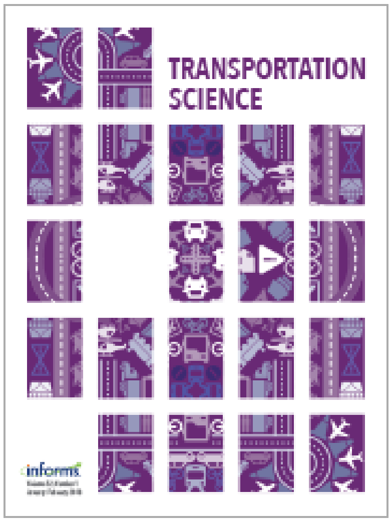

Transportation Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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TRANSPORTATION SCIENCE
Vol. 41, No. 3, August 2007, pp. 382–391
ISSN 0041-1655 | EISSN 1526-5447 | 07 | 4103 | 0382

DOI 10.1287/trsc.1060.0188

A Branch-and-Cut Algorithm for a Vendor-Managed Inventory-Routing Problem

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We consider a distribution problem in which a product has to be shipped from a supplier to several retailers over a given time horizon. Each retailer defines a maximum inventory level. The supplier monitors the inventory of each retailer and determines its replenishment policy, guaranteeing that no stockout occurs at the retailer (vendor-managed inventory policy). Every time a retailer is visited, the quantity delivered by the supplier is such that the maximum inventory level is reached (deterministic order-up-to level policy). Shipments from the supplier to the retailers are performed by a vehicle of given capacity. The problem is to determine for each discrete time instant the quantity to ship to each retailer and the vehicle route. We present a mixed-integer linear programming model and derive new additional valid inequalities used to strengthen the linear relaxation of the model. We implement a branch-and-cut algorithm to solve the model optimally. We then compare the optimal solution of the problem with the optimal solution of two problems obtained by relaxing in different ways the deterministic order-up-to level policy. Computational results are presented on a set of randomly generated problem instances.

Key words: supply chain management; vendor-managed inventory (VMI); deterministic order-up-to level; inventory-routing problem; branch and cut
History: Received: October 2005; revision received: June 2006; accepted: November 2006.

Introduction

Companies, especially large ones, have recognized that globally optimizing supply chains can yield substantial cost reductions. Information and communication technology has enabled a successful coordination of the supply chain. The scientific literature has contributed to this process with models and algorithms that optimize larger parts of the supply chain with respect to the traditional models.

The coordination of transportation management and inventory control is a problem faced by many companies. Typical examples arise in internal distribution systems, in which the supplier and the retailers represent two different echelons of the same company; and in external distribution systems, in which the supplier replenishes the retailers and ensures a given service level. The joint optimization of transportation and inventory costs has been the subject of a large number of papers. Inventory-routing models have been proposed and analyzed, for example, in Federgruen and Zipkin (1984), Dror and Ball (1987), Anily and Federgruen (1990), Bertazzi, Speranza, and Ukovich (1997), Bramel and Simchi-Levi (1997), Campbell et al. (1998), and Chan and Simchi-Levi (1998). For a more detailed discussion of these contributions, we refer to Bertazzi, Paletta, and Speranza (2002). Recent papers on inventory-routing problems are Adelman (2004), Campbell and Savelsbergh (2004), and Savelsbergh and Song (2006).

Recently, new forms of relationships in the supply chain have been adopted. One of these is the so-called vendor-managed inventory (VMI) system, in which the supplier monitors the inventory and decides the replenishment policy of each retailer. The supplier acts as a central decision maker who solves an integrated inventory-routing problem. The advantage of a VMI policy with respect to the traditional retailer managed inventory (RMI) policies lies in a more efficient resource utilization. On the one hand, the supplier can reduce its inventories while maintaining the same level of service, or can increase the level of service and reduce the transportation cost through a more uniform utilization of the transportation capacity. On the other hand, the retailers can devote fewer resources to monitoring their inventories and to placing orders, and have the guarantee that no stockout will occur. We refer to Axsäter (2001), Çetinkaya and Lee (2000), Cheung and Lee (2002), Fry, Kapuscinski, and Olsen (2001), Kleywegt, Nori, and Savelsbergh (2002, 2004), and Rabah and Mahamassani (2002) for applications of the VMI policy to systems with stochastic demand.

Bertazzi, Paletta, and Speranza (2002) have introduced a particular VMI policy, called deterministic order-up-to level policy, for an inventory-routing problem. In this problem each retailer defines a minimum and a maximum inventory level and can be visited several times during the planning horizon. The supplier monitors the inventory of each retailer and guarantees that no stockout will occur. Every time a retailer is visited, the quantity delivered is such that the maximum inventory level is reached. This policy is inspired, in a deterministic setting, by the classical order-up-to level policy, widely studied in stochastic inventory theory. We refer to Axsäter and Rosling (1994) for an overview of inventory policies in multilevel systems, to Bertazzi, Paletta, and Speranza (2002) for the first application of the deterministic order-up-to level policy to an inventory-routing problem, and to Bertazzi, Paletta, and Speranza (2005) for an application of this policy to an integrated production-distribution system. In the latter two papers, the problems were solved heuristically.

Our literature review indicates that inventory-routing is gaining in popularity, both from a practical standpoint and as a research area. However, this class of problems is rather difficult to solve, and the authors are aware of no exact algorithm capable of solving any type of inventory-routing problem of reasonable size. As a first step, we have decided to concentrate on a simplified version of the problem involving a single vehicle, although we are perfectly aware that in practice, more often than not, many vehicles are used. We believe our contribution will serve as a basis for the understanding and resolution of more realistic cases. We propose an exact branch-and-cut algorithm to solve the deterministic order-up-to level inventory-routing problem introduced in Bertazzi, Paletta, and Speranza (2002). In this problem a product is shipped from a common supplier to several retailers, where it is depleted in a deterministic and time-varying way over a given time horizon. Each retailer can be visited several times over the planning horizon. The product is shipped from the supplier to the retailers by a capacitated vehicle. Shipments can only be performed at the discrete time instants within the planning horizon, where a time instant is, for example, a day. The problem is to determine a shipping policy minimizing the sum of transportation and inventory costs both at the supplier and at the retailers in such a way that no stockout occurs. This problem is *NP*-hard, because it reduces to the traveling salesman problem if the time horizon is one time instant, inventory costs are zero, vehicle capacity is infinite, and all retailers need to be served. We introduce a mixed-integer linear programming model for the problem, derive new additional valid inequalities to strengthen the linear relaxation of the model, and present an exact branch-and-cut algorithm. Computational results are presented on a set of randomly generated problem instances to evaluate the performance of the algorithm.

