SOAP calculation note

March 20, 2018

We choose the radial basis and atomic density as

$$g_{n\ell}(r) = \sum_{k=1}^{3} \beta_{nk} r^{\ell} e^{-\alpha_{k\ell} r^2}, \tag{1}$$

$$\rho_N(\mathbf{r}) = \sum_{i=1}^{N} e^{-(\mathbf{r}-\mathbf{r}_i)^2} = \sum_{i=1}^{N} e^{-(x^2+y^2+z^2-2(xx_i+yy_i+zz_i))} e^{-(x_i^2+y_i^2+z_i^2)}. \tag{2}$$

The power spectrum what we need to calculate is given by

$$\mathbf{P}_{nn'\ell}(\mathbf{r}_i) = \sum_m c_{nlm} c^*_{n'lm}, \tag{3}$$

where

$$c_{n\ell m} = \int dV g_{n\ell}(r) \rho_N(\mathbf{r}) Y_{\ell m}(\theta, \phi). \tag{4}$$

In the calculation result, I will define the function

$$X_{ij}^{\alpha} \equiv [(x_i - iy_i)(x_j + iy_j)]^{\alpha}. \tag{5}$$

These expressions will be used in the result.

$$\mathrm{Re}[X_{ij}] = x_i x_j + y_i y_j, \tag{6}$$

$$\mathrm{Re}[X_{ij}^2] = (x_i x_j + y_i y_j)^2 - (x_i y_j - x_j y_i)^2, \tag{7}$$

$$\mathrm{Re}[X_{ij}^3] = (x_i x_j + y_i y_j)[(x_i x_j + y_i y_j)^2 - 3(x_i y_j - x_j y_i)^2], \tag{8}$$

$$\mathrm{Re}[X_{ij}^4] = (x_i x_j + y_i y_j)^4 + (x_i y_j - x_j y_i)^4 - 6(x_i x_j + y_i y_j)^2(x_i y_j - x_j y_i)^2, \tag{9}$$

$$\begin{aligned}\mathrm{Re}[X_{ij}^5] = {} & (x_i x_j + y_i y_j)[(x_i x_j + y_i y_j)^4 - 10(x_i x_j + y_i y_j)^2(x_i y_j - x_j y_i)^2 \\ & + 5(x_i y_j - x_j y_i)^4],\end{aligned} \tag{10}$$

$$\begin{aligned}\mathrm{Re}[X_{ij}^6] = {} & (x_i x_j + y_i y_j)^6 - (x_i y_j - x_j y_i)^6 - 15(x_i x_j + y_i y_j)^4(x_i y_j - x_j y_i)^2 \\ & + 15(x_i x_j + y_i y_j)^2(x_i y_j - x_j y_i)^4,\end{aligned} \tag{11}$$

$$\begin{aligned}\mathrm{Re}[X_{ij}^7] = {} & (x_i x_j + y_i y_j)[(x_i x_j + y_i y_j)^6 - 7(x_i y_j - x_j y_i)^6 \\ & - 21(x_i x_j + y_i y_j)^4(x_i y_j - x_j y_i)^2 + 35(x_i x_j + y_i y_j)^2(x_i y_j - x_j y_i)^4],\end{aligned} \tag{12}$$

$$\begin{aligned}\mathrm{Re}[X_{ij}^8] = {} & (x_i x_j + y_i y_j)^8 + (x_i y_j - x_j y_i)^8 - 28(x_i x_j + y_i y_j)^6(x_i y_j - x_j y_i)^2 \\ & - 28(x_i x_j + y_i y_j)^2(x_i y_j - x_j y_i)^6 + 70(x_i x_j + y_i y_j)^4(x_i y_j - x_j y_i)^4,\end{aligned} \tag{13}$$

$$\begin{aligned}\mathrm{Re}[X_{ij}^9] = {} & (x_i x_j + y_i y_j)[(x_i x_j + y_i y_j)^8 + 9(x_i y_j - x_j y_i)^8 \\ & - 36(x_i x_j + y_i y_j)^6(x_i y_j - x_j y_i)^2 + 126(x_i x_j + y_i y_j)^4(x_i y_j - x_j y_i)^4 \\ & - 84(x_i x_j + y_i y_j)^2(x_i y_j - x_j y_i)^6].\end{aligned} \tag{14}$$

- $\ell = 0$

Recall that the form of spherical harmonics for $\ell = 0$ is given by

$$Y_{00} = \frac{1}{2}\sqrt{\frac{1}{\pi}}. \tag{15}$$

Therefore,

$$\begin{aligned}
c_{n00} &= \frac{1}{2}\sqrt{\frac{1}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k0})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})} \\
&= \frac{1}{2}\sqrt{\frac{1}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k0})(\mathbf{r}-\frac{1}{1+\alpha_{k0}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k0}}{1+\alpha_{k0}}r_i^2} \\
&= \frac{1}{2}\sqrt{\frac{1}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k0}}{1+\alpha_{k0}}r_i^2}\int dV e^{-(1+\alpha_{k0})r^2} \\
&= \sum_{k=1}^{3}\frac{\pi\beta_{nk}}{2(1+\alpha_{k0})^{\frac{3}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k0}}{1+\alpha_{k0}}r_i^2}.
\end{aligned} \tag{16}$$

The power spectrum $\mathbf{P}_{nn'\ell}$ for $\ell = 0$ is given by

$$\begin{aligned}
\mathbf{P}_{nn'0}(\mathbf{r}_i) &= c_{n00}c^*_{n'00} \\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{\pi^2\beta_{nk}\beta^*_{n'k'}}{4(1+\alpha_{k0})^{\frac{3}{2}}(1+\alpha_{k'0})^{\frac{3}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{k0}}{1+\alpha_{k0}}r_i^2}e^{-\frac{\alpha_{k'0}}{1+\alpha_{k'0}}r_j^2}.
\end{aligned} \tag{17}$$

- $\ell = 1$

The form of spherical harmonics for $\ell = 1$ are

$$Y_{1-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\varphi}\sin\theta = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{(x-iy)}{r}, \tag{18}$$

$$Y_{10} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r}, \tag{19}$$

$$Y_{11} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\varphi}\sin\theta = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{(x+iy)}{r}. \tag{20}$$

Therefore,

$$\begin{aligned}
c_{n1-1} &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k1})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)\\
&= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k1})(\mathbf{r}-\frac{1}{1+\alpha_{k1}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}(x-iy)\\
&= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}\int dV e^{-(1+\alpha_{k1})r^2}\left(x+\frac{x_i}{1+\alpha_{k1}}-i\left(y+\frac{y_i}{1+\alpha_{k1}}\right)\right)\\
&= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}\int dV e^{-(1+\alpha_{k1})r^2}\left(\frac{x_i-iy_i}{1+\alpha_{k1}}\right)\\
&= \sum_{k=1}^{3}\frac{\sqrt{3}\pi\beta_{nk}}{2\sqrt{2}(1+\alpha_{k1})^{\frac{5}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}(x_i-iy_i). \qquad (21)
\end{aligned}$$

$$\begin{aligned}
c_{n10} &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k1})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}z\\
&= \frac{1}{2}\sqrt{\frac{3}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k1})(\mathbf{r}-\frac{1}{1+\alpha_{k1}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}z\\
&= \frac{1}{2}\sqrt{\frac{3}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}\int dV e^{-(1+\alpha_{k1})r^2}\left(z+\frac{z_i}{1+\alpha_{k1}}\right)\\
&= \frac{1}{2}\sqrt{\frac{3}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}\int dV e^{-(1+\alpha_{k1})r^2}\left(\frac{z_i}{1+\alpha_{k1}}\right)\\
&= \sum_{k=1}^{3}\frac{\sqrt{3}\pi\beta_{nk}}{2(1+\alpha_{k1})^{\frac{5}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}z_i. \qquad (22)
\end{aligned}$$

$$
\begin{aligned}
c_{n11} &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k1})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)\\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k1})(\mathbf{r}-\frac{1}{1+\alpha_{k1}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}(x+iy)\\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}\int dV e^{-(1+\alpha_{k1})r^2}\left(x+\frac{x_i}{1+\alpha_{k1}}+i\left(y+\frac{y_i}{1+\alpha_{k1}}\right)\right)\\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}\int dV e^{-(1+\alpha_{k1})r^2}\left(\frac{x_i+iy_i}{1+\alpha_{k1}}\right)\\
&= -\sum_{k=1}^{3}\frac{\sqrt{3}\pi\beta_{nk}}{2\sqrt{2}(1+\alpha_{k1})^{\frac{5}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}(x_i+iy_i).
\end{aligned}
\tag{23}
$$

