Inventory Routing Problem under Uncertainty

Zheng CUI
School of Management, Zhejiang University, Hangzhou.
zhengcui@zju.edu.cn

Daniel Zhuoyu LONG
Department of Systems Engineering and Engineering Management, Chinese University of Hong Kong, New Territories, Hong Kong. zylong@se.cuhk.edu.hk

Jin QI
Department of Industrial Engineering and Decision Analytics, Hong Kong University of Science and Technology, Kowloon, Hong Kong. jinqi@ust.hk

Lianmin ZHANG
Department of Management Science and Engineering, Nanjing University, Nanjing.
Shenzhen Research Institute of Big Data, Shenzhen. zhanglm@nju.edu.cn

We study an uncertain inventory routing problem with a finite horizon. The supplier acts as a central planner who determines the replenishment quantities and also the delivery times and routes to all retailers. We allow ambiguity in the probability distribution of each retailer's uncertain demand. Adopting a service-level viewpoint, we minimize the risk of uncertain inventory levels violating a pre-specified acceptable range. We quantify that risk using a novel decision criterion, the *Service Violation Index*, that accounts for how often and how severely the inventory requirement is violated. The solutions proposed here are adaptive in that they vary with the realization of uncertain demand. We provide algorithms to solve the problem exactly and then demonstrate the superiority of our solutions by comparing them with several benchmarks.

Key words: inventory routing, distributionally robust optimization, risk management, adjustable robust optimization, decision rule

1. Introduction

A key concept in supply chain management is the coordination of various stakeholders to minimize the overall cost. In that regard, the vendor-managed inventory (VMI) system, a relatively new business model, has attracted extensive attention in both academia and industry. In the VMI system, the supplier (vendor) behaves as a central decision maker determining not only the timing and quantities of the replenishment for all retailers, but also the delivery route. Under this arrangement, retailers benefit from reduced efforts to control inventory. Meanwhile, the supplier can improve the service level and reduce costs by using transportation capacity more efficiently. For a more detailed discussion of the VMI concept's advantages, see Waller et al. (1999), Çetinkaya and Lee (2000), Cheung and Lee (2002), and Dong and Xu (2002).

This joint management of inventory and vehicle routing gives rise to the inventory routing problem (IRP). The term IRP was first used by Golden et al. (1984) when defining a routing problem with an explicit inventory feature. Federgruen and Zipkin (1984) studied a single-period IRP and described the economic benefits of coordinating decisions related to distribution and inventory. Since then, many variants of the IRP have been proposed over the past three decades (Coelho et al. 2014a): planning horizons can be finite or infinite; excess demand can be either lost or back-ordered; and there can be either one or more than one vehicle. Scholars have used these formulations in many real-world applications of the IRP, including maritime logistics (Christiansen et al. 2011, Papageorgiou et al. 2014), the transport of oil and gas (Bell et al. 1983, Campbell and Savelsbergh 2004, Grønhaug et al. 2010), groceries (Gaur and Fisher 2004, Custódio and Oliveira 2006), perishable products (Federgruen et al. 1986), and bicycle sharing (Brinkmann et al. 2016). We refer interested readers to Andersson et al. (2010) for more on IRP applications.

Since the basic IRP incorporates a classic vehicle routing problem that is already NP-hard, it follows that the uncertain IRP must be even more complex. We next focus our review on the uncertain version of the IRP, in which the demand is uncertain. A classic way to formulate the uncertain IRP is using the Markov decision process, which assumes that the joint probability distribution of retailers' demands is known. Campbell et al. (1998) introduced a dynamic programming model in which the state is the current inventory level of each retailer and the Markov transition matrix is obtained from the known probability distribution of demand; in their model, the objective is to minimize the expected total discounted cost over an infinite horizon. This work was extended by Kleywegt et al. (2002, 2004), who solved the problem by constructing an approximation to the optimal value function. Adelman (2004) decomposed the optimal value function as the sum of single-customer inventory value functions, which are approximated by the optimal dual price derived under a linear program. Hvattum and Løkketangen (2009) and Hvattum et al. (2009) solved the same problem using heuristics based on finite scenario trees.

Another way to solve the uncertain IRP is to use stochastic programming. Federgruen and Zipkin (1984) included inventory and shortage cost in the vehicle routing problem and offered a heuristic for that problem in a scenario-based random demand environment. Coelho et al. (2014b) addressed the dynamic and uncertain inventory routing problem in the case where the distribution of demand changes over time. They gave algorithms for four solution policies whose use depends on whether demand forecasts are used and whether emergency trans-shipments are allowed. Adulyasak et al. (2015) considered the uncertain production routing problem with demand uncertainty under both two-stage and multi-stage decision processes. They minimized the average total cost by enumerating all possible scenarios and solved the problem by deriving some valid inequalities and taking a Benders decomposition approach.

The uncertain IRP literature is notable, however, for not addressing two important issues adequately. The first involves the ambiguity in demand distributions. Both the Markov decision process and stochastic programming methods assume a known probability distribution for the uncertain demand, but in practice, full distributional information is seldom available. For various reasons, such as estimation error and the lack of historical data, we may have only partial information on the probability distribution. To cope with this shortcoming, the classic robust optimization methodology has been applied to the IRP (see e.g., Aghezzaf 2008, Solyalı et al. 2012, Bertsimas et al. 2016). These authors used a polyhedron to characterize the realization of uncertain demand and then minimized the cost for the worst-case scenario. In particular, Aghezzaf (2008) assumed normally distributed demands and minimized the worst-case cost for possible realizations of demands within a confidence level. Solyalı et al. (2012) considered a robust IRP by using the "budget of uncertainty" approach proposed by Bertsimas and Sim (2003, 2004), and solved the problem using a branch-and-cut algorithm. Bertsimas et al. (2016) considered the robust IRP from the perspective of scalability; they proposed a robust and adaptive formulation that can be solved for problems with thousands of customers. However, conventional robust optimization, which only has a budgeted uncertainty set and support information, does not capture any frequency information and focuses only on the worst-case realization — an approach that some consider being unnecessarily conservative. To incorporate the frequency information, distributionally robust optimization has become popular over the last decade. For example, uncertainties can be characterized by moment information (e.g., Delage and Ye 2010, Goh and Sim 2010), distance from a reference distribution (e.g., Ben-Tal et al. 2013, Mohajerin Esfahani and Kuhn 2018), or statistical estimation (e.g., Bertsimas et al. 2018). In this paper, we propose a distributionally robust optimization framework that exploits frequency information. The resulting model allows us to make robust inventory replenishment and routing decisions that can be made immune to the effect of distributional ambiguity.

Further, as this is a multi-period problem with uncertain demand, each period's operational decisions should be adjusted in response to previous demand realization to achieve good performance. For such multi-stage problems, Ben-Tal et al. (2004) proposed an adjustable decision rule approach that models decisions as functions of the uncertain parameters' realizations. But because finding the optimal decision rule is computationally intractable, various decision rules have been proposed to solve the problem approximately by restricting the set of feasible adaptive decisions to some particular and simple functional forms (see e.g., Ben-Tal et al. 2004, Chen et al. 2008, Goh and Sim 2010, Bertsimas et al. 2011, Kuhn et al. 2011, Georghiou et al. 2015); readers are referred to the tutorial by Delage and Iancu (2015). The most popular is known as the linear decision rule, which restricts the decisions in each period to affine functions of previous periods' uncertainties. Its effectiveness has been established both computationally (e.g., Ben-Tal et al. 2005, Bertsimas et al. 2018) and analytically (e.g., Kuhn et al. 2011, Bertsimas and Goyal 2012). Bertsimas et al. (2016) is the only study that incorporates the decision rule approach in IRP. They adopted the linear decision rule and hence could solve large-scale instances. In this paper, we enhance the linear decision rule by integrating a scenario-based distributional uncertainty set and propose a new formulation to solve the uncertain IRP. In the IRP literature, two replenishment policies are commonly used to solve the deterministic IRP (Archetti et al., 2007). The first is the order-up-to level (OU) policy, which determines ordering quantities such that the maximum inventory level is reached every time a visit is made. The second is the maximum-level (ML) policy that allows the ordering quantity to be any non-negative number without exceeding the maximum inventory level. However, it is difficult to incorporate these policies into the uncertain IRP when stochastic programming is used. This paper shows that the OU policy is a special case of our linear decision rule. Further, when the minimum and maximum inventory levels are properly specified, our approach yields better performance than both the OU and ML policies.

The second issue that is largely ignored in the literature arises from the uncertain IRP's objective function. Most research on this topic minimizes the expected operational cost, which is the sum of holding cost, shortage cost, and transportation cost. However, determining the cost parameters, especially the per-unit shortage cost, is challenging in many practical cases (Brandimarte and Zotteri 2007). Even if the cost parameters can be estimated precisely, the inventory cost is a piecewise linear function and so calculating its expectation involves the sort of multi-dimensional integration that increases the computational complexity (Ardestani-Jaafari and Delage 2016). Finally, the expected cost is a risk-neutral criterion and thus fails to account for decision makers' attitudes to risk. But risk aversion is well documented empirically (as in the St. Petersburg paradox, Samuelson 1977, Colyvan 2008) and applied in inventory literature (e.g., Chen et al. 2007). So instead of optimizing the expected cost, we present an alternative approach that resolves the above issues.

Our objective function is related to the concept of service level, which is a prevalent requirement in supply chain management (Miranda and Garrido 2004, Bernstein and Federgruen 2007, Bertsimas et al. 2016). From a practical standpoint, it is extremely important for retailers to maintain inventory levels within a certain range. On the one hand, the upper limit corresponds to a retailer's capacity, e.g., storage capacity. On the other hand, running out of inventory will damage the brand image and also impose substantial costs due to emergency replenishment. It follows that establishing a lower limit will also help to control inventory levels. This concept of minimum and maximum inventory level has also been used commonly in the deterministic IRP literature, for example, the OU and ML policies. The rationale of such a service-level requirement also relates to target-driven decision making. It can be traced back to Simon (1955), who suggested that the main goal of most decision making is not to maximize returns or minimize costs but rather to achieve certain targets. Since then, a rich body of descriptive literature has confirmed that decision making is often driven by targets (Mao 1970). Indeed, in the absence of inventory control, Jaillet et al. (2016) and Zhang et al. (2019) have investigated target-driven decision making in the vehicle routing problem. It is therefore natural to assume that in uncertain IRP, the objective is to maintain the inventory within certain target levels defined by an upper and a lower bound.

When imposing such a range on inventory, there are two classic ways to model the service-level requirement. The first is to ensure that the inventory level remains within the desired range for all possible demand realizations; however, this approach is often viewed as being too conservative. The second way of modeling the service-level requirement is to minimize the probability of violation using chance-constrained optimization. But chance constraints are well known to be nonconvex and computationally intractable in most cases (e.g., Ben-Tal et al. 2009); that approach has also been criticized for being unable to capture the magnitude of a violation (Diecidue and van de Ven 2008). Hence, we introduce the *Service Violation Index* (SVI), a general decision criterion that we use to evaluate the risk of violating the service-level requirement and thus address the issues discussed so far. Our SVI decision criterion generalizes the performance measures proposed by Jaillet et al. (2016) and Zhang et al. (2019) when studying the uncertain vehicle routing problem. It can be represented in classic expected utility theory and therefore can be constructed using any utility function. In the case of a single period, the SVI is associated with the satisficing measure axiomatized by Brown and Sim (2009).

We summarize our main contributions as follows.

- Instead of minimizing the expectation of inventory and transportation cost, we study robust IRP from the service-level perspective and aim to keep the inventory level within a certain range. We propose a multi-period decision criterion that allows decision makers to take into account the risk that each retailer's uncertain inventory level in each period violates the stipulated service requirement window.

- We use a distributionally robust optimization framework to calibrate uncertain demand, by way of descriptive statistics, to derive solutions that protect against the ambiguity in demand distributions. This approach incorporates more distributional information and will yield less conservative results than the classic robust optimization approach.

- We derive adaptive inventory replenishment solutions that can be updated with observed demand by implementing a decision rule approach. Then the robust IRP can be formulated as a large-scale mixed-integer linear optimization problem. We further show that the OU policy is a special case of our adaptive inventory replenishment policy.

- We provide algorithms that can facilitate the solution procedure by exploiting the structure of the model.

The rest of our paper proceeds as follows. Section 2 defines the problem and discusses how we incorporate our proposed measure into it from the service-level perspective. Section 3 introduces the scenario-based ambiguity set to characterize the uncertain demand in a general way and the corresponding scenario-based linear decision rule. Section 4 proposes solution methods to facilitate computation. Section 5 reports results from several computational studies. We conclude in Section 6 with a brief summary.

Notations: We use $[\cdot]$ to represent a set of running indices, e.g., $[T] = \{1, \ldots, T\}$ and denote the cardinality of a set $\mathcal{S}$ as $|\mathcal{S}|$. A boldface lowercase letter, such as $\boldsymbol{x}$, represents a column vector with a comma separating each element: $\boldsymbol{x} = (x_1, \ldots, x_T)$; correspondingly, $\boldsymbol{x}^\top$ is its transpose which is a row vector. We use $(y, \boldsymbol{x}_{-i})$ to denote the vector with all elements equal to those in $\boldsymbol{x}$ except the ith element, which is equal to y; that is, $(y, \boldsymbol{x}_{-i}) = (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_T)$. In addition, given two index sets $[T]$ and $[N]$, we use $(x_n^t)_{t\in[T],n\in[N]} \in \Re^{NT}$ to represent the column vector $(\boldsymbol{x}^1,\ldots,\boldsymbol{x}^T)$ where $\boldsymbol{x}^t = (x_1^t,\ldots,x_N^t)$ for all $t \in [T]$. We use "$\leq$" to represent an element-wise comparison; for example, $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_i \leq y_i$ for all $i \in [T]$. We denote uncertain quantities by characters with "$\tilde{}$" sign, such as $\tilde{d}$, and the corresponding characters themselves, such as d, as the realization. We denote the space of random variables by $\mathcal{V}$ and let the probability space be $(\Omega,\mathcal{F},\mathbb{P})$. We assume that full information about the probability distribution $\mathbb{P}$ is not known. Instead, we know only that $\mathbb{P}$ belongs to an ambiguity set $\mathcal{P}$, which is a set of probability distributions characterized by particular descriptive statistics. An inequality between two uncertain parameters, $\tilde{t} \geq \tilde{v}$, implies statewise dominance; that is, it implies $\tilde{t}(\omega) \geq \tilde{v}(\omega)$ for all $\omega \in \Omega$. In addition, the strict inequality $\tilde{t} > \tilde{v}$ implies that there exists an $\epsilon > 0$ for which $\tilde{t} \geq \tilde{v} + \epsilon$. We let $\boldsymbol{e}_m \in \mathbb{R}^T$ be the mth standard basis vector.

2. Model

Section 2.1 defines the inventory routing problem. In Section 2.2, we introduce a new service measure to evaluate the uncertain outcome of any inventory routing decisions. Section 2.3 incorporates the new measure and the inventory routing problem.

2.1. Problem statement

We consider an uncertain inventory routing problem in which there is one supplier and N retailers over T periods. The problem is defined formally on a directed network $\mathcal{G} = ([N] \cup \{0\}, \mathcal{A})$, where node 0 is the supplier node and $\mathcal{A}$ is the set of arcs. We use (i,j) and a interchangeably to represent an arc in $\mathcal{A}$. Given any subset of nodes $\mathcal{S} \subset ([N] \cup \{0\})$, we define the arc set $\delta^+(\mathcal{S})$ as all outgoing arcs of set $\mathcal{S}$ (i.e., $\delta^+(\mathcal{S}) = \{(i,j) \in \mathcal{A} : i \in \mathcal{S}, j \in [N] \cup \{0\} \backslash \mathcal{S}\}$) and $\delta^-(\mathcal{S})$ as all incoming arcs of set $\mathcal{S}$ (i.e., $\delta^-(\mathcal{S}) = \{(i,j) \in \mathcal{A} : i \in [N] \cup \{0\} \backslash \mathcal{S}, j \in \mathcal{S}\}$). The uncertain demands at N nodes over T periods are represented by a random vector $\tilde{\boldsymbol{d}} = (\tilde{d}_n^t)_{t\in[T],n\in[N]} \in \mathcal{V}^{NT}$, i.e., $\tilde{\boldsymbol{d}} = (\tilde{\boldsymbol{d}}^1,\ldots,\tilde{\boldsymbol{d}}^T)$ where $\tilde{\boldsymbol{d}}^t = (\tilde{d}_1^t,\ldots,\tilde{d}_N^t)$ is the vector of the uncertain demands of N nodes in period $t \in [T]$. We let $\tilde{\boldsymbol{d}}^{[t]} = \left(\tilde{\boldsymbol{d}}^1,\ldots,\tilde{\boldsymbol{d}}^t\right) \in \mathcal{V}^{Nt}$ be the vector of uncertain demands of all nodes from period 1 to period t and we have $\tilde{\boldsymbol{d}}^{[T]} = \tilde{\boldsymbol{d}}$. Further, we define $\tilde{\boldsymbol{d}}^{[0]} = \emptyset$ for notational simplicity. We denote the support of $\tilde{\boldsymbol{d}}$ by $\mathcal{D}$ and assume that the transportation cost is deterministic and denoted by c_a for all $a \in \mathcal{A}$. Any excess demand is assumed to be backlogged.

We assume that the supplier uses only a single vehicle of unconstrained capacity to deliver products for notational and computational simplicity. Our model can be modified to consider the case with multiple vehicles of finite capacity by adding certain deterministic constraints. In the decision process, the supplier makes three simultaneous decisions for each period: which subset of retailers to visit, which route to use, and how many units to replenish. Ideally, all these decisions should be adaptively in response to the realization of past demands. Nevertheless, the adaptive visiting and routing decisions involve a binary decision rule, which, even if practically relevant, attracts only limited attention because of its theoretical challenge (e.g., Bertsimas and Caramanis 2007, 2010, Bertsimas and Georghiou 2015, 2018). Since this study is the first to consider such a complex uncertain IRP, we have to assume that the visiting and routing decisions are non-adaptive for computational tractability. Hence, we consider the problem in two steps. In the first step, the visiting and routing decisions for all periods are determined at the beginning of the planning horizon. They are made a priori and irrespective of demand realizations. This assumption is reasonable in practice. Visiting plans must be communicated to retailers in advance so that they can be adequately prepared. Besides, scheduling the vehicle and informing the staff at short notice is not always practically feasible (Bertsimas et al. 2016). In the second step, the supplier makes replenishment decisions at the start of each period after the demand in previous periods has been realized. In this sense, inventory decisions are made adaptively. Figure 1 summarizes the timeline of these decisions. We remark that when making the adaptive replenishment decision, the supplier already has enough flexibility when responding to the demand realizations; hence, the benefit of incorporating adaptive visiting and routing decisions becomes marginal and might not cover the loss resulting from the computational complexity incurred.

In each period $t \in [T]$, the routing decisions are denoted by $y_n^t, z_a^t \in \{0, 1\}$ for any $n \in [N] \cup \{0\}, a \in \mathcal{A}$. More specifically, $y_0^t = 1$ if the supplier's vehicle is used to replenish a subset of retailers and

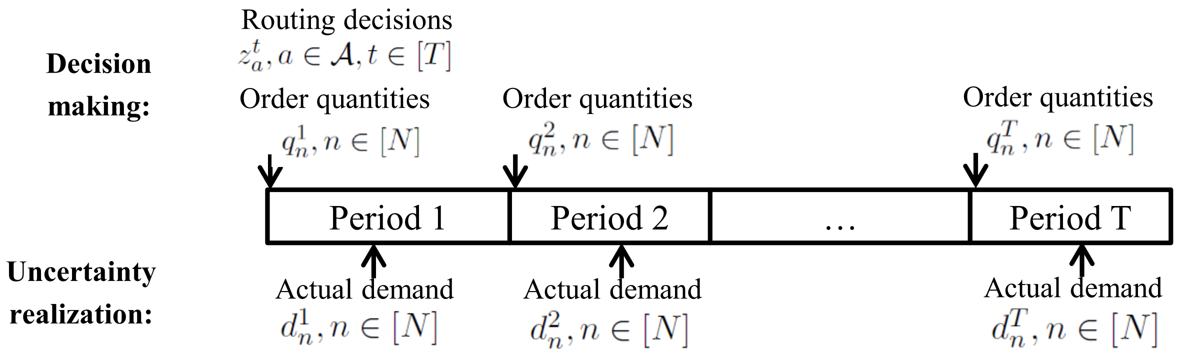

Figure 1 **Sequence of routing decisions and inventory replenishment decisions.**

$y_n^t = 1$ if retailer $n \in [N]$ is visited. Moreover, for $a \in \mathcal{A}$, we have $z_a^t = 1$ if the vehicle travels through arc a.