We then compare the optimal solution of this problem, which we call the vendor-managed inventory routing with order-up (VMIR-OU) problem, with the optimal solution of two problems obtained by relaxing in different ways the deterministic order-up-to level policy. The first problem is obtained by relaxing the constraint that every time a retailer is visited, the quantity delivered is such that the maximum level of the inventory is reached. The only constraint on the shipping quantity is that it must be not greater than the maximum inventory level. We call this problem the vendor-managed inventory-routing with maximum level (VMIR-ML) problem. The second problem is obtained by completely relaxing the order-up-to level policy—i.e., by allowing the shipping quantity to be any positive value—and is called the vendor-managed inventory-routing (VMIR) problem. The models for these two latter problems are variants of the mixed-integer linear model derived for the basic problem. Computational results are presented to evaluate the impact of the three different policies on the problem solution.

The paper is organized as follows. Section 1 presents the problem and the model. Valid inequalities are stated and proved in §2. Section 3 is devoted to the branch-and-cut algorithm, while the computational results are presented and analyzed in §4.

1. Problem Description and Formulation

We consider a logistic network in which a product is shipped from a common supplier 0 to a set $\mathcal{M} = \{1, 2, \ldots, n\}$ of retailers over a time horizon H. At each discrete time $t \in \mathcal{T} = \{1, \ldots, H\}$ a product quantity r_{0t} is produced or made available at the supplier and a quantity r_{st} is consumed at retailer $s \in \mathcal{M}$. A starting inventory level, B_0, at the supplier is given. Each retailer s defines a maximum inventory level U_s and has a given starting inventory level $I_{s0} \leq U_s$. If retailer s is visited at time t, then the product quantity x_{st} shipped to the retailer is such that the inventory level of s reaches its maximum value U_s (i.e., a deterministic order-up-to level policy is applied). More precisely, if we denote by I_{st} the inventory level of retailer s at time t, then x_{st} is either equal to $U_s - I_{st}$ if a shipment to s is performed at time t, or equal to zero otherwise. The inventory cost is charged both at the supplier and at the retailers. Denoting by h_0 the unit inventory cost at the supplier, and by B_t the inventory level at the supplier at time t, the total inventory cost at the supplier is then $\sum_{t\in\mathcal{T}'} h_0 B_t$, where $\mathcal{T}' = \mathcal{T} \cup \{H+1\}$. The time $H+1$ is included in the computation of the inventory cost in order to take into account the consequences of the operations performed at time H. Denoting by h_s the unit inventory cost of retailer $s \in \mathcal{M}$, the total inventory cost over the time horizon is then $\sum_{t\in\mathcal{T}'} h_s I_{st}$. The inventory level at the end of the time horizon can be different from the starting level; therefore, the problem is not periodic. Shipments from the supplier to the retailers can be performed at each time $t \in \mathcal{T}$ by a vehicle of capacity C capable of serving several retailers on the same route. Each vehicle route visits all retailers that are served at the same time. The transportation cost c_{ij} from vertex i to vertex j is known, and $c_{ij} = c_{ji}$, $i, j \in \mathcal{M}' = \mathcal{M} \cup \{0\}$. Therefore, letting y_{ij}^t be a binary variable equal to one if j immediately follows i in the route traveled at time t, and zero otherwise, the total transportation cost is then $\sum_{i\in\mathcal{M}'}\sum_{j\in\mathcal{M}',\, j<i}\sum_{t\in\mathcal{T}} c_{ij} y_{ij}^t$. In the VMIR-OU problem we want to determine for each time $t \in \mathcal{T}$ the product quantity x_{st} to ship to each retailer $s \in \mathcal{M}$, and a route visiting all the retailers served at time t (i.e., the values of the variables y_{ij}^t) so that the objective function

$$\sum_{t\in\mathcal{T}'} h_0 B_t + \sum_{s\in\mathcal{M}}\sum_{t\in\mathcal{T}'} h_s I_{st} + \sum_{i\in\mathcal{M}'}\sum_{j\in\mathcal{M}',\, j<i}\sum_{t\in\mathcal{T}} c_{ij} y_{ij}^t \tag{1}$$

is minimized, subject to the following constraints:

1. *Inventory definition at the supplier*: The inventory level of the supplier at time t is given by the level at time $t-1$, plus the product quantity r_{0t-1} made available at time $t-1$, minus the total quantity shipped to the retailers at time $t-1$, that is,

$$B_t = B_{t-1} + r_{0t-1} - \sum_{s\in\mathcal{M}} x_{st-1} \quad t \in \mathcal{T}', \tag{2}$$

where $r_{00} = 0$ and $x_{s0} = 0$, $s \in \mathcal{M}$.

2. *Stockout constraints at the supplier*: These constraints guarantee that for each delivery time $t \in \mathcal{T}$ the inventory level at the supplier is sufficient to ship the total quantity delivered to the retailers at time t:

$$B_t \ge \sum_{s\in\mathcal{M}} x_{st} \quad t \in \mathcal{T}. \tag{3}$$

3. *Inventory definition at the retailers*: The inventory level at time t is given by the level at time $t-1$, plus the product quantity x_{st-1} shipped from the supplier to retailer s at time $t-1$, minus the product quantity r_{st-1} consumed at time $t-1$—that is,

$$I_{st} = I_{st-1} + x_{st-1} - r_{st-1} \quad s \in \mathcal{M}\ t \in \mathcal{T}', \tag{4}$$

where $x_{s0} = r_{s0} = 0$, $s \in \mathcal{M}$.

4. *Stockout constraints at the retailers*: These constraints guarantee that for each retailer $s \in \mathcal{M}$ the inventory level I_{st} at each time $t \in \mathcal{T}'$ is nonnegative:

$$I_{st} \ge 0 \quad s \in \mathcal{M}\ t \in \mathcal{T}'. \tag{5}$$

5. *Order-up-to level constraints*: These constraints guarantee that the quantity x_{st} shipped to each retailer s at each time $t \in \mathcal{T}$ is either $U_s - I_{st}$ if s is served at time t, and zero otherwise. Let z_{st} be a binary variable equal to one if the retailer s is served at time t, and zero otherwise. Then,

$$x_{st} \ge U_s z_{st} - I_{st} \quad s \in \mathcal{M}\ t \in \mathcal{T}; \tag{6}$$

$$x_{st} \le U_s - I_{st} \quad s \in \mathcal{M}\ t \in \mathcal{T}; \tag{7}$$

$$x_{st} \le U_s z_{st} \quad s \in \mathcal{M}\ t \in \mathcal{T}. \tag{8}$$

6. *Capacity constraints*: These constraints guarantee that the total quantity of the product loaded on the vehicle at time $t \in \mathcal{T}$ does not exceed the transportation capacity:

$$\sum_{s\in\mathcal{M}} x_{st} \le C \quad t \in \mathcal{T}. \tag{9}$$

7. *Routing constraints*: These constraints guarantee that for each time $t \in \mathcal{T}$, a feasible route is determined to visit all retailers served at time t. They can be formulated as follows:

