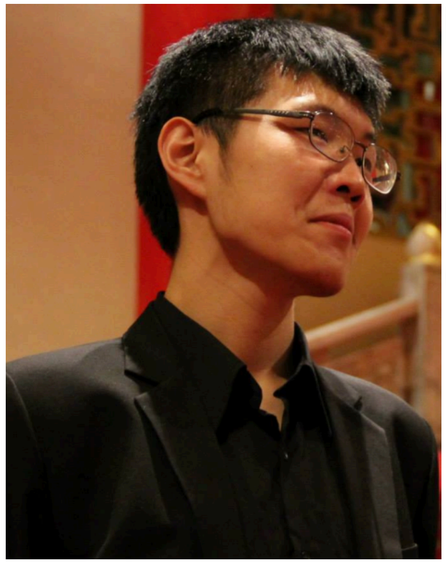
Dimension Preserving Reductions Between SVP and CVP in Different p-Norms

Divesh Aggarwal
CQT and SoC, NUS



Yanlin Chen
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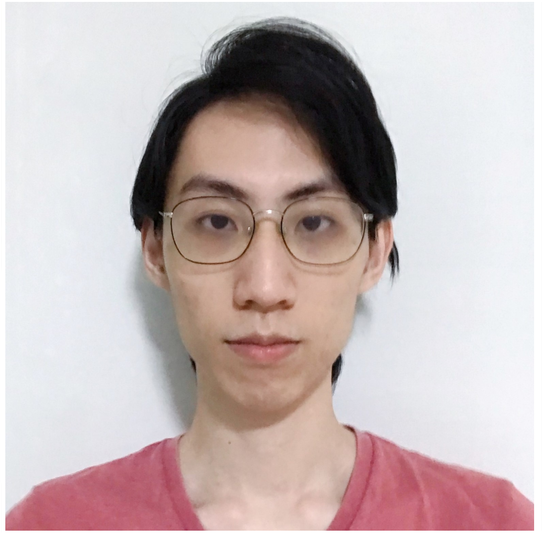
Rajendra Kumar
IITK & SoC, NUS
——> CQT, NUS



Zeyong Li
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Cornell University

SIGTACS, IIT Kanpur

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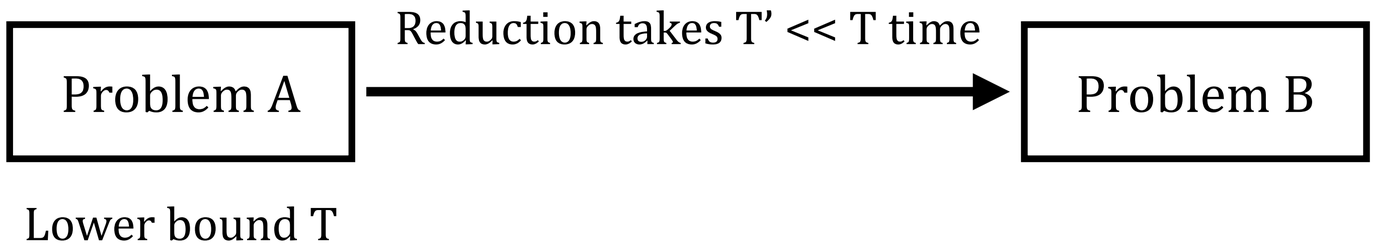
- Most popular candidate for post-quantum cryptography (classical crypto system secure even after quantum computer)

Exponential Time Reductions

- Polynomial Time Reduction are used to prove the NP-Hardness of the problem.

- Exponential Time Reductions:
  - If we believe problem is exponentially hard.
  - Implies the fine-grained hardness.



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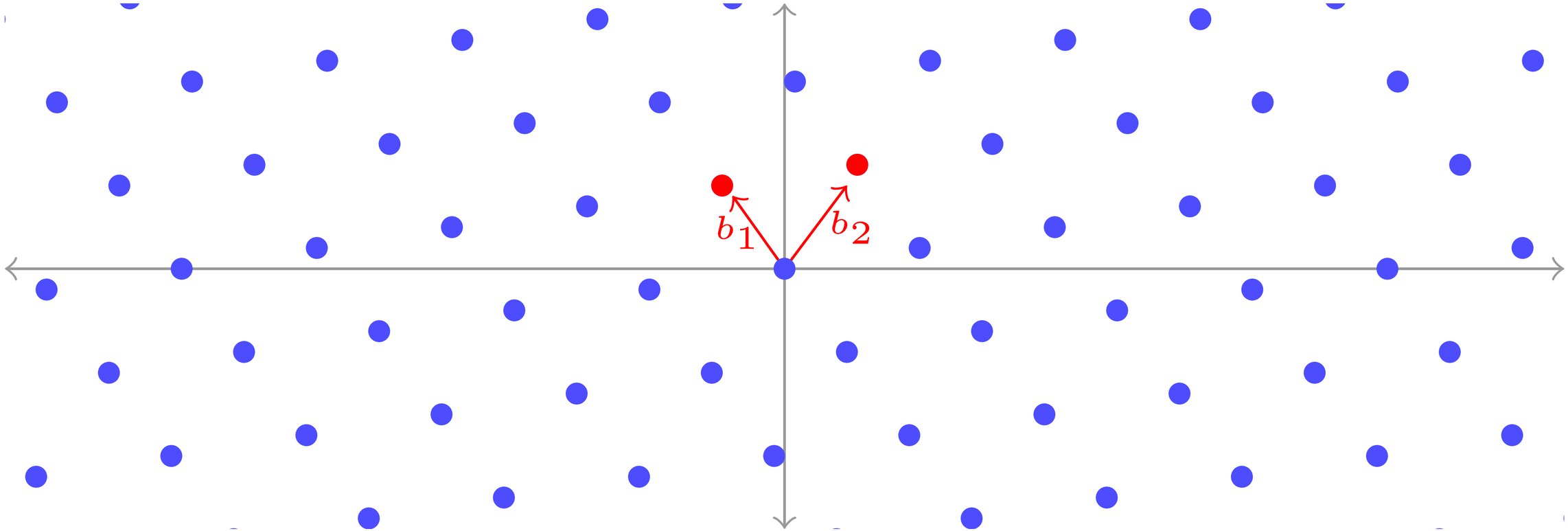

