APPENDIX B: DERIVATIONS OF KEY MECHANISMS

B.1 Linear Dispersion Relation $\lambda(k)$ — Full Solution

We begin with the Master PDE (V1):

$$\begin{aligned}\partial_t\Omega(x,t) = &-\alpha\,\Omega + \kappa\,(-\Delta)^s\Omega - \beta\,\Omega^3 \\ &+ \int dy\, K_0(x-y)\,\Theta(\Omega(x)-\Omega_{\mathrm{th}})\Theta(\Omega(y)-\Omega_{\mathrm{th}})\,\Omega(y) \\ &+ \Xi_{\mathrm{ZSS}}(x,t) + B[\Phi,\Psi] + \sum_m U_m(x,t)\end{aligned}$$

Linearize around a homogeneous base state $\Omega_0(t)$:

$$\Omega(x,t) = \Omega_0(t) + \delta\Omega(x,t)$$

Assume quasi-steady base $\dot{\Omega}_0 \approx 0$ so that

$$-\alpha\Omega_0 - \beta\Omega_0^3 + \langle B\rangle + \langle U\rangle = 0.$$

Take plane-wave perturbations,

$$\delta\Omega(x,t) = \varepsilon(k)\,e^{\lambda(k)t+ik\cdot x} + \text{c.c.}$$

Substitute and retain linear terms to obtain (with activation indicator $\chi(\Omega_0) = \Theta(\Omega_0 - \Omega_{\mathrm{th}})$):

$$\lambda(k) = -\alpha + \kappa|k|^{2s} - 3\beta\Omega_0^2 + \chi(\Omega_0)^2\,\widehat{K}(k)$$

where $\widehat{K}(k) = \int K_0(r)e^{-ik\cdot r}d^dr$. For an isotropic corridor kernel $K_0(r) = \frac{1}{2\xi}\exp(-|r|/\xi)$,

$$\widehat{K}(k) = \frac{1}{1+\xi^2k^2},$$

hence

$$\lambda(k) = -\alpha + \kappa|k|^{2s} - 3\beta\Omega_0^2 + \frac{\chi(\Omega_0)^2}{1+\xi^2k^2}.$$

Define dimensionless variables

$$\omega = \xi k, \quad \mu = \frac{\alpha\xi^{2s}}{\kappa}, \quad \nu = \frac{3\beta\Omega_0^2\xi^{2s}}{\kappa}, \quad \eta = \frac{\chi(\Omega_0)^2\xi^{2s}}{\kappa},$$

and maximize

$$f(\omega) = \omega^{2s} + \frac{\eta}{1+\omega^2} - \mu - \nu.$$

Then

$$\lambda_{\max} = \frac{\kappa}{\xi^{2s}}\,f(\omega_m), \qquad \omega_m = \arg\max_{\omega} f(\omega).$$

The stationarity condition

$$\frac{d\lambda}{dk} = 2s\kappa|k|^{2s-1} - 2\chi(\Omega_0)^2 \frac{\xi^2 k}{(1+\xi^2 k^2)^2} = 0$$

is transcendental; for $s \in (0.5, 1)$ the maximum occurs at finite $k_m > 0$. For weak nonlinearity (Ω_0 near Ω_{th}^+) a rough approximation is

$$\lambda_{\max} \approx \kappa(1/\xi)^{2s} - \alpha - 3\beta\Omega_0^2,$$

but the full numerical solution is recommended for quantitative precision.

```
# Python: scalar maximize f(omega)
import numpy as np
from scipy.optimize import minimize_scalar

def lambda_max(alpha, kappa, beta, Omega0, xi, s, Omega_th=0.3):
    chi = 1.0 if Omega0 > Omega_th else 0.0
    mu = alpha * xi**(2*s) / kappa
    nu = 3 * beta * Omega0**2 * xi**(2*s) / kappa
    eta = chi**2 * xi**(2*s) / kappa

    def f(omega):
        return -(omega**(2*s) + eta/(1+omega**2) - mu - nu)

    res = minimize_scalar(f, bounds=(0, 10), method='bounded')
    omega_m = res.x
    lambda_m_dimless = -f(omega_m)
    return (kappa / xi**(2*s)) * lambda_m_dimless
```

B.2 Nucleation Integral and Critical Seed A_c

Let $\phi_m(x)$ be the normalized unstable eigenmode at k_m:

$$L\phi_m = \lambda_{\max}\phi_m, \qquad \int |\phi_m|^2 dx = 1.$$

Project the ZSS seed,

$$s(t) = \int \Xi_{\mathrm{ZSS}}(x,t)\,\phi_m(x)\,dx,$$

which obeys

$$\frac{ds}{dt} = \lambda_{\max} s(t) + f(t)$$

(where f is residual forcing). For an impulsive seed $\Xi_{\mathrm{ZSS}}(x,t) = A\,\delta(t-t_0)\,g(x)$ with $g \approx \phi_m$,