(a) If at least one retailer $s \in \mathcal{M}$ is visited at time t, then the route traveled at time t has to "visit" the supplier. Let z_{0t} be a binary variable equal to one if the supplier is visited at time t and zero otherwise; then,

$$\sum_{s\in\mathcal{M}} x_{st} \le C z_{0t} \quad t \in \mathcal{T}. \tag{10}$$

(b) If deliveries are made at time t (i.e., z_{it} is equal to one for some $i \in \mathcal{M}'$), then the route traveled at time t has to contain one arc entering every vertex i of the route and one arc leaving every i:

$$\sum_{j\in\mathcal{M}',\, j<i} y_{ij}^t + \sum_{j\in\mathcal{M}',\, j>i} y_{ji}^t = 2 z_{it} \quad i \in \mathcal{M}'\ t \in \mathcal{T}. \tag{11}$$

(c) Subtours elimination constraints (e.g., Fischetti, Salazar-González, and Toth 1998; Gendreau, Laporte, and Semet 1998):

$$\sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S},\, j<i} y_{ij}^t \le \sum_{i\in\mathcal{S}} z_{it} - z_{kt} \quad \mathcal{S} \subseteq \mathcal{M}\ t \in \mathcal{T} \tag{12}$$

for some $k \in \mathcal{S}$.

8. *Nonnegativity and integrality constraints*:

$$x_{st} \ge 0 \quad s \in \mathcal{M}\ t \in \mathcal{T}. \tag{13}$$

$$y_{ij}^t \in \{0, 1\} \quad i \in \mathcal{M}\ j \in \mathcal{M},\ j < i\ t \in \mathcal{T}. \tag{14}$$

$$y_{i0}^t \in \{0, 1, 2\} \quad i \in \mathcal{M}\ t \in \mathcal{T}. \tag{15}$$

$$z_{it} \in \{0, 1\} \quad i \in \mathcal{M}'\ t \in \mathcal{T}. \tag{16}$$

The VMIR-ML problem is modeled without the order-up-to level constraints (6) and (8) and maintains the constraint (7), which imposes that the shipping quantity must be not greater than the maximum inventory level. The VMIR problem is modeled without any constraint on the shipping quantity, i.e., without constraints (6)–(8).

Note that the problem in which several products have to be shipped from the supplier to the retailers can easily be handled by these models. It suffices to replace each retailer s with a set of retailers, one for each product used by the retailer s, to set to zero the transportation cost between vertices of the same set, and to set the transportation cost between vertices of different retailer sets equal to the transportation cost between the corresponding retailers.

2. Valid Inequalities

In this section we derive a set of new valid inequalities for the VMIR-OU, VMIR-ML, and the VMIR problems. Inequalities known in the literature for the traveling salesman problem and for vehicle-routing problems are not valid, mainly because it is not known which retailers will be visited at a given time. The inequalities we present exploit the special structure of the VMIR-OU, VMIR-ML, and VMIR problems and will be used in the branch-and-cut algorithm described in the next section. Their effectiveness will be discussed later in §4, the section devoted to the computational results.

Theorem 1. *The inequalities*

$$I_{st} \geq (1 - z_{st}) r_{st} \quad s \in \mathcal{M} \ t \in \mathcal{T} \tag{17}$$

are valid for the VMIR-OU, VMIR-ML, and VMIR problems.

Proof. Constraints (17) imply that if the retailer s is not served at time t—i.e., $z_{st} = 0$—then the inventory level I_{st} at the retailer s at time t is at least equal to the product quantity r_{st} consumed by s at time t, while $I_{st} \geq 0$ otherwise. □

Theorem 2. *The inequalities*

$$I_{st-k} \geq \left(\sum_{j=0}^{k} r_{st-j}\right)\left(1 - \sum_{j=0}^{k} z_{st-j}\right) \quad s \in \mathcal{M} \ t \in \mathcal{T} \ k = 0, 1, \ldots, t-1 \tag{18}$$

are valid for the VMIR-OU, VMIR-ML, and VMIR problems.

Proof. These inequalities extend the inequalities (17) to the case where, given k, s is not served in the times $t-k, t-k+1, \ldots, t$. Therefore, if $\sum_{j=0}^{k} z_{st-j} = 0$, then $I_{st-k} \geq \sum_{j=0}^{k} r_{st-j}$. Otherwise, $I_{st-k} \geq 0$. □

Theorem 3. *The inequalities*

$$I_{st} \geq U_s z_{st-k} - \sum_{j=t-k}^{t-1} r_{sj} \quad s \in \mathcal{M} \ t \in \mathcal{T} \ k = 1, 2, \ldots, t-1 \tag{19}$$

are valid for the VMIR-OU problem.

Proof. Given k, if $t-k$ is the last time retailer s was visited before time t, i.e., $z_{st-k} = 1$ and $z_{sp} = 0$ for $t-k+1 \leq p < t$, then $I_{st} = U_s - \sum_{j=t-k}^{t-1} r_{sj}$, because of (4). Otherwise, if s has been served at $t-k$ and also before t, then $I_{st} > U_s - \sum_{j=t-k}^{t-1} r_{sj}$. □

Theorem 4. *The inequalities*

$$\sum_{j=1}^{t} z_{sj} \geq \left\lceil \frac{\sum_{j=1}^{t-1} r_{sj} - I_{s0}}{U_s} \right\rceil \quad s \in \mathcal{M} \ t \in \mathcal{T} \tag{20}$$

are valid for the VMIR-OU and VMIR-ML problems.

Proof. The supplier has to ship at least $\sum_{j=1}^{t-1} r_{sj} - I_{s0}$ to retailer s up to time t, to avoid stockout at s. Because the maximum shipping quantity is U_s, then s has to be visited at least $\lceil \sum_{j=1}^{t-1} r_{sj} - I_{s0}/U_s \rceil$ times up to time t. □

Theorem 5. *The inequalities*

$$\sum_{s \in \mathcal{M}} \left(U_s - I_{s0} + \sum_{j=1}^{t-1} r_{sj}\right) z_{st} \leq tC \quad t \in \mathcal{T} \tag{21}$$

are valid for the VMIR-OU problem.

Proof. Bertazzi, Paletta, and Speranza (2002) showed that if the retailer s is served at time t, then the total quantity delivered to s up to time t is $U_s - I_{s0} + \sum_{j=1}^{t-1} r_{sj}$, independently of the times at which s is served. Because one vehicle of capacity C is available for each time $t \in \mathcal{T}$, then the total quantity delivered up to time t to the retailers served at time t cannot be larger than tC. □

Theorem 6. *The inequalities*

$$z_{st} \leq z_{0t} \quad s \in \mathcal{M} \ t \in \mathcal{T} \tag{22}$$

are valid for the VMIR-OU, VMIR-ML, and VMIR problems.