The power spectrum for $\ell = 1$ is

$$
\begin{aligned}
\mathbf{P}_{nn'1}(\mathbf{r}_i) &= \sum_m c_{n1m}c^*_{n'1m}\\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{3\pi^2\beta_{nk}\beta^*_{n'k'}}{4(1+\alpha_{k1})^{\frac{5}{2}}(1+\alpha_{k'1})^{\frac{5}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{k1}}{1+\alpha_{k1}}r_i^2}e^{-\frac{\alpha_{k'1}}{1+\alpha_{k'1}}r_j^2}\left[\mathrm{Re}[X_{ij}]+z_iz_j\right].
\end{aligned}
\tag{24}
$$

- $\ell = 2$

The form of spherical harmonics for $\ell = 2$ are

$$Y_{2-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{-2i\varphi}\sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x-iy)^2}{r^2}, \tag{25}$$

$$Y_{2-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{-i\varphi}\sin\theta\cos\theta = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x-iy)z}{r^2}, \tag{26}$$

$$Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta-1) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{(2z^2-x^2-y^2)}{r^2}, \tag{27}$$

$$Y_{21} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\varphi}\sin\theta\cos\theta = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)z}{r^2}, \tag{28}$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\varphi}\sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{r^2}. \tag{29}$$

Therefore,

$$\begin{aligned}
c_{n2-2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k2})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2 \\
&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k2})(\mathbf{r}-\frac{1}{1+\alpha_{k2}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x-iy)^2 \\
&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\left(x+\frac{x_i}{1+\alpha_{k2}}-i\left(y+\frac{y_i}{1+\alpha_{k2}}\right)\right)^2 \\
&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\left(x^2-y^2+\frac{(x_i-iy_i)^2}{(1+\alpha_{k2})^2}\right) \\
&= \sum_{k=1}^{3}\frac{\sqrt{15}\pi\beta_{nk}}{4\sqrt{2}(1+\alpha_{k2})^{\frac{7}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x_i-iy_i)^2. \qquad (30)
\end{aligned}$$

$$\begin{aligned}
c_{n2-1} &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k2})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)z \\
&= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k2})(\mathbf{r}-\frac{1}{1+\alpha_{k2}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x-iy)z \\
&= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\left(x+\frac{x_i}{1+\alpha_{k2}}-i\left(y+\frac{y_i}{1+\alpha_{k2}}\right)\right)\left(z+\frac{z_i}{1+\alpha_{k2}}\right) \\
&= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\frac{(x_i-iy_i)z_i}{(1+\alpha_{k2})^2} \\
&= \sum_{k=1}^{3}\frac{\sqrt{15}\pi\beta_{nk}}{2\sqrt{2}(1+\alpha_{k2})^{\frac{7}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x_i-iy_i)z_i. \qquad (31)
\end{aligned}$$

$$
\begin{aligned}
c_{n20} &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k2})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(2z^2-x^2-y^2) \\
&= \frac{1}{4}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k2})(\mathbf{r}-\frac{1}{1+\alpha_{k2}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(2z^2-x^2-y^2) \\
&= \frac{1}{4}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2} \\
&\times\left(2\left(z+\frac{z_i}{1+\alpha_{k2}}\right)^2-\left(x+\frac{x_i}{1+\alpha_{k2}}\right)^2-\left(y+\frac{y_i}{1+\alpha_{k2}}\right)^2\right) \\
&= \frac{1}{4}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\left(2z^2-x^2-y^2+\frac{2z_i^2-x_i^2-y_i^2}{(1+\alpha_{k2})^2}\right) \\
&= \sum_{k=1}^{3}\frac{\sqrt{5}\pi\beta_{nk}}{4(1+\alpha_{k2})^{\frac{7}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(2z_i^2-x_i^2-y_i^2). \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
c_{n21} &= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k2})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)z \\
&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k2})(\mathbf{r}-\frac{1}{1+\alpha_{k2}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x+iy)z \\
&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\left(x+\frac{x_i}{1+\alpha_{k2}}+i\left(y+\frac{y_i}{1+\alpha_{k2}}\right)\right)\left(z+\frac{z_i}{1+\alpha_{k2}}\right) \\
&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\frac{(x_i+iy_i)z_i}{(1+\alpha_{k2})^2} \\
&= -\sum_{k=1}^{3}\frac{\sqrt{15}\pi\beta_{nk}}{2\sqrt{2}(1+\alpha_{k2})^{\frac{7}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x_i+iy_i)z_i. \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
c_{n22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k2})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2 \\
&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k2})(\mathbf{r}-\frac{1}{1+\alpha_{k2}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x+iy)^2 \\
&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\left(x+\frac{x_i}{1+\alpha_{k2}}+i\left(y+\frac{y_i}{1+\alpha_{k2}}\right)\right)^2 \\
&= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}\int dV e^{-(1+\alpha_{k2})r^2}\left(x^2-y^2+\frac{(x_i+iy_i)^2}{(1+\alpha_{k2})^2}\right) \\
&= \sum_{k=1}^{3}\frac{\sqrt{15}\pi\beta_{nk}}{4\sqrt{2}(1+\alpha_{k2})^{\frac{7}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}(x_i+iy_i)^2. \qquad (34)
\end{aligned}
$$

The power spectrum for $\ell = 2$ is

$$
\begin{aligned}
\mathbf{P}_{nn'2}(\mathbf{r}_i) &= \sum_m c_{n2m}c^*_{n'2m} \\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{5\pi^2\beta_{nk}\beta^*_{n'k'}}{16(1+\alpha_{k2})^{\frac{7}{2}}(1+\alpha_{k'2})^{\frac{7}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{k2}}{1+\alpha_{k2}}r_i^2}e^{-\frac{\alpha_{k'2}}{1+\alpha_{k'2}}r_j^2}\left[3\mathrm{Re}[X_{ij}^2]+12z_iz_j\mathrm{Re}[X_{ij}]\right. \\
&\left.+(2z_i^2-x_i^2-y_i^2)(2z_j^2-x_j^2-y_j^2)\right]. \qquad (35)
\end{aligned}
$$

- $\ell = 3$

The form of spherical harmonics for $\ell = 3$ are

$$Y_{3-3} = \frac{1}{8}\sqrt{\frac{35}{\pi}}e^{-3i\varphi}\sin^3\theta = \frac{1}{8}\sqrt{\frac{35}{\pi}}\frac{(x-iy)^3}{r^3}, \qquad (36)$$

$$Y_{3-2} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}e^{-2i\varphi}\sin^2\theta\cos\theta = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\frac{(x-iy)^2z}{r^3}, \qquad (37)$$

$$Y_{3-1} = \frac{1}{8}\sqrt{\frac{21}{\pi}}e^{-i\varphi}\sin\theta(5\cos^2\theta-1) = \frac{1}{8}\sqrt{\frac{21}{\pi}}\frac{(x-iy)(4z^2-x^2-y^2)}{r^3}, \qquad (38)$$

$$Y_{30} = \frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\theta-3\cos\theta) = \frac{1}{4}\sqrt{\frac{7}{\pi}}\frac{z(2z^2-3x^2-3y^2)}{r^3}, \qquad (39)$$

$$Y_{31} = -\frac{1}{8}\sqrt{\frac{21}{\pi}}e^{i\varphi}\sin\theta(5\cos^2\theta-1) = -\frac{1}{8}\sqrt{\frac{21}{\pi}}\frac{(x+iy)(4z^2-x^2-y^2)}{r^3}, \qquad (40)$$

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}e^{2i\varphi}\sin^2\theta\cos\theta = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\frac{(x+iy)^2z}{r^3}, \qquad (41)$$

$$Y_{33} = -\frac{1}{8}\sqrt{\frac{35}{\pi}}e^{3i\varphi}\sin^3\theta = -\frac{1}{8}\sqrt{\frac{35}{\pi}}\frac{(x+iy)^3}{r^3}. \qquad (42)$$