Retailer n's replenishment quantity is denoted by $q_n^t\left(\tilde{\boldsymbol{d}}^{[t-1]}\right)$, where the decision q_n^t is a measurable function that maps from $\mathbb{R}^{N(t-1)}$ to $\mathbb{R}$. Without loss of generality (WLOG), we assume that the system has no inventory at the start of the planning horizon: $x_n^0 = 0$. For retailer n in period t, we use $x_n^t\left(\tilde{\boldsymbol{d}}^{[t]}\right)$ to represent the ending inventory. The inventory dynamics of retailer n can therefore be calculated as

$$x_n^t\left(\boldsymbol{d}^{[t]}\right) = x_n^{t-1}\left(\boldsymbol{d}^{[t-1]}\right) + q_n^t\left(\boldsymbol{d}^{[t-1]}\right) - d_n^t = \sum_{m \in [t]}\left(q_n^m\left(\boldsymbol{d}^{[m-1]}\right) - d_n^m\right), \qquad \forall t \in [T], \boldsymbol{d} \in \mathcal{D}. \quad (1)$$

We adopt a target-based decision-making approach for the IRP. Specifically, for the routing decisions, we have a budget B^t for the transportation cost in each period $t \in [T]$. For the replenishment decisions, we require the period-t ending inventory level $x_n^t\left(\tilde{\boldsymbol{d}}^{[t]}\right)$ to be within a pre-specified interval known as the requirement window: $[\underline{\tau}_n^t, \overline{\tau}_n^t]$. The requirement window can be determined by practical considerations. For example, $\underline{\tau}_n^t = 0$ indicates a strong preference for avoiding stockouts, and an upper bound $\overline{\tau}_n^t$ can be specified in accordance with the storage capacity which, if exceeded, would incur a high holding cost.

We summarize the decision variables and parameters in Table 1.

The joint constraints for the routing and replenishment decisions are presented as follows,

Parameters	**Description**
$c_a, a \in \mathcal{A}$	Transportation cost along the arc a;
$B^t, t \in [T]$	Transportation cost budget in period t;
$\tilde{d}_n^t, t \in [T], n \in [N]$	Uncertain demand of retailer n in period t.
Decision variables	**Description**
$y_0^t \in \{0,1\}, t \in [T]$	$y_0^t = 1$ if the vehicle is used in period t ($=0$ otherwise);
$y_n^t \in \{0,1\}, t \in [T], n \in [N]$	$y_n^t = 1$ if retailer n is replenished in period t ($=0$ otherwise);
$z_a^t \in \{0,1\}, a \in \mathcal{A}$	$z_a^t = 1$ if the arc a is traversed in period t ($=0$ otherwise);
$q_n^t : \mathbb{R}^{N(t-1)} \to \mathbb{R}, t \in [T], n \in [N]$	$q_n^t\left(\tilde{\boldsymbol{d}}^{[t-1]}\right)$ is the replenishment quantity of retailer n in period t;
$x_n^t : \mathbb{R}^{Nt} \to \mathbb{R}, t \in [T], n \in [N]$	$x_n^t\left(\tilde{\boldsymbol{d}}^{[t]}\right)$ is the inventory level of retailer n at the end of period t.

Table 1 **Parameters and decisions used in the formulations (1) - (3)**

$$
\begin{aligned}
& 0 \le q_n^t\left(\boldsymbol{d}^{[t-1]}\right) \le M y_n^t, \qquad \forall n \in [N], t \in [T], \boldsymbol{d} \in \mathcal{D}, \\
& (y_n^t)_{t \in [T], n \in [N]} \in \mathcal{Y}.
\end{aligned} \tag{2}
$$

Here as M is a large number, the first constraint uses the Big-M method to infer that the ordering decision can be made only when retailer n is served. In the second constraint, $\mathcal{Y}$ is the set of all feasible visiting decisions and defined as

$$
\mathcal{Y} = \left\{ (y_n^t)_{t\in[T],n\in[N]} \;\middle|\; \begin{array}{lll}
\exists\, (y_0^t)_{t\in[T]}, (z_a^t)_{t\in[T],a\in\mathcal{A}} & \text{satisfying} & \\
y_n^t \le y_0^t, & \forall t \in [T], n \in [N] & \text{(a)} \\
\sum_{a\in\delta^+(\{n\})} z_a^t = y_n^t, & \forall t \in [T], n \in [N] \cup \{0\} & \text{(b)} \\
\sum_{a\in\delta^-(\{n\})} z_a^t = y_n^t, & \forall t \in [T], n \in [N] \cup \{0\} & \text{(c)} \\
\sum_{a\in\delta^+(\mathcal{S})} z_a^t \ge y_k^t, \quad \text{??} & \forall t \in [T], \mathcal{S} \subseteq [N], |\mathcal{S}| \ge 2, k \in \mathcal{S} & \text{(d)} \\
\sum_{a\in\mathcal{A}} c_a z_a^t \le B^t, & \forall t \in [T] & \text{(e)} \\
y_0^t, y_n^t, z_a^t \in \{0,1\}, & \forall t \in [T], n \in [N], a \in \mathcal{A}. &
\end{array} \right\}. \tag{3}
$$

Constraints (3a)-(3d) are the standard routing constraints. Specifically, constraint (3a) ensures that, if any retailer n is served in period t, then the route in that period must "visit" the supplier at node 0. The constraints (3b) and (3c) guarantee that each visited node n has only one arc entering n and one arc exiting n. Constraint (3d) represents the subtour elimination constraint. With constraint (3e), the transportation cost in each period cannot exceed the pre-specified budget.

Our primary concern now is to evaluate the risk that the uncertain inventory level fails to satisfy the requirement. We propose a new performance measure to evaluate this risk and then present a comprehensive model.

2.2. Service Violation Index

We now introduce a service quality measure to evaluate the risk of service violations in terms of the inventory level. Inspired by Chen et al. (2015) and Jaillet et al. (2016), we propose a new and more general index to evaluate this risk. Our construction of this index is based on classic convex risk measures.

In particular, given an I-dimensional random vector $\tilde{\boldsymbol{x}} \in \mathcal{V}^I$, we must evaluate the risk that $\tilde{\boldsymbol{x}}$ realizes to be outside the required window $[\underline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}] \subseteq \mathbb{R}^I$. Toward that end, we propose the following concept of the Service Violation Index. We first define a function $v_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\cdot)$ as

$$v_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\boldsymbol{x}) = \max\{\boldsymbol{x} - \overline{\boldsymbol{\tau}}, \underline{\boldsymbol{\tau}} - \boldsymbol{x}\} = (\max\{x_i - \overline{\tau}_i, \underline{\tau}_i - x_i\})_{i \in [I]},$$

which is the violation of $\boldsymbol{x}$ with regard to the requirement window $[\underline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}]$. Specifically, a negative value of violation, e.g., $v_{0,3}(2) = -1$, implies that a change can still be made, in any arbitrary direction, without violating the requirement window.

Definition 1 *A function $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\cdot) : \mathcal{V}^I \to [0, \infty]$ is called a Service Violation Index (SVI) if, for all $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}} \in \mathcal{V}^I$, it satisfies the following properties.*

← state wise comparison.

1. *Monotonicity: $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) \geq \rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{y}})$ if $v_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) \geq v_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{y}})$.*
2. *Satisficing:*
   (a) Attainment content: $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = 0$ if $v_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) \leq \boldsymbol{0}$;
   (b) Starvation aversion: $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \infty$ if there exists an $i \in [I]$ such that $v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i) > 0$.
3. *Convexity: $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\lambda\tilde{\boldsymbol{x}} + (1-\lambda)\tilde{\boldsymbol{y}}) \leq \lambda\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) + (1-\lambda)\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{y}})$ for any $\lambda \in [0, 1]$.*

4. *Positive homogeneity:* $\rho_{\lambda\underline{\boldsymbol{\tau}},\lambda\overline{\boldsymbol{\tau}}}(\lambda\tilde{\boldsymbol{x}}) = \lambda\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}})$ *for any* $\lambda \geq 0$.

5. *Dimension-wise additivity:* $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \sum_{i\in[I]} \rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}((\tilde{x}_i, \boldsymbol{w}_{-i}))$ *for any* $\boldsymbol{w} \in [\underline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}]$.

6. *Order invariance:* $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \rho_{\boldsymbol{P}\underline{\boldsymbol{\tau}},\boldsymbol{P}\overline{\boldsymbol{\tau}}}(\boldsymbol{P}\tilde{\boldsymbol{x}})$ *for any permutation matrix* $\boldsymbol{P}$.

7. *Left continuity:* $\lim_{a\downarrow 0} \rho_{-\boldsymbol{\infty},\boldsymbol{0}}(\tilde{\boldsymbol{x}} - a\boldsymbol{1}) = \rho_{-\boldsymbol{\infty},\boldsymbol{0}}(\tilde{\boldsymbol{x}})$, *where* $(-\boldsymbol{\infty})$ *is the vector with all elements as* $-\infty$.

We define the SVI such that a low value indicates a low risk of violating the requirement window. Monotonicity states that if a given violation is always greater than another, then the former will be associated with higher risk and less preferred. The satisficing property specifies the riskiness in two extreme cases. Under attainment content, if every realization of the uncertain attributes lies within the requirement window, there is no risk of violation. By contrast, starvation aversion implies that risk takes the highest value when at least one attribute always violates the requirement window. Our convexity property mimics the preference for diversification in risk management. Positive homogeneity enables the cardinal nature, the motivation for which can be found in Artzner et al. (1999). Dimension-wise additivity and order invariance imply that overall risk is the aggregation of individual risks and insensitivity to the sequence of dimensions, respectively. Both properties are justified in a multi-stage setting by Chen et al. (2015). Finally, for solutions to be tractable, we need left continuity; together with other properties, it also allows us to represent the SVI in terms of a convex risk measure as follows.

Theorem 1 *A function* $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\cdot) : \mathcal{V}^I \to [0, \infty]$ *is a SVI if and only if it has the representation as*

$$\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \inf \left\{ \sum_{i\in[I]} \alpha_i \,\middle|\, \mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right) \leq 0, \alpha_i > 0,\ \forall i \in [I] \right\}, \tag{4}$$

where we define $\inf \emptyset = \infty$ *by convention and* $\mu(\cdot) : \mathcal{V} \to \mathbb{R}$ *is a normalized convex risk measure, i.e.,* $\mu(\cdot)$ *satisfies the following properties for all* $\tilde{x}, \tilde{y} \in \mathcal{V}$.

1. *Monotonicity:* $\mu(\tilde{x}) \geq \mu(\tilde{y})$ *if* $\tilde{x} \geq \tilde{y}$.

2. *Cash invariance:* $\mu(\tilde{x} + w) = \mu(\tilde{x}) + w$ *for any* $w \in \mathbb{R}$.

3. *Convexity:* $\mu(\lambda\tilde{x} + (1-\lambda)\tilde{y}) \leq \lambda\mu(\tilde{x}) + (1-\lambda)\mu(\tilde{y})$ *for any* $\lambda \in [0, 1]$.

4. *Normalization:* $\mu(0) = 0$.

Conversely, given an SVI $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\cdot)$*, the underlying convex risk measure is given by*

$$\mu(\tilde{x}) = \min\{a \,|\rho_{-\boldsymbol{\infty},\boldsymbol{0}}((\tilde{x}-a)\boldsymbol{e}_1) \le 1\}. \tag{5}$$

For the sake of brevity, we present all proofs in the Appendix.

Apart from illustrating the connection to convex risk measures, Theorem 1 also provides a simple way to construct the SVI from the convex risk measure. We now illustrate this through some interesting examples.

Example 1 (CVaR-based SVI) *One popular example of the convex risk measure is Conditional Value at Risk, which was developed by Rockafellar and Uryasev (2000, 2002). For any* $\eta \in [0,1)$*, it is defined as*

$$CVaR_\eta(\tilde{x}) = \inf_a \left\{ a + \frac{1}{1-\eta} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[(\tilde{x}-a)^+ \right] \right\}.$$

When we choose $\mu(\tilde{x}) = CVaR_\eta(\tilde{x})$*, the corresponding SVI can be represented as follows:*

$$\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \inf \left\{ \sum_{i\in[I]} \alpha_i \,\middle|\, \exists a \in \mathbb{R}, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i} + a \right)^+ \right] \le a(1-\eta), \alpha_i > 0, \forall i \in [I] \right\}.$$

Example 2 (Utility-based SVI) *Given any non-decreasing convex utility function normalized by* $u(0) = 0$ *and* $1 \in \partial u(0)$*, the function* $\mu_u(\cdot) : \mathcal{V} \to \mathbb{R}$ *defined as*

$$\mu_u(\tilde{x}) = \inf_\eta \left\{ \eta \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u(\tilde{x}-\eta) \right] \le 0 \right\}, \tag{6}$$

can be shown as a convex risk measure and is called the shortfall risk measure (Föllmer and Schied 2002). The corresponding SVI, which we refer to as the utility-based SVI, can be represented as follows.

$$\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \inf \left\{ \sum_{i\in[I]} \alpha_i \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i} \right) \right] \le 0, \alpha_i > 0, \forall i \in [I] \right\}. \tag{7}$$

Here, $\partial u(0)$ represents the partial derivative of the function u at 0, and the operation $\sup_{\mathbb{P}\in\mathcal{P}}$ in above examples allows us to take robustness into account. We present this proof in Appendix.

Remark Jaillet et al. (2016) propose a decision criterion, called the Requirements Violation Index (RVI), to quantify the risk associated with the violation of requirement windows. Our work differs from theirs in two ways. First, the RVI captures the risk of violating each side of the windows separately and then takes the maximum risk for the two sides as the risk of violating the window. Instead, we consider the violation from both sides jointly. Consider a single-dimensional case as an example, where $[\underline{\tau}, \bar{\tau}] = [0, 10]$, and $\tilde{x}$ is either -2 or 12 with equal probability. Then the RVI takes a finite value since from the perspective of either side, the risk is moderate. Nevertheless, our SVI would be infinite since there is a strictly positive violation of the window in both scenarios. For the IRP, our SVI is more appropriate since we need to incorporate any violation of the windows, not just on one side, on all sample paths. Second, while the RVI is based on exponential utility only, our utility-based SVI is constructed on general utility functions and hence includes the RVI as a special case when there is only one side to the requirement. It is also worthwhile to mention that being based on an exponential utility function, the RVI can be applied to the vehicle routing problems in Jaillet et al. (2016), where the random variables are independent and the targeting of uncertain performance (i.e., uncertain arrival time) is linear in the binary decision variables. Nevertheless, it cannot lead to computational tractability in our IRP where neither of the two requirements is met.

2.3. Model for IRP

Since the SVI is a general concept without an explicit form, we next use the utility-based SVI, in which the utility is a piecewise linear function, as the evaluation criterion to present our IRP formulation. We formalize it with the following assumption.

Assumption 1 *The decision maker minimizes the utility-based SVI, defined in Equation (7), for the violation of the ending inventories, where u is a piecewise linear utility function* $u(x) = \max_{k\in[K]}\{a_k x + b_k\}$ *with* $a_k \geq 0$, b_k *for all* $k \in [K]$ *and normalized by* $u(0) = 0$ *and* $1 \in \partial u(0)$.

Notice that due to the high complexity of the IRP, we need to select the objective function such that it does not add significantly to the computational burden. Therefore, we adopt this piecewise linear utility u, which preserves the optimization problem's linear structure and hence eases the computational burden even while accounting for risk. In practice, a piecewise linear utility can be used to approximate any increasing convex utility. The number of pieces depends on how precise the decision maker wants this approximation to be. In addition, the choice of the underlying utility function can follow standard approaches in expected utility theory, such as by questionnaire (see e.g., Grable and Lytton 1999).

Note that by Equation (7) and with u being the piecewise utility function in Assumption 1, the utility-base SVI can be reformulated as

$$\begin{aligned}\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) &= \inf\left\{\sum_{i\in[I]}\alpha_i \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\left[\max_{k\in[K]}\left\{a_k\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}+b_k\right\}\right]\leq 0, \alpha_i>0, \forall i\in[I]\right\}\\ &= \inf\left\{\sum_{i\in[I]}\alpha_i \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\left[\max_{k\in[K]}\left\{a_k v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)+b_k\alpha_i\right\}\right]\leq 0, \alpha_i>0, \forall i\in[I]\right\},\end{aligned}$$

where the equality follows from $\alpha_i > 0$, $\forall i \in [I]$. Therefore, with Assumption 1, we evaluate the overall risk of inventory violation for all nodes $n \in [N]$ in all periods $t \in [T]$ and present our model as follows.

$$\boxed{\begin{aligned} Z^* = \inf\ & \sum_{t\in[T]}\sum_{n\in[N]}\alpha_n^t \\ \text{s.t. } & \sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\left[\max_{k\in[K]}\left\{a_k v_{\underline{\tau}_n^t,\overline{\tau}_n^t}\left(x_n^t\left(\tilde{\boldsymbol{d}}^{[t]}\right)\right)+b_k\alpha_n^t\right\}\right]\leq 0, && \forall n\in[N], t\in[T] \\ & x_n^t\left(\boldsymbol{d}^{[t]}\right) = \sum_{m\in[t]}\left(q_n^m\left(\boldsymbol{d}^{[m-1]}\right)-d_n^m\right), && \forall n\in[N], t\in[T], \boldsymbol{d}\in\mathcal{D} \\ & 0\leq q_n^t\left(\boldsymbol{d}^{[t-1]}\right)\leq M y_n^t, && \forall n\in[N], t\in[T], \boldsymbol{d}\in\mathcal{D} \\ & \alpha_n^t\geq\epsilon, && \forall n\in[N], t\in[T] \\ & (y_n^t)_{t\in[T],n\in[N]}\in\mathcal{Y}. \end{aligned}} \tag{8}$$

Instead of using the constraint $\alpha_n^t > 0$ as in the definition (7) directly, here we use $\alpha_n^t \geq \epsilon$ so that the feasible set will be closed. Indeed, we can always choose a positive ϵ small enough that optimality is not compromised (Chen et al. 2015).

3. Scenario-based uncertainties and affinely adaptive policies

The main problem (8) involves an inner optimization over a set of possible distributions, i.e., the operation of $\sup_{\mathbb{P}\in\mathcal{P}}$ in the first constraint. Therefore, the optimization procedure inevitably depends on the availability of information about the uncertain demand. In this section, we propose to use a robust stochastic optimization approach by defining $\mathcal{P}$ to be a scenario-based uncertainty set. This extends the approach by Chen et al. (2020) and integrates the scenario-tree-based stochastic optimization and distributionally robust optimization into a unified framework.

We consider that in each period $t\in[T]$, there is a random scenario $\tilde{s}_t$ with a total of S_t realizations. This scenario is categorical, in the sense that it can take any realization within the set $[S_t]$. For example, $\tilde{s}_t$ can be the uncertain weather with $S_t = 2$, $\tilde{s}_t = 1$ indicating a sunny day and $\tilde{s}_t = 2$ implying a rainy day. We denote a set

$$\mathcal{S} = \left\{ \boldsymbol{s} \in \Re^T : s_t \in [S_t], t \in [T] \right\}$$

as the set of all possible scenarios in the scenario tree, and label the elements in $\mathcal{S}$ with an arbitrary order such that $\mathcal{S} = \{\boldsymbol{s}^w : w \in [W]\}$ (obviously $W = \Pi_{t\in[T]} S_t$). Conditioning on the realization of the random scenario, the demand $\tilde{\boldsymbol{d}}^t$ could have different distributional information. Formally, the scenario-based ambiguity set $\mathcal{P}$ is defined as follows,

$$\mathcal{P} = \left\{ \mathbb{P} \left| \begin{array}{lll} \mathbb{P}\left(\underline{\boldsymbol{d}}^{t,s} \le \tilde{\boldsymbol{d}}^t \le \overline{\boldsymbol{d}}^{t,s} \;\middle|\; \tilde{s}_t = s \right) = 1, & \forall s \in [S_t], t\in[T] & \text{(a)} \\ \mathbb{E}_{\mathbb{P}}\left[\tilde{\boldsymbol{d}}^t \;\middle|\; \tilde{s}_t = s \right] = \boldsymbol{\mu}^{t,s}, & \forall s \in [S_t], t\in[T] & \text{(b)} \\ \mathbb{E}_{\mathbb{P}}\left[\left| \tilde{\boldsymbol{d}}^t - \boldsymbol{\mu}^{t,s} \right| \;\middle|\; \tilde{s}_t = s \right] \le \boldsymbol{\sigma}^{t,s}, & \forall s \in [S_t], t\in[T] & \text{(c)} \\ \mathbb{E}_{\mathbb{P}}\left[\left| \sum_{n\in\mathcal{N}_h} \frac{\tilde{d}^t_n - \mu^{t,s}_n}{\sigma^{t,s}_n} \right| \;\middle|\; \tilde{s}_t = s \right] \le \epsilon^{t,s}_h, & \forall s \in [S_t], t\in[T], h\in[H] & \text{(d)} \\ \mathbb{P}\left(\tilde{\boldsymbol{s}} = \boldsymbol{s}^w \right) = p_w, & \forall w \in [W] & \text{(e)} \\ \boldsymbol{p} \in \Xi & & \text{(f)} \end{array} \right. \right\}. \qquad (9)$$

In period $t\in[T]$, given the uncertain scenario $\tilde{s}_t$ realizing as $s\in[S_t]$, constraints (9a)-(9c) specify the conditional bound (i.e., $\underline{\boldsymbol{d}}^{t,s}$, $\overline{\boldsymbol{d}}^{t,s}$), mean (i.e., $\boldsymbol{\mu}^{t,s}$), and bound for its mean absolute deviation (i.e., $\boldsymbol{\sigma}^{t,s}$) of $\tilde{\boldsymbol{d}}^t$. Similar to the standard deviation, the mean absolute deviation in (9c) provides a direct measure of the uncertain demand's dispersion about its mean; in addition, it can be obtained in a closed form for many common distributions (see e.g., Pham-Gia and Hung, 2001). For example, if $\tilde{d}_n^t$ is distributed uniformly on $[0,1]$, then the mean absolute deviation takes the value $1/4$. If $\tilde{d}_n^t$ follows a normal distribution, its mean absolute deviation amounts to its standard deviation multiplied by $\sqrt{2/\pi}$.