Proof. If any retailer s is visited at time t—i.e., $z_{st} = 1$—then the supplier has to be included in the route traveled at time t, i.e., $z_{0t} = 1$. □

Theorem 7. *The inequalities*

$$y_{i0}^{t} \leq 2 z_{it} \quad i \in \mathcal{M} \ t \in \mathcal{T} \tag{23}$$

$$y_{ij}^{t} \leq z_{it} \quad i \in \mathcal{M} \ j \in \mathcal{M} \ t \in \mathcal{T} \tag{24}$$

are valid for the VMIR-OU, VMIR-ML, and VMIR problems.

Proof. If supplier 0 is the successor of retailer i in the route traveled at time t, i.e., $y_{i0}^t = 1$ or $y_{i0}^t = 2$, then i has to be visited at time t, i.e., $z_{it} = 1$. If retailer j is the successor of retailer i in the route traveled at time t, i.e., $y_{ij}^t = 1$, then i has to be visited at time t, i.e., $z_{it} = 1$. □

The latter inequalities are referred to as logical inequalities, as they are inspired by the logical cuts introduced by Fischetti, Salazar-González, and Toth (1998) and Gendreau, Laporte, and Semet (1998).

3. Branch-and-Cut Algorithm

We now describe the branch-and-cut algorithm we have developed to solve the problems optimally. To solve the VMIR-OU problem, we have implemented the model defined by (1)–(16), excluding the subtour elimination constraints (12) but including the valid inequalities (17)–(19) and (22)–(24). As later shown in §4, we have conducted a set of experiments showing that the excluded classes of inequalities are not effective. In another preliminary set of experiments, we have observed that the dynamic management of the inequalities, introduced whenever violated, slowed down the execution. For this reason, we included all the inequalities in the root node. To solve the VMIR-ML problem, we have implemented the model defined by (1)–(16), excluding the order-up-to level constraints (6) and (8) and the subtour elimination constraints (12), but including the constraints (7), the valid inequalities (17), (18), and (22)–(24). To solve the VMIR problem, we have implemented the model defined by (1)–(16), excluding the order-up-to level constraints (6)–(8) and the subtour elimination constraints (12), but including the valid inequalities (17), (18), and (22)–(24). The subtour elimination constraints (12) were introduced with $k = \arg\max_j \{z_{jt}\}$ by using the separation algorithm of Padberg and Rinaldi (1991). This algorithm is called at each node of the branch-and-cut tree. Whenever it identifies a violated subtour elimination constraint, this constraint is added to the current subproblem, which is then reoptimized; otherwise, branching occurs at the current node. Branching is performed in priority on variables z_{jt}, and then on y_{ij}^t variables. The search is developed according to a best bound first strategy.

The heuristic of Bertazzi, Paletta, and Speranza (2002) defines an initial upper bound. In this algorithm, the retailers are ranked in nondecreasing order of the average number of time units needed to consume the quantity U_s, and the retailers with the same number of time units are ranked in nonincreasing order of U_s. In the initialization phase of the algorithm, a feasible solution of the problem is constructed by means of an iterative procedure that inserts a retailer at each iteration. When retailer s is considered, a set of delivery times is determined by solving a shortest-path problem on an acyclic network in which every vertex is a possible delivery time. Then, for each of the selected delivery times, the retailer is inserted in the route traveled by the vehicle by applying the cheapest insertion criterion. In the second phase of the algorithm, the current solution is improved iteratively. At each iteration, a pair of retailers is temporarily removed from the current solution. Then, each of the removed retailers is inserted in the current solution. If this reduces the total cost, the solution is modified accordingly. The second step of the algorithm is repeated as long as a total cost improvement is obtained.

A few remarks are now in order for the VMIR-OU problem. First, inequalities (12), with $k = \arg\max_j \{z_{jt}\}$, proved to be much more effective than the traditional subtours constraints:

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S},\, j<i} y_{ij}^t \leq |\mathcal{S}| - 1 \quad \mathcal{S} \subseteq \mathcal{M} \ t \in \mathcal{T}. \tag{25}$$

Second, lower bounds known for the vehicle routing problem (VRP) (see Toth and Vigo 2002) cannot be applied to our problem because of the quantity variables x_{it}. The same is true for the classical VRP valid inequalities. Third, we tested a branching strategy commonly used for the VRP: If at a time t there exists a subset $\mathcal{S} \subseteq \mathcal{M}$ for which $0 < \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S},\, j<i} y_{ij}^t < 2$, then create two branches with $\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S},\, j<i} y_{ij}^t = 0$ and $\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S},\, j<i} y_{ij}^t \geq 2$. The former equality means that no vertex in $\mathcal{S}$ is visited at time t. Thus, in the first branch we also add the equalities $z_{it} = 0$, $i \in \mathcal{S}$. The latter equality means that at least one vertex in $\mathcal{S}$ is visited at time t. Similarly, in the second branch we also add the inequalities $\sum_{i \in \mathcal{S}} z_{it} \geq 1$, $t \in \mathcal{T}$. However, this branching strategy proved to be ineffective for our problem. Fourth, we tested the following branching strategy. Given a fractional variable z_{it}, create two branches with z_{it} fixed at zero and one. In the branch with $z_{it} = 1$, we also add the following inequalities:

$$\sum_{k=1}^{t} z_{sk} \geq \left\lceil 1 - \frac{I_{s0}}{U_s} + \frac{\sum_{k=1}^{t} r_{sk-1}}{U_s} \right\rceil \quad s \in \mathcal{M}.$$

Then, as we know that:

$$\sum_{s \in \mathcal{M}} \left(U_s - I_{s0} + \sum_{k=1}^{t} r_{sk-1} \right) z_{st} \leq tC \quad t \in \mathcal{T}, \tag{26}$$

we fix to zero the variables z_{jt} causing a violation of constraint (26) on the basis of the variables already fixed at one in the node. When there is no z_{it} fractional variable, the solver chooses a variable y_{ij}^t. This branching strategy also proved to be ineffective.