Therefore,

$$\begin{aligned}
c_{n3-3} &= \frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k3})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^3 \\
&= \frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k3})(\mathbf{r}-\frac{1}{1+\alpha_{k3}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x-iy)^3 \\
&= \frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\left(x+\frac{x_i}{1+\alpha_{k3}}-i\left(y+\frac{y_i}{1+\alpha_{k3}}\right)\right)^3 \\
&= \frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\frac{(x_i-iy_i)^3}{(1+\alpha_{k3})^3} \\
&= \sum_{k=1}^{3}\frac{\sqrt{35}\pi\beta_{nk}}{8(1+\alpha_{n3})^{\frac{9}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x_i-iy_i)^3. \qquad (43)
\end{aligned}$$

$$\begin{aligned}
c_{n3-2} &= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k3})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2 z \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k3})(\mathbf{r}-\frac{1}{1+\alpha_{k3}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x-iy)^2 z \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\left(x+\frac{x_i}{1+\alpha_{k3}}-i\left(y+\frac{y_i}{1+\alpha_{k3}}\right)\right)^2\left(z+\frac{z_i}{1+\alpha_{k3}}\right) \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\frac{(x_i-iy_i)^2 z_i}{(1+\alpha_{k3})^3} \\
&= \sum_{k=1}^{3}\frac{\sqrt{105}\pi\beta_{nk}}{4\sqrt{2}(1+\alpha_{k3})^{\frac{9}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x_i-iy_i)^2 z_i. \qquad (44)
\end{aligned}$$

$$
\begin{aligned}
c_{n3-1} &= \frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k3})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)(4z^2-x^2-y^2) \\
&= \frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k3})(\mathbf{r}-\frac{1}{1+\alpha_{k3}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x-iy)(4z^2-x^2-y^2) \\
&= \frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\left(x+\frac{x_i}{1+\alpha_{k3}}-i\left(y+\frac{y_i}{1+\alpha_{k3}}\right)\right) \\
&\times\left[4\left(z+\frac{z_i}{1+\alpha_{k3}}\right)^2-\left(x+\frac{x_i}{1+\alpha_{k3}}\right)^2-\left(y+\frac{y_i}{1+\alpha_{k3}}\right)^2\right] \\
&= \frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\frac{(x_i-iy_i)(4z_i^2-x_i^2-y_i^2)}{(1+\alpha_{k3})^3} \\
&= \sum_{k=1}^{3}\frac{\sqrt{21}\pi\beta_{nk}}{8(1+\alpha_{k3})^{\frac{9}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x_i-iy_i)(4z_i^2-x_i^2-y_i^2). \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
c_{n30} &= \frac{1}{4}\sqrt{\frac{7}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k3})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}z(2z^2-3x^2-3y^2) \\
&= \frac{1}{4}\sqrt{\frac{7}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k3})(\mathbf{r}-\frac{1}{1+\alpha_{k3}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}z(2z^2-3x^2-3y^2) \\
&= \frac{1}{4}\sqrt{\frac{7}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\left(z+\frac{z_i}{1+\alpha_{k3}}\right) \\
&\times\left[2\left(z+\frac{z_i}{1+\alpha_{k3}}\right)^2-3\left(x+\frac{x_i}{1+\alpha_{k3}}\right)^2-3\left(y+\frac{y_i}{1+\alpha_{k3}}\right)^2\right] \\
&= \frac{1}{4}\sqrt{\frac{7}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\frac{z_i(2z_i^2-3x_i^2-3y_i^2)}{(1+\alpha_{k3})^3} \\
&= \sum_{k=1}^{3}\frac{\sqrt{7}\pi\beta_{nk}}{4(1+\alpha_{k3})^{\frac{9}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}z_i(2z_i^2-3x_i^2-3y_i^2). \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
c_{n31} &= -\frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k3})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)(4z^2-x^2-y^2)\\
&= -\frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k3})(\mathbf{r}-\frac{1}{1+\alpha_{k3}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x+iy)(4z^2-x^2-y^2)\\
&= -\frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\left(x+\frac{x_i}{1+\alpha_{k3}}+i\left(y+\frac{y_i}{1+\alpha_{k3}}\right)\right)\\
&\times\left[4\left(z+\frac{z_i}{1+\alpha_{k3}}\right)^2-\left(x+\frac{x_i}{1+\alpha_{k3}}\right)^2-\left(y+\frac{y_i}{1+\alpha_{k3}}\right)^2\right]\\
&= -\frac{1}{8}\sqrt{\frac{21}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\frac{(x_i+iy_i)(4z_i^2-x_i^2-y_i^2)}{(1+\alpha_{k3})^3}\\
&= -\sum_{k=1}^{3}\frac{\sqrt{21}\pi\beta_{nk}}{8(1+\alpha_{k3})^{\frac{9}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x_i+iy_i)(4z_i^2-x_i^2-y_i^2). \qquad (47)
\end{aligned}
$$

$$
\begin{aligned}
c_{n32} &= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k3})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2 z\\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k3})(\mathbf{r}-\frac{1}{1+\alpha_{k3}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x+iy)^2 z\\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\left(x+\frac{x_i}{1+\alpha_{k3}}+i\left(y+\frac{y_i}{1+\alpha_{k3}}\right)\right)^2\left(z+\frac{z_i}{1+\alpha_{k3}}\right)\\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\frac{(x_i+iy_i)^2 z_i}{(1+\alpha_{k3})^3}\\
&= \sum_{k=1}^{3}\frac{\sqrt{105}\pi\beta_{nk}}{4\sqrt{2}(1+\alpha_{k3})^{\frac{9}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x_i+iy_i)^2 z_i. \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
c_{n33} &= -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k3})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^3 \\
&= -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k3})(\mathbf{r}-\frac{1}{1+\alpha_{k3}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x+iy)^3 \\
&= -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\left(x+\frac{x_i}{1+\alpha_{k3}}+i\left(y+\frac{y_i}{1+\alpha_{k3}}\right)\right)^3 \\
&= -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}\int dV e^{-(1+\alpha_{k3})r^2}\frac{(x_i+iy_i)^3}{(1+\alpha_{k3})^3} \\
&= -\sum_{k=1}^{3}\frac{\sqrt{35}\pi\beta_{nk}}{8(1+\alpha_{n3})^{\frac{9}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k3}}{1+\alpha_{k3}}r_i^2}(x_i+iy_i)^3. \qquad (49)
\end{aligned}
$$

The power spectrum for $\ell = 3$ is

$$
\begin{aligned}
\mathbf{P}_{nn'3}(\mathbf{r}_i) &= \sum_m c_{n3m}c^*_{n'3m} \\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{7\pi^2\beta_{nk}\beta^*_{n'k'}}{16(1+\alpha_{n3})^{\frac{9}{2}}(1+\alpha_{n'3})^{\frac{9}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{n3}}{1+\alpha_{n3}}r_i^2}e^{-\frac{\alpha_{n'3}}{1+\alpha_{n'3}}r_j^2}\left[\frac{5}{2}\mathrm{Re}[X_{ij}^3]+15z_iz_j\mathrm{Re}[X_{ij}^2]\right. \\
&\left.+\frac{3}{2}(4z_i^2-x_i^2-y_i^2)(4z_j^2-x_j^2-y_j^2)\mathrm{Re}[X_{ij}]+z_iz_j(2z_i^2-3x_i^2-3y_i^2)(2z_j^2-3x_j^2-3y_j^2)\right]. \qquad (50)
\end{aligned}
$$