The information in (9d) characterizes the correlation between the random demands at nodes in a given set $\mathcal{N}_h$, $\forall h \in [H]$. In practice, the respective demands of these retailers may be correlated because retailers, in general, are geographically dispersed within a nearby region and serve common customers. As an illustrative example, we consider $\mathcal{N}_1 = [N]$ and have $\mathbb{E}_{\mathbb{P}}\left[\left|\sum_{n\in[N]} \frac{\tilde{d}_n^t - \mu_n^{t,s}}{\sigma_n^{t,s}}\right| \middle| \tilde{s}_t = s\right] \leq \epsilon_1^{t,s}$. Conditioning on the scenario $\tilde{s}_t = s$, this bounds on the variation of total normalized demand over all retailers in period t. We will show later that our problem remains tractable with this joint dispersion information. This representation is important for the computation since with conventional approaches that characterize the correlation effect, for example, the covariance matrix, the inventory management problem is computationally challenging even when it is a single-period problem.

Regarding the discrete uncertain scenarios, (9e) specifies the probability distribution for each scenario tree. Moreover, by (9f) we allow this probability distribution, $\boldsymbol{p}$, to have ambiguity as well, and we only know it belongs to a polyhedron Ξ.

Incorporating a random scenario tree in the ambiguity set $\mathcal{P}$ enhances the modeling power substantially. One example is that it can capture a multimodal distribution. We refer interested readers to Chen et al. (2020) for a more detailed discussion. We note that in Chen et al. (2020), there is only one random scenario that is fully realized in the first period, and its realization determines the information of uncertainties in all periods. Alternatively, in the uncertainty set $\mathcal{P}$ defined by Equation (9), we have an uncertain scenario s_t in each period t, and the distributional information for uncertainties $\tilde{\boldsymbol{d}}^t$ would only be affected by the uncertain scenario in the same period, i.e., $\tilde{s}_t$. This representation can recover the ambiguity set in Chen et al. (2020), by choosing Ξ to enforce $\tilde{s}_t = \tilde{s}_1$

for all $t \in [T]$ and hence the distributional information for the demand in all periods only depends on $\tilde{s}_1$. As we will show later, since our ambiguity set has a random scenario in each period, the non-anticipativity of the decision can be modeled explicitly without additional constraints, which is not the case for the ambiguity set in Chen et al. (2020). Moreover, this scenario-based ambiguity set $\mathcal{P}$ also includes the empirical distribution as a special case by choosing $\underline{\boldsymbol{d}}^{t,s} = \bar{\boldsymbol{d}}^{t,s} = \boldsymbol{\mu}^{t,s}$ and hence in each scenario there is only one possible demand realization. In the rest of the paper, we focus only on the general ambiguity set $\mathcal{P}$ defined in Equation (9).

Like many other tractable adaptive robust optimization models, for the sake of tractability, we require the replenishment decisions to be affinely dependent on the past demand realization, which is the so-called linear decision rule approach. Nevertheless, unlike the classic approaches where the linear coefficients are universally identical for all demand realizations, with the scenario-based ambiguity set $\mathcal{P}$, we let the linear coefficients be dependent on the realization of the random scenarios.

Formally, we denote $\boldsymbol{s}^{[t]} = (s_1, \ldots, s_t)$ for any $t \in [T]$ and $\boldsymbol{s}^{[0]} = \emptyset$, and let the replenishment decisions be

$$q_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}) = q_{n,0}^t(\boldsymbol{s}^{[t-1]}) + \sum_{i=1}^{t-1} \left(\boldsymbol{q}_{n,i}^t(\boldsymbol{s}^{[t-1]})\right)^\top \boldsymbol{d}^i, \qquad \forall (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}},$$

where $\hat{\mathcal{D}}$ is the set of all possible realizations of $(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{d}})$ and can be represented as

$$\hat{\mathcal{D}} = \left\{ (\boldsymbol{s}, \boldsymbol{d}) \ : \ \text{for each } t \in [T],\ s_t \in [S_t] \text{ with } \underline{\boldsymbol{d}}^{t,s_t} \le \boldsymbol{d}^t \le \bar{\boldsymbol{d}}^{t,s_t} \right\}.$$

With this scenario-based linear decision rule, we can reformulate Problem (8) as follows,

$$
\begin{aligned}
Z_S^* = \inf \ & \sum_{t\in[T]} \sum_{n\in[N]} \alpha_n^t, \\
\text{s.t.} \ & \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[\max_{k\in[K]} \left\{ a_k v_{\underline{\tau}_n^t, \bar{\tau}_n^t} \left(x_n^t \left(\tilde{\boldsymbol{s}}^{[t]}, \tilde{\boldsymbol{d}}^{[t]} \right) \right) + b_k \alpha_n^t \right\} \right] \le 0, \ \forall n \in [N], t \in [T] && \text{(a)} \\
& x_n^t \left(\boldsymbol{s}^{[t]}, \boldsymbol{d}^{[t]} \right) = \sum_{m\in[t]} \left(q_n^m \left(\boldsymbol{s}^{[m-1]}, \boldsymbol{d}^{[m-1]} \right) - d_n^m \right), \quad \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}} && \text{(b)} \\
& 0 \le q_n^t \left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]} \right) \le M y_n^t, \quad \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}} && \text{(c)} \\
& q_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}) = q_{n,0}^t(\boldsymbol{s}^{[t-1]}) + \sum_{i=1}^{t-1} \left(\boldsymbol{q}_{n,i}^t(\boldsymbol{s}^{[t-1]}) \right)^\top \boldsymbol{d}^i, \ \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}} && \text{(d)} \\
& \alpha_n^t \ge \epsilon, \quad \forall n \in [N], t \in [T] && \text{(e)} \\
& (y_n^t)_{t\in[T], n\in[N]} \in \mathcal{Y}. && \text{(f)}
\end{aligned}
\tag{10}
$$

We now demonstrate that, with the scenario-based ambiguity set $\mathcal{P}$ defined by Equation (9), the constraints in Problem (10) can be transformed into linear ones.

Proposition 1 *Let $\mathcal{P}$ be defined in the equation (9). Then the constraints (10a)-(10d) can be reformulated as a set of linear constraints.*

The equivalent linear constraints are presented in the Appendix when proving Proposition 1.

With Proposition 1, our uncertain IRP (10) can be transformed to a deterministic optimization problem with an equivalent linear structure. Its computational tractability is further enhanced later in Section 4 and demonstrated by the computational studies in Section 5.

Relationship with classic policies

Although our linear decision rule approach, enforced by the constraint (10d), leads to an approximate solution to the uncertain IRP, we now show that it does indeed improve two commonly used ordering policies in the classic IRP.

We start from the order-up-to level (OU) policy, where once visited, the ordering quantity in period t, node n is determined as

$$
q_n^t \left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]} \right) = U_n - x_n^{t-1} \left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]} \right) \qquad \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}}, \tag{11}
$$

with $U_n \geq x_n^0$ being a given constant to represent the maximum inventory level of retailer n (Archetti et al. 2007). It implies that as long as a node is visited, we will replenish the stock up to the maximum level. Formally, we define the IRP under the OU policy being the same as Problem (10), except that the constraint (10d) is replaced by the following constraints,

$$\begin{aligned} q_n^t\left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}\right) &\geq U_n y_n^t - x_n^{t-1}\left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}\right) \ \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}}, \\ q_n^t\left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}\right) &\leq U_n - x_n^{t-1}\left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}\right) \quad \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}}, \end{aligned} \tag{12}$$

which, together with (10c), enforce the ordering quantity in (11) for all visits. We denote the corresponding optimal value as Z_{OU}. In other words, Z_{OU} is the optimal SVI under the OU policy.

Proposition 2 *The OU policy defined in the inequalities (10c) and (12) is an affine function of the uncertain demand. Hence, we have $Z_S^* \leq Z_{OU}$.*

Another policy of interest in the IRP is the maximum-level (ML) policy, where the ordering quantity in period t, node n is constrained by

$$q_n^t\left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}\right) \leq U_n - x_n^{t-1}\left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}\right) \qquad \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}}. \tag{13}$$

This indicates that after replenishment, the inventory cannot exceed the maximum level. Unlike the OU policy, the ML policy implies no explicit dependence between the ordering decision and the past realizations of uncertainty. Hence, typically it does not ease the computational burden. Nevertheless, in our IRP with the SVI, the ML policy can be shown to perform equally with the OU policy and hence is dominated by our linear decision rule approach when the maximum level U_n satisfies certain conditions. As in the case of the OU policy, we define the IRP under the ML policy to be identical to Problem (10), except that (10d) is replaced by (13), and let the corresponding optimal value be Z_{ML}. That means that Z_{ML} is the optimal SVI under the ML policy.

Proposition 3 *If $U_n \leq \overline{\tau}_n^t + \min_{s \in [S_t]} \underline{d}_n^{t,s}$ $\forall n \in [N], t \in [T]$, we have $Z_S^* \leq Z_{OU} = Z_{ML}$.*

4. Algorithm

In this section, we focus on facilitating the solution procedure for our IRP (10).

Exclusion of suboptimal routes

As discussed in Section 1, we consider the uncertain IRP from the service point of view. Consequently, while the classic IRP minimizes the transportation cost and hence involves the vehicle routing problem as a subproblem, our IRP only requires the routing decision to satisfy the constraint imposed by a transportation cost budget. Driven by this budget constraint, our model can be shown to ease the computational complexity. We first define a set of maximal visiting nodes as follows.

Definition 2 *Given any transportation cost budget $B \in \Re_+$, we call the set $\mathcal{H}_B \subseteq 2^{[N]}$ the set of maximal visiting nodes with B, and define it as*

$$\mathcal{H}_B = \left\{ \mathcal{C} \subseteq [N] \mid \mathcal{C} \in \hat{\mathcal{H}}_B,\ \nexists \hat{\mathcal{C}} \in \hat{\mathcal{H}}_B \text{ with } \mathcal{C} \subset \hat{\mathcal{C}} \right\}$$

where

$$\hat{\mathcal{H}}_B = \{\mathcal{C} \subseteq [N] \mid \exists \text{a route visiting all nodes in } \mathcal{C} \cup \{0\} \text{ with transportation cost no more than } B\}.$$

We now show that in Problem (10), by exploiting the structure of maximal visiting nodes, we can avoid using the set $\mathcal{Y}$ and hence do not need to solve the embedded vehicle routing problems.

Definition 3 *For any $t \in [T]$, we denote the cardinality of $\mathcal{H}_{B^t}$ by h_t, and label the elements in $\mathcal{H}_{B^t}$ arbitrarily such that*

$$\mathcal{H}_{B^t} = \{\mathcal{C}_1^t, \mathcal{C}_2^t, \ldots, \mathcal{C}_{h_t}^t\};$$

moreover, $\forall n \in [N], i \in [h_t]$, we denote the constant $u_{i,n}^t = 1$ if $n \in \mathcal{C}_i^t$ and $u_{i,n}^t = 0$ otherwise.

Given the pre-determined parameters $u_{i,n}^t, i \in [h_t], t \in [T], n \in [N]$, we can reformulate the routing constraints by the following proposition.

Proposition 4 *In Problem (10), the optimal value remains unchanged if we replace the constraint (10f) by the following set of constraints,*

$$\begin{aligned}
&\sum_{i=1}^{h_t} l_i^t = 1, && \forall t \in [T] \\
&y_n^t = \sum_{i=1}^{h_t} l_i^t u_{i,n}^t, && \forall t \in [T], n \in [N] \\
&l_i^t \in \{0,1\}, && \forall i \in [h_t], t \in [T].
\end{aligned} \tag{14}$$

Unlike $\mathcal{Y}$, which includes all feasible routing decisions, the set $\mathcal{H}_{B^t}$ excludes all sets of visiting nodes if they are a subset of another set. Hence, the number of feasible routing decisions to consider is reduced substantially. Further, because of the low cardinality, the set $\mathcal{H}_{B^t}$ can be derived by an offline process before the joint optimization of routing and replenishment decisions. Then, the computational burden can be eased in many cases.

We now provide the offline process to generate $\mathcal{H}_{B^t}$ for any given $t \in [T]$, and the constraints (14) can be obtained correspondingly.

Algorithm 1: Generation of the set of maximal visiting nodes $\mathcal{H}_B$

Input: Budget $B \in \Re_+$.

1. Initialize $n = N$, $\mathcal{I} = \{[N]\}$, $\mathcal{H}_B = \emptyset$ and $k_n = 1$.

2. For each element $\mathcal{C} \in \mathcal{I}$, solve a traveling salesman problem defined on $(\mathcal{C} \cup \{0\}, \mathcal{A}(\mathcal{C} \cup \{0\}))$. Label the elements in $\mathcal{I}$ and define $l_n \in \{0, 1, \ldots, k_n\}$ such that $\mathcal{I} = \left\{\mathcal{C}_1^n, \ldots, \mathcal{C}_{k_n}^n\right\}$ and $\mathcal{C}_1^n, \ldots, \mathcal{C}_{l_n}^n$ are with traveling salesman cost no greater than B, $\mathcal{C}_{l_n+1}^n, \ldots, \mathcal{C}_{k_n}^n$ are with traveling salesman cost strictly greater than B. Update $\mathcal{H}_B = \mathcal{H}_B \bigcup \{\mathcal{C}_1^n, \ldots, \mathcal{C}_{l_n}^n\}$.

3. If $l_n < k_n$, update $\mathcal{I} = \{\mathcal{C} : |\mathcal{C}| = n-1, \mathcal{C} \subset \mathcal{C}_i^n \text{ with } i \in \{l_n+1, \ldots, k_n\}, \mathcal{C} \notin \mathcal{H}_B\}$, $n = n-1$ and $k_n = |\mathcal{I}|$. Then go to Step 2. If $l_n = k_n$, terminate.

Output: $\mathcal{H}_B$.

Proposition 5 *Given any cost budget $B \in \mathbb{R}_+$ as an input, the output of Algorithm 1 is the set of maximal visiting nodes $\mathcal{H}_B$ defined in Definition 2.*

Algorithm 1 is not of polynomial time and the complexity depends highly on the cost budget B. We acknowledge that there are some values of B for which the associated $\mathcal{H}_B$ has high cardinality and hence Algorithm 1 takes time. Practically, our computational examples in Section 5 show that for a wide range of values that B can take, Algorithm 1 can be terminated efficiently and return a set with low cardinality. As a result, the optimization problem based on constraints (14) is much more efficient than that based on typical routing formulation (i.e., $\mathcal{Y}$).

Exclusion of infeasible routes

By exploiting the structure of the SVI, we can also apply valid arc reduction techniques to exclude some routing decisions, hence reducing the size of the solution space and expediting the optimization procedure. Specifically, given any node $n \in [N]$, period $t \in [T-1]$ and $t_\delta \in [T-t]$, we define

$$\begin{aligned} Z_n^{t,t_\delta} = \min_{\hat{U}_n^t \in \mathbb{R}} \quad & 0 \\ \text{s.t.} \quad & \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[v_{\underline{\tau}_n^i, \bar{\tau}_n^i} \left(x_n^i \left(\tilde{\boldsymbol{s}}^{[i]}, \tilde{\boldsymbol{d}}^{[i]} \right) \right) \right] \leq 0, && \forall i \in \{t+1, \ldots, t+t_\delta\} \\ & x_n^i \left(\boldsymbol{s}^{[t]}, \boldsymbol{d}^{[t]} \right) = \hat{U}_n^t - \sum_{m=t}^{i} d_n^m, && \forall i \in \{t+1, \ldots, t+t_\delta\}, (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}}. \end{aligned} \tag{15}$$

Here $\hat{U}_n^t$ is a decision variable and represents the inventory order-up-to level in period t at node n. Problem (15) describes the problem that for the next t_δ periods after period t, whether the expected violation is positive or not if node n is visited in period t but no future visit will be made. If all of these expected violations are non-positive, then Problem (15) is feasible and $Z_n^{t,t_\delta} = 0$; otherwise $Z_n^{t,t_\delta} = \infty$.

Like Proposition 1, the constraints in Problem (15) can be reformulated equivalently as a set of deterministic linear constraints and solved efficiently by an offline procedure. We let

$$k_n^t = \begin{cases} T - t + 1, & \text{if } Z_n^{t,t_\delta} = 0, \forall t_\delta \in [T-t], \\ \min\{t_\delta \ : \ Z_n^{t,t_\delta} = \infty\}, & \text{otherwise}, \end{cases}$$

and then establish valid inequalities as follows for Problem (10).

Proposition 6 *The following constraints are valid inequalities for Problem (10),*

$$\sum_{i=t+1}^{t+k_n^t} y_n^i \geq 1, \qquad \forall n \in [N], t \in [T-1] \text{ with } t + k_n^t \leq T.$$

We have shown that our IRP under uncertainty can be formulated as a mixed integer-linear optimization problem and have also introduced different ways to expedite its computation. Then, we can use the Benders (1962) decomposition approach (for additional details, see Rahmaniani et al. 2017) to solve it efficiently.

5. Computational Study

In this section, we carry out computational studies to show that our proposed model is practically solvable and can yield an attractive solution under uncertainty. The program is coded in python and performed on a PC with an Intel Core i7-9700 3.00 GHz CPU and 8.00 GB RAM using CPLEX 12.10.0. The data and source code can be found in the electronic companion.

5.1. Comparison between our approaches and benchmark approaches

We evaluate the benefit of our approach by comparing the quality of solutions from six approaches. We use the approach proposed by Bertsimas et al. (2016) as our benchmark approach since it also considers the service level requirement without explicitly specifying the holding cost and backlogging cost. We compare it with a model that minimizes the probability of inventory violation, and four variants of our SVI method. For simplicity, in all variants of the SVI model, we choose $u(x) = \max\{-1, x\}$ as the underlying utility function. We next introduce these six approaches briefly.

Robust transportation cost minimization approach (Cost-R) This approach is proposed in Bertsimas et al. (2016). In the model, each customer n has a maximum inventory capacity, U_n, and no stockout is allowed. Nevertheless, in the Cost-R model, the inventory requirement window is a deterministic constraint and has to be satisfied for all demand realizations in an ambiguity set $\mathcal{U}$, which is constructed as follows:

$$\mathcal{U} = \left\{ \boldsymbol{d} \left| \left| \frac{\sum\limits_{t\in[T]} (d_n^t - \mu_n)}{\sigma_n \sqrt{T}} \right| \leq \Gamma_n, \mu_n - \phi\sigma_n \leq d_n^t \leq \mu_n + \phi\sigma_n, \forall n \in [N], t \in [T] \right. \right\}. \tag{16}$$

Here μ_n and σ_n can be considered to be the mean and standard deviation of the demand of retailer n, Γ_n and ϕ represent the controls on the size of the uncertainty set. In this experiment, we select

Γ_n such that $\left|\sum_{t\in[T]} (d_n^t - \mu_n)/\sigma_n\sqrt{T}\right| \le \Gamma_n$ is with probability 95% for the samples in the training data. The value of ϕ will be varied to provide comprehensive comparisons.

For tractability, this approach also adopts a linear decision rule method. The aim is to minimize the total transportation cost. Formally, the model is defined as follows,

$$
\begin{aligned}
\min\ & \sum_{t\in[T]}\sum_{\theta\in[\Theta]} c_\theta y_\theta^t \\
\text{s.t. } & 0 \le x_n^t\left(\boldsymbol{d}^{[t]}\right) \le U_n, && \forall \boldsymbol{d}\in\mathcal{U}, t\in[T], n\in[N] \\
& 0 \le q_n^t\left(\boldsymbol{d}^{[t-1]}\right) \le M\sum_{\theta\in[\Theta]} u_{\theta,n} y_\theta^t, && \forall \boldsymbol{d}\in\mathcal{U}, t\in[T], n\in[N] \\
& q_n^t\left(\boldsymbol{d}^{[t-1]}\right) = q_{n,0}^t + \sum_{i\in[t-1]} (\boldsymbol{q}_{n,i}^t)^\top \boldsymbol{d}^i, && \forall \boldsymbol{d}\in\mathcal{U}, t\in[T], n\in[N] \\
& x_n^t\left(\boldsymbol{d}^{[t]}\right) = \sum_{m\in[t]}\left(q_n^m\left(\boldsymbol{d}^{[m-1]}\right) - d_n^m\right), && \forall \boldsymbol{d}\in\mathcal{U}, t\in[T], n\in[N] \\
& \sum_{\theta\in[\Theta]} y_\theta^t \le 1, && \forall t\in[T] \\
& y_\theta^t \in \{0,1\}, && \forall \theta\in[\Theta], t\in[T].
\end{aligned}
$$

Here Θ is the number of all possible routes. For each route $\theta\in[\Theta]$, the parameter $u_{\theta,n}$ is 1 if it visits node n and 0 otherwise; correspondingly, c_θ is the transportation cost if selecting the route θ. M is a sufficiently large scalar to allow the use of the big-M method.