Table 1 VMIR-OU—Average Computational Results on 160 Randomly Generated Instances

n	H	h_s	h_0	Seconds	Cuts	Nodes	%Gap	%UB error	%LP
5	3	[0.01, 0.05]	0.03	0.0	11.8	1.4	0.00	2.88	52.43
10	3	[0.01, 0.05]	0.03	0.0	57.2	34.8	0.00	0.78	41.45
15	3	[0.01, 0.05]	0.03	1.0	105.6	59.2	0.00	2.56	34.02
20	3	[0.01, 0.05]	0.03	7.4	217.8	131.8	0.00	3.83	26.00
25	3	[0.01, 0.05]	0.03	27.8	376.8	184.8	0.00	2.99	34.75
30	3	[0.01, 0.05]	0.03	79.6	504.0	300.0	0.00	3.60	35.24
35	3	[0.01, 0.05]	0.03	208.6	635.2	478.8	0.00	4.46	32.92
40	3	[0.01, 0.05]	0.03	686.4	1,088.0	843.6	0.00	6.46	32.38
45	3	[0.01, 0.05]	0.03	1,049.6	1,269.8	777.0	0.00	7.60	30.21
50	3	[0.01, 0.05]	0.03	3,572.4	2,141.2	1,454.6	0.19	5.81	27.79
5	3	[0.1, 0.5]	0.3	0.0	12.0	3.8	0.00	1.31	22.24
10	3	[0.1, 0.5]	0.3	0.0	54.0	39.2	0.00	1.74	13.50
15	3	[0.1, 0.5]	0.3	1.8	142.8	82.2	0.00	2.18	9.99
20	3	[0.1, 0.5]	0.3	7.6	223.4	119.0	0.00	3.30	6.79
25	3	[0.1, 0.5]	0.3	24.4	329.8	158.4	0.00	1.06	11.26
30	3	[0.1, 0.5]	0.3	101.6	594.2	386.2	0.00	1.21	7.45
35	3	[0.1, 0.5]	0.3	237.0	734.0	465.2	0.00	2.25	7.06
40	3	[0.1, 0.5]	0.3	561.6	1,053.6	655.0	0.00	2.26	6.76
45	3	[0.1, 0.5]	0.3	1,189.6	1,414.8	791.6	0.00	2.49	6.07
50	3	[0.1, 0.5]	0.3	2,842.8	1,638.8	1,685.2	0.00	1.57	5.90
5	6	[0.01, 0.05]	0.03	0.0	44.8	69.8	0.00	1.64	3.55
10	6	[0.01, 0.05]	0.03	5.0	200.6	215.0	0.00	1.36	4.73
15	6	[0.01, 0.05]	0.03	29.2	384.4	365.8	0.00	4.27	4.63
20	6	[0.01, 0.05]	0.03	558.0	958.6	4,112.2	0.00	2.95	3.51
25	6	[0.01, 0.05]	0.03	1,063.0	1,268.8	1,983.6	0.00	6.19	4.51
30	6	[0.01, 0.05]	0.03	3,216.0	1,747.4	3,410.4	0.00	4.64	4.34
5	6	[0.1, 0.5]	0.3	0.0	45.2	82.80	0.00	0.34	2.92
10	6	[0.1, 0.5]	0.3	6.0	216.6	255.40	0.00	1.87	2.92
15	6	[0.1, 0.5]	0.3	31.6	377.6	457.20	0.00	1.20	2.73
20	6	[0.1, 0.5]	0.3	530.2	1,037.0	3,170.80	0.00	2.09	2.12
25	6	[0.1, 0.5]	0.3	1,018.2	1,264.2	1,841.60	0.00	2.12	2.33
30	6	[0.1, 0.5]	0.3	4,099.8	1,933.0	4,267.00	0.00	2.56	2.20

4. Computational Results

The branch-and-cut algorithm described in §3 was implemented in C++ by using ILOG Concert 2 and CPLEX 9.0, and run on an Intel Pentium IV 2.8 GHz and 1 GB RAM personal computer with a maximum running time of two hours.

The test instances were generated on the basis of the following data:

time horizon H: 3, 6;

number of retailers n: $5k$, with $k = 1, 2, \ldots, 10$ when $H = 3$, and $k = 1, 2, \ldots, 6$ when $H = 6$;

product quantity r_{st} consumed by retailer s at time t: constant over time, i.e., $r_{st} = r_s$, $t \in \mathcal{T}$, and randomly generated as an integer number in the interval $[10, 100]$;

product quantity r_{0t} made available at the supplier at time t: $\sum_{s\in\mathcal{M}} r_s$;

maximum inventory level U_s at retailer s: $r_s g_s$, where g_s is randomly selected from the set $\{2, 3\}$ and represents the number of time units needed in order to consume the quantity U_s;

starting inventory level B_0 at the supplier: $\sum_{s\in\mathcal{M}} U_s$;

starting inventory level I_{s0} at the retailer s: $U_s - r_s$;

inventory cost at retailer $s \in \mathcal{M}$, h_s: randomly generated in the intervals $[0.01, 0.05]$ and $[0.1, 0.5]$;

inventory cost at the supplier h_0: 0.03 when h_s is generated in $[0.01, 0.05]$ and 0.3 when h_s is generated in $[0.1, 0.5]$;

transportation capacity C: $\frac{3}{2}\sum_{s\in\mathcal{M}} r_s$;

transportation cost c_{ij}: $\lfloor\sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2}\rfloor$, where the points (X_i, Y_i) and (X_j, Y_j) are obtained by randomly generating each coordinate as an integer number in the interval $[0, 500]$.

In all cases, random selections were performed in accordance with a uniform distribution. The generated instances and the computational results are available at the following URL: www-c.eco.unibs.it/~bertazzi/abls.zip.

The computational results are shown in Tables 1–10. Table 1 provides average results for the VMIR-OU problem on five instances generated for each combination of n, H, an interval for the generation of h_s, and h_0, for a total of 160 instances. Columns 1–4 give the parameters of the instance. In particular, the first column gives the number of retailers, the second the time horizon, the third the range of the

Table 2 VMIR-ML—Average Computational Results on 160 Randomly Generated Instances