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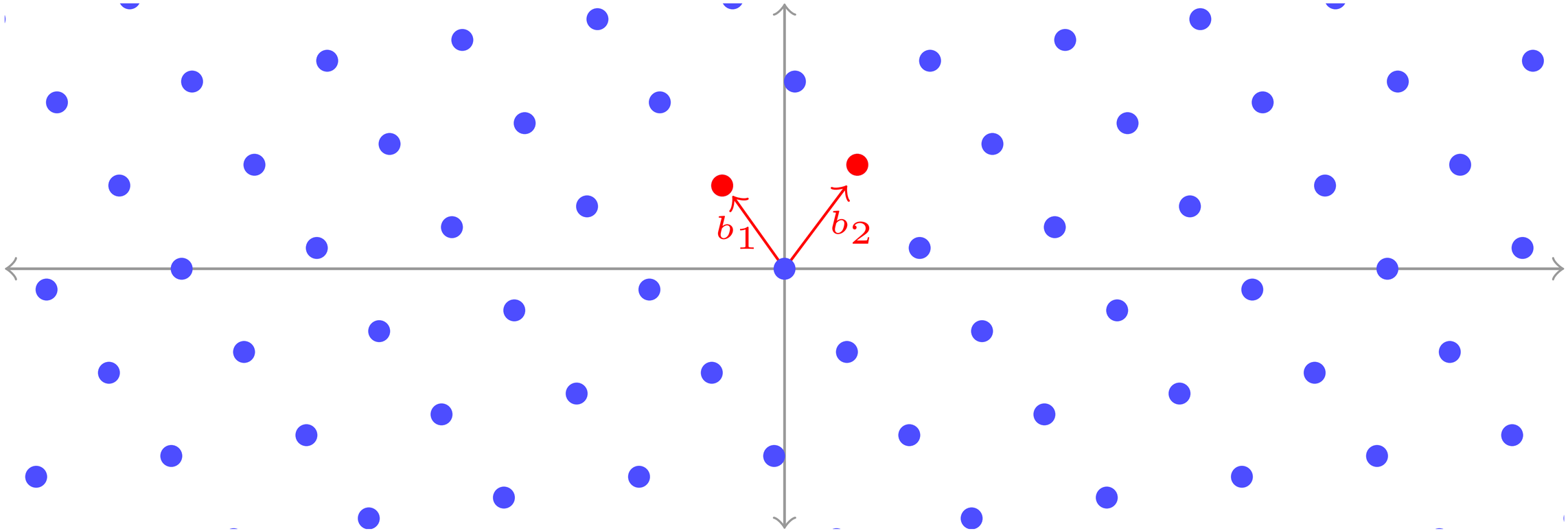

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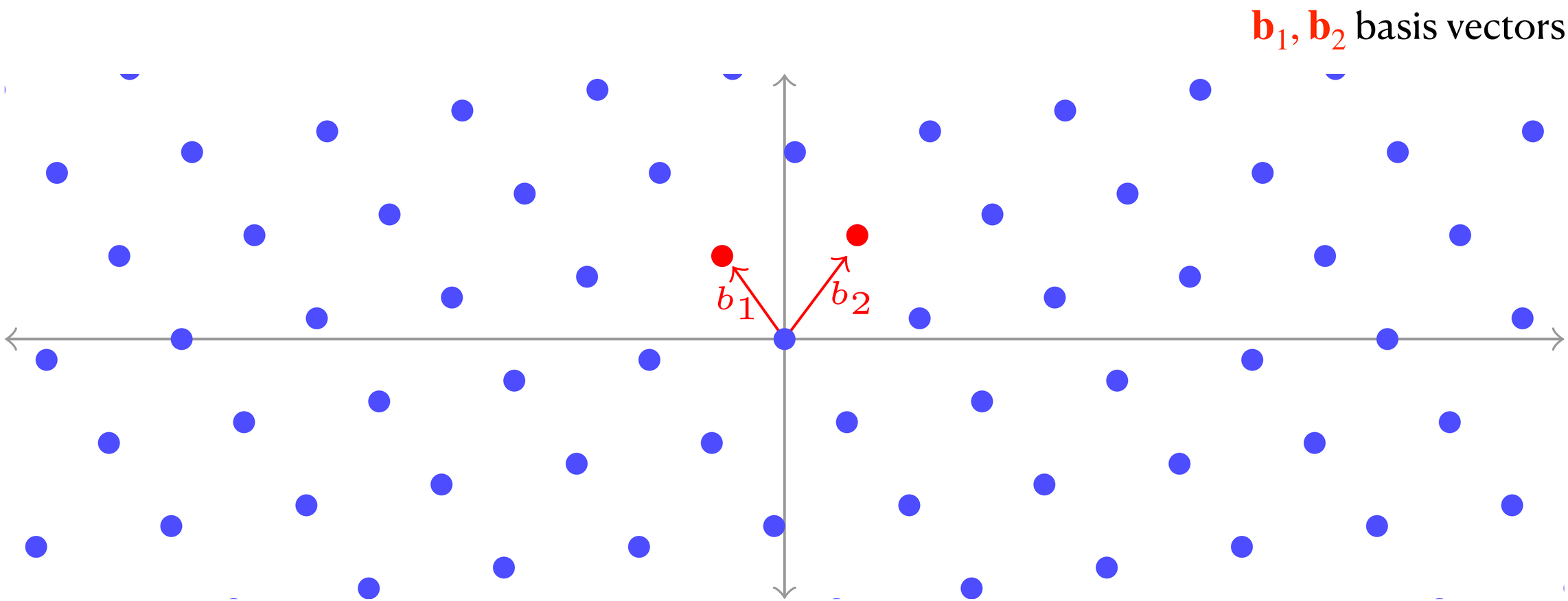

