Supplementary Material: Leader-Follower Density Control of Spatial Dynamics in Large-Scale Multi-Agent Systems

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Abstract

In this document we report supplementary material for the manuscript "Leader-Follower Density Control of Spatial Dynamics in Large-Scale Multi-Agent Systems" which is currently under review. We present an extended version of Appendix A and B, two figures comparing periodic and non-periodic repulsive interaction kernels for different values of the characteristic length scale, and one extra numerical validation in a scenario where the desired reference density for the followers is bi-modal.

I. Periodization of the interaction kernel

Periodic interaction kernels f are obtained from the periodization of standard non-periodic kernels $\hat{f}$,

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(x + 2k\pi). \tag{S1}$$

A. Repulsive kernel

The non-periodic repulsive kernel is in the form

$$\hat{f}(x) = \mathrm{sgn}(x)\mathrm{exp}\left(-\frac{|x|}{L}\right). \tag{S2}$$

Note that we utilize a length-scale L while fixing the domain to $[-\pi, \pi]$. Periodization leads to

$$f(x) = \sum_{k=-\infty}^{\infty} \mathrm{sgn}(x + 2k\pi)\mathrm{exp}\left(-\frac{|x + 2k\pi|}{L}\right). \tag{S3}$$

By separating the infinite series into two other infinite series based on the sign of $x + 2k\pi$ and computing each of these series individually leads to

$$f(x) = \frac{1}{\exp\left(\frac{2\pi}{L}\right) - 1}\mathrm{sgn}(x)\left[\exp\left(\frac{2\pi - |x|}{L}\right) - \exp\left(\frac{|x|}{L}\right)\right]. \tag{S4}$$

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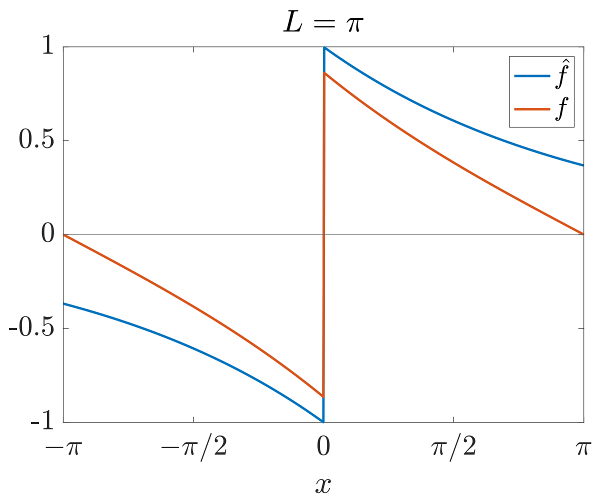
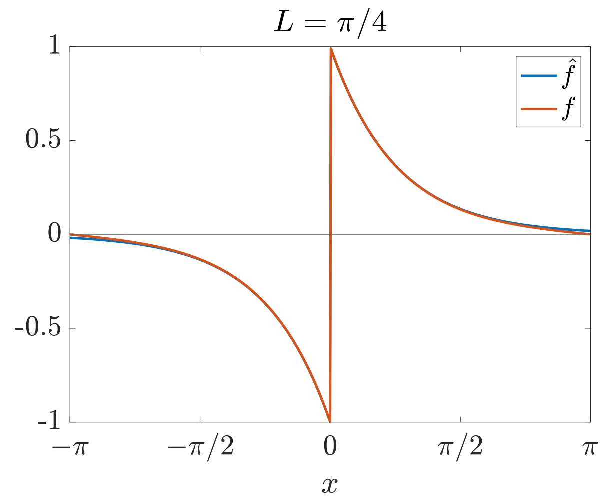


(a)



(b)

Fig. S1: Non-periodic VS periodic repulsive interaction kernel. (a) $L = \pi$, (b) $L = \pi/4$.

In Fig. S1, we report the representation of the non-periodic and periodic repulsive kernel for two different values of the characteristic interaction length scale L. It can be noticed that, as L is decreased, letting the amplitude of interactions decays faster, the mismatch between the non-periodic and periodic kernel becomes smaller.