- $\ell = 4$

The form of spherical harmonics for $\ell = 4$ are

$$Y_{4-4} = \frac{3}{16}\sqrt{\frac{35}{2\pi}}e^{-4i\varphi}\sin^4\theta = \frac{3}{16}\sqrt{\frac{35}{2\pi}}\frac{(x-iy)^4}{r^4} \tag{51}$$

$$Y_{4-3} = \frac{3}{8}\sqrt{\frac{35}{\pi}}e^{-3i\varphi}\sin^3\theta\cos\theta = \frac{3}{8}\sqrt{\frac{35}{\pi}}\frac{(x-iy)^3 z}{r^4}, \tag{52}$$

$$Y_{4-2} = \frac{3}{8}\sqrt{\frac{5}{2\pi}}e^{-2i\varphi}\sin^2\theta(7\cos^2\theta - 1) = \frac{3}{8}\sqrt{\frac{5}{2\pi}}\frac{(x-iy)^2(7z^2-r^2)}{r^4}, \tag{53}$$

$$Y_{4-1} = \frac{3}{8}\sqrt{\frac{5}{\pi}}e^{-i\varphi}\sin\theta(7\cos^3\theta - 3\cos\theta) = \frac{3}{8}\sqrt{\frac{5}{\pi}}\frac{(x-iy)z(7z^2-3r^2)}{r^4}, \tag{54}$$

$$Y_{40} = \frac{3}{16}\sqrt{\frac{1}{\pi}}(35\cos^4\theta - 30\cos^2\theta + 3) = \frac{3}{16}\sqrt{\frac{1}{\pi}}\frac{(35z^4 - 30z^2r^2 + 3r^4)}{r^4}, \tag{55}$$

$$Y_{41} = -\frac{3}{8}\sqrt{\frac{5}{\pi}}e^{i\varphi}\sin\theta(7\cos^3\theta - 3\cos\theta) = -\frac{3}{8}\sqrt{\frac{5}{\pi}}\frac{(x+iy)z(7z^2-3r^2)}{r^4}, \tag{56}$$

$$Y_{42} = \frac{3}{8}\sqrt{\frac{5}{2\pi}}e^{2i\varphi}\sin^2\theta(7\cos^2\theta - 1) = \frac{3}{8}\sqrt{\frac{5}{2\pi}}\frac{(x+iy)^2(7z^2-r^2)}{r^4}, \tag{57}$$

$$Y_{43} = -\frac{3}{8}\sqrt{\frac{35}{\pi}}e^{3i\varphi}\sin^3\theta\cos\theta = -\frac{3}{8}\sqrt{\frac{35}{\pi}}\frac{(x+iy)^3 z}{r^4}, \tag{58}$$

$$Y_{44} = \frac{3}{16}\sqrt{\frac{35}{2\pi}}e^{4i\varphi}\sin^4\theta = \frac{3}{16}\sqrt{\frac{35}{2\pi}}\frac{(x+iy)^4}{r^4}. \tag{59}$$

Therefore,

$$\begin{aligned}
c_{n4-4} &= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2 + r_i^2 - 2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^4 \\
&= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x-iy)^4 \\
&= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x + \frac{x_i}{1+\alpha_{k4}} - i\left(y + \frac{y_i}{1+\alpha_{k4}}\right)\right)^4 \\
&= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x^4 + y^4 - 6x^2y^2 + \frac{(x_i - iy_i)^4}{(1+\alpha_{k4})^4}\right) \\
&= \sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i - iy_i)^4.
\end{aligned} \tag{60}$$

$$
\begin{aligned}
c_{n4-3} &= \frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^3 z \\
&= \frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x-iy)^3 z \\
&= \frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x+\frac{x_i}{1+\alpha_{k4}}-i\left(y+\frac{y_i}{1+\alpha_{k4}}\right)\right)^3\left(z+\frac{z_i}{1+\alpha_{k4}}\right) \\
&= \frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\frac{(x_i-iy_i)^3 z_i}{(1+\alpha_{k4})^4} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{8(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i-iy_i)^3 z_i. \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
c_{n4-2} &= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2(7z^2-r^2) \\
&= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x-iy)^2(7z^2-r^2) \\
&= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x+\frac{x_i}{1+\alpha_{k4}}-i\left(y+\frac{y_i}{1+\alpha_{k4}}\right)\right)^2 \\
&\times\left[7\left(z+\frac{z_i}{1+\alpha_{k4}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k4})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k4}}\right)\right] \\
&= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\frac{(x_i-iy_i)^2(7z_i^2-r_i^2)}{(1+\alpha_{k4})^4} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{5}\pi\beta_{nk}}{8\sqrt{2}(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i-iy_i)^2(7z_i^2-r_i^2). \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
c_{n4-1} &= \frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)z(7z^2-3r^2) \\
&= \frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x-iy)z(7z^2-3r^2) \\
&= \frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x+\frac{x_i}{1+\alpha_{k4}}-i\left(y+\frac{y_i}{1+\alpha_{k4}}\right)\right) \\
&\times\left(z+\frac{z_i}{1+\alpha_{k4}}\right)\left[7\left(z+\frac{z_i}{1+\alpha_{k4}}\right)-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k4})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k4}}\right)\right] \\
&= \frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\frac{(x_i-iy_i)z_i(7z_i^2-3r_i^2)}{(1+\alpha_{k4})^4} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{5}\pi\beta_{nk}}{8(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i-iy_i)z_i(7z_i^2-3r_i^2). \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
c_{n40} &= \frac{3}{16}\sqrt{\frac{1}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(35z^4-30z^2r^2+3r^4) \\
&= \frac{3}{16}\sqrt{\frac{1}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(35z^4-30z^2r^2+3r^4) \\
&= \frac{3}{16}\sqrt{\frac{1}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left[35\left(z+\frac{z_i}{1+\alpha_{k4}}\right)^4\right. \\
&-30\left(z+\frac{z_i}{1+\alpha_{k4}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k4})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k4}}\right) \\
&\left.+3\left(r^2+\frac{r_i^2}{(1+\alpha_{k4})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k4}}\right)^2\right] \\
&= \frac{3}{16}\sqrt{\frac{1}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\frac{35z_i^4-30z_i^2r_i^2+3r_i^4}{(1+\alpha_{k4})^4} \\
&= \sum_{k=1}^{3}\frac{3\pi\beta_{nk}}{16(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(35z_i^4-30z_i^2r_i^2+3r_i^4). \qquad (64)
\end{aligned}
$$

$$
\begin{aligned}
c_{n41} &= -\frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)z(7z^2-3r^2) \\
&= -\frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x+iy)z(7z^2-3r^2) \\
&= -\frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x+\frac{x_i}{1+\alpha_{k4}}+i\left(y+\frac{y_i}{1+\alpha_{k4}}\right)\right) \\
&\times\left(z+\frac{z_i}{1+\alpha_{k4}}\right)\left[7\left(z+\frac{z_i}{1+\alpha_{k4}}\right)-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k4})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k4}}\right)\right] \\
&= -\frac{3}{8}\sqrt{\frac{5}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\frac{(x_i+iy_i)z_i(7z_i^2-3r_i^2)}{(1+\alpha_{k4})^4} \\
&= -\sum_{k=1}^{3}\frac{3\sqrt{5}\pi\beta_{nk}}{8(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i+iy_i)z_i(7z_i^2-3r_i^2). \qquad (65)
\end{aligned}
$$

$$
\begin{aligned}
c_{n42} &= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2(7z^2-r^2) \\
&= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x+iy)^2(7z^2-r^2) \\
&= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x+\frac{x_i}{1+\alpha_{k4}}+i\left(y+\frac{y_i}{1+\alpha_{k4}}\right)\right)^2 \\
&\times\left[7\left(z+\frac{z_i}{1+\alpha_{k4}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k4})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k4}}\right)\right] \\
&= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\frac{(x_i+iy_i)^2(7z_i^2-r_i^2)}{(1+\alpha_{k4})^4} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{5}\pi\beta_{nk}}{8\sqrt{2}(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i+iy_i)^2(7z_i^2-r_i^2). \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
c_{n43} &= -\frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^3 z \\
&= -\frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x+iy)^3 z \\
&= -\frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x+\frac{x_i}{1+\alpha_{k4}}+i\left(y+\frac{y_i}{1+\alpha_{k4}}\right)\right)^3\left(z+\frac{z_i}{1+\alpha_{k4}}\right) \\
&= -\frac{3}{8}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\frac{(x_i+iy_i)^3 z_i}{(1+\alpha_{k4})^4} \\
&= -\sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{8(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i+iy_i)^3 z_i. \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
c_{n44} &= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k4})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^4 \\
&= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k4})(\mathbf{r}-\frac{1}{1+\alpha_{k4}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x+iy)^4 \\
&= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x+\frac{x_i}{1+\alpha_{k4}}+i\left(y+\frac{y_i}{1+\alpha_{k4}}\right)\right)^4 \\
&= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}\int dV e^{-(1+\alpha_{k4})r^2}\left(x^4+y^4-6x^2y^2+\frac{(x_i+iy_i)^4}{(1+\alpha_{k4})^4}\right) \\
&= \sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{k4})^{\frac{11}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k4}}{1+\alpha_{k4}}r_i^2}(x_i+iy_i)^4. \qquad (68)
\end{aligned}
$$