n	H	h_s	h_0	Seconds	Cuts	Nodes	%Gap
5	3	[0.01, 0.05]	0.03	0.0	11.8	5.2	0.00
10	3	[0.01, 0.05]	0.03	0.4	41.4	86.4	0.00
15	3	[0.01, 0.05]	0.03	1.4	87.6	74.6	0.00
20	3	[0.01, 0.05]	0.03	4.8	121.8	88.8	0.00
25	3	[0.01, 0.05]	0.03	10.2	137.2	119.2	0.00
30	3	[0.01, 0.05]	0.03	31.0	197.8	192.8	0.00
35	3	[0.01, 0.05]	0.03	89.0	213.4	363.8	0.00
40	3	[0.01, 0.05]	0.03	106.6	339.6	98.0	0.00
45	3	[0.01, 0.05]	0.03	408.0	432.2	630.8	0.00
50	3	[0.01, 0.05]	0.03	4,169.6	1,968.0	3,093.4	0.98
5	3	[0.1, 0.5]	0.3	0.0	8.2	3.6	0.00
10	3	[0.1, 0.5]	0.3	0.0	40.8	23.2	0.00
15	3	[0.1, 0.5]	0.3	0.4	56.6	18.6	0.00
20	3	[0.1, 0.5]	0.3	3.8	119.0	60.2	0.00
25	3	[0.1, 0.5]	0.3	6.8	118.8	34.2	0.00
30	3	[0.1, 0.5]	0.3	32.4	252.8	127.8	0.00
35	3	[0.1, 0.5]	0.3	71.4	336.0	110.4	0.00
40	3	[0.1, 0.5]	0.3	132.8	368.0	122.6	0.00
45	3	[0.1, 0.5]	0.3	329.2	530.0	258.4	0.00
50	3	[0.1, 0.5]	0.3	1,196.8	895.4	1,038.0	0.00
5	6	[0.01, 0.05]	0.03	0.0	42.6	102.0	0.00
10	6	[0.01, 0.05]	0.03	23.8	270.0	1,579.4	0.00
15	6	[0.01, 0.05]	0.03	40.0	351.6	652.4	0.00
20	6	[0.01, 0.05]	0.03	763.0	1,180.0	3,667.2	0.00
25	6	[0.01, 0.05]	0.03	319.6	766.0	614.8	0.00
30	6	[0.01, 0.05]	0.03	2,584.4	1,527.8	2,636.4	0.00
5	6	[0.1, 0.5]	0.3	0.0	36.4	61.0	0.00
10	6	[0.1, 0.5]	0.3	5.8	174.4	249.4	0.00
15	6	[0.1, 0.5]	0.3	27.6	339.4	324.0	0.00
20	6	[0.1, 0.5]	0.3	248.0	783.8	1,154.4	0.00
25	6	[0.1, 0.5]	0.3	214.8	628.8	306.8	0.00
30	6	[0.1, 0.5]	0.3	1,570.0	1,299.8	1,380.8	0.00

Table 3 VMIR—Average Computational Results on 160 Randomly Generated Instances

n	H	h_s	h_0	Seconds	Cuts	Nodes	%Gap
5	3	[0.01, 0.05]	0.03	0.0	9.0	6.8	0.00
10	3	[0.01, 0.05]	0.03	0.0	26.6	47.4	0.00
15	3	[0.01, 0.05]	0.03	1.6	80.4	120.6	0.00
20	3	[0.01, 0.05]	0.03	23.8	165.6	1,020.8	0.00
25	3	[0.01, 0.05]	0.03	16.2	158.8	266.6	0.00
30	3	[0.01, 0.05]	0.03	38.4	244.2	268.6	0.00
35	3	[0.01, 0.05]	0.03	48.0	183.2	150.4	0.00
40	3	[0.01, 0.05]	0.03	412.0	554.2	1,130.2	0.00
45	3	[0.01, 0.05]	0.03	428.8	408.0	612.8	0.00
50	3	[0.01, 0.05]	0.03	2,797.4	1,316.8	2,378.4	0.62
5	3	[0.1, 0.5]	0.3	0.0	8.4	6.2	0.00
10	3	[0.1, 0.5]	0.3	0.0	26.2	7.6	0.00
15	3	[0.1, 0.5]	0.3	1.0	84.2	32.0	0.00
20	3	[0.1, 0.5]	0.3	7.6	182.4	180.6	0.00
25	3	[0.1, 0.5]	0.3	21.8	234.8	231.2	0.00
30	3	[0.1, 0.5]	0.3	65.6	370.4	353.0	0.00
35	3	[0.1, 0.5]	0.3	66.6	270.8	145.4	0.00
40	3	[0.1, 0.5]	0.3	628.6	904.6	1,212.0	0.00
45	3	[0.1, 0.5]	0.3	1,905.2	1,290.0	1,199.8	0.41
50	3	[0.1, 0.5]	0.3	3,253.8	1,684.8	1,657.6	0.39
5	6	[0.01, 0.05]	0.03	0.6	64.6	309.4	0.00
10	6	[0.01, 0.05]	0.03	80.4	441.0	5,394.6	0.00
15	6	[0.01, 0.05]	0.03	456.0	1,133.8	9,583.8	0.00
20	6	[0.01, 0.05]	0.03	2,733.8	1,728.2	14,834.2	2.46
25	6	[0.01, 0.05]	0.03	5,028.4	2,717.6	17,124.8	1.48
30	6	[0.01, 0.05]	0.03	6,169.0	2,773.8	9,544.6	7.80
5	6	[0.1, 0.5]	0.3	0.2	50.4	165.4	0.00
10	6	[0.1, 0.5]	0.3	22.0	300.6	1,575.0	0.00
15	6	[0.1, 0.5]	0.3	355.4	942.6	7,116.0	0.00
20	6	[0.1, 0.5]	0.3	4,460.4	1,905.2	21,715.8	0.57
25	6	[0.1, 0.5]	0.3	4,283.2	2,298.4	9,439.8	0.99
30	6	[0.1, 0.5]	0.3	6,435.8	2,499.0	7,149.8	1.65

unit inventory cost at the retailers, and the fourth the unit inventory cost at the supplier. Columns 5–10 show the results. In particular, Column 5 gives the average CPU time (seconds); Column 6 the average number of cuts (corresponding to the subtours elimination constraints); Column 7 the average number of nodes of the branch-and-bound tree; Column 8 gives the average percent gap between the cost of the incumbent solution with respect to the cost of the best LP relaxation, when the optimal solution is not obtained within the time limit; the percent gap on the only instance which was not solved optimally within the time limit of two hours is 0.99%. Column 9 gives the average percent error generated by the heuristic of Bertazzi, Paletta, and Speranza (2002) with respect to the optimal cost (or to the cost of the best LP relaxation in the case when the optimum is not found). Finally, Column 10 provides the average percent increase of the optimal cost of the LP relaxation (1)–(16), excluding the subtours elimination constraints (12), when the valid inequalities (17)–(19) and (22)–(24) are added at the root, with respect to the optimal cost of the LP relaxation (1)–(16), excluding the subtours elimination constraints (12).

When the time horizon H is equal to 3, all instances with up to 50 retailers but could be solved in less than 2 hours, while when the time horizon is 6, only instances with not more than 30 retailers could be solved in less than 2 hours. The last column shows that the impact of the valid inequalities on the LP value is stronger when the horizon is short and the inventory cost is low.

Tables 2 and 3 provide the same results shown in Table 1 for the VMIR-ML and VMIR problems, respectively, with the only exception being that Columns 9 and 10 are missing. While the VMIR-OU and the VMIR-ML require a similar computational time, the VMIR seems to be more difficult. The computational time required by the VMIR is larger, and a smaller number of instances could be solved to optimality within the time limit.