Minimizing violation probability (MVP) Since we focus on the service level, a natural benchmark is to minimize the probability of an inventory requirement violation, that is, to minimize $\frac{1}{NT}\sum_{n\in[N]}\sum_{t\in[T]}\mathbb{P}(v_{\underline{\tau}_n^t,\bar{\tau}_n^t}(\tilde{x}_n^t) > 0)$. Because this MVP model is highly intractable, we will use the sampling average approximation to derive the optimal solution. More specifically, we use the training data as the sample paths, the number of which is denoted by M_c. We let $d_n^{t,m}$ be the demand in period t, node n in the sample path m, $m\in[M_c]$. Then the problem of MVP can be formulated as

$$
\begin{aligned}
\min \quad & \frac{1}{M_c NT} \sum_{m\in[M_c]} \sum_{t\in[T]} \sum_{n\in[N]} I_n^{t,m} \\
\text{s.t.} \quad & I_n^{t,m} \geq \frac{x_n^{t,m} - \overline{\tau}_n^t}{M}, && \forall m \in [M_c], n \in [N], t \in [T] \\
& I_n^{t,m} \geq \frac{\underline{\tau}_n^t - x_n^{t,m}}{M}, && \forall m \in [M_c], n \in [N], t \in [T] \\
& I_n^{t,m} \in \{0,1\}, && \forall m \in [M_c], n \in [N], t \in [T] \\
& x_n^{t,m} = \sum_{i\in[t]} \left(q_n^i - d_n^{i,m} \right), && \forall m \in [M_c], n \in [N], t \in [T] \\
& 0 \leq q_n^t \leq M y_n^t, && \forall n \in [N], t \in [T] \\
& (y_n^t)_{t\in[T], n\in[N]} \in \mathcal{Y}.
\end{aligned}
$$

Here M is a sufficiently large scalar for the use of the big-M method.

Robust approach with scenario information (SVI-RS) The third approach, which we dub the SVI-RS model, derives the optimal solution based on Problem (10) with the ambiguity set (9).

Robust approach without scenario information (SVI-R) The fourth approach is similar to the previous SVI-RS approach, except that we do not incorporate any scenario information. In other words, we let $S_t = 1$ and $\tilde{s}_t = 1$ almost surely for any day t.

Stochastic approach (SVI-S) The fifth approach, which we call the SVI-S model, is the SVI approach with an exact discrete probability distribution. As mentioned in Section 3, the discrete distribution can also be considered to be a special case of the ambiguity set (9) and hence the solution procedure simply follows Sections 3 and 4.

SVI with the order-up-to level policy (SVI-OU) The last approach adopts an order-up-to level policy instead of a linear decision rule. The problem is defined as like Problem (10), except that the constraint (10d) is replaced by constraints (12) and we do not incorporate any scenario information.

Since we assume the initial inventory is 0, the replenishment quantity for the first period is $q_n^1 = U_n$ for any $n \in [N]$. For $t \in [T] \setminus \{1\}$, once the retailer has been visited, the replenishment quantity is $q_n^t = \sum_{i=t'}^{t-1} d_n^i$, where t' is the time when the last replenishment happened. However, the problem is still not solvable with this formulation since q_n^t involves the unknown index t'. Thus we introduce an auxiliary variable $\hat{q}_n^{t,i}$ to indicate whether the demand of period i is included in the replenishment quantity of period t and $c_n^{t,i}$ to denote whether replenishment is made from period $i+1$ to $t-1$. Specifically, $c_n^{t,i}=0$ if no replenishment is made from period $i+1$ to $t-1$, otherwise $c_n^{t,i}=1$. We formulate the SVI model when the replenishment follows such OU policy as follows.

Proposition 7 *When the replenishment follows the OU policy, the inventory routing problem under SVI can be formulated as the following optimization problem.*

$$
\begin{aligned}
\inf \ & \sum_{t\in[T]}\sum_{n\in[N]} \alpha_n^t \\
s.t.\ & \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[\max_{k\in[K]}\left\{a_k v_{\underline{\tau}_n^t,\bar{\tau}_n^t}\left(x_n^t\left(\tilde{\boldsymbol{d}}^{[t]}\right)\right)+b_k\alpha_n^t\right\}\right]\le 0, && \forall n\in[N], t\in[T] \\
& x_n^t\left(\boldsymbol{d}^{[t]}\right)=\sum_{m\in[t]}\left(q_n^m\left(\boldsymbol{d}^{[m-1]}\right)-d_n^m\right), && \forall n\in[N], t\in[T], \boldsymbol{d}\in\mathcal{D} \\
& q_n^1=U_n, && \forall n\in[N] \\
& q_n^t\left(\boldsymbol{d}^{[t-1]}\right)=\sum_{i=1}^{t-1}\hat{q}_n^{t,i}d_n^i, && \forall n\in[N], t\in[T]\setminus\{1\}, \boldsymbol{d}\in\mathcal{D} \quad (a) \\
& \hat{q}_n^{t,t-1}=y_n^t, && \forall n\in[N], t\in[T]\setminus\{1\} \quad (b) \\
& 0\le \hat{q}_n^{t,i}\le y_n^t, && \forall n\in[N], i\in[t-2], t\in[T]\setminus\{1,2\} \quad (c) \\
& y_n^t-c_n^{t,i}\le \hat{q}_n^{t,i}\le 1-c_n^{t,i}, && \forall n\in[N], i\in[t-2], t\in[T]\setminus\{1,2\} \quad (d) \\
& y_n^j\le c_n^{t,i}\le 1, && \forall n\in[N], j\in[t-1]\setminus[i] \quad (e) \\
& 0\le c_n^{t,i}\le \sum_{j=i+1}^{t-1} y_n^j, && \forall n\in[N], i\in[t-2], t\in[T]\setminus\{1,2\} \quad (f) \\
& \alpha_n^t\ge\epsilon, && \forall n\in[N], t\in[T] \\
& (y_n^t)_{t\in[T],n\in[N]}\in\mathcal{Y}.
\end{aligned}
\tag{17}
$$

For the case of $y_n^t=0$, i.e., retailer n is not visited at period t, then for all $i\in[t-1]$, we have the coefficient $\hat{q}_n^{t,i}=0$ by constraints (17b) and (17c), and thus $q_n^t=0$ based on constraint (17a). For the case of $y_n^t=1$, by (17b), $\hat{q}_n^{t,t-1}=1$, which means that once the node has been visited, the replenishment quantity includes at least the demand of the preceding period. Moreover, for all $i\in[t-2]$, we have $\hat{q}_n^{t,i}=1-c_n^{t,i}$ by (17d). This implies that if no replenishment is made in period $i+1$ to $t-1$, i.e., $c_n^{t,i}=0$, then $\hat{q}_n^{t,i}=1$ and the demand quantity d_n^i is included in q_n^t. Constraints

(17e) and (17f) indicate that $c_n^{t,i} = 1$ as long as the node is visited at least once in the period $i+1$ to $t-1$ and $c_n^{t,i} = 0$ otherwise (when $\sum_{j=i+1}^{t-1} y_n^j = 0$).

Based on Proposition 1, we can transform this problem into a linear optimization problem.

5.1.1. On a real data set To assess the quality of our solutions, we work on an instance characterized by real-world data from a major oil company. This company is responsible for supplying fuel to a set of filling stations. It needs to make daily decisions on which station to refuel, how much to refuel, and which route to follow to visit those stations. We consider 10 filling stations and let $T = 4$ in this study to guarantee that we can derive solutions for all approaches within a reasonable time (larger instances will be tested later to demonstrate the computational efficiency). Figure 2 shows the relative location of the supplier (node 0) and filling stations. The transportation cost between any two nodes is considered to be proportional to their Euclidean distance. We have demand data for 182 days, ranging from February 20th, 2019 to August 20th, 2019. We use the data from the first four months as "training data" and extract the distributional information for the demands. The data in the last two months are used as "testing data" for out-of-sample testing.

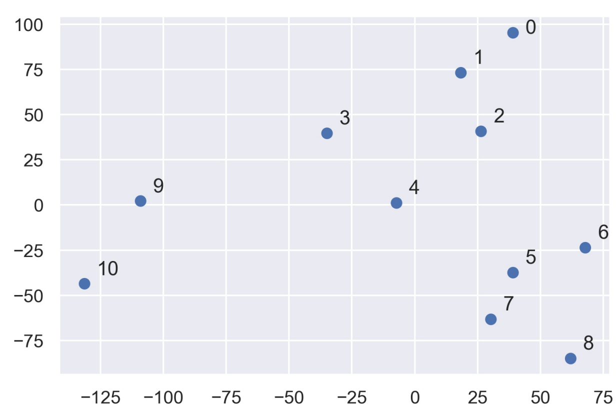

Figure 2 **Locations of the supplier and 10 filling stations.**

For SVI-RS, noticing that a primary feature affecting oil demand is the weather, we use the weather on each day as the random scenario. It has two realizations $S_t = 2$, and $\tilde{s}_t = 1$ stands for rainy and $\tilde{s}_t = 2$ stands for not rainy for any day t. In the 182 days with demand data, there are 37 days with significant precipitation and the remaining 145 days without. Moreover, in the ambiguity set defined in (9), we choose $H = 1$ and $\mathcal{N}_1 = [N]$, which indicates that the correlation information is imposed for all retailers. The coefficients in (9) are obtained from the training data.

We use the same inventory requirement window for all six approaches. In particular, in Cost-R model, we set the capacity $U_n = 2\mu_n, \forall n \in [N]$; correspondingly, in the remaining models, we set $\underline{\tau}_n^t = 0, \overline{\tau}_n^t = U_n, \forall n \in [N], t \in [T]$. To conduct a fair comparison, we vary the value of ϕ, which controls the size of the ambiguity set in Equation (16), for the Cost-R model. Given ϕ, we first solve the Cost-R model for its optimal solution and obtain the corresponding transportation cost in each period. Then we use these costs as the budgets, B^t, for the remaining five approaches to solve for their optimal solutions. We use the testing data to compare the quality of these solutions. Hence, all comparisons between the six approaches are based on the same transportation cost (or budget for the transportation cost). We plot the comparison of the quality of these solutions with respect to the parameter ϕ in Figure 3.

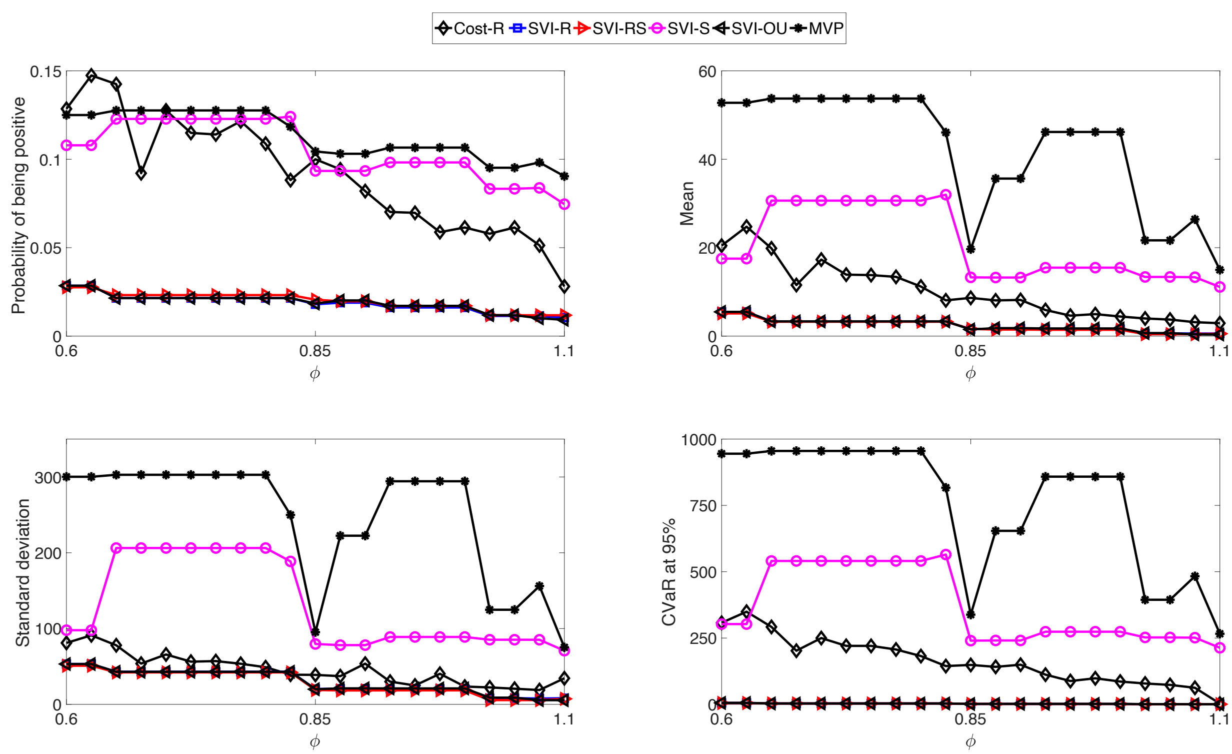

Figure 3 Comparison of the violation of inventory requirement between solutions from the five approaches.

Our comparison focuses on the violation of the inventory requirement window. In particular, for each solution, based on the testing data, we aggregate the violation samples (i.e., $\max\{\tilde{x}_n^t - U_n, 0 - \tilde{x}_n^t, 0\}$ at node n and period t) for all $n \in [N]$ and $t \in [T]$ and take them as samples of the overall violation of this solution. Based on the aggregated samples, we calculate the statistics of the overall violation, including the probability of being positive (Prob), the expected value (Mean), the standard deviation (Std), and the Conditional Value at Risk (CVaR) at the 95th percentiles. For all the four criteria, a smaller value indicates a better performance.

In Figure 3, SVI-RS, SVI-R, and SVI-OU dominate the other approaches, because, with the same transportation cost (or even slightly lower than that in Cost-R), their solutions exhibit better performance in all four criteria. That illustrates the advantage of our SVI approach in meeting the inventory requirement window. Noticing that in Figure 3, SVI-RS, SVI-R, and SVI-OU perform very similarly in most cases. Indeed, their performances are different, but the difference is small compared with the magnitude of that from the MVP approach. The details of the values are listed in Appendix. Later in Section 5.1.2, we will further demonstrate the difference using synthetic data. It is worth mentioning that there is no dominant relationship between SVI-R and SVI-RS. The reason is that in our data set, the difference between the demand in different weather conditions (the scenario we consider in this experiment) is not very great. Nevertheless, SVI-RS is a more general framework than SVI-R and we can easily construct examples where SVI-RS dominates SVI-R. We also notice that SVI-S does not perform well, sometimes even worse than Cost-R. That implies the need to consider robustness when characterizing the uncertainties.

5.1.2. On synthetic data Since the performances of SVI-RS, SVI-R, and SVI-OU are very similar using the real data set, we next compare these models with synthetic data to show the value of scenario information and the superiority of our decision rule over the order-up-to level policy. We consider the same network consisting of 10 filling stations with $T = 5$ planning periods. We first calculate the minimum cost to visit all retailers based on the conventional traveling salesman problem and denote this minimum cost as TSP. We assume that the initial inventory level of each retailer is zero. Hence, the budget for the first period is set as TSP such that all retailers can be served. For all subsequent periods, we let the budget be $\kappa \times TSP$ with κ being a given constant. To investigate the trade-off between service quality and transportation cost, κ is chosen from the set $\{0.8, 0.85, 0.9, 0.95\}$.

In all periods, all the retailers share a common inventory level requirement window $[0, U]$, and the random demand $\tilde{d}_n^t$ follows an i.i.d. distribution for any $n \in [N], t \in [T]$. For the SVI-RS model, we assume there are $S_t = 2$ possible realizations of the random scenario for each period $t \in [T]$ and both realizations happen with equal probability. Then the total number of scenarios in the scenario tree is $W = 2^5 = 32$ and $p_w = 1/32, \forall w \in [W]$. For each scenario $s \in [S_t]$, we first generate $100,000$ samples of data from a truncated normal distribution, which is obtained by bounding the normal distribution $\mathcal{N}(\mu_s, \sigma_s)$ to the support $\{d_n^t : \mu_s - 3\sigma_s \le d_n^t \le \mu_s + 3\sigma_s\}$. Then the parameters for the ambiguity set in (9) are obtained from the sample data. Hence, the real probability distribution is a multi-modal distribution, a mixture of two truncated normal distributions. We also choose $H = 1$ and $\mathcal{N}_1 = [N]$. For the SVI-R model and SVI-OU model, we do not consider the scenario information, and the parameters of the ambiguity set are obtained by pooling the above sample data. We first derive the optimal solution for each approach and then generate $20,000$ sample demand paths to test the quality of solutions.

κ	Model	Prob	Mean	Std	CVaR
0.80	SVI-RS	11.99%	0.65	2.39	9.78
	SVI-R	15.30%	0.79	2.40	9.76
	SVI-OU	18.43%	0.97	2.62	10.39
0.85	SVI-RS	10.15%	0.54	2.19	8.92
	SVI-R	13.25%	0.68	2.23	9.18
	SVI-OU	15.69%	0.82	2.42	9.81
0.90	SVI-RS	6.64%	0.33	1.73	6.46
	SVI-R	8.80%	0.43	1.79	7.39
	SVI-OU	10.21%	0.51	1.93	8.06
0.95	SVI-RS	4.89%	0.23	1.42	0.23
	SVI-R	6.39%	0.28	1.44	5.52
	SVI-OU	7.45%	0.35	1.61	6.47

Table 2 Comparison of the violation of inventory requirement between SVI-RS, SVI-R, and SVI-OU approaches.

Table 2 states the results with $\mu_1 = 15$, $\mu_2 = 30$, $\sigma_1 = \sigma_2 = 5$ and $U = 40$. As the transportation budget increases (higher κ), the supplier has more flexibility when choosing routes, which reduces the risk of violation. For each κ, the performance of SVI-RS is the best of the three approaches in all the criteria, which stresses the value of scenario information. Though both SVI-R and SVI-OU use the same ambiguity set without scenario information, SVI-R still dominates SVI-OU because the linear decision rule gives more flexibility than the order-up-to level policy.

We further vary the parameter μ_2 and calculate the average performance. We let μ_2 take $16, 20, 24, 28, 30$ while keeping $\mu_1 = 15$. Given four different levels of budgets and five possible μ_2, there are in total 20 instances. For each instance, we normalize the performance of these models by that of our SVI-RS model. The average performances of all instances are presented in Table 3.

Model	Prob	Mean	Std	CVaR
SVI-RS	1.00	1.00	1.00	1.00
SVI-R	1.24	1.41	1.22	6.70
SVI-OU	1.34	1.50	1.25	8.46

Table 3 Average performances among solutions from SVI-RS, SVI-R, and SVI-OU approaches.

Moreover, among these 20 instances, 17 instances show that the SVI-RS model is the best for all criteria. The remaining 3 instances are all from the setting with $\mu_2 = 16$, which indicates that the scenario information may not be helpful in improving the performance when the difference between the demand in different scenarios is not very great. Nonetheless, there is only a subtle difference between these models in this case. The detailed results are provided in the Appendix.

Though we choose the function $u(x) = \max\{-1, x\}$ as the utility function for all variants of the SVI model, we have tested other utility functions, which are with a similar shape but a higher number of linear segments, and observed similar results. It demonstrates the robustness of our model with respect to the number of linear segments of utility functions. A sample of the results is presented in the Appendix.

5.2. Computational efficiency

In this subsection, we demonstrate the computational efficiency using the networks given by Archetti et al. (2007), which are commonly used in the literature (Coelho et al. 2014a). For any given network, we let the budget be $\kappa \times TSP$ in all periods except that the first period has the budget as TSP. Our experiment considers two cases: $\kappa = 0.7$ and $\kappa = 0.9$. The random demand $\tilde{d}_n^t$ follows an i.i.d. uniform distribution in $[10, 40]$, $\forall n \in [N], t \in [T]$. We do not consider any scenario information and calibrate the parameters $\boldsymbol{\mu}^t$, $\boldsymbol{\sigma}^t$, ϵ^t for the ambiguity based on the uniform distribution described above. In all periods, all retailers share a common inventory level requirement window $[0, 40]$. We test the computation time for the IRP based on all 5 networks with 15 nodes and 5 networks with 20 nodes. The number of periods for the IRP and the corresponding computation time are presented in Table 4.

(N,T)	$(20,6)$		$(20,5)$		$(15,9)$		$(15,8)$		$(15,7)$	
κ	0.7	0.9	0.7	0.9	0.7	0.9	0.7	0.9	0.7	0.9
Number of instances solvable within 2 hours	4	1	5	5	5	1	5	2	5	4
Average computation time for solvable instances (seconds)	1526	639	410	560	2689	963	1079	3013	323	1224

Table 4 Computation efficiency

For networks with 20 nodes, we can solve for IRPs with no more than five periods efficiently. For problems with six periods, the solvability depends on the specification of the instances and the parameter κ. For networks with 15 nodes, we can solve most IRP instances with seven periods. Further, by comparing the number of solvable instances, we conclude that the instances with $\kappa = 0.9$ are in general more computationally challenging than those with $\kappa = 0.7$. The reason is that when $\kappa = 0.9$, the budget for the transportation cost is higher, and hence there is more flexibility. That implies a bigger solution space and leads to a longer computation time.