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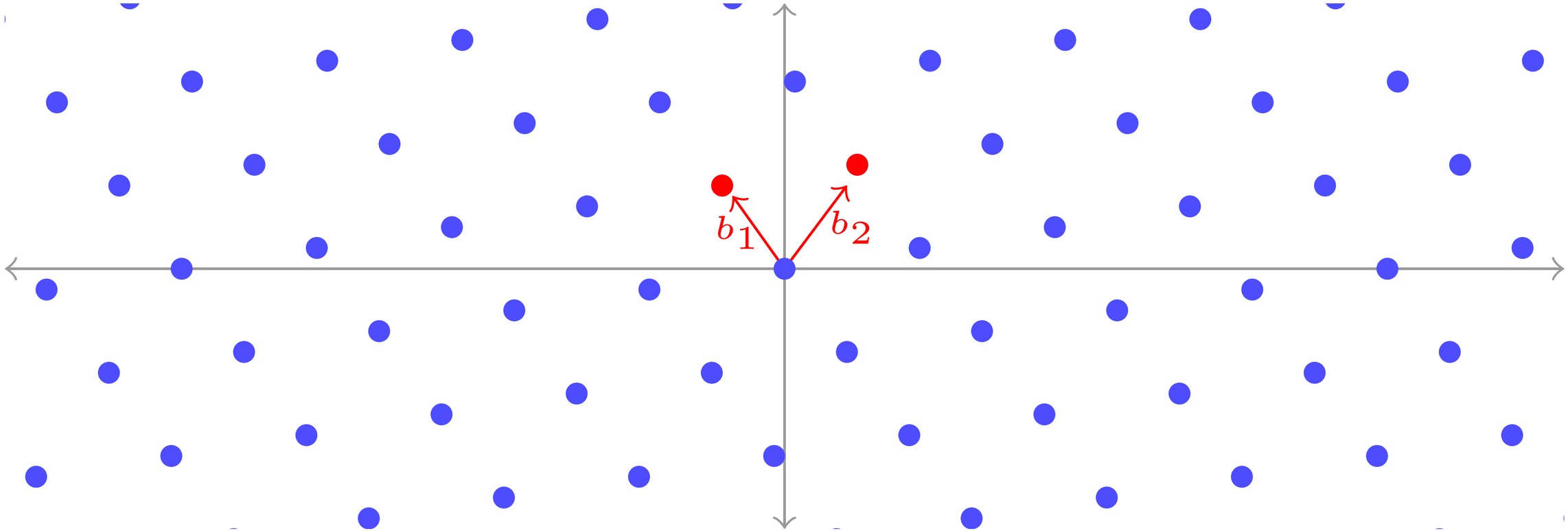
- Specified by a basis $\mathbf{b}_1, \cdots, \mathbf{b}_n$, linearly independent vectors.
- $\mathscr{L} = \{z_1\mathbf{b}_1 + \cdots + z_n\mathbf{b}_n \,|\, z_i \in \mathbb{Z}\}$, rank=$n$, dimension=$m$.

$\mathbf{b}_1, \mathbf{b}_2$ basis vectors



Notations

- ℓ_p norm:

$$p \in [1,\infty),\ \|\mathbf{v}\|_p := \left(\sum_{i=1}^{m} |v_i|^p \right)^{1/p}$$

$$p = \infty,\ \|\mathbf{v}\|_\infty := \max_i |v_i| .$$

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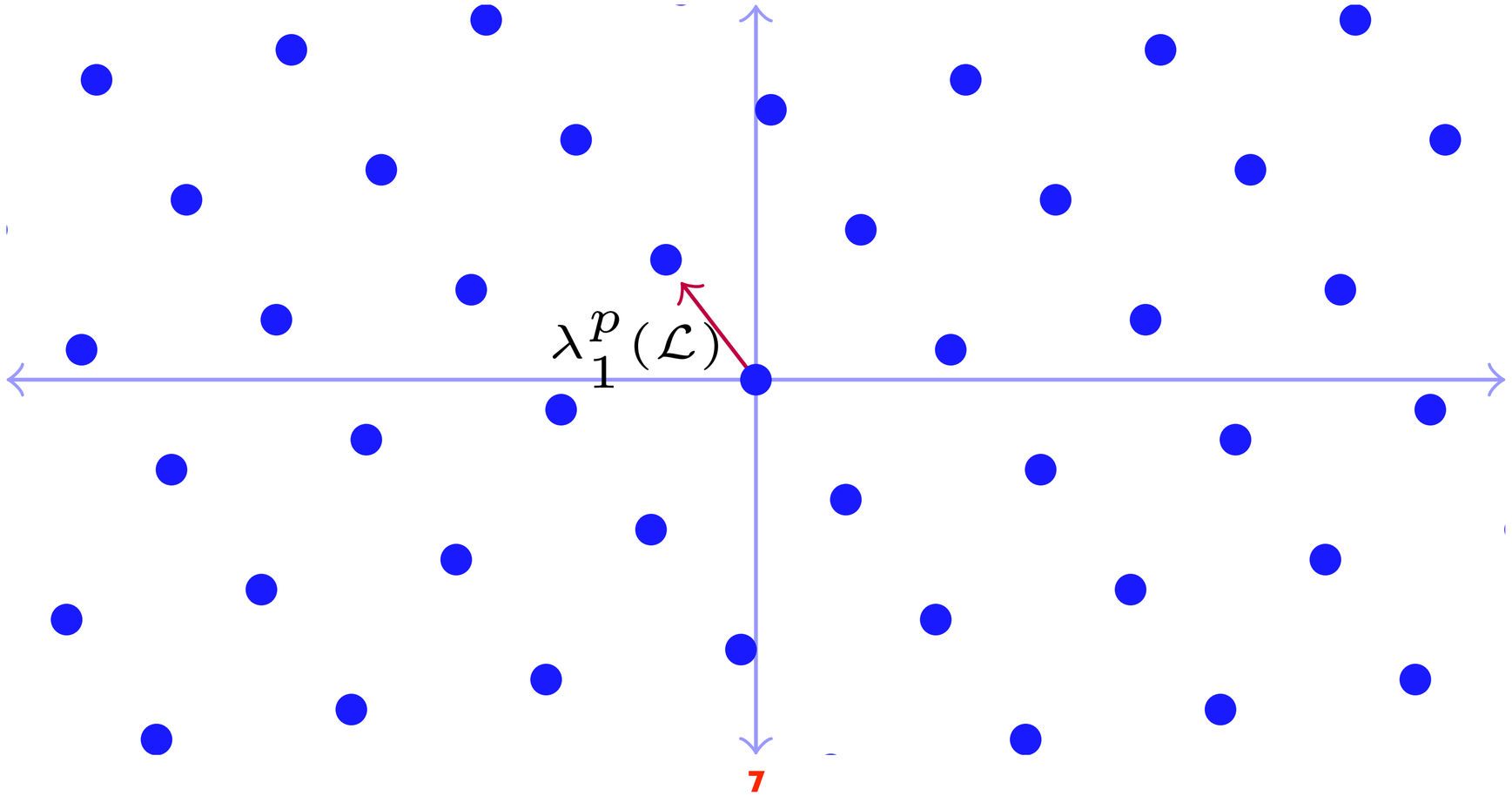