Tables 4–6 show the average results aggregated in four classes of instances. The first class contains the instances with time horizon $H = 3$ and low inventory cost ($h_s \in [0.01, 0.05]$ and $h_0 = 0.03$). The second class

Table 4 VMIR-OU—Average Computational Results on Four Classes of Instances

H	h_s	h_0	Seconds	Cuts	Nodes	%Gap	%UB error
3	[0.01, 0.05]	0.03	19.3	212.2	118.7	0.00	2.77
3	[0.1, 0.5]	0.3	22.6	226.0	131.5	0.00	1.80
6	[0.01, 0.05]	0.03	811.9	767.4	1,692.8	0.00	3.51
6	[0.1, 0.5]	0.3	947.6	812.3	1,679.1	0.00	1.70

Table 6 VMIR—Average Computational Results on Four Classes of Instances

H	h_s	h_0	Seconds	Cuts	Nodes	%Gap
3	[0.01, 0.05]	0.03	13.3	114.1	288.5	0.00
3	[0.1, 0.5]	0.3	16.0	151.1	135.1	0.00
6	[0.01, 0.05]	0.03	2,411.4	1,476.5	9,465.2	1.96
6	[0.1, 0.5]	0.3	2,592.8	1,332.7	7,860.3	0.54

contains the instances with time horizon $H = 3$ and high inventory cost ($h_s \in [0.1, 0.5]$ and $h_0 = 0.3$), while the third class contains the instances with time horizon $H = 6$ and low inventory cost. Finally, the fourth class contains the instances with time horizon $H = 6$ and high inventory cost. The tables are organized as Table 1, with the only exception being the number of retailers n. The figures are computed by considering instances up to 30 retailers. These tables give evidence to the fact that the length of the time horizon has a dramatic impact on the time needed to solve an instance.

As already mentioned, in the branch-and-cut algorithm all valid inequalities are included in the LP relaxation at the root node, for efficiency reasons. To test the effectiveness of the valid inequalities presented in §2 for the VMIR-OU problem, we conducted a set of tests on instances with time horizon $H = 6$ and high inventory cost by introducing the inequalities dynamically, whenever violated. Table 7 gives the number of times each valid inequality is introduced during the algorithm. The results show that inequalities (20) and (21) are not effective. Table 8 allows us to compare the average CPU time and the average number of nodes obtained when the problem is solved including the valid inequalities (17)–(19) and (22)–(24) with respect to the case in which no valid inequality is included in the model. The results show that the valid inequalities are effective. The average CPU time and the average number of nodes on all the instances almost triple when the valid inequalities are not included in the model.

Finally, Tables 9 and 10 show the percent variation of the total cost and of the cost components of the VMIR-ML problem with respect to the VMIR-OU problem and of the VMIR problem with respect to the VMIR-ML problem, respectively. In particular, Columns 1–4 give the parameters of the instance. Column 5 shows the average percent decrease of the total cost on five instances. Columns 6–8 show the average percent variation of the inventory cost at the supplier (IS), the inventory cost at the retailers (IR), and the transportation cost (T), respectively.

These two tables allow us to evaluate the impact on the total cost of different policies for the inventory management in a coordinated supply chain. Let us first comment on the impact of the relaxation of the order-up-to level constraint, by observing Table 9. We recall that in the VMIR-OU problem, whenever a shipment is performed, the exact quantity to be shipped is determined as the difference between the maximum and the present inventory level. In the VMIR-ML problem the quantity can be smaller, if convenient. Only the maximum inventory level must be satisfied. Table 9 shows that the savings due to the relaxation of the order-up-to level constraint ranges from 4.12% to 26.06% and depends on the specific instance. The saving is larger when the horizon is longer and the inventory cost is lower (or, in other words, the transportation cost is higher). This can be explained by observing that the saving is mainly due to a reduction of the transportation cost. A longer horizon offers more possibilities for the coordination of the shipments. In almost all the tested instances, the total saving is due to a large reduction of the transportation cost, a more limited reduction of the inventory cost of the supplier and an increase of the inventory costs of the retailers. Further savings can be obtained if no

Table 5 VMIR-ML—Average Computational Results on Four Classes of Instances

H	h_s	h_0	Seconds	Cuts	Nodes	%Gap
3	[0.01, 0.05]	0.03	8.0	99.6	94.5	0.00
3	[0.1, 0.5]	0.3	7.2	99.4	44.6	0.00
6	[0.01, 0.05]	0.03	621.8	689.7	1,542.0	0.00
6	[0.1, 0.5]	0.3	344.4	543.8	579.4	0.00

Table 7 VMIR-OU—Effectiveness of the Valid Inequalities

n	(17)	(18)	(19)	(20)	(21)	(22)	(23)–(24)
5	16	23	0	0	0	18	42
5	17	30	0	0	0	22	43
10	43	57	0	0	0	47	169
10	44	60	0	0	0	40	135
15	61	80	0	0	0	67	219
15	63	84	0	0	0	69	198
20	91	122	12	0	0	99	549
20	80	112	0	0	0	102	419
25	102	138	0	0	0	131	417
25	117	161	14	0	0	128	657
30	144	219	118	0	0	153	935
30	140	191	40	0	0	150	700

Table 8 VMIR-OU—Effectiveness of the Valid Inequalities

n	CPU with ineq.	CPU without ineq.	Nodes with ineq.	Nodes without ineq.
5	0	0	102	91
5	0	0	128	104
10	11	11	573	548
10	7	8	223	348
15	22	29	238	354
15	31	49	350	656
20	1,536	3,334	10,171	23,499
20	354	361	1,975	2,235
25	578	724	1,082	1,306
25	1,732	4,274	2,984	11,076
30	4,947	25,576	5,578	27,210
30	4,661	12,910	4,316	14,846

constraint is set on the quantity to be shipped. This can be seen from Table 10, where the savings of the VMIR problem with respect to the VMIR-ML problem are shown. Obviously, in this case the inventory levels at the retailers may become very high and even unacceptable.

Table 9 Percent Variation of the Total Cost and of the Cost Components of VMIR-ML with Respect to VMIR-OU

