 2	~~-1~~ 3	~~-1~~ 2

-1 => No prime Factors

36 = 2x2x3x3

$\frac{36}{2}$ = 18

{2, 2, 3, 3}

SPF

$\frac{18}{2} = 9$

$\frac{9}{3} = 3$

$\frac{3}{3} = \boxed{1}$

T.C. $\boxed{N \log \log N}$