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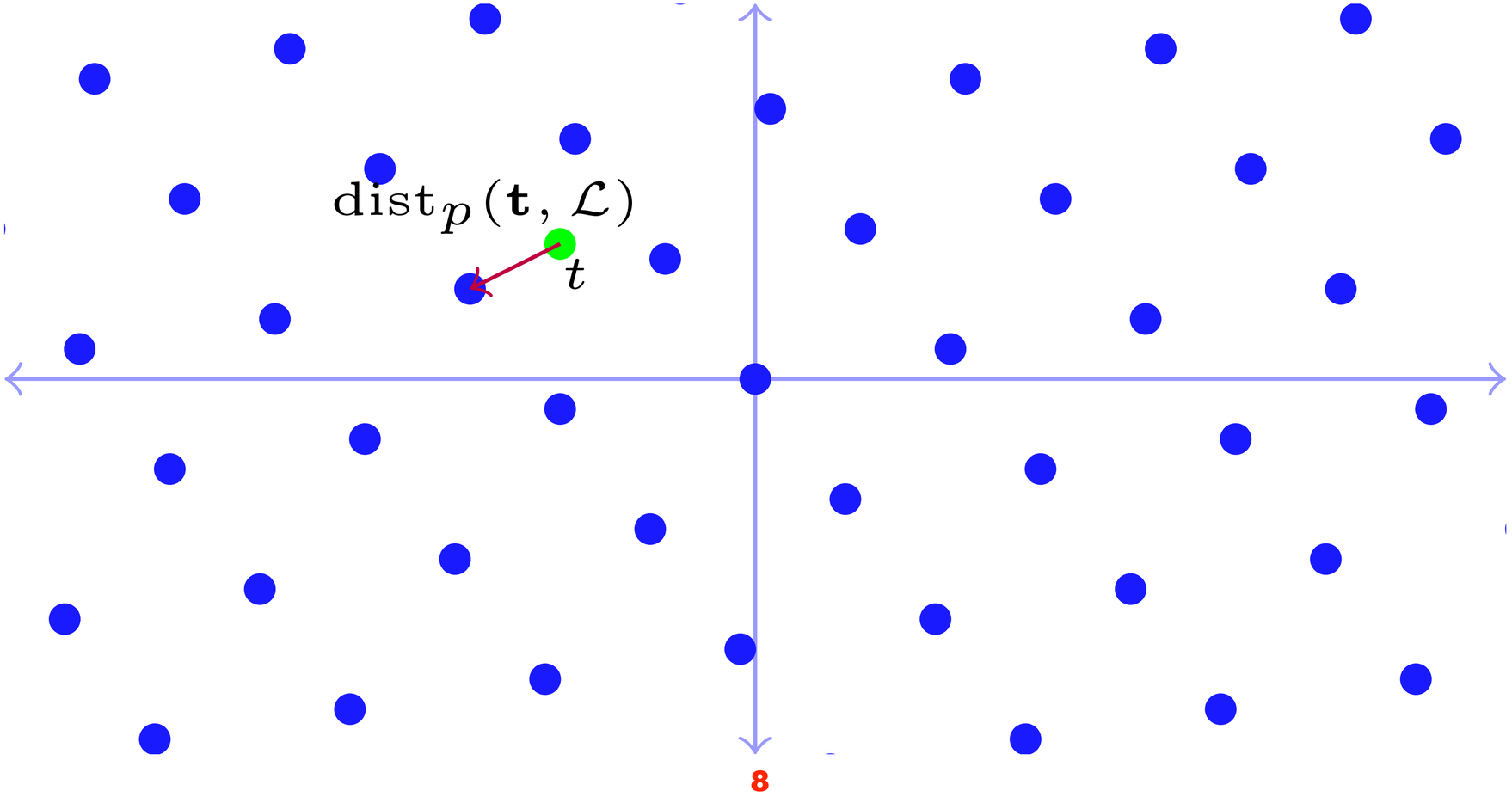