n	H	h_s	h_0	Total cost	IS	IR	T
5	3	[0.01, 0.05]	0.03	−13.23	−2.17	4.19	−14.14
10	3	[0.01, 0.05]	0.03	−15.94	−1.50	0.05	−17.99
15	3	[0.01, 0.05]	0.03	−13.68	−1.28	−0.30	−15.69
20	3	[0.01, 0.05]	0.03	−14.73	1.46	−5.14	−17.57
25	3	[0.01, 0.05]	0.03	−14.60	−2.09	4.97	−17.95
30	3	[0.01, 0.05]	0.03	−10.97	−3.51	5.01	−13.77
35	3	[0.01, 0.05]	0.03	−10.89	0.10	−3.50	−13.84
40	3	[0.01, 0.05]	0.03	−11.21	−0.74	−2.82	−14.30
45	3	[0.01, 0.05]	0.03	−9.89	−0.96	−0.03	−13.06
50	3	[0.01, 0.05]	0.03	−11.04	−3.55	7.48	−14.59
5	3	[0.1, 0.5]	0.3	−8.80	−2.43	4.36	−13.92
10	3	[0.1, 0.5]	0.3	−8.55	−1.48	1.25	−17.99
15	3	[0.1, 0.5]	0.3	−6.92	−3.99	8.29	−15.60
20	3	[0.1, 0.5]	0.3	−6.40	−1.44	1.37	−16.76
25	3	[0.1, 0.5]	0.3	−5.96	−3.93	7.91	−16.71
30	3	[0.1, 0.5]	0.3	−4.98	−4.70	6.27	−13.16
35	3	[0.1, 0.5]	0.3	−4.46	−3.00	3.89	−12.69
40	3	[0.1, 0.5]	0.3	−4.72	−3.06	1.98	−13.12
45	3	[0.1, 0.5]	0.3	−4.12	−4.15	5.84	−11.71
50	3	[0.1, 0.5]	0.3	−4.36	−1.69	−2.55	−12.27
5	6	[0.01, 0.05]	0.03	−26.06	−7.13	21.15	−28.24
10	6	[0.01, 0.05]	0.03	−22.18	−2.63	4.02	−24.40
15	6	[0.01, 0.05]	0.03	−21.64	−6.22	19.94	−24.61
20	6	[0.01, 0.05]	0.03	−21.78	−6.53	17.93	−24.96
25	6	[0.01, 0.05]	0.03	−22.73	−6.70	17.23	−26.38
30	6	[0.01, 0.05]	0.03	−21.27	−5.58	12.86	−25.71
5	6	[0.1, 0.5]	0.3	−15.69	−5.92	15.64	−28.10
10	6	[0.1, 0.5]	0.3	−12.53	−4.16	9.52	−24.40
15	6	[0.1, 0.5]	0.3	−10.76	−6.16	16.05	−23.68
20	6	[0.1, 0.5]	0.3	−10.66	−7.26	19.58	−24.73
25	6	[0.1, 0.5]	0.3	−10.61	−7.91	21.65	−26.40
30	6	[0.1, 0.5]	0.3	−9.03	−6.05	13.87	−25.49

Table 10 Percent Variation of the Total Cost and of the Cost Components of VMIR with Respect to VMIR-ML

n	H	h_s	h_0	Total cost	IS	IR	T
5	3	[0.01, 0.05]	0.03	−0.06	−3.71	6.37	0.00
10	3	[0.01, 0.05]	0.03	−0.50	−11.28	19.89	0.00
15	3	[0.01, 0.05]	0.03	0.00	0.00	0.00	0.00
20	3	[0.01, 0.05]	0.03	−0.74	−20.50	41.23	0.64
25	3	[0.01, 0.05]	0.03	−0.75	−11.46	13.86	0.67
30	3	[0.01, 0.05]	0.03	−2.57	−26.85	31.36	0.85
35	3	[0.01, 0.05]	0.03	−1.65	−29.61	41.80	2.62
40	3	[0.01, 0.05]	0.03	−2.38	−28.74	33.63	1.67
45	3	[0.01, 0.05]	0.03	−2.73	−28.87	33.71	1.95
50	3	[0.01, 0.05]	0.03	−2.24	−24.43	22.50	2.10
5	3	[0.1, 0.5]	0.3	−0.96	−10.01	13.41	1.29
10	3	[0.1, 0.5]	0.3	−3.18	−28.90	51.05	6.35
15	3	[0.1, 0.5]	0.3	−5.08	−31.37	37.67	11.76
20	3	[0.1, 0.5]	0.3	−7.17	−30.53	44.61	6.00
25	3	[0.1, 0.5]	0.3	−8.31	−31.88	42.33	10.02
30	3	[0.1, 0.5]	0.3	−12.77	−38.70	35.42	11.71
35	3	[0.1, 0.5]	0.3	−13.30	−39.56	42.19	9.62
40	3	[0.1, 0.5]	0.3	−13.21	−39.65	38.31	10.30
45	3	[0.1, 0.5]	0.3	−13.86	−39.28	29.84	14.64
50	3	[0.1, 0.5]	0.3	−13.48	−40.81	43.22	14.11
5	6	[0.01, 0.05]	0.03	−14.51	−21.65	49.63	−15.83
10	6	[0.01, 0.05]	0.03	−18.63	−29.52	81.55	−21.14
15	6	[0.01, 0.05]	0.03	−22.49	−25.78	70.17	−26.69
20	6	[0.01, 0.05]	0.03	−23.82	−26.44	67.22	−28.46
25	6	[0.01, 0.05]	0.03	−23.72	−26.27	66.34	−28.76
30	6	[0.01, 0.05]	0.03	−23.51	−37.16	81.54	−28.75
5	6	[0.1, 0.5]	0.3	−9.73	−22.92	48.28	−13.56
10	6	[0.1, 0.5]	0.3	−10.91	−35.56	69.50	−15.07
15	6	[0.1, 0.5]	0.3	−12.35	−39.50	67.92	−14.40
20	6	[0.1, 0.5]	0.3	−13.66	−40.27	60.81	−15.39
25	6	[0.1, 0.5]	0.3	−13.40	−41.90	57.37	−11.86
30	6	[0.1, 0.5]	0.3	−17.72	−50.19	61.73	−9.14

We can conclude that the order-up-to level policy is a very popular, but at the same time quite rigid, inventory-management policy. More flexible policies may generate substantial savings. The company management will have to evaluate the organizational impact of the various policies and find the most suitable solution.

Conclusions

While classical vehicle-routing problems have received considerable attention by researchers, inventory-routing problems still represent a class of relatively new problems. Jointly optimizing transportation and inventory costs leads to cost reductions with respect to the more traditional approach consisting of separately optimizing transportation costs and inventory costs. In this article we have presented a branch-and-cut algorithm for a vendor-managed inventory problem. This algorithm is the first exact approach ever proposed for an inventory-routing problem and uses a set of valid inequalities that exploit the structure of the problem. The effectiveness of the derived valid inequalities and of the branch-and-cut algorithm was tested through a set of computational experiments. Instances with up to 50 customers were solved optimally with a time horizon equal to three. When the time horizon is equal to six, the largest instances that could be solved involve 30 customers. We believe that some of the valid inequalities we presented may be used for other inventory-routing problems.

The availability of an exact algorithm has allowed us to compare different inventory-management policies. The algorithm was used to solve three different problems associated with different inventory-management policies. We have shown that by relaxing the constraints on the shipment quantities, substantial savings can be achieved. Future research should be devoted to the multiple-vehicle version of the problem and to the exact solution of other inventory-routing problems.

Acknowledgments

This work was partially supported by the Canadian Natural and Engineering Research Council under Grant OPG0039682. The authors wish to thank Ivan Luzzi for his precious help in the computational part at an early stage of this work. The authors also wish to thank two anonymous referees for their useful suggestions.

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