5.3. Large scale uncertain IRP

To demonstrate the computational efficiency of our approach, we solve a problem with a larger data set collected from the same oil company. This data set includes the locations of 109 filling stations (retailers) and one supplier. For each retailer, we collect data for the daily demand over 300 days. The demand data is partitioned into two parts: the first 200 days are used to calculate parameters for the ambiguity set and the remaining 100 days are used to construct sample paths for the out-of-sample test. Since tractability is a major concern for the large-scale uncertain IRP, we make two assumptions to simplify the problem.

All retailers are first grouped into several "clusters" or "neighborhoods", and we assume that, once a retailer is replenished, all retailers in the same neighborhood will be replenished in the same period. The partition of retailers can be made in several ways: 1) with current knowledge of the company's operations, the supplier can manually select retailers that are typically served together; 2) clustering methods, such as k-means, can be used to group the retailers based on their location. Though this assumption weakens the flexibility of the model, it significantly reduces the feasible space of routes, and hence we can scale to large problem sizes.

The second simplification is that we do not consider that the demand of different retailers may be correlated, i.e, $\mathcal{N}_h = \emptyset$ in (9d). We further assume that the replenishment decisions for each retailer depend on the retailer's own demand, instead of all retailers' demand. Consequently, the number of decision variables in the decision rule method is reduced significantly.

We consider the problem with a planning horizon $T = 30$ and compare the performance of the Cost-R and the SVI-R model. Both models use the same inventory requirement window, $\underline{\tau}_n^t = 0, \overline{\tau}_n^t = U_n = 6\overline{d}_n, \forall n \in [N], t \in [T]$, where $\overline{d}_n$ is the upper bound of the demand for retailer n and is derived from the training data set. For the Cost-R model, we set $\phi = 1.5$. We first calculate the minimum cost to visit all 109 retailers and denote it as the TSP. Then we assume the feasible route set used in the Cost-R model to be all routes that satisfy the first simplifying assumption and are associated with a cost lower than $\kappa \times TSP$, $\kappa \in \{0.4, 0.5, 0.6, 0.7\}$. We then derive the optimal solution for the Cost-R model and the associated transportation costs of each period. The corresponding costs are used as the budgets B^t for the SVI-R model, with which the set of maximal visiting nodes $\mathcal{H}_{B^t}$ defined in Definition 2 can be found. It allows us to further eliminate the sub-optimal routes for the SVI-R model and makes the problem tractable. This is also one of the reasons why the SVI-R model can be solved faster than the Cost-R model. The comparison results are shown in Table 5.

κ	Model	Prob	Mean	Std	CVaR	Gap	Time (seconds)
0.4	Cost-R	6.51%	1913	12529	37886	0.0%	2988
	SVI-R	3.27%	492	4541	492	0.0%	1983
0.5	Cost-R	6.31%	1544	11116	30634	1.5%	>7200
	SVI-R	2.88%	449	4326	449	0.0%	1144
0.6	Cost-R	6.33%	1682	11817	33334	3.0%	>7200
	SVI-R	2.59%	366	3683	366	0.0%	3103
0.7	Cost-R	6.39%	1549	10360	30617	8.6%	>7200
	SVI-R	2.67%	453	4498	453	0.0%	974

Table 5 Performance comparison for Cost-R and SVI-R model with $N = 109$ and $T = 30$.

The last two columns present the duality gaps output by the CPLEX solver (if optimality is not yet achieved because of the time limit, i.e., 2 hours) and the computation times. By Table 5, the SVI-R approach still leads to a solution with better performance than the Cost-R model, and the SVI-R model provides adequate protection against unfavorable violations while retaining the tractability of underlying problems. It is worth mentioning that for the Cost-R model with $\kappa = 0.5, 0.6, 0.7$, the performance with a larger budget (i.e., larger κ) does not dominate that with a lower one. The reason is that the Cost-R model cannot be solved to optimality for these values of κ.

6. Conclusions

We use a distributionally robust optimization approach to study the uncertain inventory routing problem. After proposing a new decision criterion to evaluate the associated risk, we develop methods geared to solving for the optimal affine policies. Our model can easily be extended to incorporate various operational features (e.g., production), and its computational efficiency can be improved by employing some well-developed heuristics (e.g., restriction to a subset of routes). We undertake computational studies to illustrate the strength of our method.

There are several possible directions to explore in the future. Firstly, more efficient algorithms or heuristics can be developed to expedite the computation. In our paper, since our focuses are the performance measures for evaluating the risk and the characterization of uncertainties, we only use Benders decomposition to solve the reformulated mixed-integer linear optimization problem. Further investigations can be done to check whether classic algorithms for solving the deterministic IRP can be used, or whether certain valid inequalities can be developed to solve a larger scale of uncertain IRP. For example, after incorporating risk aversion in the classic vehicle routing problem by Jaillet et al. (2016), an elegant algorithm was designed by Adulyasak and Jaillet (2016) to facilitate that specific solution procedure. Secondly, the binary decision rule approach may be adopted so that the visiting decisions can also be adaptive. Unfortunately, because of the computational challenge, the study of the binary decision rule is quite limited. Current studies can only solve examples with a few binary decisions. We will further propose different formulations to incorporate the binary decision rule and test the performance.

Appendix

Proof of Theorem 1.

We first prove the "if" direction. If μ is a convex risk measure, we prove that the $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}})$ defined in the equation (4) is an SVI by showing its satisfaction of the properties in Definition 1.

1. Monotonicity: If $v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i) \geq v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{y}_i), \forall i \in [I]$, then by the monotonicity of μ, we have

$$\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right) \geq \mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{y}_i)}{\alpha_i}\right), \qquad \forall \alpha_i > 0.$$

Hence, $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) \geq \rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{y}})$.

2(a). Attainment content: If $v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i) \leq 0, \forall i \in [I]$, we have for any $\alpha_i > 0$,

$$\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right) \leq \mu(0) = 0,$$

Hence, $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = 0$.

2(b). Starvation aversion: If $\exists i \in [I]$ such that $v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i) > 0$, then for any $\alpha_i > 0$, we have that $v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)/\alpha_i > 0$. Hence, $\exists \epsilon > 0$ such that $v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)/\alpha_i \geq \epsilon$ and

$$\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right) \geq \mu(\epsilon) = \mu(0) + \epsilon = \epsilon > 0,$$

Therefore, $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \infty$.

3. Convexity: Given any $\tilde{\boldsymbol{x}} \in \mathcal{V}^I$, we define a set of $\boldsymbol{\alpha}$ as

$$S(\tilde{\boldsymbol{x}}) = \left\{ \boldsymbol{\alpha} \,\middle|\, \mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right) \leq 0, \alpha_i > 0, \ \forall i \in [I] \right\}.$$

We consider any $\boldsymbol{\alpha}^x \in S(\tilde{\boldsymbol{x}})$, $\boldsymbol{\alpha}^y \in S(\tilde{\boldsymbol{y}})$, and define $\boldsymbol{\alpha}^\lambda = \lambda\boldsymbol{\alpha}^x + (1-\lambda)\boldsymbol{\alpha}^y$ for any $\lambda \in [0,1]$. Based on the definition of $v_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) = \max\{\tilde{\boldsymbol{x}} - \overline{\boldsymbol{\tau}}, \underline{\boldsymbol{\tau}} - \tilde{\boldsymbol{x}}\}$, the function $v_{\underline{\tau}_i,\overline{\tau}_i}(\cdot)$ is convex. Hence,

$$\begin{aligned}
\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i)}{\alpha_i^\lambda}\right) &\leq \mu\left(\frac{\lambda v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i) + (1-\lambda)v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{y}_i)}{\alpha_i^\lambda}\right) \\
&= \mu\left(\frac{\lambda\alpha_i^x}{\alpha_i^\lambda}\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i^x} + \frac{(1-\lambda)\alpha_i^y}{\alpha_i^\lambda}\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{y}_i)}{\alpha_i^y}\right) \\
&\leq o\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i^x}\right) + (1-o)\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{y}_i)}{\alpha_i^y}\right) \\
&\leq 0,
\end{aligned}$$

where $o = \lambda\alpha_i^x/\alpha_i^\lambda \in [0,1]$. The first inequality holds because of the convexity of $v_{\underline{\tau}_i,\overline{\tau}_i}(\cdot)$ and the monotonicity of $\mu(\cdot)$, the second inequality is due to the convexity of μ and the last inequality holds based on the definition of $S(\tilde{\boldsymbol{x}})$ and $S(\tilde{\boldsymbol{y}})$. Hence, for any $\lambda \in [0,1], \boldsymbol{\alpha}^x \in S(\tilde{\boldsymbol{x}}), \boldsymbol{\alpha}^y \in S(\tilde{\boldsymbol{y}})$, $\boldsymbol{\alpha}^\lambda \in S(\lambda\tilde{\boldsymbol{x}} + (1-\lambda)\tilde{\boldsymbol{y}})$ and we get

$$\begin{aligned}
\lambda\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}})+(1-\lambda)\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{y}}) &= \lambda\inf\left\{\sum_{i\in[I]}\alpha_i^x, \boldsymbol{\alpha}^x\in S(\tilde{\boldsymbol{x}})\right\}+(1-\lambda)\inf\left\{\sum_{i\in[I]}\alpha_i^y, \boldsymbol{\alpha}^y\in S(\tilde{\boldsymbol{y}})\right\}\\
&= \inf\left\{\sum_{i\in[I]}\left(\lambda\alpha_i^x+(1-\lambda)\alpha_i^y\right), \boldsymbol{\alpha}^x\in S(\tilde{\boldsymbol{x}}), \boldsymbol{\alpha}^y\in S(\tilde{\boldsymbol{y}})\right\}\\
&\geq \inf\left\{\sum_{i\in[I]}\alpha_i, \boldsymbol{\alpha}\in S(\lambda\tilde{\boldsymbol{x}}+(1-\lambda)\tilde{\boldsymbol{y}})\right\}\\
&= \rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\lambda\tilde{\boldsymbol{x}}+(1-\lambda)\tilde{\boldsymbol{y}})
\end{aligned}$$

4. Positive homogeneity: For all $\lambda>0$,

$$\begin{aligned}
\rho_{\lambda\underline{\boldsymbol{\tau}},\lambda\overline{\boldsymbol{\tau}}}(\lambda\tilde{\boldsymbol{x}}) &= \inf\left\{\sum_{i\in[I]}\alpha_i\,\middle|\,\mu\left(\frac{v_{\lambda\underline{\tau}_i,\lambda\overline{\tau}_i}(\lambda\tilde{x}_i)}{\alpha_i}\right)\leq 0, \alpha_i>0, \forall i\in[I]\right\}\\
&= \inf\left\{\sum_{i\in[I]}\alpha_i\,\middle|\,\mu\left(\frac{\lambda v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right)\leq 0, \alpha_i>0, \forall i\in[I]\right\}\\
&= \inf\left\{\sum_{i\in[I]}\alpha_i\,\middle|\,\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i/\lambda}\right)\leq 0, \alpha_i>0, \forall i\in[I]\right\}\\
&= \inf\left\{\sum_{i\in[I]}\lambda\beta_i\,\middle|\,\mu\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\beta_i}\right)\leq 0, \beta_i>0, \forall i\in[I]\right\}\\
&= \lambda\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}).
\end{aligned}$$

5. Dimension-wise additivity: It is obvious due to representation (4).

6. Order invariance: It is obvious due to representation (4).

7. Left continuity: Denote $\alpha^*=\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}),\ \alpha_i^*=\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{x}_i\boldsymbol{e}_i),\ i\in[I]$. By dimension-wise additivity, we have $\alpha^*=\sum_{i\in[I]}\alpha_i^*$. We next prove the left continuity in two cases.

Case 1: If $\alpha_i^*\in\mathbb{R}_+,\forall i\in[I]$, then $\alpha^*\in\mathbb{R}_+$. We need to show that $\forall\epsilon>0$, there exists $\overline{\delta}>0$ such that $\forall\delta\in\left(0,\overline{\delta}\right)$, $|\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-\delta\mathbf{1})-\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}})|\leq\epsilon$. Since $v_{-\boldsymbol{\infty},\mathbf{0}}(\boldsymbol{x}-\delta\mathbf{1})=\boldsymbol{x}-\delta\mathbf{1}\leq\boldsymbol{x}=v_{-\boldsymbol{\infty},\mathbf{0}}(\boldsymbol{x})$, by Monotonicity, we have $\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-\delta\mathbf{1})\leq\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}})$. Hence, we only need to prove $\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-\delta\mathbf{1})\geq\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}})-\epsilon=\alpha^*-\epsilon$. The case of $\epsilon\geq\alpha^*$ is trivial. For $\epsilon\in(0,\alpha^*)$, we define $\mathcal{I}=\{\,i\in[I]|\,\alpha_i^*>0\}$, and $\bar{\mathcal{I}}=[I]/\mathcal{I}=\{\,i\in[I]|\,\alpha_i^*=0\}$. Then we have $\epsilon<\alpha^*=\sum_{i\in\mathcal{I}}\alpha_i^*$. Hence, we can find a vector $\{\epsilon_i\}_{i\in\mathcal{I}}$ such that $\sum_{i\in\mathcal{I}}\epsilon_i=\epsilon$ and $\epsilon_i\in(0,\alpha_i^*),\ \forall i\in\mathcal{I}$. Choose $\overline{\delta}=\min_{i\in\mathcal{I}}(\alpha_i^*-\epsilon_i)\mu(\frac{v_{-\infty,0}(\tilde{x}_i)}{\alpha_i^*-\epsilon_i})$. Since $\alpha_i^*=\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{x}_i\boldsymbol{e}_i)=\inf\{\alpha>0\,|\mu(v_{-\infty,0}(\tilde{x}_i)/\alpha)\leq 0\},\ \forall i\in\mathcal{I}$, $\overline{\delta}$ is strictly positive. Considering $\forall\delta\in(0,\overline{\delta})$ and $i\in\mathcal{I}$, we have

$$\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i-\delta)}{\alpha_i^*-\epsilon_i}\right)=\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)-\delta}{\alpha_i^*-\epsilon_i}\right)=\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)}{\alpha_i^*-\epsilon_i}\right)-\frac{\delta}{\alpha_i^*-\epsilon_i}>\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)}{\alpha_i^*-\epsilon_i}\right)-\frac{\overline{\delta}}{\alpha_i^*-\epsilon_i}\geq 0,$$

where the first equality holds because of the fact that $\forall x\in\mathbb{R},\ v_{-\infty,0}(x)=\max\{-\infty-x,x-0\}=x$, the second equality holds because of cash invariance, and the last inequality holds due to the definition of $\overline{\delta}$. By

Lemma 1 in Chen et al. (2015) and the representation (4), we know $\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}_i-\delta)\boldsymbol{e}_i)\geq\alpha_i^*-\epsilon_i$. For $i\in\bar{\mathcal{I}}$, obviously we have $\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}_i-\delta)\boldsymbol{e}_i)=0$. Therefore,

$$\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-\delta\mathbf{1})=\sum_{i\in\mathcal{I}}\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}_i-\delta)\boldsymbol{e}_i)\geq\sum_{i\in\mathcal{I}}(\alpha_i^*-\epsilon_i)=\alpha^*-\epsilon.$$

Case 2: If $\exists i\in[I]$ such that $\alpha_i^*=\infty$, then $\alpha^*=\infty$. Suppose $\lim_{a\downarrow 0}\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-a\mathbf{1})$ is not ∞, i.e. $\lim_{a\downarrow 0}\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-a\mathbf{1})=\alpha^o\in\mathbb{R}_+$. It implies that $\forall\epsilon>0$, $\exists\overline{\delta}>0$ such that $\forall\delta\in\left(0,\overline{\delta}\right)$, we have $\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-\delta\mathbf{1})\in[\alpha^o-\epsilon,\alpha^o+\epsilon]$. Since $\alpha_i^*=\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{x}_i\boldsymbol{e}_i)=\infty$, we have $\mu(\frac{v_{-\infty,0}(\tilde{x}_i)}{\alpha})>0,\ \forall\alpha>0$. Hence, we can choose

$$\delta=\min\left\{\frac{\overline{\delta}}{2},\hat{\alpha}\left(\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)}{\hat{\alpha}}\right)-\eta\right)\right\}\in(0,\overline{\delta}),$$

where $\hat{\alpha}=\alpha^o+2\epsilon$, and η is any real number in $(0,\mu(v_{-\infty,0}(\tilde{x}_i)/\hat{\alpha}))$. As $\delta\in\left(0,\overline{\delta}\right)$, we still have $\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}_i-\delta)\boldsymbol{e}_i)\leq\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-\delta\mathbf{1})\leq\alpha^o+\epsilon$. So by Lemma 1 in Chen et al. (2015), $\mu\left(v_{-\infty,0}(\tilde{x}_i-\delta)/\hat{\alpha}\right)\leq 0$ since $\hat{\alpha}>\alpha^o+\epsilon$. We can also get that

$$\begin{aligned}\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i-\delta)}{\hat{\alpha}}\right)&=\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)-\delta}{\hat{\alpha}}\right)\\&=\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)}{\hat{\alpha}}\right)-\frac{\delta}{\hat{\alpha}}\geq\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)}{\hat{\alpha}}\right)-\left(\mu\left(\frac{v_{-\infty,0}(\tilde{x}_i)}{\hat{\alpha}}\right)-\eta\right)=\eta>0,\end{aligned}$$

which contradicts with $\mu\left(v_{-\infty,0}(\tilde{x}_i-\delta)/\hat{\alpha}\right)\leq 0$. So the assumption is false, $\lim_{a\downarrow 0}\rho_{-\boldsymbol{\infty},\mathbf{0}}(\tilde{\boldsymbol{x}}-a\mathbf{1})=\infty$.

We next prove the "only if" direction. First, we need to show function μ defined by the equation (5) is a convex risk measure. Notice that $\forall\boldsymbol{x}\in\mathbb{R}^I,\ v_{-\boldsymbol{\infty},\mathbf{0}}(\boldsymbol{x})=\max\{-\boldsymbol{\infty}-\boldsymbol{x},\boldsymbol{x}-\mathbf{0}\}=\boldsymbol{x}$.

1. Monotonicity: If $\tilde{x}\geq\tilde{y}$, then $v_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}-a)\boldsymbol{e}_1)\geq v_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{y}-a)\boldsymbol{e}_1)$ for any $a\in\mathbb{R}$. Hence, based on the monotonicity property, we have $\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}-a)\boldsymbol{e}_1)\geq\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{y}-a)\boldsymbol{e}_1)$. According to the definition of $\mu(\cdot)$, we get $\mu(\tilde{x})\geq\mu(\tilde{y})$.