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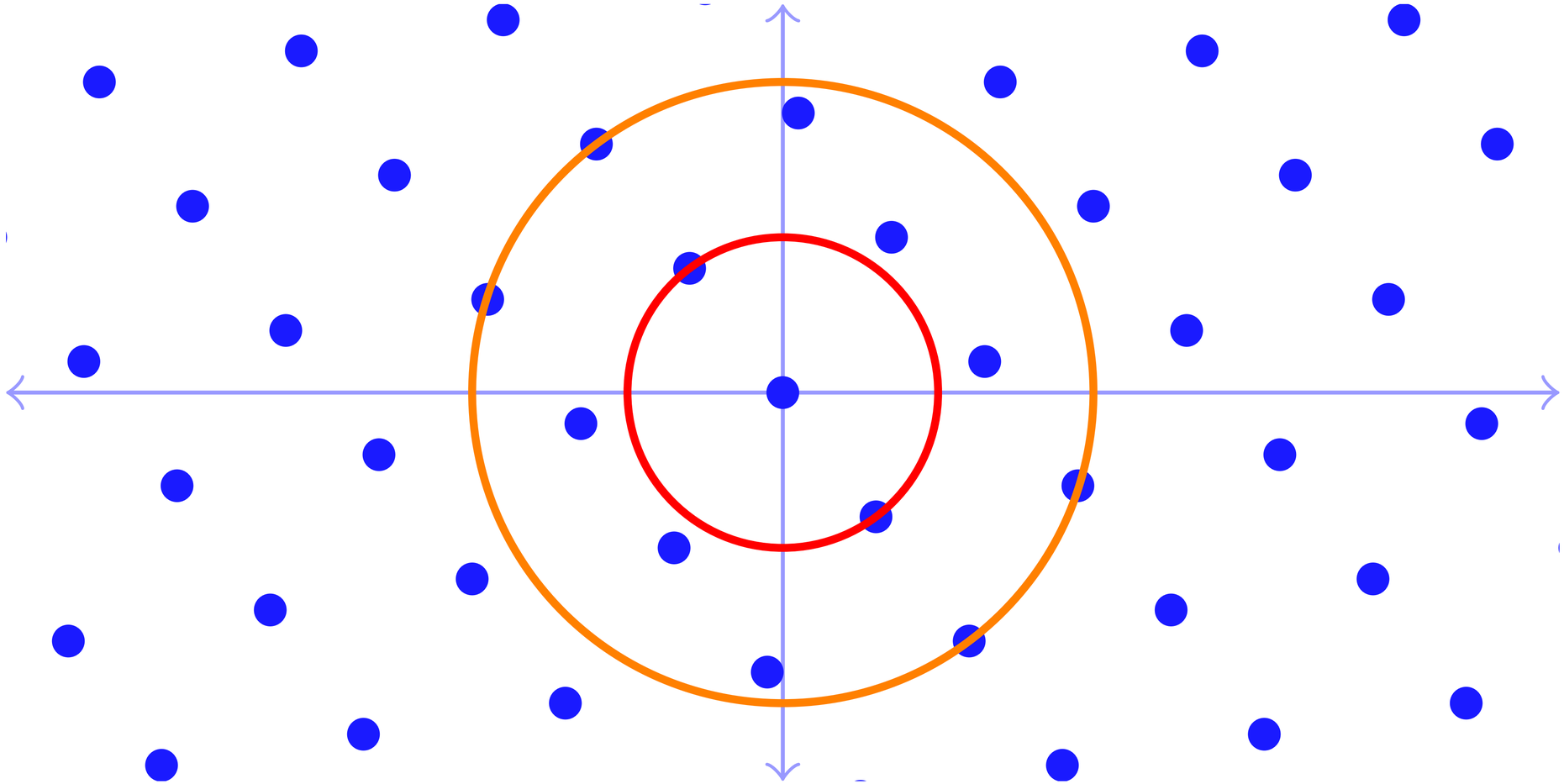
- ℓ_p ball, $r\mathcal{B}_p^m := \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_p \leq r\}$.



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Shortest Vector Problem

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Shortest Vector Problem (γ-SVP$_p$) : Given a basis $\mathbf{B}$ of lattice $\mathscr{L}$, find a lattice vector $\mathbf{v}$ such that,

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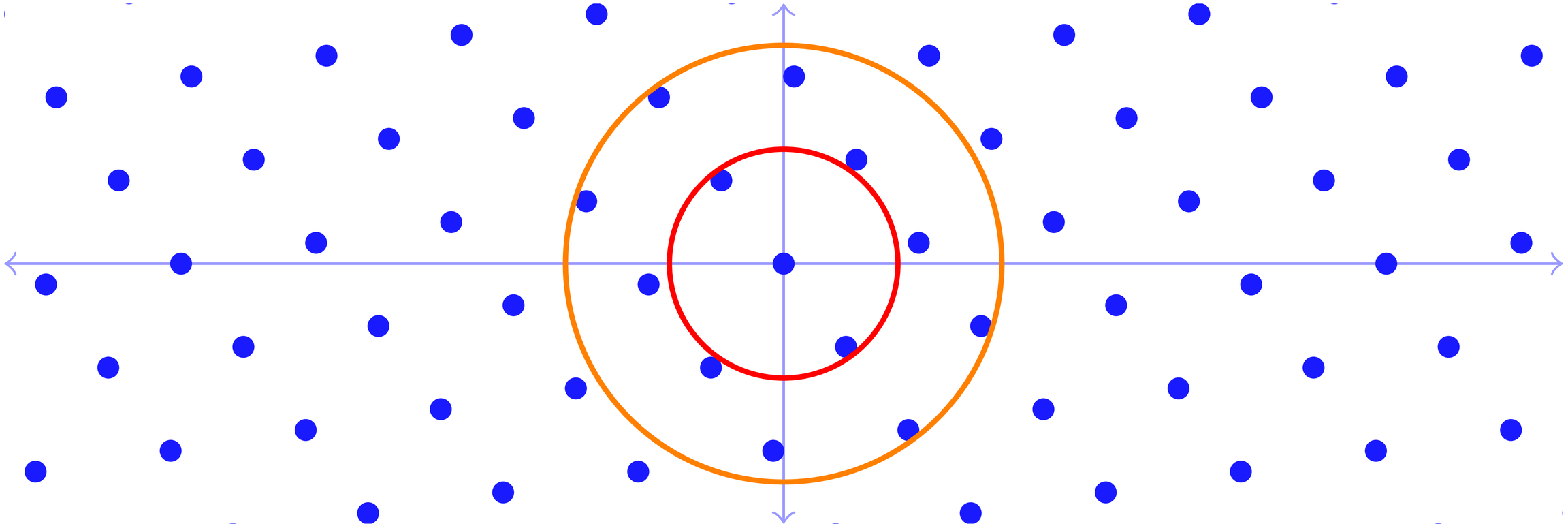