The power spectrum for $\ell = 4$ is

$$
\begin{aligned}
\mathbf{P}_{nn'4}(\mathbf{r}_i) &= \sum_m c_{n4m} c^*_{n'4m} \\
&= \sum_{k=1}^{3} \sum_{k'=1}^{3} \frac{9\pi^2 \beta_{nk} \beta^*_{n'k'}}{256(1+\alpha_{n4})^{\frac{11}{2}} (1+\alpha_{n'4})^{\frac{11}{2}}} \sum_{i=1}^{N} \sum_{j=1}^{N} e^{-\frac{\alpha_{n4}}{1+\alpha_{n4}} r_i^2} e^{-\frac{\alpha_{n'4}}{1+\alpha_{n'4}} r_j^2} \big[35\mathrm{Re}[X_{ij}^4] + 280 z_i z_j \mathrm{Re}[X_{ij}^3] \\
&+20(7z_i^2 - r_i^2)(7z_j^2 - r_j^2)\mathrm{Re}[X_{ij}^2] + 40 z_i z_j (7z_i^2 - 3r_i^2)(7z_j^2 - 3r_j^2)\mathrm{Re}[X_{ij}] \\
&+(35z_i^4 - 30z_i^2 r_i^2 + 3r_i^4)(35z_j^4 - 30z_j^2 r_j^2 + 3r_j^4)\big] \,. \qquad (69)
\end{aligned}
$$

- $\ell = 5$

The form of spherical harmonics for $\ell = 5$ are

$$Y_{5-5} = \frac{3}{32}\sqrt{\frac{77}{\pi}}e^{-5i\varphi}\sin^5\theta = \frac{3}{32}\sqrt{\frac{77}{\pi}}\frac{(x-iy)^5}{r^5} \tag{70}$$

$$Y_{5-4} = \frac{3}{16}\sqrt{\frac{385}{2\pi}}e^{-4i\varphi}\sin^4\theta\cos\theta = \frac{3}{16}\sqrt{\frac{385}{2\pi}}\frac{(x-iy)^4 z}{r^5} \tag{71}$$

$$Y_{5-3} = \frac{1}{32}\sqrt{\frac{385}{\pi}}e^{-3i\varphi}\sin^3\theta(9\cos^2\theta - 1) = \frac{1}{32}\sqrt{\frac{385}{\pi}}\frac{(x-iy)^3(9z^2-r^2)}{r^5}, \tag{72}$$

$$Y_{5-2} = \frac{1}{8}\sqrt{\frac{1155}{2\pi}}e^{-2i\varphi}\sin^2\theta(3\cos^3\theta - \cos\theta) = \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\frac{(x-iy)^2 z(3z^2-r^2)}{r^5}, \tag{73}$$

$$Y_{5-1} = \frac{1}{16}\sqrt{\frac{165}{2\pi}}e^{-i\varphi}\sin\theta(21\cos^4\theta - 14\cos^2\theta + 1) = \frac{1}{16}\sqrt{\frac{165}{2\pi}}\frac{(x-iy)(21z^4-14z^2r^2+r^4)}{r^5}, \tag{74}$$

$$Y_{50} = \frac{1}{16}\sqrt{\frac{11}{\pi}}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta) = \frac{1}{16}\sqrt{\frac{11}{\pi}}\frac{z(63z^4-70z^2r^2+15r^4)}{r^5}, \tag{75}$$

$$Y_{51} = -\frac{1}{16}\sqrt{\frac{165}{2\pi}}e^{i\varphi}\sin\theta(21\cos^4\theta - 14\cos^2\theta + 1) = -\frac{1}{16}\sqrt{\frac{165}{2\pi}}\frac{(x+iy)(21z^4-14z^2r^2+r^4)}{r^5}, \tag{76}$$

$$Y_{52} = \frac{1}{8}\sqrt{\frac{1155}{2\pi}}e^{2i\varphi}\sin^2\theta(3\cos^3\theta - \cos\theta) = \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\frac{(x+iy)^2 z(3z^2-r^2)}{r^5}, \tag{77}$$

$$Y_{53} = -\frac{1}{32}\sqrt{\frac{385}{\pi}}e^{3i\varphi}\sin^3\theta(9\cos^2\theta - 1) = -\frac{1}{32}\sqrt{\frac{385}{\pi}}\frac{(x+iy)^3(9z^2-r^2)}{r^5}, \tag{78}$$

$$Y_{54} = \frac{3}{16}\sqrt{\frac{385}{2\pi}}e^{4i\varphi}\sin^4\theta\cos\theta = \frac{3}{16}\sqrt{\frac{385}{2\pi}}\frac{(x+iy)^4 z}{r^5} \tag{79}$$

$$Y_{55} = -\frac{3}{32}\sqrt{\frac{77}{\pi}}e^{5i\varphi}\sin^5\theta = -\frac{3}{32}\sqrt{\frac{77}{\pi}}\frac{(x+iy)^5}{r^5} \tag{80}$$

Therefore,

$$
\begin{aligned}
c_{n5-5} &= \frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^5 \\
&= \frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x-iy)^5 \\
&= \frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}-i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^5 \\
&= \frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i-iy_i)^5}{(1+\alpha_{k5})^5} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{77}\pi\beta_{nk}}{32(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i-iy_i)^5. \qquad (81)
\end{aligned}
$$

$$
\begin{aligned}
c_{n5-4} &= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^4 z \\
&= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x-iy)^4 z \\
&= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}-i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^4\left(z+\frac{z_i}{1+\alpha_{k5}}\right) \\
&= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i-iy_i)^4 z_i}{(1+\alpha_{k5})^5} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{385}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i-iy_i)^4 z_i. \qquad (82)
\end{aligned}
$$

$$
\begin{aligned}
c_{n5-3} &= \frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^3(9z^2-r^2) \\
&= \frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x-iy)^3(9z^2-r^2) \\
&= \frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dVe^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}-i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^3 \\
&\times\left[9\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)\right] \\
&= \frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dVe^{-(1+\alpha_{k5})r^2}\frac{(x_i-iy_i)^3(9z_i^2-r_i^2)}{(1+\alpha_{k5})^5} \\
&= \sum_{k=1}^{3}\frac{\sqrt{385}\pi\beta_{nk}}{32(1+\alpha_{n5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i-iy_i)^3(9z_i^2-r_i^2). \qquad (83)
\end{aligned}
$$

$$
\begin{aligned}
c_{n5-2} &= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2z(3z^2-r^2) \\
&= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x-iy)^2z(3z^2-r^2) \\
&= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dVe^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}-i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^2\left(z+\frac{z_i}{1+\alpha_{k5}}\right) \\
&\times\left[3\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)\right] \\
&= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dVe^{-(1+\alpha_{k5})r^2}\frac{(x_i-iy_i)^2z_i(3z_i^2-r_i^2)}{(1+\alpha_{n5})^5} \\
&= \sum_{k=1}^{3}\frac{\sqrt{1155}\pi\beta_{nk}}{8\sqrt{2}(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i-iy_i)^2z_i(3z_i^2-r_i^2). \qquad (84)
\end{aligned}
$$