2. Cash invariance: Notice that $\forall w\in\mathbb{R}$,

$$\begin{aligned}\mu(\tilde{x}+w)&=\min\{a\,|\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}+w-a)\boldsymbol{e}_1)\leq 1\}\\&=\min\{a+w\,|\rho_{-\boldsymbol{\infty},\mathbf{0}}((\tilde{x}-a)\boldsymbol{e}_1)\leq 1\}\\&=\mu(\tilde{x})+w\end{aligned}$$

3. Convexity:

$$\begin{aligned}&\rho_{-\boldsymbol{\infty},\mathbf{0}}\left((\lambda\tilde{x}+(1-\lambda)\tilde{y}-\lambda\mu(\tilde{x})-(1-\lambda)\mu(\tilde{y}))\,\boldsymbol{e}_1\right)\\\leq\ &\lambda\rho_{-\boldsymbol{\infty},\mathbf{0}}\left((\tilde{x}-\mu(\tilde{x}))\,\boldsymbol{e}_1\right)+(1-\lambda)\rho_{-\boldsymbol{\infty},\mathbf{0}}\left((\tilde{y}-\mu(\tilde{y}))\,\boldsymbol{e}_1\right)\\\leq\ &\lambda+(1-\lambda)\\=\ &1,\end{aligned}$$

where the first inequality follows from the convexity of $\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}})$, and the second inequality holds based on the definition of μ. Hence we have

$$\mu\left(\lambda\tilde{x}+(1-\lambda)\tilde{y}\right)=\min\left\{a\,\middle|\,\rho_{-\boldsymbol{\infty},\boldsymbol{0}}\left((\lambda\tilde{x}+(1-\lambda)\tilde{y}-a)\boldsymbol{e}_1\right)\leq 1\right\}\leq\lambda\mu(\tilde{x})+(1-\lambda)\mu(\tilde{y}).$$

4. Normalization: $\rho_{-\boldsymbol{\infty},\boldsymbol{0}}\left(-a\boldsymbol{e}_1\right)$ is 0 if $a\geq 0$, and it is ∞ if $a<0$. Therefore

$$\mu(0)=\min\left\{a\,\middle|\,\rho_{-\boldsymbol{\infty},\boldsymbol{0}}\left(-a\boldsymbol{e}_1\right)\leq 1\right\}=0.$$

Finally, we need to show for any SVI defined as (4), the equation (5) holds. With Left continuity of SVI, the minimum in (5) is achievable. Therefore,

$$\begin{aligned}
&\min\left\{a\,\middle|\,\rho_{-\boldsymbol{\infty},\boldsymbol{0}}\left((\tilde{x}-a)\,\boldsymbol{e}_1\right)\leq 1\right\}\\
=\;&\min\left\{a\,\middle|\,\sum_{i\in[I]}\alpha_i\leq 1,\mu\left(\frac{v_{-\infty,0}(\tilde{x}-a)}{\alpha_1}\right)\leq 0,\ \mu\left(\frac{v_{-\infty,0}(0)}{\alpha_j}\right)\leq 0,\ j=2,\ldots,I,\ \alpha_i>0,\ i\in[I]\right\}\\
=\;&\min\left\{a\,\middle|\,\alpha\leq 1,\mu\left(\frac{v_{-\infty,0}(\tilde{x}-a)}{\alpha}\right)\leq 0,\ \alpha>0\right\}\\
=\;&\min\left\{a\,\middle|\,\mu\left(v_{-\infty,0}(\tilde{x}-a)\right)\leq 0\right\}\\
=\;&\min\left\{a\,\middle|\,\mu\left(\tilde{x}-a\right)\leq 0\right\}\\
=\;&\min\left\{a\,\middle|\,\mu\left(\tilde{x}\right)\leq a\right\}\\
=\;&\mu(\tilde{x}),
\end{aligned}$$

where the second equality is due to $\mu(v_{-\infty,0}(0)/\alpha_j)\leq 0,\ \forall\alpha_j>0$, the fourth equality holds for the definition of function v and the fifth equality holds for the cash invariance of μ. In particular, the third equality is due to the equivalence between condition A) $\mu(\tilde{w})\leq 0$ and condition B) $\exists\alpha\in(0,1]$ such that $\mu(\tilde{w}/\alpha)\leq 0$, which we prove as follows. The direction of "A⇒B" is trivial. To see the direction of "B⇒A", we observe that for any such $\alpha\in(0,1]$,

$$\mu(\tilde{w})=\mu\left(\alpha\frac{\tilde{w}}{\alpha}+(1-\alpha)0\right)\leq\alpha\mu\left(\frac{\tilde{w}}{\alpha}\right)+(1-\alpha)\mu(0)=\alpha\mu\left(\frac{\tilde{w}}{\alpha}\right)\leq 0,$$

where inequalities are due to the convexity of μ and the condition B), respectively. Q.E.D.

Proof of Example 2.

The shortfall risk measure without $\sup_{\mathbb{P}\in\mathcal{P}}$ has been shown to be a convex risk measure (Föllmer and Schied 2002). Incorporating distributional ambiguity, it is straightforward to show our shortfall risk measure to be a convex risk measure as well.

By the definition of shortfall risk measure (6), the SVI defined by (4) can be formulated as

$$\begin{aligned}
\rho_{\underline{\boldsymbol{\tau}},\overline{\boldsymbol{\tau}}}(\tilde{\boldsymbol{x}}) &= \inf\left\{ \sum_{i\in[I]} \alpha_i \,\middle|\, \mu_u\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right) \le 0, \alpha_i > 0,\ \forall i \in [I] \right\} \\
&= \inf\left\{ \sum_{i\in[I]} \alpha_i \,\middle|\, \inf_{\eta\in\mathbb{R}}\left\{ \eta \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i} - \eta\right)\right] \le 0 \right\} \le 0, \alpha_i > 0,\ \forall i \in [I] \right\} \\
&= \inf\left\{ \sum_{i\in[I]} \alpha_i \,\middle|\, \exists \eta_i \le 0,\ \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i} - \eta_i\right)\right] \le 0,\ \alpha_i > 0,\ \forall i \in [I] \right\} \\
&= \inf\left\{ \sum_{i\in[I]} \alpha_i \,\middle|\, \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right)\right] \le 0, \alpha_i > 0, \forall i \in [I] \right\}.
\end{aligned}$$

The last equality holds due to the equivalence between condition A) $\exists \eta \le 0$ such that $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i} - \eta\right)\right] \le 0$ and condition B) $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u\left(\frac{v_{\underline{\tau}_i,\overline{\tau}_i}(\tilde{x}_i)}{\alpha_i}\right)\right] \le 0$. "B⇒A" is trivial by setting $\eta = 0$. The direction "A⇒B" is due to the monotonicity of u. Q.E.D.

Proof of Proposition 1.

We note that the constraints (10b) and (10d) together define the random variables $x_n^t\left(\tilde{\boldsymbol{s}}^{[t]}, \tilde{\boldsymbol{d}}^{[t]}\right)$ used in (10a), the constraints (10d) define the variables $q_n^t\left(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}\right)$ used in (10c). Therefore, in the reformulation, we can provide a set of constraints equivalent to (10a), (10b) and (10d) together, and another set of constraints equivalent to (10c) and (10d) together. Note that Ξ is a polyhedron containing the probability mass and hence we represent it as $\Xi = \{\boldsymbol{p} : \boldsymbol{Z}\boldsymbol{p} \le \boldsymbol{k}, \boldsymbol{p} \ge \boldsymbol{0}\}$ for certain $\boldsymbol{Z}, \boldsymbol{k}$.

We first focus on the equivalent constraints for (10a), (10b) and (10d). In the probability space, conditioning on $\tilde{\boldsymbol{s}} = \boldsymbol{s}^w$, we denote the probability distribution as $\mathbb{P}_w$, $\forall w \in [W]$. Then, considering any given $n \in [N]$, $t \in [T]$, by the law of total probability, the constraint (10a) becomes

$$\sup_{\boldsymbol{p}:\boldsymbol{Z}\boldsymbol{p}\le\boldsymbol{k},\boldsymbol{p}\ge\boldsymbol{0}} \sum_{w\in[W]} p_w f(\mathcal{P}_w) \le 0, \tag{18}$$

where

$$\begin{aligned}
f(\mathcal{P}_w) = \sup\quad & \mathbb{E}_{\mathbb{P}_w}\left(\max_{k\in[K]}\left\{a_k v_{\underline{\tau}_n^t,\overline{\tau}_n^t}\left(x_n^t\left(\boldsymbol{s}^{w,[t]}, \tilde{\boldsymbol{d}}^{[t]}\right)\right) + b_k \alpha_n^t\right\}\right) \\
\text{s.t.}\quad & \mathbb{P}_w\left(\underline{\boldsymbol{d}}^{\tau,s_\tau^w} \le \tilde{\boldsymbol{d}}^\tau \le \overline{\boldsymbol{d}}^{\tau,s_\tau^w}\right) = 1, && \forall \tau \in [T] \\
& \mathbb{E}_{\mathbb{P}_w}\left[\tilde{\boldsymbol{d}}^\tau\right] = \boldsymbol{\mu}^{\tau,s_\tau^w}, && \forall \tau \in [T] \\
& \mathbb{E}_{\mathbb{P}_w}\left[\left|\tilde{\boldsymbol{d}}^\tau - \boldsymbol{\mu}^{\tau,s_\tau^w}\right|\right] \le \boldsymbol{\sigma}^{\tau,s_\tau^w}, && \forall \tau \in [T] \\
& \mathbb{E}_{\mathbb{P}_w}\left[\left|\sum_{i\in\mathcal{N}_h} \frac{\tilde{d}_i^\tau - \mu_i^{\tau,s_\tau^w}}{\sigma_i^{\tau,s_\tau^w}}\right|\right] \le \epsilon_h^{\tau,s_\tau^w}, && \forall \tau \in [T], h \in [H]
\end{aligned}.$$

Note that by strong duality, the constraint (18) is equivalent to the following set of constraints,

$$\boldsymbol{k}^\top \boldsymbol{c} \le 0,$$

$$\boldsymbol{Z}_w^\top \boldsymbol{c} \geq f(\mathcal{P}_w) \qquad \forall w \in [W], \tag{19}$$
$$\boldsymbol{c} \geq 0.$$

Now it remains to represent constraints (19) in linear constraints. For any distinct w_1 and w_2, the calculation of $f(\mathcal{P}_{w_1})$ and $f(\mathcal{P}_{w_2})$ are independent. Therefore, for notational simplicity, we drop the subscript/superscript of w in the reformulation of $f(\mathcal{P}_w)$ and simply write $f = f(\mathcal{P}_w)$. Hence, we have

$$\begin{aligned}
f = \sup \quad & \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k v_{\underline{\tau}_n^t,\overline{\tau}_n^t}\left(x_n^t\left(\boldsymbol{s}^{[t]},\tilde{\boldsymbol{d}}^{[t]}\right)\right)+b_k\alpha_n^t\right\}\right) & \\
\text{s.t.} \quad & \mathbb{P}\left(\underline{\boldsymbol{d}}^{\tau,s_\tau} \leq \tilde{\boldsymbol{d}}^\tau \leq \overline{\boldsymbol{d}}^{\tau,s_\tau}\right)=1, & \forall \tau\in[T] \\
& \mathbb{E}_{\mathbb{P}}\left[\tilde{\boldsymbol{d}}^\tau\right]=\boldsymbol{\mu}^{\tau,s_\tau}, & \forall \tau\in[T] \\
& \mathbb{E}_{\mathbb{P}}\left[\left|\tilde{\boldsymbol{d}}^\tau-\boldsymbol{\mu}^{\tau,s_\tau}\right|\right]\leq\boldsymbol{\sigma}^{\tau,s_\tau}, & \forall \tau\in[T] \\
& \mathbb{E}_{\mathbb{P}}\left[\left|\sum_{i\in\mathcal{N}_h}\frac{\tilde{d}_i^\tau-\mu_i^{\tau,s_\tau}}{\sigma_i^{\tau,s_\tau}}\right|\right]\leq\epsilon_h^{\tau,s_\tau}, & \forall \tau\in[T],h\in[H].
\end{aligned}$$

By strong duality (Bertsimas et al. 2019, Chen et al. 2020), we have

$$\begin{aligned}
f = \inf \quad & \sum_{\tau\in[T]}\left(r_{n,0}^{t,\tau}+\left(\boldsymbol{\mu}^{\tau,s_\tau}\right)^\top\boldsymbol{r}_n^{t,\tau}+\left(\boldsymbol{\sigma}^{\tau,s_\tau}\right)^\top\boldsymbol{u}_n^{t,\tau}+\sum_{h\in[H]}\epsilon_h^{\tau,s_\tau}v_{n,h}^{t,\tau}\right) & \\
\text{s.t.} \quad & \sum_{\tau\in[T]}\left(r_{n,0}^{t,\tau}+\left(\boldsymbol{d}^\tau\right)^\top\boldsymbol{r}_n^{t,\tau}+\left(\left|\boldsymbol{d}^\tau-\boldsymbol{\mu}^{\tau,s_\tau}\right|\right)^\top\boldsymbol{u}_n^{t,\tau}+\sum_{h\in[H]}\left|\sum_{i\in\mathcal{N}_h}\frac{d_i^\tau-\mu_i^{\tau,s_\tau}}{\sigma_i^{\tau,s_\tau}}\right|v_{n,h}^{t,\tau}\right) & \\
& \quad \geq \max_{k\in[K]}\left\{a_k v_{\underline{\tau}_n^t,\overline{\tau}_n^t}\left(x_n^t\left(\boldsymbol{s}^{[t]},\boldsymbol{d}^{[t]}\right)\right)+b_k\alpha_n^t\right\}, \forall \underline{\boldsymbol{d}}^{\tau,s_\tau}\leq\boldsymbol{d}^\tau\leq\overline{\boldsymbol{d}}^{\tau,s_\tau}, \tau\in[T] & (a) \\
& r_{n,0}^{t,\tau}\in\mathbb{R},\boldsymbol{r}_n^{t,\tau}\in\mathbb{R}^N,\boldsymbol{u}_n^{t,\tau}\in\mathbb{R}_+^N,v_{n,h}^{t,\tau}\in\mathbb{R}_+, \qquad \forall\tau\in[T],h\in[H] &
\end{aligned} \tag{20}$$

With (20), we can represent (19) as linear constraints, except that (20a) has a nonlinear terms and involves an infinite number of $\boldsymbol{d}^\tau$, which we address as follows. By the definition of $v_{\underline{\tau}_n^t,\overline{\tau}_n^t}(x) = \max\{x-\overline{\tau}_n^t, \underline{\tau}_n^t - x\}$, the constraints (20a) are equivalent to the following two sets of constraints,

$$\begin{aligned}
& \sum_{\tau\in[T]}\left(r_{n,0}^{t,\tau}+\left(\boldsymbol{d}^\tau\right)^\top\boldsymbol{r}_n^{t,\tau}+\left(\left|\boldsymbol{d}^\tau-\boldsymbol{\mu}^{\tau,s_\tau}\right|\right)^\top\boldsymbol{u}_n^{t,\tau}+\sum_{h\in[H]}\left|\sum_{i\in\mathcal{N}_h}\frac{d_i^\tau-\mu_i^{\tau,s_\tau}}{\sigma_i^{\tau,s_\tau}}\right|v_{n,h}^{t,\tau}\right) \\
& \quad \geq a_k\left(x_n^t\left(\boldsymbol{s}^{[t]},\boldsymbol{d}^{[t]}\right)-\overline{\tau}_n^t\right)+b_k\alpha_n^t, \forall k\in[K], \underline{\boldsymbol{d}}^{\tau,s_\tau}\leq\boldsymbol{d}^\tau\leq\overline{\boldsymbol{d}}^{\tau,s_\tau}, \tau\in[T] \qquad (a) \\
& \sum_{\tau\in[T]}\left(r_{n,0}^{t,\tau}+\left(\boldsymbol{d}^\tau\right)^\top\boldsymbol{r}_n^{t,\tau}+\left(\left|\boldsymbol{d}^\tau-\boldsymbol{\mu}^{\tau,s_\tau}\right|\right)^\top\boldsymbol{u}_n^{t,\tau}+\sum_{h\in[H]}\left|\sum_{i\in\mathcal{N}_h}\frac{d_i^\tau-\mu_i^{\tau,s_\tau}}{\sigma_i^{\tau,s_\tau}}\right|v_{n,h}^{t,\tau}\right) \\
& \quad \geq a_k\left(\underline{\tau}_n^t-x_n^t\left(\boldsymbol{s}^{[t]},\boldsymbol{d}^{[t]}\right)\right)+b_k\alpha_n^t, \forall k\in[K], \underline{\boldsymbol{d}}^{\tau,s_\tau}\leq\boldsymbol{d}^\tau\leq\overline{\boldsymbol{d}}^{\tau,s_\tau}, \tau\in[T] \qquad (b)
\end{aligned} \tag{21}$$

For any given $k \in [K]$, we now reformulate (21a) further. Specifically, combining (10b) and (10d),

$$
\begin{aligned}
x_n^t\left(\boldsymbol{s}^{[t]},\boldsymbol{d}^{[t]}\right) &= \sum_{\tau\in[t]}\left(q_n^\tau\left(\boldsymbol{s}^{[\tau-1]},\boldsymbol{d}^{[\tau-1]}\right)-d_n^\tau\right)\\
&= \sum_{\tau\in[t]}\left(q_{n,0}^\tau(\boldsymbol{s}^{[\tau-1]})+\sum_{m\in[\tau-1]}\left(\boldsymbol{q}_{n,m}^\tau(\boldsymbol{s}^{[\tau-1]})\right)^\top\boldsymbol{d}^m-\boldsymbol{e}_n^\top\boldsymbol{d}^\tau\right)\\
&= \sum_{\tau\in[t]}\left(\sum_{m\in[t]\setminus[\tau]}q_{n,\tau}^m\left(\boldsymbol{s}^{[m-1]}\right)-\boldsymbol{e}_n\right)^\top\boldsymbol{d}^\tau+\sum_{\tau\in[t]}q_{n,0}^\tau(\boldsymbol{s}^{[\tau-1]}).
\end{aligned}\tag{22}
$$

Inserting the above (22) into the constraints (21a) for a given $k\in[K]$, we have

$$
\begin{aligned}
\inf_{\underline{\boldsymbol{d}}^{\tau,s_\tau}\le\boldsymbol{d}^\tau\le\overline{\boldsymbol{d}}^{\tau,s_\tau},\tau\in[T]}&\left\{\sum_{\tau\in[t]}\left(\boldsymbol{r}_n^{t,\tau}-a_k\sum_{m\in[t]\setminus[\tau]}q_{n,\tau}^m\left(\boldsymbol{s}^{[m-1]}\right)+a_k\boldsymbol{e}_n\right)^\top\boldsymbol{d}^\tau+\sum_{\tau\in[T]\setminus[t]}\left(\boldsymbol{r}_n^{t,\tau}\right)^\top\boldsymbol{d}^\tau\right.\\
&\left.+\sum_{\tau\in[T]}\left(\left(|\boldsymbol{d}^\tau-\boldsymbol{\mu}^{\tau,s_\tau}|\right)^\top\boldsymbol{u}_n^{t,\tau}+\sum_{h\in[H]}\left|\sum_{i\in\mathcal{N}_h}\frac{d_i^\tau-\mu_i^{\tau,s_\tau}}{\sigma_i^{\tau,s_\tau}}\right|v_{n,h}^{t,\tau}\right)\right\}\\
&\ge a_k\left(\sum_{\tau\in[t]}q_{n,0}^\tau(\boldsymbol{s}^{[\tau-1]})-\bar{\tau}_n^t\right)+b_k\alpha_n^t-\sum_{\tau\in[T]}r_{n,0}^{t,\tau}.
\end{aligned}\tag{23}
$$

The left-hand side involves a minimization problem. By strong duality, it equals to

$$
\begin{aligned}
\sup\ &\sum_{\tau\in[T]}\left(\left(\underline{\boldsymbol{d}}^{\tau,s_\tau}\right)^\top\boldsymbol{f}_n^{t,\tau}-\left(\overline{\boldsymbol{d}}^{\tau,s_\tau}\right)^\top\boldsymbol{g}_n^{t,\tau}+\left(\boldsymbol{\mu}^{\tau,s_\tau}\right)^\top\left(\boldsymbol{l}_n^{t,\tau}-\boldsymbol{h}_n^{t,\tau}\right)+\sum_{h\in[H]}\sum_{i\in\mathcal{N}_h}\frac{\mu_i^{\tau,s_\tau}}{\sigma_i^{\tau,s_\tau}}\left(p_{n,h}^{t,\tau}-o_{n,h}^{t,\tau}\right)\right)\\
\text{s.t. }\ &\boldsymbol{f}_n^{t,\tau}-\boldsymbol{g}_n^{t,\tau}+\boldsymbol{l}_n^{t,\tau}-\boldsymbol{h}_n^{t,\tau}+\sum_{h\in[H]}\sum_{i\in\mathcal{N}_h}\frac{\boldsymbol{e}_i}{\sigma_i^{\tau,s_\tau}}\left(p_{n,h}^{t,\tau}-o_{n,h}^{t,\tau}\right)=\\
&\qquad\qquad\boldsymbol{r}_n^{t,\tau}-a_k\sum_{m\in[t]/[\tau]}q_{n,\tau}^m\left(\boldsymbol{s}^{[m-1]}\right)+a_k\boldsymbol{e}_n \qquad \forall\tau\in[t]\ (a)\\
&\boldsymbol{f}_n^{t,\tau}-\boldsymbol{g}_n^{t,\tau}+\boldsymbol{l}_n^{t,\tau}-\boldsymbol{h}_n^{t,\tau}+\sum_{h\in[H]}\sum_{i\in\mathcal{N}_h}\frac{\boldsymbol{e}_i}{\sigma_i^{\tau,s_\tau}}\left(p_{n,h}^{t,\tau}-o_{n,h}^{t,\tau}\right)=\boldsymbol{r}_n^{t,\tau} \qquad \forall\tau\in[T]\setminus[t]\ (b)\\
&\boldsymbol{l}_n^{t,\tau}+\boldsymbol{h}_n^{t,\tau}=\boldsymbol{u}_n^{t,\tau} \qquad \forall\tau\in[T]\ (c)\\
&p_{n,h}^{t,\tau}+o_{n,h}^{t,\tau}=v_{n,h}^{t,\tau} \qquad \forall\tau\in[T],h\in[H]\ (d)\\
&\boldsymbol{f}_n^{t,\tau},\boldsymbol{g}_n^{t,\tau},\boldsymbol{l}_n^{t,\tau},\boldsymbol{h}_n^{t,\tau}\in\mathbb{R}_+^N \qquad \forall\tau\in[T]\ (e)\\
&p_{n,h}^{t,\tau},o_{n,h}^{t,\tau}\in\mathbb{R}_+ \qquad \forall\tau\in[T],h\in[H]\ (f)
\end{aligned}\tag{24}
$$

Consequently, the constraints (23) are equivalent to the following constraints:

$$
\begin{cases}
\displaystyle\sum_{\tau\in[T]}\left(\left(\underline{\boldsymbol{d}}^{\tau,s_\tau}\right)^\top\boldsymbol{f}_n^{t,\tau}-\left(\overline{\boldsymbol{d}}^{\tau,s_\tau}\right)^\top\boldsymbol{g}_n^{t,\tau}+\left(\boldsymbol{\mu}^{\tau,s_\tau}\right)^\top\left(\boldsymbol{l}_n^{t,\tau}-\boldsymbol{h}_n^{t,\tau}\right)+\sum_{h\in[H]}\sum_{i\in\mathcal{N}_h}\frac{\mu_i^{\tau,s_\tau}}{\sigma_i^{\tau,s_\tau}}\left(p_{n,h}^{t,\tau}-o_{n,h}^{t,\tau}\right)\right)\\
\qquad\qquad\displaystyle\ge a_k\left(\sum_{\tau\in[t]}q_{n,0}^\tau(\boldsymbol{s}^{[\tau-1]})-\bar{\tau}_n^t\right)+b_k\alpha_n^t-\sum_{\tau\in[T]}r_{n,0}^{t,\tau}\\
(24a)\ -\ (24f)
\end{cases}
$$

We hence finish reformulating constraints (21a) equivalently into a set of finite linear constraints. The constraints (21b) can be reformulated in the same way.