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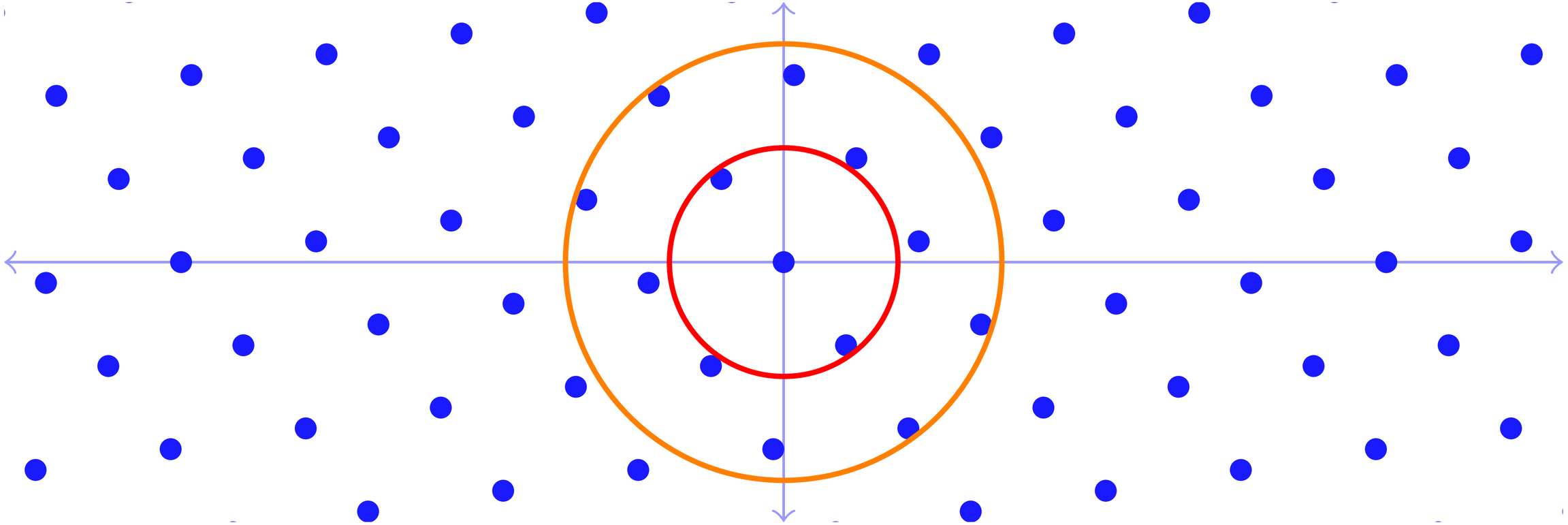

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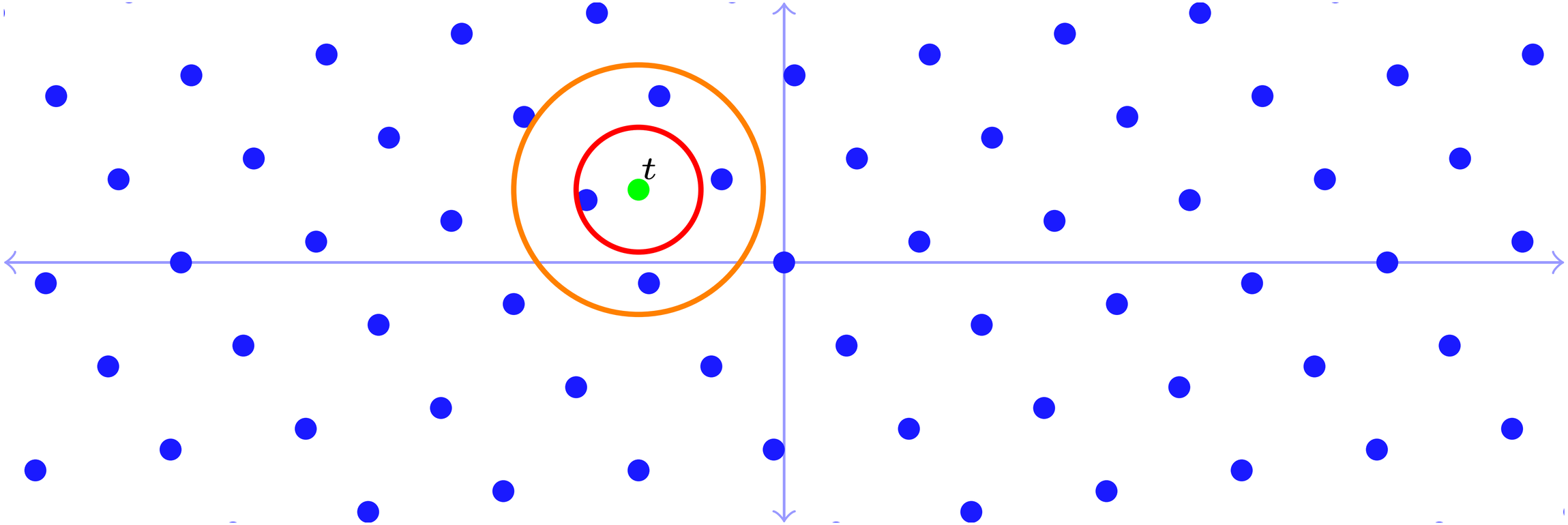

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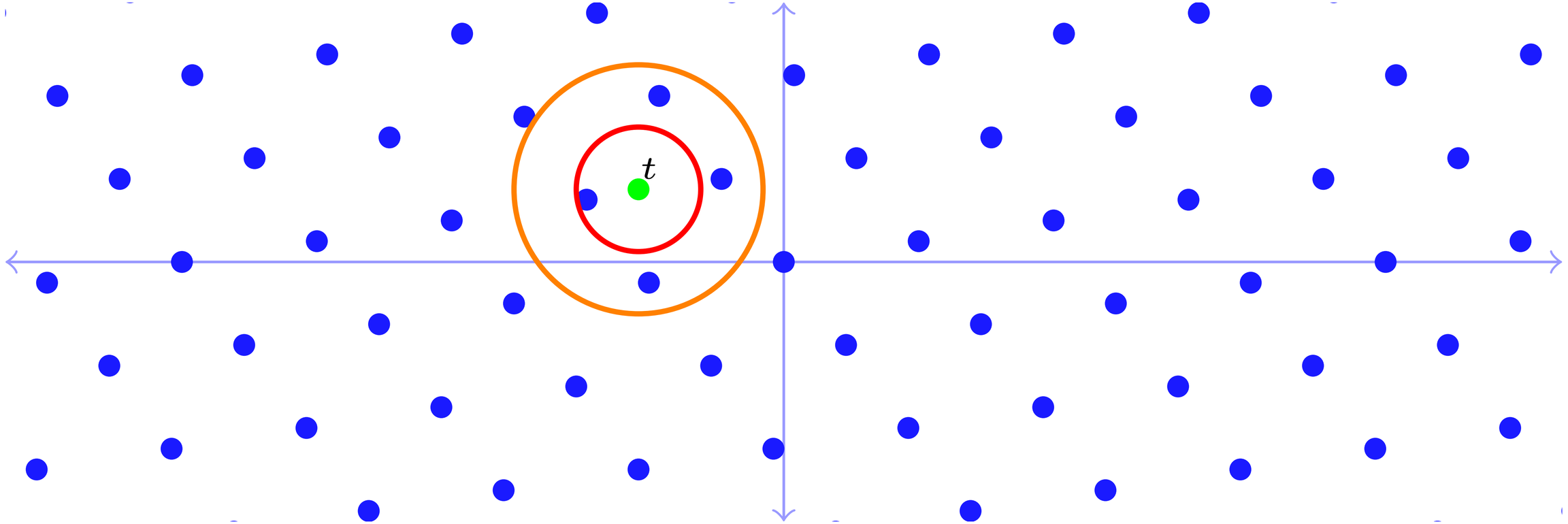