$$
\begin{aligned}
c_{n5-1} &= \frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)(21z^4-14z^2r^2+r^4)\\
&= \frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x-iy)(21z^4-14z^2r^2+r^4)\\
&= \frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}-i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)\\
&\times\left[21\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^4-14\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)\right.\\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)^2\right]\\
&= \frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i-iy_i)(21z_i^4-14z_i^2r_i^2+r_i^4)}{(1+\alpha_{k5})^5}\\
&= \sum_{k=1}^{3}\frac{\sqrt{165}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i-iy_i)(21z_i^4-14z_i^2r_i^2+r_i^4).
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
c_{n50} &= \frac{1}{16}\sqrt{\frac{11}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}z(63z^4-70z^2r^2+15r^4)\\
&= \frac{1}{16}\sqrt{\frac{11}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}z(63z^4-70z^2r^2+15r^4)\\
&= \frac{1}{16}\sqrt{\frac{11}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(z+\frac{z_i}{1+\alpha_{k5}}\right)\\
&\times\left[63\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^4-70\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)\right.\\
&\left.+15\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)^2\right]\\
&= \frac{1}{16}\sqrt{\frac{11}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{z_i(63z_i^4-70z_i^2r_i^2+15r_i^4)}{(1+\alpha_{k5})^5}\\
&= \sum_{k=1}^{3}\frac{\sqrt{11}\pi\beta_{nk}}{16(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}z_i(63z_i^4-70z_i^2r_i^2+15r_i^4).
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
c_{n51} &= -\frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)(21z^4-14z^2r^2+r^4)\\
&= -\frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x+iy)(21z^4-14z^2r^2+r^4)\\
&= -\frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}+i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)\\
&\times\left[21\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^4-14\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)\right.\\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)^2\right]\\
&= -\frac{1}{16}\sqrt{\frac{165}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i+iy_i)(21z_i^4-14z_i^2r_i^2+r_i^4)}{(1+\alpha_{k5})^5}\\
&= -\sum_{k=1}^{3}\frac{\sqrt{165}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i+iy_i)(21z_i^4-14z_i^2r_i^2+r_i^4).
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
c_{n52} &= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2z(3z^2-r^2)\\
&= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x+iy)^2z(3z^2-r^2)\\
&= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}+i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^2\left(z+\frac{z_i}{1+\alpha_{k5}}\right)\\
&\times\left[3\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)\right]\\
&= \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i+iy_i)^2z_i(3z_i^2-r_i^2)}{(1+\alpha_{n5})^5}\\
&= \sum_{k=1}^{3}\frac{\sqrt{1155}\pi\beta_{nk}}{8\sqrt{2}(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i+iy_i)^2z_i(3z_i^2-r_i^2).
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
c_{n53} &= -\frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^3(9z^2-r^2)\\
&= -\frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x+iy)^3(9z^2-r^2)\\
&= -\frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}+i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^3\\
&\times\left[9\left(z+\frac{z_i}{1+\alpha_{k5}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k5})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k5}}\right)\right]\\
&= -\frac{1}{32}\sqrt{\frac{385}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i+iy_i)^3(9z_i^2-r_i^2)}{(1+\alpha_{k5})^5}\\
&= -\sum_{k=1}^{3}\frac{\sqrt{385}\pi\beta_{nk}}{32(1+\alpha_{n5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i+iy_i)^3(9z_i^2-r_i^2). \qquad (89)
\end{aligned}
$$

$$
\begin{aligned}
c_{n54} &= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^4 z\\
&= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x+iy)^4 z\\
&= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}+i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^4\left(z+\frac{z_i}{1+\alpha_{k5}}\right)\\
&= \frac{3}{16}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i+iy_i)^4 z_i}{(1+\alpha_{k5})^5}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{385}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i+iy_i)^4 z_i. \qquad (90)
\end{aligned}
$$

$$
\begin{aligned}
c_{n55} &= -\frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k5})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^5 \\
&= -\frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k5})(\mathbf{r}-\frac{1}{1+\alpha_{k5}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x+iy)^5 \\
&= -\frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\left(x+\frac{x_i}{1+\alpha_{k5}}+i\left(y+\frac{y_i}{1+\alpha_{k5}}\right)\right)^5 \\
&= -\frac{3}{32}\sqrt{\frac{77}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}\int dV e^{-(1+\alpha_{k5})r^2}\frac{(x_i+iy_i)^5}{(1+\alpha_{k5})^5} \\
&= -\sum_{k=1}^{3}\frac{3\sqrt{77}\pi\beta_{nk}}{32(1+\alpha_{k5})^{\frac{13}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k5}}{1+\alpha_{k5}}r_i^2}(x_i+iy_i)^5. \qquad (91)
\end{aligned}
$$

The power spectrum for $\ell = 5$ is

$$
\begin{aligned}
\mathbf{P}_{nn'5}(\mathbf{r}_i) &= \sum_m c_{n5m}c^*_{n'5m} \\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{11\pi^2\beta_{nk}\beta^*_{n'k'}}{256(1+\alpha_{n5})^{\frac{13}{2}}(1+\alpha_{n'5})^{\frac{13}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{n5}}{1+\alpha_{n5}}r_i^2}e^{-\frac{\alpha_{n'5}}{1+\alpha_{n'5}}r_j^2}\Big[\frac{63}{2}\mathrm{Re}[X^5_{ij}]+315z_iz_j\mathrm{Re}[X^4_{ij}] \\
&\frac{35}{2}(9z_i^2-r_i^2)(9z_j^2-r_j^2)\mathrm{Re}[X^3_{ij}]+420z_iz_j(3z_i^2-r_i^2)(3z_j^2-r_j^2)\mathrm{Re}[X^2_{ij}] \\
&+15(21z_i^4-14z_i^2r_i^2+r_i^4)(21z_j^4-14z_j^2r_j^2+r_j^4)\mathrm{Re}[X_{ij}] \\
&+(63z_i^4-70z_i^2r_i^2+15r_i^4)(63z_j^4-70z_j^2r_j^2+15r_j^4)z_iz_j\Big]. \qquad (92)
\end{aligned}
$$

- $\ell = 6$

The form of spherical harmonics for $\ell = 6$ are

$$Y_{6-6} = \frac{1}{64}\sqrt{\frac{3003}{\pi}} e^{-6i\varphi} \sin^6\theta = \frac{1}{64}\sqrt{\frac{3003}{\pi}} \frac{(x-iy)^6}{r^6}, \tag{93}$$

$$Y_{6-5} = \frac{3}{32}\sqrt{\frac{1001}{\pi}} e^{-5i\varphi} \sin^5\theta\cos\theta = \frac{3}{32}\sqrt{\frac{1001}{\pi}} e^{-5i\varphi} \frac{(x-iy)^5 z}{r^6}, \tag{94}$$

$$Y_{6-4} = \frac{3}{32}\sqrt{\frac{91}{2\pi}} e^{-4i\varphi} \sin^4\theta(11\cos^2\theta - 1) = \frac{3}{32}\sqrt{\frac{91}{2\pi}} \frac{(x-iy)^4(11z^2 - r^2)}{r^6}, \tag{95}$$

$$Y_{6-3} = \frac{1}{32}\sqrt{\frac{1365}{\pi}} e^{-3i\varphi} \sin^3\theta(11\cos^3\theta - 3\cos\theta) = \frac{1}{32}\sqrt{\frac{1365}{\pi}} \frac{(x-iy)^3 z(11z^2 - 3r^2)}{r^6}, \tag{96}$$

$$Y_{6-2} = \frac{1}{64}\sqrt{\frac{1365}{\pi}} e^{-2i\varphi} \sin^2\theta(33\cos^4\theta - 18\cos^2\theta + 1) = \frac{1}{64}\sqrt{\frac{1365}{\pi}} \frac{(x-iy)^2(33z^4 - 18z^2r^2 + r^4)}{r^6}, \tag{97}$$

$$Y_{6-1} = \frac{1}{16}\sqrt{\frac{273}{2\pi}} e^{-i\varphi} \sin\theta(33\cos^5\theta - 30\cos^3\theta + 5\cos\theta) = \frac{1}{16}\sqrt{\frac{273}{2\pi}} \frac{(x-iy)z(33z^4 - 30z^2r^2 + 5r^4)}{r^6}, \tag{98}$$

$$Y_{60} = \frac{1}{32}\sqrt{\frac{13}{\pi}} (231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5) = \frac{1}{32}\sqrt{\frac{13}{\pi}} \frac{(231z^6 - 315z^4r^2 + 105z^2r^4 - 5r^6)}{r^6}, \tag{99}$$

$$Y_{61} = -\frac{1}{16}\sqrt{\frac{273}{2\pi}} e^{i\varphi} \sin\theta(33\cos^5\theta - 30\cos^3\theta + 5\cos\theta) = -\frac{1}{16}\sqrt{\frac{273}{2\pi}} \frac{(x+iy)z(33z^4 - 30z^2r^2 + 5r^4)}{r^6}, \tag{100}$$