II. Deconvolution

For the periodic repulsive kernel we derived in (S4), the convolution $\phi = f * \rho$ can be expanded as

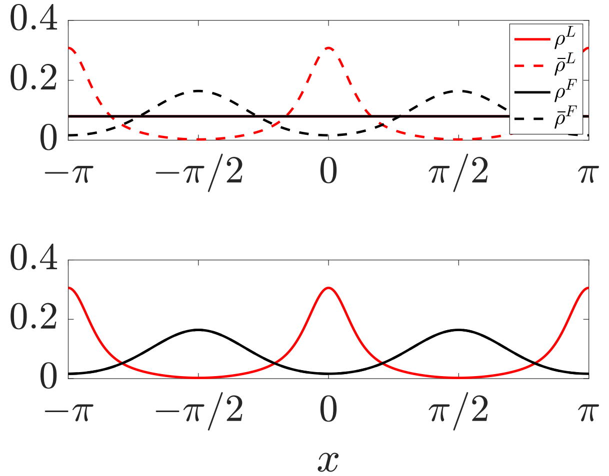
$$\begin{aligned}\phi(x) = (f * \rho)(x) = \frac{1}{\exp\left(\frac{2\pi}{L}\right) - 1}\Bigg[&\exp\left(\frac{2\pi - x}{L}\right)\int_{-\pi}^{x} \exp\left(\frac{y}{L}\right)\rho(y)\,\mathrm{d}y \\ &- \exp\left(\frac{x}{L}\right)\int_{-\pi}^{x} \exp\left(-\frac{y}{L}\right)\rho(y)\,\mathrm{d}y - \exp\left(\frac{2\pi + x}{L}\right)\int_{x}^{\pi} \exp\left(-\frac{y}{L}\right)\rho(y)\,\mathrm{d}y \\ &+ \exp\left(-\frac{x}{L}\right)\int_{x}^{\pi} \exp\left(\frac{y}{L}\right)\rho(y)\,\mathrm{d}y\Bigg]. \end{aligned} \tag{S5}$$

This expression can be derived twice in space. By taking the first spatial derivative, one gets

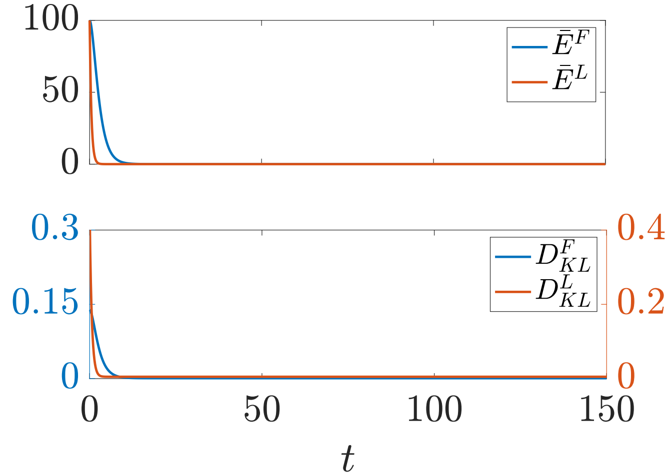
$$\begin{aligned}\phi_x(x) = \frac{1}{L\left(\exp\left(\frac{2\pi}{L}\right) - 1\right)}\Bigg[&- \exp\left(\frac{2\pi - x}{L}\right)\int_{-\pi}^{x} \exp\left(\frac{y}{L}\right)\rho(y)\,\mathrm{d}y - \exp\left(\frac{x}{L}\right)\int_{-\pi}^{x} \exp\left(-\frac{y}{L}\right)\rho(y)\,\mathrm{d}y \\ &- \exp\left(\frac{2\pi + x}{L}\right)\int_{x}^{\pi} \exp\left(-\frac{y}{L}\right)\rho(y)\,\mathrm{d}y - \exp\left(-\frac{x}{L}\right)\int_{x}^{\pi} \exp\left(\frac{y}{L}\right)\rho(y)\,\mathrm{d}y\Bigg] + 2\rho(x), \end{aligned} \tag{S6}$$