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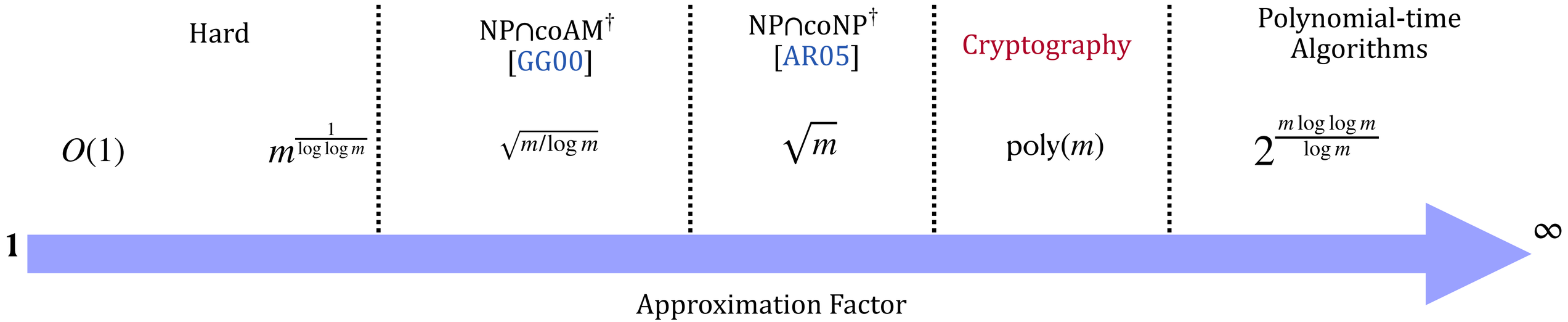
Approximation of SVP and CVP



† Only for ℓ_2 norm. Extend to ℓ_p norm by [RR06]

Reduction between Lattice Problems

Polynomial time reductions

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- When $p > 2$, there is super constant increment in dimension.

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- $2^{(1-\epsilon)n}$ lower bound for small constant approximation of $\mathbf{CVP}_p$ and $\mathbf{SVP}_\infty$ assuming (gap) Strong Exponential time Hypothesis [BGS17; AS18; Aggarwal, Bennett, Golovnev, Stephens-Davidowitz '20].

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- Uses some specific properties of the $O(1)$-$\mathbf{SVP}_2$ algorithm.

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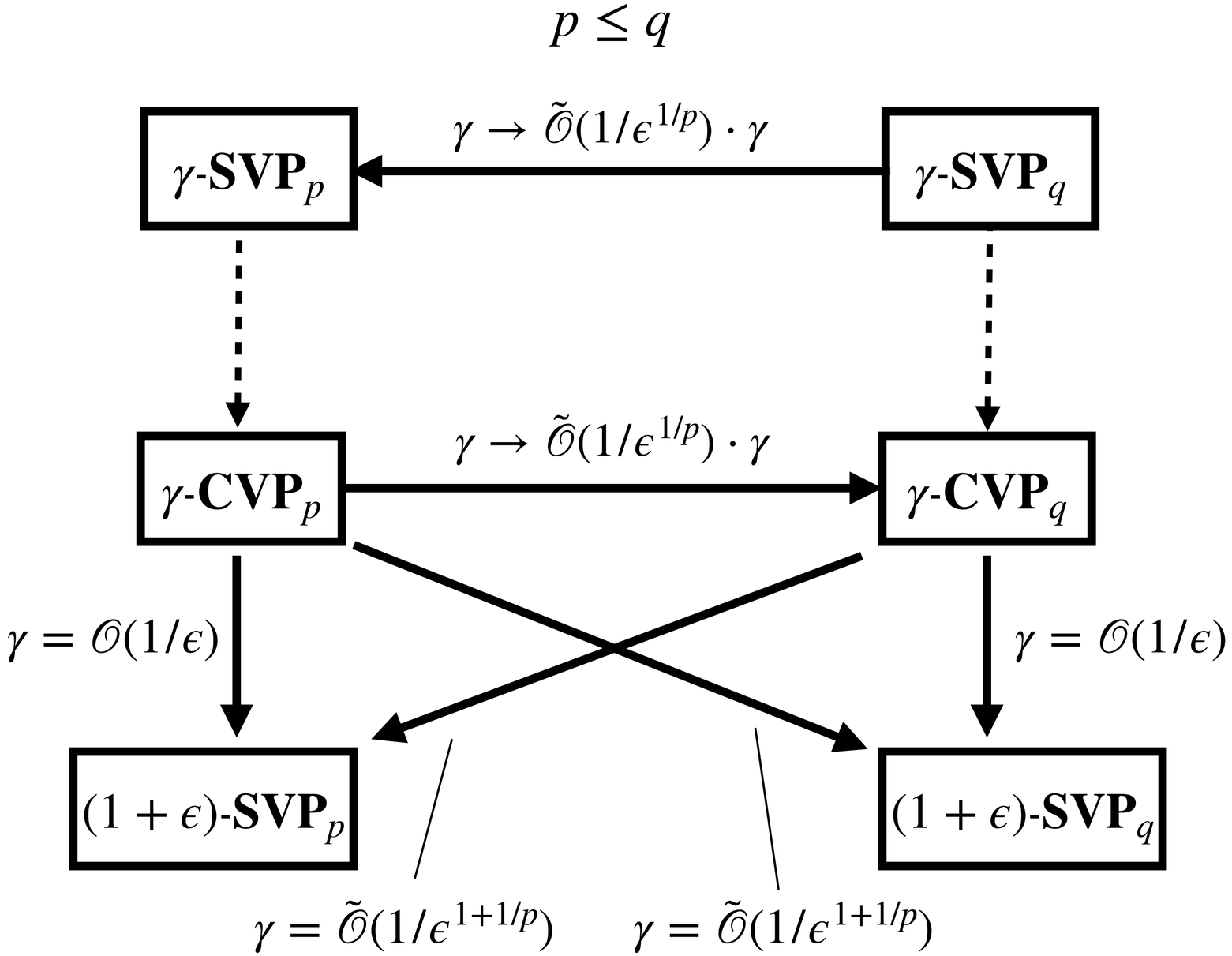
Either preserves the rank and dimension or increases them by atmost one.