Now it remains to reformulate the constraints (10c) and (10d). For any $n \in [N]$ and $t \in [T]$, inserting (10d) in (10c), we can equivalently have:

$$\begin{cases} \inf\limits_{(\boldsymbol{s},\boldsymbol{d})\in\hat{\mathcal{D}}} q^t_{n,0}(\boldsymbol{s}^{[t-1]}) + \sum\limits_{i=1}^{t-1} \left(q^t_{n,i}(\boldsymbol{s}^{[t-1]})\right)^\top \boldsymbol{d}^i \geq 0, \\ \sup\limits_{(\boldsymbol{s},\boldsymbol{d})\in\hat{\mathcal{D}}} q^t_{n,0}(\boldsymbol{s}^{[t-1]}) + \sum\limits_{i=1}^{t-1} \left(q^t_{n,i}(\boldsymbol{s}^{[t-1]})\right)^\top \boldsymbol{d}^i \leq M y^t_n, \end{cases}$$

$$\Longrightarrow \begin{cases} q^t_{n,0}(\boldsymbol{s}^{w,[t-1]}) + \sum\limits_{i\in[t-1]} \left(\left(\underline{\boldsymbol{d}}^{i,s^w_i}\right)^\top \underline{\boldsymbol{\beta}}^{t,i}_{n,w} - \left(\overline{\boldsymbol{d}}^{i,s^w_i}\right)^\top \overline{\boldsymbol{\beta}}^{t,i}_{n,w} \right) \geq 0, & \forall w \in [W] \\ q^t_{n,0}(\boldsymbol{s}^{w,[t-1]}) + \sum\limits_{i\in[t-1]} \left(\left(\overline{\boldsymbol{d}}^{i,s^w_i}\right)^\top \overline{\boldsymbol{\gamma}}^{t,i}_{n,w} - \left(\underline{\boldsymbol{d}}^{i,s^w_i}\right)^\top \underline{\boldsymbol{\gamma}}^{t,i}_{n,w} \right) \leq M y^t_n, & \forall w \in [W] \\ q^t_{n,i}(\boldsymbol{s}^{w,[t-1]}) + \overline{\boldsymbol{\beta}}^{t,i}_{n,w} - \underline{\boldsymbol{\beta}}^{t,i}_{n,w} = 0, & \forall i \in [t-1], w \in [W] \\ q^t_{n,i}(\boldsymbol{s}^{w,[t-1]}) - \overline{\boldsymbol{\gamma}}^{t,i}_{n,w} + \underline{\boldsymbol{\gamma}}^{t,i}_{n,w} = 0, & \forall i \in [t-1], w \in [W] \\ \overline{\boldsymbol{\beta}}^{t,i}_{n,w}, \underline{\boldsymbol{\beta}}^{t,i}_{n,w}, \overline{\boldsymbol{\gamma}}^{t,i}_{n,w}, \underline{\boldsymbol{\gamma}}^{t,i}_{n,w} \in \mathbb{R}^N_+ & \forall i \in [t-1], w \in [W] \end{cases}$$

Q.E.D.

Proof of Proposition 2.

We drop the dependence on the scenario s_t in this proof for notational simplicity. Since the OU policy determines the ordering quantity for each retailer, we analyze each retailer separately. Given any visiting decision of retailer n as $\hat{\boldsymbol{y}}_n \in \{0,1\}^T$, we use $1 \leq k_1 < k_2 < \ldots < k_l \leq T$ to represent the visiting sequence as

$$\hat{y}^t_n = \begin{cases} 1, \ t \in \{k_1, k_2, \ldots, k_l\} \\ 0, \ t \in [T] \backslash \{k_1, k_2, \ldots, k_l\}. \end{cases}$$

With this visiting decision, we can write the ordering quantity in period $k_i, i = 1, 2, \ldots, l$.

If $k_1 = 1$,

$$q^{k_1}_n = U_n - x^0_n.$$

If $k_1 > 1$,

$$q^{k_1}_n = U_n - x^{k_1-1}_n(\boldsymbol{d}_n) = U_n - \left(x^0_n - \sum_{m=1}^{k_1-1} d^m_n\right) = U_n + \sum_{m=1}^{k_1-1} d^m_n - x^0_n.$$

And $\forall k_i, i = 2, \ldots, l$,

$$q^{k_i}_n(\boldsymbol{d}_n) = U_n - x^{k_i-1}_n(\boldsymbol{d}_n) = U_n - \left(U_n - \sum_{m=k_{i-1}}^{k_i-1} d^m_n\right) = \sum_{m=k_{i-1}}^{k_i-1} d^m_n.$$

Then the ordering quantity in period $k_i, i = 1, 2, \ldots, l$ can be written as an affine function of realized demand.

Since the OU policy is a special case of the linear decision rule, we have $Z^*_S \leq Z_{OU}$. Q.E.D.

Proof of Proposition 3.

We drop the dependence on the scenario s_t in this proof for notational simplicity. When $U_n \leq \overline{\tau}_n^t + \underline{d}_n^t$, the inventory levels at the end of period t under either the OU or the ML policy always satisfy the upper bound of the inventory requirement window as $x_n^t \leq U_n - \tilde{d}_n^t \leq \overline{\tau}_n^t$. Hence, under the OU(ML) policy, the risk of violating the inventory requirement window only lies on the lower bound. Hence, we will always raise the inventory level to U_n and we have $Z_{ML} = Z_{OU}$. The proof is completed. Q.E.D.

Proof of Proposition 4.

We denote the set $\hat{\mathcal{Y}}$ as the feasible set for $\{y_n^t\}_{n\in[N],t\in[T]}$ that satisfies the constraints (14) and by replacing the constraint (10f), the problem is

$$
\begin{aligned}
\hat{Z}_S^* = \inf\ & \sum_{t\in[T]}\sum_{n\in[N]} \alpha_n^t \\
\text{s.t. } & \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\max_{k\in[K]}\left\{a_k v_{\underline{\tau}_n^t,\overline{\tau}_n^t}\left(x_n^t\left(\tilde{\boldsymbol{s}}^{[t]},\tilde{\boldsymbol{d}}^{[t]}\right)\right)+b_k\alpha_n^t\right\}\right)\leq 0 \quad \forall n\in[N],t\in[T] && \text{(a)}\\
& x_n^t\left(\boldsymbol{s}^{[t]},\boldsymbol{d}^{[t]}\right)=\sum_{m\in[t]}\left(q_n^m\left(\boldsymbol{s}^{[m-1]},\boldsymbol{d}^{[m-1]}\right)-d_n^m\right) \quad \forall n\in[N],t\in[T],(\boldsymbol{s},\boldsymbol{d})\in\hat{\mathcal{D}} && \text{(b)}\\
& 0\leq q_n^t\left(\boldsymbol{s}^{[t-1]},\boldsymbol{d}^{[t-1]}\right)\leq My_n^t \quad \forall n\in[N],t\in[T],(\boldsymbol{s},\boldsymbol{d})\in\hat{\mathcal{D}} && \text{(c)}\\
& q_n^t(\boldsymbol{s}^{[t-1]},\boldsymbol{d}^{[t-1]})=q_{n,0}^t(\boldsymbol{s}^{[t-1]})+\sum_{i=1}^{t-1}\left(\boldsymbol{q}_{n,i}^t(\boldsymbol{s}^{[t-1]})\right)^\top\boldsymbol{d}^i \quad \forall n\in[N],t\in[T],(\boldsymbol{s},\boldsymbol{d})\in\hat{\mathcal{D}} && \text{(d)}\\
& \alpha_n^t\geq\epsilon \quad \forall n\in[N],t\in[T] && \text{(e)}\\
& (y_n^t)_{t\in[T],n\in[N]}\in\hat{\mathcal{Y}} && \text{(f).}
\end{aligned}
\tag{25}
$$

We first show $\hat{\mathcal{Y}}\subseteq\mathcal{Y}$. Observing that given any feasible choice of $\left\{(\hat{y}_n^t)_{n\in[N],t\in[T]},(\hat{l}_i^t)_{i\in[h_t],t\in[T]}\right\}$ that satisfies constraint (14), for each $t\in[T]$, there should be one $\hat{l}_i^t$ equal to 1 and others equal to 0 within $(\hat{l}_i^t)_{i\in[h_t]}$. Without loss of generality, we assume $\hat{l}_1^t=1$ and $\hat{l}_i^t=0,\forall i\in[h_t]\setminus[1],t\in[T]$. Then, $\hat{y}_n^t=\sum_{i=1}^{h_t}\hat{l}_i^t u_{i,n}^t=u_{1,n}^t$. By the definition of $u_{1,n}^t$, for each $t\in[T]$, there exists a route that can visit all nodes in set $\mathcal{C}_1^t\cup\{0\}$ with the transportation cost no more than B^t, which means the objective value of the following traveling salesman problem (26) is less than B^t.

$$
\begin{aligned}
\min \quad & \sum_{a \in \mathcal{A}\left(\mathcal{C}_1^t\right)} c_a z_a^t \\
\text{s.t.} \quad & \sum_{a \in \delta^+(\{n\})} z_a^t = 1, \ \forall n \in \mathcal{C}_1^t \cup \{0\} \\
& \sum_{a \in \delta^-(\{n\})} z_a^t = 1, \ \forall n \in \mathcal{C}_1^t \cup \{0\} \\
& \sum_{a \in \delta^+(\mathcal{S})} z_a^t \geq 1, \quad \forall \mathcal{S} \subseteq \mathcal{C}_1^t, |\mathcal{S}| \geq 2 \\
& z_a^t \in \{0,1\}, \qquad \forall n \in \mathcal{C}_1^t, a \in \mathcal{A}\left(\mathcal{C}_1^t\right).
\end{aligned}
\tag{26}
$$

Denote the optimal solution of problem (26) as $\hat{z}_a^t, \forall a \in \mathcal{A}(\mathcal{C}_1^t)$. Then for $\forall t \in [T]$, there exists a solution $y_0^t = 1$, $y_n^t = \hat{y}_n^t, \forall n \in [N]$, $z_a^t = \hat{z}_a^t, \forall a \in \mathcal{A}(\mathcal{C}_1^t)$, $z_a^t = 0, \forall a \in \mathcal{A} \setminus \mathcal{A}(\mathcal{C}_1^t)$ that satisfies the constraints (3a) - (3e), i.e., $\{\hat{y}_n^t\}_{n \in [N], t \in [T]} \in \mathcal{Y}$ as well. Since $\hat{\mathcal{Y}} \subseteq \mathcal{Y}$, we have $\hat{Z}_S^* \geq Z_S^*$.

Next, we prove $\hat{Z}_S^* \leq Z_S^*$ by showing that given the optimal solution of problem (10), we can construct a feasible solution of Problem (25). Assume $\left\{\overline{\alpha}_n^t, \overline{q}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}), \overline{x}_n^t\left(\boldsymbol{s}^{[t]}, \boldsymbol{d}^{[t]}\right), \overline{y}_n^t\right\}_{n \in [N], t \in [T]}$ is the optimal solution of Problem (10). Then, for each $t \in [T]$, we can define a set of node $\overline{\mathcal{C}}^t = \{n \in [N] \mid \overline{y}_n^t = 1\}$, and there is a route that can visit all nodes in $\overline{\mathcal{C}}^t$ with a cost less than B^t. By Definition 2, we can find a set $\mathcal{C}^t \in \mathcal{H}_{B^t}$ such that $\overline{\mathcal{C}}^t \subseteq \mathcal{C}^t$. Without loss of generality, we let it be indexed as $\mathcal{C}_1^t$. Then we can construct a feasible solution $\left\{\hat{\alpha}_n^t, \hat{q}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}), \hat{x}_n^t\left(\boldsymbol{s}^{[t]}, \boldsymbol{d}^{[t]}\right), \hat{y}_n^t, \hat{l}_i^t\right\}_{i \in [h_t], n \in [N], t \in [T]}$ for problem (25) by letting $\hat{\alpha}_n^t = \overline{\alpha}_n^t$, $\hat{q}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}) = \overline{q}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]})$, $\hat{x}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}) = \overline{x}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]})$, $\forall n \in [N], t \in [T]$ and $\hat{y}_n^t = 1, \forall n \in \mathcal{C}_1^t$, $\hat{y}_n^t = 0, \forall n \in [N] \setminus \mathcal{C}_1^t$, $\hat{l}_1^t = 1$, $\hat{l}_i^t = 0, \forall i \in [h_t] \setminus [1]$ for $\forall t \in [T]$. It is easy to see that (25a), (25b), (25d), (25e) hold. For (25c), we have

$$
0 \leq \hat{q}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}) = \overline{q}_n^t(\boldsymbol{s}^{[t-1]}, \boldsymbol{d}^{[t-1]}) \leq M \overline{y}_n^t \leq M \hat{y}_n^t, \qquad \forall n \in [N], t \in [T], (\boldsymbol{s}, \boldsymbol{d}) \in \hat{\mathcal{D}}.
$$

Therefore, $\hat{Z}_S^* \leq Z_S^*$. Combining with the first part, we have $\hat{Z}_S^* = Z_S^*$. Q.E.D.

Proof of Proposition 5.

We denote the output of Algorithm 1 as $\bar{\mathcal{H}}_B$, we need to prove $\bar{\mathcal{H}}_B = \mathcal{H}_B$, where the latter is defined in Definition 2. We note that, by definition, for any $\mathcal{C} \in \bar{\mathcal{H}}_B$, $\mathcal{C}$ is with TSP cost no greater than B and hence $\mathcal{C} \in \hat{\mathcal{H}}_B$. It implies $\bar{\mathcal{H}}_B \subseteq \hat{\mathcal{H}}_B$. We avoid the trivial case by assuming $[N]$ is with TSP cost greater than B, otherwise $\bar{\mathcal{H}}_B = \mathcal{H}_B = \{[N]\}$.

We first prove $\mathcal{H}_B \subseteq \bar{\mathcal{H}}_B$. Consider any $\mathcal{C} \in \mathcal{H}_B$. Then by the definition of $\mathcal{H}_B$, we can find $i \in [N] \setminus \mathcal{C}$ such that $\hat{C} = \mathcal{C} \cup \{i\}$ is with TSP cost greater than B. By the procedure of Algorithm 1, at certain iteration:

when start with Step 2 we have $\hat{\mathcal{C}} \in \mathcal{I}$ and after the update of $\mathcal{I}$ at Step 3 we have $\mathcal{C} \in \mathcal{I}$. Going to the next iteration, we will then include $\mathcal{C}$ into $\bar{\mathcal{H}}_B$.

We now prove $\bar{\mathcal{H}}_B \subseteq \mathcal{H}_B$ by contradiction. Assume that there exists $\mathcal{C} \in \bar{\mathcal{H}}_B$ but $\mathcal{C} \notin \mathcal{H}_B$. Note that $\mathcal{C} \in \bar{\mathcal{H}}_B$ and $\bar{\mathcal{H}}_B \subseteq \hat{\mathcal{H}}_B$ imply $\mathcal{C} \in \hat{\mathcal{H}}_B$ which, together with $\mathcal{C} \notin \mathcal{H}_B$, implies there exists $\hat{\mathcal{C}} \in \mathcal{H}_B$ such that $\mathcal{C} \subset \hat{\mathcal{C}}$. As we have previously proved $\mathcal{H}_B \subseteq \bar{\mathcal{H}}_B$, we now have $\hat{\mathcal{C}} \in \bar{\mathcal{H}}_B$. However, $\mathcal{C}, \hat{\mathcal{C}} \in \bar{\mathcal{H}}_B$ and $\mathcal{C} \subset \hat{\mathcal{C}}$ contradict with the way we update $\mathcal{I}$ in Step 3 of Algorithm 1. Q.E.D.

Proof of Proposition 6.

Suppose $\sum_{i=t+1}^{t+k_n^t} y_n^i < 1$ for certain $n \in [N], t \in [T-1]$ with $t + k_n^t \leq T$, i.e. $y_n^i = 0, \forall i \in \{t+1, \ldots, t+k_n^t\}$. By the definition of k_n^t, $Z_n^{t,t_\delta} = \infty$ and problem (15) is infeasible. That means there exists $i \in \{t+1, \ldots, t+k_n^t\}$ such that $\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[v_{\underline{\tau}_n^i, \bar{\tau}_n^i} \left(x_n^i \left(\tilde{\boldsymbol{s}}^{[i]}, \tilde{\boldsymbol{d}}^{[i]} \right) \right) \right] > 0$. Hence, $\exists \hat{\mathbb{P}} \in \mathcal{P}$ such that $\mathbb{E}_{\hat{\mathbb{P}}} \left[v_{\underline{\tau}_n^i, \bar{\tau}_n^i} \left(x_n^i \left(\tilde{\boldsymbol{s}}^{[i]}, \tilde{\boldsymbol{d}}^{[i]} \right) \right) \right] > 0$. Then,

$$
\begin{aligned}
& \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u \left(v_{\underline{\tau}_n^i, \bar{\tau}_n^i} \left(x_n^i \left(\tilde{\boldsymbol{s}}^{[i]}, \tilde{\boldsymbol{d}}^{[i]} \right) \right) \right) \right] \\
\geq\ & \mathbb{E}_{\hat{\mathbb{P}}} \left[u \left(v_{\underline{\tau}_n^i, \bar{\tau}_n^i} \left(x_n^i \left(\tilde{\boldsymbol{s}}^{[i]}, \tilde{\boldsymbol{d}}^{[i]} \right) \right) \right) \right] \\
\geq\ & u \left(\mathbb{E}_{\hat{\mathbb{P}}} \left[v_{\underline{\tau}_n^i, \bar{\tau}_n^i} \left(x_n^i \left(\tilde{\boldsymbol{s}}^{[i]}, \tilde{\boldsymbol{d}}^{[i]} \right) \right) \right] \right) \\
>\ & 0,
\end{aligned}
$$

where the second inequality holds due to the Jensen's inequality and the last inequality holds because $u(\cdot)$ is a non-decreasing function with $u(0) = 0$ and $1 \in \partial u(0)$. This indicates that the problem (10) is infeasible if $\sum_{i=t+1}^{t+k_n^t} y_n^i < 1$. Q.E.D.

