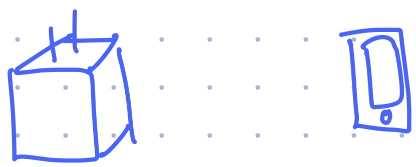
Σ

problem: m goods, n agents; want to "fairly" divide goods



Envy-free: $\forall i,\ v_i(X_i) \geq v_i(X_j)$

↳ 1 iPhone, 2 people $v_i = v_j = \$1000$

EF1: $\forall i,\ v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for some $g \in X_j$

EF1: 1 does not envy 2

2 does not envy 1 after removing good valued at \$2

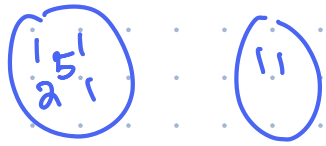
$v(X_j) = \sum_{x \in X_j} v(x)$

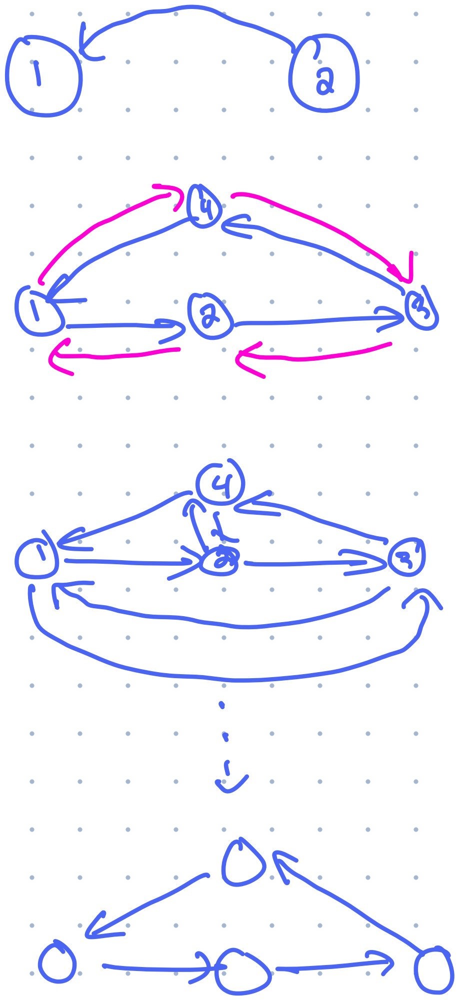
$v(g_1) = \$10 \quad v(g_2) = \$5 \quad v(g_3) = \$1$

$v(\{g_1, g_3\}) = 10 + 1 = 11$

Envy Graph

add a vertex for each agent
$i \Rightarrow j$ if i envies j





By rotating bundles, all edges disappear

After a rotation, amount of envy only decreases

Since ≥ 1 edge involved in cycle deleted, after enough rotations graph is a DAG

Obs: sinks envy noone
source not envied by anybody

Claim: giving good to src cause at most EF1 only from other agents

Proof: can remove added good and envy goes away $\Rightarrow$ EF1 def

while $\exists$ unalloc. goods
give good to src
eliminate cycles

10^{10} 5 5

Envy free up to any item: $\forall i,\ v_i(X_i) \geq v_i(X_j \setminus \{g\})\ \forall g \in X_j$

EFX ↑ agent i must no longer envy j after removing their least fav. good from j's bundle

identically ordered valuations

$v_{i1} \leq v_{i2} \leq \cdots \leq v_{im} \quad \forall i$

	v_1	v_2
g_1	ε	1
g_2	4	2
g_3	4	$7-\varepsilon$

$g_{1.5}$ $v=3$

$g_1\ g_2\ g_{1.5}$ g_3

$v_i(X_i) = \varepsilon + 3 + 4$

$v_2(X_1) = \varepsilon + 3 + 4$

$v_2(X_2) = 7-\varepsilon$

$v_2(X_1 \setminus g) = 7 > 7-\varepsilon$

For goods $n, \ldots, 1$
assign g_i to src
rotate bundles

Consider agents i and j.
i src.

$v_j(X_j) \geq v_i(X_i)$

$\Rightarrow v_j(X_j) \geq v_i(X_i \cup \{g\} \setminus \{g\})$

EFX!

existence: identical valuations

$\lceil \frac{m}{n} \rceil$ rounds $m = kn$

1. topologically sort G_E
2. allow each agent to pick fav. good
3. eliminate cycles

d_i: max difference between valuations of consecutive items for a given agent i

σ_i: min

$d = \max(d)$

$\sigma = \min(\sigma)$

$\rho = \frac{\sigma}{d}$

after round, $v_j(X_i) - v_j(X_j) \leq dn$

$\left(1 - \frac{4n^2}{\rho m^2}\right)$ - EFX $\quad$ 0.618