Our Result: Exponential time reductions



- Dimension and rank increase by at most one.
- Dotted lines represent polynomial time reduction by Goldreich, Micciancio, Safra, Seifert ’99.
- Solid lines represent $2^{\epsilon m}$ time reductions .

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- Improves on Kannan's celebrated polynomial time reduction from $\sqrt{m}$-$\mathbf{CVP}_2$ to $\mathbf{SVP}_2$.

Reduction from CVP_q to SVP_p

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- Eisenbrand and Venzin observed that it suffices to simultaneously find a lattice vector close to $k\mathbf{t}$ and a lattice vector close to $(k-1)\mathbf{t}$.

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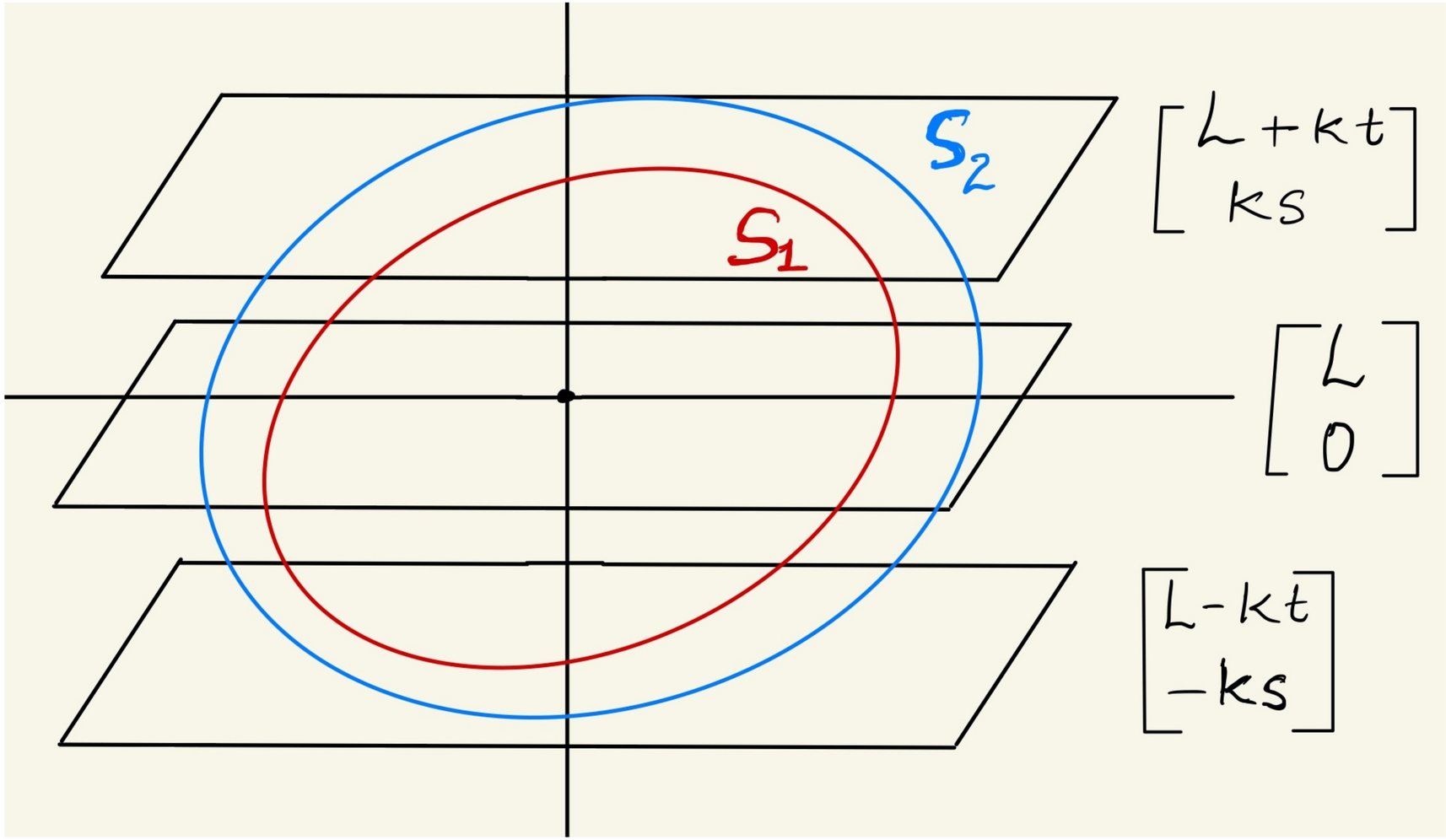
- We use generalisation of this to any ℓ_p norm, to sample a uniformly random lattice vector.

Proof sketch

- $\exists r > 0,\ r' = r - m^{1/p-1/q}\mathrm{dist}_q(\mathbf{t}, \mathscr{L}) - s,\ S_1 = r'\mathscr{B}_p^m,\ S_2 = r\mathscr{B}_p^m$ such that $\frac{|S_2|}{|S_1|} \leq 2^{\epsilon m}$.
- $S \subseteq S_1 \cap (\mathscr{L} + k'\mathbf{t}, k's)$
- $T = S + (-\mathbf{v} + \mathbf{t}, s) \in S_2$
- $2^{2\epsilon m}|S| = 2^{2\epsilon m}|T| \geq |S_2|$



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Thanks!

Questions?