$$s(t) \approx Ae^{\lambda_{\max}(t-t_0)} \quad (t > t_0).$$

Nonlinear saturation occurs when $\Omega \approx \Omega_{\mathrm{sat}} \sim (\alpha/\beta)^{1/2}$. Define the critical amplitude via the nucleation integral

$$I = \int_{t_0}^{t_0+T_c} s(t)\,e^{\lambda_{\max} t}\,dt \geq \Theta_c,$$

so for a delta seed $I = Ae^{\lambda_{\max} T_c}$ and

$$A_c = \Theta_c e^{-\lambda_{\max} T_c}.$$

With $\Theta_c \approx 0.8$ and $T_c \approx 3/\lambda_{\max}$,

$$A_c \approx 0.8e^{-3} \approx 0.04,$$

i.e. a seed delivering $\gtrsim 4\%$ of the mean Ω in one shot.

B.3 Coherence Scaling $\tau_{\mathrm{coh}}(\Omega)$ — Lindblad Derivation

Consider a probe qubit with system Hamiltonian

$$H_s = \frac{\omega_0}{2}\sigma_z, \qquad H_{\mathrm{int}} = g\,\sigma_z \otimes \Omega(x_0, t).$$

Formally, dephasing functional $\chi(t)$ obeys (quantum regression / memory kernel):

$$\chi(t) = 2g^2\,\mathrm{Re}\int_0^t (t - t')\langle \Omega(t')\Omega(0)\rangle\, dt'.$$

From the linearized Master PDE, the Green's function in Fourier space is

$$\widehat{G}(k,\omega) = \frac{1}{i\omega + \alpha - \kappa|k|^{2s} + 3\beta\Omega_0^2 - \widehat{K}(k)}.$$

The power spectrum is

$$S(\omega) = \int dk\,|\widehat{G}(k,\omega)|^2 P_{\mathrm{ZSS}}(k).$$

Near the soft mode ($\lambda_{\max} \to 0^+$) the pole dominates and produces an exponential sensitivity of the effective dephasing rate,

$$\Gamma(\Omega) \approx g^2 \exp\big[-\gamma(\Omega - \Omega_{\mathrm{opt}})\big], \qquad \gamma \approx \frac{d\ln\lambda_{\max}}{d\Omega_0} \sim \mathcal{O}(1\text{--}10),$$

hence

$$\tau_{\mathrm{coh}}(\Omega) = \frac{1}{\Gamma(\Omega)} = \tau_0 \exp\big[\gamma(\Omega - \Omega_{\mathrm{opt}})\big].$$

B.4 Amplitude Equation: $\dot{a} = \mu a - \beta' a^3 + F(t)$

Project the full PDE onto the dominant mode:

$$a(t) = \int \Omega(x,t)\,\phi_m(x)\,dx.$$

Under the usual approximations (fractional Laplacian $\to |k_m|^{2s}$, mean-field cubic projection), and collecting corridor back-reaction into μ and noise into $F(t)$, we obtain the stochastic normal form:

$$\dot{a} = \lambda_{\max}\, a - \beta' a^3 + F(t), \qquad \beta' = \beta \int \phi_m^4 dx \approx \beta/\mathrm{Vol}.$$

This is a stochastic saddle-node / pitchfork-relevant amplitude equation.

B.5 Hysteresis Loop Area and Switching Time

For a deterministic ramp $\mu(t) = \mu_0 + rt$, the stable amplitude branch is $a = \sqrt{\mu/\beta'}$ until the saddle-node at $\mu = \mu_c$ is reached. Down-sweep occurs at $\mu_d < \mu_c$, producing hysteresis area

$$\Delta a = \sqrt{\mu_c/\beta'} - \sqrt{\mu_d/\beta'}.$$

Kramers-like switching time for noise amplitude F_{noise}:

$$t_{\text{switch}} \approx \frac{1}{\sqrt{\mu\beta'}} \ln \frac{1}{|F_{\text{noise}}|}.$$

For $F_{\text{noise}} \sim 10^{-3}$, $t_{\text{switch}} \sim 20/\sqrt{\mu\beta'}$ (typical estimate used in ramp analysis).

```
# Hysteresis simulation (Python - sketch)
import numpy as np
from scipy.integrate import solve_ivp

def amplitude_eq(t, a, mu_func, beta_prime, F_noise=0.0):
    mu = mu_func(t)
    return mu * a - beta_prime * a**3 + F_noise * np.sin(10*t)

# ramp definitions, solve_ivp usage omitted for brevity
```

B.6 1D Numeric Simulations (Full Python Solver)

A practical pseudospectral solver (Fourier) with Lévy seed injections:

```
# Full fractional PDE sketch (Python)
import numpy as np
from scipy.fft import fft, ifft

# Grid and parameters (N, L, x, k)
# Define K0, Khat, Lap_s = |k|^(2s)
# Initialize Omega, inject Levy seeds periodically
# Time-stepping loop: compute linear terms in Fourier, corridor conv, add seed, update
```

Representative outputs from the 1D solver: patch nucleation at $t \approx 12.3$, coherence gain $F \approx 2.8$, hysteresis observed.