where we used that

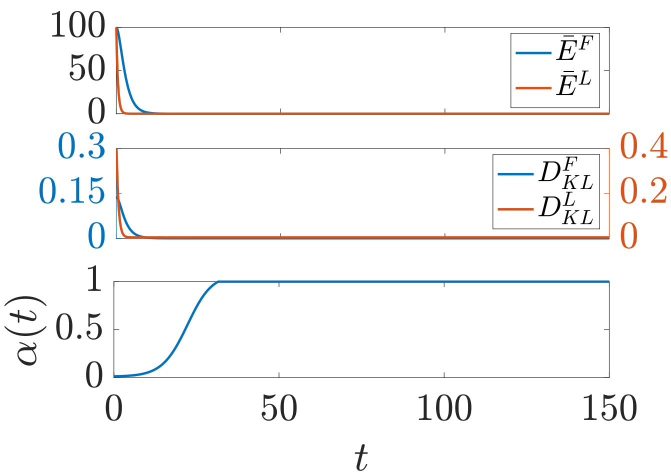
$$\left(\int_{f_1(x)}^{f_2(x)} g(x)\,\mathrm{d}x\right)_x = g(f_2(x))(f_2(x))_x - g(f_1(x))(f_1(x))_x. \tag{S7}$$



(a)



(b)



(c)

Fig. S2: Multimodal trial: (a) initial and final densities; (b) time evolution of the percentage error and KL divergences using the feed-forward control scheme; (c) time evolution of percentage error, KL divergences, and α using the reference-governor scheme.

By further differentiating (S6) in space, we get

$$\phi_{xx}(x) = \frac{1}{L^2\left(\exp\left(\frac{2\pi}{L}\right)-1\right)}\left[\exp\left(\frac{2\pi-x}{L}\right)\int_{-\pi}^{x}\exp\left(\frac{y}{L}\right)\rho(y)\,\mathrm{d}y - \exp\left(\frac{x}{L}\right)\int_{-\pi}^{x}\exp\left(-\frac{y}{L}\right)\rho(y)\,\mathrm{d}y \right.$$
$$\left. - \exp\left(\frac{2\pi+x}{L}\right)\int_{x}^{\pi}\exp\left(-\frac{y}{L}\right)\rho(y)\,\mathrm{d}y + \exp\left(-\frac{x}{L}\right)\int_{x}^{\pi}\exp\left(\frac{y}{L}\right)\rho(y)\,\mathrm{d}y\right] + 2\rho_x(x) \quad \text{(S8)}$$

Hence, we can rewrite

$$\phi_{xx}(x) = \frac{\phi(x)}{L^2} + 2\rho_x(x). \quad \text{(S9)}$$

Thus, by integration, we can retrieve ρ as follows:

$$\rho(x) = \frac{1}{2}\int \left(\phi_{xx}(x) - \frac{\phi(x)}{L^2}\right) \mathrm{d}x + B, \tag{S10}$$

where B is an arbitrary constant.

III. Numerical validation for multimodal desired densities

In this section we report an extra simulation trial for the feed-forward and reference-governor scheme discussed in Sec. VI and VII of the main manuscript.

We consider as desired followers' density the mixture of two von Mises distributions, that is

$$\bar{\rho}^F(x) = \frac{M^F}{2}\left[\frac{\exp\left(\kappa_1 \cos(x-\mu_1)\right)}{2\pi I_0(\kappa_1)} + \frac{\exp\left(\kappa_2 \cos(x-\mu_2)\right)}{2\pi I_0(\kappa_2)}\right], \tag{S11}$$

where κ_1, κ_2 and μ_1, μ_2 are the concentration coefficients and means, respectively, of the two modes, and I_0 is the modified Bessel function of the first kind of order 0. We fix $\kappa_1 = \kappa_2 = 3$, and different means, $\mu_1 = -\pi/2$ and $\mu_2 = \pi/2$. To guarantee feasibility, we fixed the leaders' mass to be $M^L = 0.5$. Moreover, we consider $D = 0.05$ and $L = \pi$.

Results are reported in Fig. S2. Specifically, we show the initial and final displacement of the leaders' and followers' densities, resulting in the same steady-state profile with both the control techniques. Then, in Fig. S2b and S2c (upper panel), we report the time evolution of the percentage errors and KL divergences using respectively the feed-forward and the reference-governor control schemes. In Fig. S2c (bottom panel), we show the time evolution of the control function α selected according to (87) of the main manuscript.