$$Y_{62} = \frac{1}{64}\sqrt{\frac{1365}{\pi}} e^{2i\varphi} \sin^2\theta(33\cos^4\theta - 18\cos^2\theta + 1) = \frac{1}{64}\sqrt{\frac{1365}{\pi}} \frac{(x+iy)^2(33z^4 - 11z^2r^2 + r^4)}{r^6}, \tag{101}$$

$$Y_{63} = -\frac{1}{32}\sqrt{\frac{1365}{\pi}} e^{3i\varphi} \sin^3\theta(11\cos^3\theta - 3\cos\theta) = -\frac{1}{32}\sqrt{\frac{1365}{\pi}} \frac{(x+iy)^3 z(11z^2 - 3r^2)}{r^6}, \tag{102}$$

$$Y_{64} = \frac{3}{32}\sqrt{\frac{91}{2\pi}} e^{4i\varphi} \sin^4\theta(11\cos^2\theta - 1) = \frac{3}{32}\sqrt{\frac{91}{2\pi}} \frac{(x+iy)^4(11z^2 - r^2)}{r^6}, \tag{103}$$

$$Y_{65} = -\frac{3}{32}\sqrt{\frac{1001}{\pi}} e^{5i\varphi} \sin^5\theta\cos\theta = -\frac{3}{32}\sqrt{\frac{1001}{\pi}} \frac{(x+iy)^5 z}{r^6}, \tag{104}$$

$$Y_{66} = \frac{1}{64}\sqrt{\frac{3003}{\pi}} e^{6i\varphi} \sin^6\theta = \frac{1}{64}\sqrt{\frac{3003}{\pi}} \frac{(x+iy)^6}{r^6}. \tag{105}$$

Therefore,

$$
\begin{aligned}
c_{n6-6} &= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^6\\
&= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x-iy)^6\\
&= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}-i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^6\\
&= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x^6-y^6+\frac{(x_i-iy_i)^6}{(1+\alpha_{k6})^6}\right)\\
&= \sum_{k=1}^{3}\frac{\sqrt{3003}\pi\beta_{nk}}{64(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i-iy_i)^6, \qquad (106)
\end{aligned}
$$

$$
\begin{aligned}
c_{n6-5} &= \frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{n6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^5 z\\
&= \frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x-iy)^5 z\\
&= \frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(z+\frac{z_i}{1+\alpha_{k6}}\right)\\
&\times\left(x+\frac{x_i}{1+\alpha_{n6}}-i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^5\\
&= \frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i-iy_i)^5 z_i}{(1+\alpha_{k6})^6}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{1001}\pi\beta_{nk}}{32(1+\alpha_{n6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i-iy_i)^5 z_i, \qquad (107)
\end{aligned}
$$

$$
\begin{aligned}
c_{n6-4} &= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^4(11z^2-r^2) \\
&= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x-iy)^4(11z^2-r^2) \\
&= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}-i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^4 \\
&\times\left[11\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\right] \\
&= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i-iy_i)^4(11z_i^2-r_i^2)}{(1+\alpha_{k6})^6} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{91}\pi\beta_{nk}}{32\sqrt{2}(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i-iy_i)^4(11z_i^2-r_i^2), \qquad (108)
\end{aligned}
$$

$$
\begin{aligned}
c_{n6-3} &= \frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^3z(11z^2-3r^2) \\
&= \frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x-iy)^3z(11z^2-3r^2) \\
&= \frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}-i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^3 \\
&\times\left(z+\frac{z_i}{1+\alpha_{k6}}\right)\left[11\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\right] \\
&= \frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i-iy_i)^3z_i(11z_i^2-3r_i^2)}{(1+\alpha_{k6})^6} \\
&= \sum_{k=1}^{3}\frac{\sqrt{1365}\pi\beta_{nk}}{32(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i-iy_i)^3z_i(11z_i^2-3r_i^2), \qquad (109)
\end{aligned}
$$

$$
\begin{aligned}
c_{n6-2} &= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2(33z^4-18z^2r^2+r^4)\\
&= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x-iy)^2(33z^4-18z^2r^2+r^4)\\
&= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}-i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^2\\
&\times\left[33\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^4-18\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\right.\\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)^2\right]\\
&= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i-iy_i)^2(33z_i^4-18z_i^2r_i^2+r_i^4)}{(1+\alpha_{k6})^6}\\
&= \sum_{k=1}^{3}\frac{\sqrt{1365}\pi\beta_{nk}}{64(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i-iy_i)^2(33z_i^4-18z_i^2r_i^2+r_i^4),
\end{aligned}
\tag{110}
$$

$$
\begin{aligned}
c_{n6-1} &= \frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)z(33z^4-30z^2r^2+5r^4)\\
&= \frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x-iy)z(33z^4-30z^2r^2+5r^4)\\
&= \frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}-i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)\left(z+\frac{z_i}{1+\alpha_{k6}}\right)\\
&\times\left[33\left(z+\frac{z_i}{1+\alpha_{n6}}\right)^4-30\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\right.\\
&\left.+5\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)^2\right]\\
&= \frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i-iy_i)z_i(33z_i^4-30z_i^2r_i^2+5r_i^4)}{(1+\alpha_{k6})^6}\\
&= \sum_{k=1}^{3}\frac{\sqrt{273}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{n6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i-iy_i)z_i(33z_i^4-30z_i^2r_i^2+5r_i^4),
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
c_{n60} &= \frac{1}{32}\sqrt{\frac{13}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(231z^6-315z^4r^2+105z^2r^4-5r^6)\\
&= \frac{1}{32}\sqrt{\frac{13}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(231z^6-315z^4r^2+105z^2r^4-5r^6)\\
&= \frac{1}{32}\sqrt{\frac{13}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dVe^{-(1+\alpha_{k6})r^2}\Bigg[231\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^6\\
&-315\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)^2\\
&+105\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)^4\\
&-5\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)^6\Bigg]\\
&= \frac{1}{32}\sqrt{\frac{13}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dVe^{-(1+\alpha_{k6})r^2}\frac{(231z_i^6-315z_i^4r_i^2+105z_i^2r_i^4-5r_i^6)}{(1+\alpha_{k6})^6}\\
&= \sum_{k=1}^{3}\frac{\sqrt{13}\pi\beta_{nk}}{32(1+\alpha_{n6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(231z_i^6-315z_i^4r_i^2+105z_i^2r_i^4-5r_i^6),
\end{aligned}
\tag{112}
$$

$$
\begin{aligned}
c_{n61} &= -\frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)z(33z^4-30z^2r^2+5r^4)\\
&= -\frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x+iy)z(33z^4-30z^2r^2+5r^4)\\
&= -\frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dVe^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}+i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)\left(z+\frac{z_i}{1+\alpha_{k6}}\right)\\
&\times\Bigg[33\left(z+\frac{z_i}{1+\alpha_{n6}}\right)^4-30\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\\
&+5\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)^2\Bigg]\\
&= -\frac{1}{16}\sqrt{\frac{273}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dVe^{-(1+\alpha_{k6})r^2}\frac{(x_i+iy_i)z_i(33z_i^4-30z_i^2r_i^2+5r_i^4)}{(1+\alpha_{k6})^6}\\
&= -\sum_{k=1}^{3}\frac{\sqrt{273}\pi\beta_{nk}}{16\sqrt{2}(1+\alpha_{n6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i+iy_i)z_i(33z_i^4-30z_i^2r_i^2+5r_i^4),
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
c_{n62} &= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2(33z^4-18z^2r^2+r^4)\\
&= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x+iy)^2(33z^4-18z^2r^2+r^4)\\
&= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}+i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^2\\
&\times\left[33\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^4-18\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\right.\\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)^2\right]\\
&= \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i+iy_i)^2(33z_i^4-18z_i^2r_i^2+r_i^4)}{(1+\alpha_{k6})^6}\\
&= \sum_{k=1}^{3}\frac{\sqrt{1365}\pi\beta_{nk}}{64(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i+iy_i)^2(33z_i^4-18z_i^2r_i^2+r_i^4), \quad (114)
\end{aligned}
$$