Comparison results for Section 5.1.1

ϕ	Model	Prob	Mean	Std	CVaR	ϕ	Model	Prob	Mean	Std	CVaR
1.100	Cost-R	2.81%	2.92	34.42	2.92	0.825	Cost-R	8.82%	8.09	39.16	144.34
	SVI-RS	1.18%	0.55	7.32	0.55		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.05%	0.59	8.13	0.59		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	0.92%	0.37	5.48	0.37		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	7.46%	11.12	70.83	213.25		SVI-S	12.41%	31.98	188.59	564.67
	MVP	9.04%	14.99	75.21	266.19		MVP	11.84%	46.06	249.92	816.69
1.075	Cost-R	5.13%	3.15	18.88	62.44	0.800	Cost-R	10.88%	11.21	49.01	182.10
	SVI-RS	1.18%	0.44	5.60	0.44		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.14%	0.58	7.67	0.58		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	1.01%	0.39	5.54	0.39		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	8.38%	13.32	85.21	251.38		SVI-S	12.28%	30.64	206.26	540.52
	MVP	9.82%	26.41	156.12	483.05		MVP	12.76%	53.73	302.87	955.20
1.050	Cost-R	6.14%	3.74	20.64	73.39	0.775	Cost-R	12.15%	13.40	53.56	206.75
	SVI-RS	1.18%	0.44	5.60	0.44		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.14%	0.61	8.17	0.61		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	1.18%	0.69	8.87	0.69		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	8.33%	13.38	85.26	252.26		SVI-S	12.28%	30.64	206.26	540.52
	MVP	9.52%	21.66	124.83	394.29		MVP	12.76%	53.73	302.87	955.20
1.025	Cost-R	5.79%	3.99	22.36	78.63	0.750	Cost-R	11.40%	13.80	57.08	220.40
	SVI-RS	1.18%	0.44	5.60	0.44		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.14%	0.61	8.17	0.61		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	1.18%	0.69	8.87	0.69		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	8.33%	13.38	85.26	252.26		SVI-S	12.28%	30.64	206.26	540.52
	MVP	9.52%	21.66	124.83	394.29		MVP	12.76%	53.73	302.87	955.20
1.000	Cost-R	6.14%	4.43	23.47	85.96	0.725	Cost-R	11.49%	13.90	56.34	220.50
	SVI-RS	1.71%	1.43	18.46	1.43		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.62%	1.58	20.45	1.58		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	1.71%	1.73	21.02	1.73		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	9.82%	15.48	88.90	273.79		SVI-S	12.28%	30.64	206.26	540.52
	MVP	10.66%	46.17	294.35	858.35		MVP	12.76%	53.73	302.87	955.20
0.975	Cost-R	5.88%	4.97	40.37	98.25	0.700	Cost-R	12.76%	17.31	65.81	249.91
	SVI-RS	1.71%	1.43	18.46	1.43		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.62%	1.58	20.45	1.58		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	1.71%	1.73	21.02	1.73		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	9.82%	15.48	88.90	273.79		SVI-S	12.28%	30.64	206.26	540.52
	MVP	10.66%	46.17	294.35	858.35		MVP	12.76%	53.73	302.87	955.20
0.950	Cost-R	6.97%	4.63	24.91	87.98	0.675	Cost-R	9.21%	11.55	53.77	202.18
	SVI-RS	1.71%	1.43	18.46	1.43		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.62%	1.58	20.45	1.58		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	1.71%	1.73	21.02	1.73		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	9.82%	15.48	88.90	273.79		SVI-S	12.28%	30.64	206.26	540.52
	MVP	10.66%	46.17	294.35	858.35		MVP	12.76%	53.73	302.87	955.20
0.925	Cost-R	7.02%	5.90	29.99	112.00	0.650	Cost-R	14.25%	19.85	77.94	291.93
	SVI-RS	1.71%	1.43	18.46	1.43		SVI-RS	2.32%	3.26	42.14	3.26
	SVI-R	1.62%	1.58	20.45	1.58		SVI-R	2.15%	3.33	43.00	3.33
	SVI-OU	1.71%	1.73	21.02	1.73		SVI-OU	2.15%	3.33	43.00	3.33
	SVI-S	9.82%	15.48	88.90	273.79		SVI-S	12.28%	30.64	206.26	540.52
	MVP	10.66%	46.17	294.35	858.35		MVP	12.76%	53.73	302.87	955.20
0.900	Cost-R	8.20%	8.18	53.73	149.45	0.625	Cost-R	14.74%	24.75	91.11	349.31
	SVI-RS	1.97%	1.48	18.34	1.48		SVI-RS	2.76%	5.13	50.96	5.13
	SVI-R	1.89%	1.68	20.63	1.68		SVI-R	2.85%	5.48	53.24	5.48
	SVI-OU	2.02%	1.80	21.11	1.80		SVI-OU	2.85%	5.48	53.24	5.48
	SVI-S	9.34%	13.25	78.15	240.79		SVI-S	10.79%	17.52	97.84	302.58
	MVP	10.31%	35.63	222.50	654.05		MVP	12.50%	52.77	300.26	944.87
0.875	Cost-R	9.43%	8.08	37.22	141.39	0.600	Cost-R	12.85%	20.46	81.04	307.56
	SVI-RS	1.97%	1.48	18.34	1.48		SVI-RS	2.76%	5.13	50.96	5.13
	SVI-R	1.89%	1.68	20.63	1.68		SVI-R	2.85%	5.48	53.24	5.48
	SVI-OU	2.02%	1.80	21.11	1.80		SVI-OU	2.85%	5.48	53.24	5.48
	SVI-S	9.34%	13.25	78.15	240.79		SVI-S	10.79%	17.52	97.84	302.58
	MVP	10.31%	35.63	222.50	654.05		MVP	12.50%	52.77	300.26	944.87
0.850	Cost-R	10.00%	8.65	38.87	148.64						
	SVI-RS	2.06%	1.63	18.98	1.63						
	SVI-R	1.80%	1.56	20.24	1.56						
	SVI-OU	1.84%	1.51	19.96	1.51						
	SVI-S	9.34%	13.29	79.67	240.11						
	MVP	10.44%	19.66	95.28	338.17						

Table 6 **Comparison of the violation of inventory requirement between solutions from the six approaches.**

Comparison results for Section 5.1.2

κ	μ_2	Model	Prob	Mean	Std	CVaR	κ	μ_2	Model	Prob	Mean	Std	CVaR
0.8	16	SVI-Rs	1.00	1.00	1.00	1.00	0.9	16	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	0.93	0.90	0.93	0.90			SVI-R	1.00	0.99	0.99	0.99
		SVI-OU	0.93	0.90	0.93	0.90			SVI-OU	1.00	0.99	0.99	0.99
	20	SVI-Rs	1.00	1.00	1.00	1.00		20	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.81	2.23	1.60	43.50			SVI-R	1.83	2.25	1.62	2.25
		SVI-OU	1.81	2.23	1.60	43.47			SVI-OU	1.81	2.23	1.60	2.23
	24	SVI-Rs	1.00	1.00	1.00	1.00		24	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.09	1.60	1.52	1.58			SVI-R	1.09	1.60	1.52	1.60
		SVI-OU	1.13	1.62	1.52	1.60			SVI-OU	1.13	1.62	1.51	1.62
	28	SVI-Rs	1.00	1.00	1.00	1.00		28	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.02	1.01	1.00	1.00			SVI-R	1.09	1.06	1.02	1.06
		SVI-OU	1.24	1.19	1.06	1.06			SVI-OU	1.24	1.19	1.06	1.16
	30	SVI-Rs	1.00	1.00	1.00	1.00		30	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.28	1.23	1.01	1.00			SVI-R	1.33	1.28	1.03	1.14
		SVI-OU	1.54	1.51	1.10	1.06			SVI-OU	1.54	1.51	1.12	1.25
0.85	16	SVI-Rs	1.00	1.00	1.00	1.00	0.95	16	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	0.92	0.90	0.94	0.90			SVI-R	1.00	1.00	1.00	1.00
		SVI-OU	0.92	0.90	0.94	0.90			SVI-OU	1.00	1.00	1.00	1.00
	20	SVI-Rs	1.00	1.00	1.00	1.00		20	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.81	2.23	1.60	44.53			SVI-R	1.81	2.24	1.61	2.24
		SVI-OU	1.81	2.22	1.60	44.41			SVI-OU	1.81	2.24	1.61	2.24
	24	SVI-Rs	1.00	1.00	1.00	1.00		24	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.09	1.60	1.52	1.60			SVI-R	1.08	1.58	1.51	1.58
		SVI-OU	1.12	1.61	1.51	32.21			SVI-OU	1.12	1.60	1.51	1.60
	28	SVI-Rs	1.00	1.00	1.00	1.00		28	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.07	1.05	1.01	1.02			SVI-R	1.05	1.02	1.00	1.02
		SVI-OU	1.25	1.19	1.06	1.08			SVI-OU	1.24	1.20	1.07	1.20
	30	SVI-Rs	1.00	1.00	1.00	1.00		30	SVI-Rs	1.00	1.00	1.00	1.00
		SVI-R	1.30	1.25	1.02	1.03			SVI-R	1.31	1.24	1.01	24.02
		SVI-OU	1.55	1.51	1.10	1.10			SVI-OU	1.52	1.52	1.13	28.19

Table 7 **Comparison of SVI-RS, SVI-R, and SVI-OU with synthetic data**

Tests with different utility functions

In this work, we assume that the decision-maker minimizes the utility-based SVI, which is defined with a piecewise linear utility function. The piecewise linear utility can be used to approximate any convex utility. Here, we show that the performance of the SVI model would not change significantly as we use more linear segments.

We assume that the underlying utility is an exponential utility function $u(x)=e^x-1$, which is widely used in the literature. We will use different piecewise linear functions to approximate the function $u(x)=e^x-1$. The utility function $u_1(x)=\max\{-1,x\}$ we used in the numerical study can serve as an approximation to $u(x)$. The first piece of linear functions $y(x)=-1$ is an asymptote and $u(x)$ becomes arbitrarily close to it for x approaching $-\infty$. The second piece of linear functions $y(x)=x$ is tangent to $u(x)$ at $x=0$. Then we can find a utility function $u_2(x)$ by adding another piece of linear function which is tangent to $u(x)$ at $x=1$. It would be $u_2(x)=\max\{-1,x,e^1x-1\}$. By the same way, we add pieces of linear functions that are tangent to $u(x)$ at $x=2$ and $x=3$, and the resulted utility functions are $u_3(x)=\max\{-1,x,e^1x-1,e^2x-e^2-1\}$ and $u_4(x)=\max\{-1,x,e^1x-1,e^2x-e^2-1,e^3x-2e^3-1\}$, respectively.

We conduct the tests with the above utility functions. The parameters of the problem are the same as those used in Section 5.1.2. A sample of results is presented in Table 8 for illustration.

κ	Utility function	Prob	Mean	Std	CVaR		κ	Utility function	Prob	Mean	Std	CVaR
0.80	u_1	15.30%	0.79	2.40	9.76		0.90	u_1	8.80%	0.43	1.79	7.39
	u_2	14.59%	0.82	2.52	10.26			u_2	7.38%	0.38	1.71	7.00
	u_3	14.46%	0.82	2.51	10.24			u_3	7.32%	0.38	1.71	6.97
	u_4	14.46%	0.82	2.51	10.24			u_4	7.32%	0.38	1.71	6.97
0.85	u_1	13.25%	0.68	2.23	9.18		0.95	u_1	6.39%	0.28	1.44	5.52
	u_2	12.11%	0.65	2.22	9.19			u_2	5.24%	0.26	1.42	5.19
	u_3	11.99%	0.64	2.21	9.17			u_3	5.17%	0.26	1.41	5.14
	u_4	11.99%	0.64	2.21	9.17			u_4	5.17%	0.26	1.41	5.14

Table 8 Performance of the SVI-R model with different utility functions.

The performances of the SVI-R model with different utility functions are similar. Moreover, the results are quite representative of our comprehensive tests. It demonstrates the robustness of our model with respect to the number of linear segments of utility functions.

References

Adelman D (2004) A price-directed approach to stochastic inventory/routing. *Operations Research* **52**(4): 499-514.

Adulyasak Y, Cordeau J F, Jans R (2015) Benders decomposition for production routing under demand uncertainty. *Operations Research* **63**(4): 851-867.

Adulyasak Y, Jaillet P (2016) Models and algorithms for stochastic and robust vehicle routing with deadlines. *Transportation Science* **50**(2): 608-626.

Aghezzaf E H (2008) Robust distribution planning for supplier-managed inventory agreements when demand rates and travel times are stationary. *Journal of the Operational Research Society* **59**(8): 1055-1065.

Andersson H, Hoff A, Christiansen M, Løkketangen A (2010) Industrial aspects and literature survey: Combined inventory management and routing. *Computers & Operations Research* **37**(9): 1515-1536.

Archetti C, Bertazzi L, Laporte G, Speranza M G (2007) A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Science* **41**(3): 382-391.

Ardestani-Jaafari A, Delage E (2016) Robust optimization of sums of piecewise linear functions with application to inventory problems. *Operations Research* **64**(2): 474-494.

Artzner P, Delbaen F, Eber J M, Heath D (1999) Coherent measures of risk. *Mathematical Finance* **9**(3): 203-228.

Bell W J, Dalberto L M, Fisher M L, et al. (1983) Improving the distribution of industrial gases with an on-line computerized routing and scheduling optimizer. *Interfaces* **13**(6): 4-23.

Benders J F (1962) Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik* **4**(1): 238-252.

Ben-Tal A, Den Hertog D, De Waegenaere A, Melenberg B, Rennen G (2013) Robust solutions of optimization problems affected by uncertain probabilities. *Management Science* **59**(2): 341-357.

Ben-Tal A, El Ghaoui L, Nemirovski A (2009) *Robust Optimization.* Princeton University Press.

Ben-Tal A, Golany B, Nemirovski A, Vial J P (2005) Retailer-supplier flexible commitments contracts: A robust optimization approach. *Manufacturing & Service Operations Management* **7**(3): 248-271.

Ben-Tal A, Goryashko A, Guslitzer E, Nemirovski A (2004) Adjustable robust solutions of uncertain linear programs. *Mathematical Programming* **99**(2): 351-376.

Bernstein F, Federgruen A (2007) Coordination mechanisms for supply chains under price and service competition. *Manufacturing & Service Operations Management* **9**(3): 242-262.

Bertsimas D, Caramanis C (2010) Finite adaptability in multistage linear optimization. *IEEE Transactions on Automatic Control*, **55**(12), 2751-2766.

Bertsimas D, Caramanis C (2007) Adaptability via sampling. Decision and Control, 2007 46th IEEE Conference on. IEEE, 2007: 4717-4722.

Bertsimas D, Georghiou A (2015) Design of near optimal decision rules in multistage adaptive mixed-integer optimization. *Operations Research* **63**(3): 610-627.

Bertsimas D, Georghiou A (2018) Binary decision rules for multistage adaptive mixed-integer optimization. *Mathematical Programming* **167**(2): 395-433.

Bertsimas D, Goyal V (2012) On the power and limitations of affine policies in two-stage adaptive optimization. *Mathematical Programming* **134**(2): 491-531.

Bertsimas D, Gupta S, Tay J (2016) Scalable Robust and Adaptive Inventory Routing. *Available on Optimization Online.*

Bertsimas D, Gupta V, Kallus N (2018) Robust sample average approximation. *Mathematical Programming*, **171**(1): 217-282.

Bertsimas D, Iancu D A, Parrilo P A (2011) A hierarchy of near-optimal policies for multistage adaptive optimization. *IEEE Transactions on Automatic Control* **56**(12): 2809-2824.

Bertsimas D, Sim M (2003) Robust discrete optimization and network flows. *Mathematical Programming* **98**(1): 49-71.

Bertsimas D, Sim M (2004) The price of robustness. *Operations Research* **52**(1): 35-53.

Bertsimas D, Sim M, Zhang M (2019) Adaptive distributionally robust optimization. *Management Science* **65**(2): 604-618.

Brandimarte P, Zotteri G (2007) *Introduction to Distribution Logistics.* John Wiley & Sons.

Brinkmann J, Ulmer M W, Mattfeld D C (2016) Inventory routing for bike sharing systems. *Transportation Research Procedia* **19**: 316-327.

Brown D B, Sim M (2009) Satisficing measures for analysis of risky positions. *Management Science* **55**(1): 71-84.

Campbell A, Clarke L, Kleywegt A, Savelsbergh M (1998) The inventory routing problem. *Fleet Management and Logistics.* Springer, Boston, MA, 95-113.

Campbell A M, Savelsbergh M W P (2004) A decomposition approach for the inventory-routing problem. *Transportation Science* **38**(4): 488-502.

Çetinkaya S, Lee C Y (2000) Stock replenishment and shipment scheduling for vendor-managed inventory systems. *Management Science* **46**(2): 217-232.

Chen L G, Long D Z, Sim M (2015) On Dynamic Decision Making to Meet Consumption Targets. *Operations Research* **63**(5): 1117-1130.

Chen X, Sim M, Simchi-Levi D, Sun P (2007) Risk aversion in inventory management. *Operations Research* **55**(5): 828-842.

Chen X, Sim M, Sun P, Zhang J (2008) A linear decision-based approximation approach to stochastic programming. *Operations Research* **56**(2): 344-357.

Chen Z, Sim M, Xiong P (2020) Robust stochastic optimization made easy with RSOME. *Management Science* **66**(8): 3329-3339.

Cheung K L, Lee H L (2002) The inventory benefit of shipment coordination and stock rebalancing in a supply chain. *Management Science* **48**(2): 300-306.

Christiansen M, Fagerholt K, Flatberg T, et al. (2011) Maritime inventory routing with multiple products: A case study from the cement industry. *European Journal of Operational Research* **208**(1): 86-94.

Coelho L C, Cordeau J F, Laporte G (2014a) Thirty years of inventory routing. *Transportation Science* **48**(1): 1-19.

Coelho L C, Cordeau J F, Laporte G (2014b) Heuristics for dynamic and stochastic inventory-routing. *Computers & Operations Research* **52**: 55-67.

Colyvan M (2008) Relative expectation theory. *The Journal of philosophy* **105**(1), 37-44.

Custódio A L, Oliveira R C (2006) Redesigning distribution operations: a case study on integrating inventory management and vehicle routes design. *International Journal of Logistics: Research and Applications* **9**(2): 169-187.

Delage E, Iancu D A (2015) Robust multistage decision making. *INFORMS Tutorials in Operations Research*, 20-46.

Delage E, Ye Y (2010) Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations Research* **58**(3): 595-612.

Diecidue E, van de Ven J (2008) Aspiration level, probability of success and failure, and expected utility. *International Economic Review* **49**: 683-700.

Dong Y, Xu K (2002) A supply chain model of vendor managed inventory. *Transportation Research Part E: Logistics and Transportation Review* **38**(2): 75-95.

Esfahani P M, Kuhn D (2018) Data-driven distributionally robust optimization using the Wasserstein metric: Performance guarantees and tractable reformulations. *Mathematical Programming* **171**(1): 115-166.

Federgruen A, Prastacos G, Zipkin P H (1986) An allocation and distribution model for perishable products. *Operations Research* **34**(1): 75-82.

Federgruen A, Zipkin P (1984) A combined vehicle routing and inventory allocation problem. *Operations Research* **32**(5): 1019-1037.

Föllmer H, Schied A (2002) Convex measures of risk and trading constraints. *Finance and Stochastics* **6**(4): 429-447.

Gaur V, Fisher M L (2004) A periodic inventory routing problem at a supermarket chain. *Operations Research* **52**(6): 813-822.

Georghiou A, Wiesemann W, Kuhn D (2015) Generalized decision rule approximations for stochastic programming via liftings. *Mathematical Programming* **152**(1-2): 301-338.

Goh J, Sim M (2010) Distributionally robust optimization and its tractable approximations. *Operations Research* **58**(4-part-1): 902-917.

Golden B, Assad A, Dahl R (1984) Analysis of a large scale vehicle routing problem with an inventory component. *Large Scale Systems* **7**(2-3): 181-190.

Grable J E, Lytton R H (1999) Financial risk tolerance revisited: The development of a risk assessment instrument. *Financial Services Review* **8**: 163-181.

Grønhaug R, Christiansen M, Desaulniers G, Desrosiers J (2010) A branch-and-price method for a liquefied natural gas inventory routing problem. *Transportation Science* **44**(3): 400-415.

Hvattum L M, Løkketangen A (2009) Using scenario trees and progressive hedging for stochastic inventory routing problems. *Journal of Heuristics* **15**(6): 527-557.

Hvattum L M, Løkketangen A, Laporte G (2009) Scenario tree-based heuristics for stochastic inventory-routing problems. *INFORMS Journal on Computing* **21**(2): 268-285.

Jaillet P, Qi J, Sim M (2016) Routing optimization under uncertainty. *Operations Research* **64**(1): 186-200.

Kleywegt A J, Nori V S, Savelsbergh M W P (2002) The stochastic inventory routing problem with direct deliveries. *Transportation Science* **36**(1): 94-118.

Kleywegt A J, Nori V S, Savelsbergh M W P (2004) Dynamic programming approximations for a stochastic inventory routing problem. *Transportation Science* **38**(1): 42-70.

Kuhn D, Wiesemann W, Georghiou A (2011) Primal and dual linear decision rules in stochastic and robust optimization. *Mathematical Programming* **130**(1): 177-209.

Mao J C T (1970) Survey of capital budgeting: Theory and practice. *The Journal of Finance* **25**(2): 349-360.

Miranda P A, Garrido R A (2004) Incorporating inventory control decisions into a strategic distribution network design model with stochastic demand. *Transportation Research Part E: Logistics and Transportation Review* **40**(3): 183-207.

Papageorgiou D J, Cheon M S, Nemhauser G, Sokol J (2014) Approximate dynamic programming for a class of long-horizon maritime inventory routing problems. *Transportation Science* **49**(4): 870-885.

Pham-Gia T, Hung T L (2001) The mean and median absolute deviations. *Mathematical and Computer Modelling* **34**(7-8): 921-936.

Rahmaniani R, Crainic T G, Gendreau M, et al. (2017) The Benders decomposition algorithm: A literature review. *European Journal of Operational Research* **259**(3): 801-817.

Rockafellar R T, Uryasev S (2000) Optimization of conditional value-at-risk. *Journal of risk* **2**: 21-42.

Rockafellar R T, Uryasev S (2002) Conditional value-at-risk for general loss distributions. *Journal of banking & finance* **26**(7): 1443-1471.

Samuelson P A (1977) St. Petersburg paradoxes: Defanged, dissected, and historically described. *Journal of Economic Literature* **15**(1), 24-55.

Simon H A (1955) A behavioral model of rational choice. *The Quarterly Journal of Economics* **69**(1): 99-118.

Solyalı O, Cordeau J F, Laporte G (2012) Robust inventory routing under demand uncertainty. *Transportation Science* **46**(3): 327-340.

Waller M, Johnson M E, Davis T (1999) Vendor-managed inventory in the retail supply chain. *Journal of Business Logistics* **20**(1): 183-203.

Zhang Y, Baldacci R, Sim M, et al. (2019) Routing optimization with time windows under uncertainty. *Mathematical Programming* **175**(1): 263-305.