$$
\begin{aligned}
c_{n63} &= -\frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^3z(11z^2-3r^2)\\
&= -\frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x+iy)^3z(11z^2-3r^2)\\
&= -\frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}+i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^3\\
&\times\left(z+\frac{z_i}{1+\alpha_{k6}}\right)\left[11\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\right]\\
&= -\frac{1}{32}\sqrt{\frac{1365}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i+iy_i)^3z_i(11z_i^2-3r_i^2)}{(1+\alpha_{k6})^6}\\
&= -\sum_{k=1}^{3}\frac{\sqrt{1365}\pi\beta_{nk}}{32(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i+iy_i)^3z_i(11z_i^2-3r_i^2), \quad (115)
\end{aligned}
$$

$$
\begin{aligned}
c_{n64} &= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^4(11z^2-r^2)\\
&= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x+iy)^4(11z^2-r^2)\\
&= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}+i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^4\\
&\times\left[11\left(z+\frac{z_i}{1+\alpha_{k6}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k6})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k6}}\right)\right]\\
&= \frac{3}{32}\sqrt{\frac{91}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i+iy_i)^4(11z_i^2-r_i^2)}{(1+\alpha_{k6})^6}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{91}\pi\beta_{nk}}{32\sqrt{2}(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i+iy_i)^4(11z_i^2-r_i^2), \qquad (116)
\end{aligned}
$$

$$
\begin{aligned}
c_{n6-5} &= -\frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{n6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^5 z\\
&= -\frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x+iy)^5 z\\
&= -\frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(z+\frac{z_i}{1+\alpha_{k6}}\right)\\
&\times\left(x+\frac{x_i}{1+\alpha_{n6}}+i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^5\\
&= -\frac{3}{32}\sqrt{\frac{1001}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\frac{(x_i+iy_i)^5 z_i}{(1+\alpha_{k6})^6}\\
&= -\sum_{k=1}^{3}\frac{3\sqrt{1001}\pi\beta_{nk}}{32(1+\alpha_{n6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i+iy_i)^5 z_i, \qquad (117)
\end{aligned}
$$

$$
\begin{aligned}
c_{n6-6} &= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k6})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^6 \\
&= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k6})(\mathbf{r}-\frac{1}{1+\alpha_{k6}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x+iy)^6 \\
&= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x+\frac{x_i}{1+\alpha_{k6}}+i\left(y+\frac{y_i}{1+\alpha_{k6}}\right)\right)^6 \\
&= \frac{1}{64}\sqrt{\frac{3003}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}\int dV e^{-(1+\alpha_{k6})r^2}\left(x^6-y^6+\frac{(x_i+iy_i)^6}{(1+\alpha_{k6})^6}\right) \\
&= \sum_{k=1}^{3}\frac{\sqrt{3003}\pi\beta_{nk}}{64(1+\alpha_{k6})^{\frac{15}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}(x_i+iy_i)^6, \qquad (118)
\end{aligned}
$$

The power spectrum for $\ell = 6$ is

$$
\begin{aligned}
\mathbf{P}_{nn'6}(\mathbf{r}_i) &= \sum_m c_{n6m}c^*_{n'6m} \\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{13\pi^2\beta_{nk}\beta^*_{n'k'}}{1024(1+\alpha_{k6})^{\frac{15}{2}}(1+\alpha_{k'6})^{\frac{15}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{k6}}{1+\alpha_{k6}}r_i^2}e^{-\frac{\alpha_{k'6}}{1+\alpha_{k'6}}r_j^2}\left[\frac{231}{2}\mathrm{Re}[X_{ij}^6]\right. \\
&+1386z_iz_j\mathrm{Re}[X_{ij}^5]+63(11z_i^2-r_i^2)(11z_j^2-r_j^2)\mathrm{Re}[X_{ij}^4] \\
&+210z_iz_j(11z_i^2-3r_i^2)(11z_j^2-3r_j^2)\mathrm{Re}[X_{ij}^3] \\
&+\frac{105}{2}(33z_i^4-18z_i^2r_i^2+r_i^4)(33z_j^4-18z_j^2r_j^2+r_j^4)\mathrm{Re}[X_{ij}^2] \\
&+84z_iz_j(33z_i^4-30z_i^2r_i^2+5r_i^4)(33z_j^4-30z_j^2r_j^2+5r_j^4)\mathrm{Re}[X_{ij}] \\
&\left.+(231z_i^6-315z_i^4r_i^2+105z_i^2r_i^4-5r_i^6)(231z_j^6-315z_j^4r_j^2+105z_j^2r_j^4-5r_j^6)\right]. \qquad (119)
\end{aligned}
$$

- $\ell = 7$

The form of spherical harmonics for $\ell = 7$ are

$$Y_{7-7} = \frac{3}{64}\sqrt{\frac{715}{2\pi}}e^{-7i\varphi}\sin^7\theta = \frac{3}{64}\sqrt{\frac{715}{2\pi}}\frac{(x-iy)^7}{r^7} \tag{120}$$

$$Y_{7-6} = \frac{3}{64}\sqrt{\frac{5005}{\pi}}e^{-6i\varphi}\sin^6\theta\cos\theta = \frac{3}{64}\sqrt{\frac{5005}{\pi}}\frac{(x-iy)^6 z}{r^7}, \tag{121}$$

$$Y_{7-5} = \frac{3}{64}\sqrt{\frac{385}{2\pi}}e^{-5i\varphi}\sin^5\theta(13\cos^2\theta - 1) = \frac{3}{64}\sqrt{\frac{385}{2\pi}}\frac{(x-iy)^5(13z^2 - r^2)}{r^7}, \tag{122}$$

$$Y_{7-4} = \frac{3}{32}\sqrt{\frac{385}{2\pi}}e^{-4i\varphi}\sin^4\theta(13\cos^3\theta - 3\cos\theta) = \frac{3}{32}\sqrt{\frac{385}{2\pi}}\frac{(x-iy)^4 z(13z^2 - 3r^2)}{r^7}, \tag{123}$$

$$\begin{aligned} Y_{7-3} &= \frac{3}{64}\sqrt{\frac{35}{2\pi}}e^{-3i\varphi}\sin^3\theta(143\cos^4\theta - 66\cos^2\theta + 3) \\ &= \frac{3}{64}\sqrt{\frac{35}{2\pi}}\frac{(x-iy)^3(143z^4 - 66z^2r^2 + 3r^4)}{r^7}, \end{aligned} \tag{124}$$

$$\begin{aligned} Y_{7-2} &= \frac{3}{64}\sqrt{\frac{35}{\pi}}e^{-2i\varphi}\sin^2\theta(143\cos^5\theta - 110\cos^3\theta + 15\cos\theta) \\ &= \frac{3}{64}\sqrt{\frac{35}{\pi}}\frac{(x-iy)^2 z(143z^4 - 110z^2r^2 + 15r^4)}{r^7}, \end{aligned} \tag{125}$$

$$\begin{aligned} Y_{7-1} &= \frac{1}{64}\sqrt{\frac{105}{2\pi}}e^{-i\varphi}\sin\theta(429\cos^6\theta - 495\cos^4\theta + 135\cos^2\theta - 5) \\ &= \frac{1}{64}\sqrt{\frac{105}{2\pi}}\frac{(x-iy)(429z^6 - 495z^4r^2 + 135z^2r^4 - 5r^6)}{r^7}, \end{aligned} \tag{126}$$

$$\begin{aligned} Y_{70} &= \frac{1}{32}\sqrt{\frac{15}{\pi}}(429\cos^7\theta - 693\cos^5\theta + 315\cos^3\theta - 35\cos\theta) \\ &= \frac{1}{32}\sqrt{\frac{15}{\pi}}\frac{z(429z^6 - 693z^4r^2 + 315z^2r^4 - 35r^6)}{r^7}, \end{aligned} \tag{127}$$

$$\begin{aligned} Y_{71} &= -\frac{1}{64}\sqrt{\frac{105}{2\pi}}e^{i\varphi}\sin\theta(429\cos^6\theta - 495\cos^4\theta + 135\cos^2\theta - 5) \\ &= -\frac{1}{64}\sqrt{\frac{105}{2\pi}}\frac{(x+iy)(429z^6 - 495z^4r^2 + 135z^2r^4 - 5r^6)}{r^7}, \end{aligned} \tag{128}$$

$$\begin{aligned} Y_{72} &= \frac{3}{64}\sqrt{\frac{35}{\pi}}e^{2i\varphi}\sin^2\theta(143\cos^5\theta - 110\cos^3\theta + 15\cos\theta) \\ &= \frac{3}{64}\sqrt{\frac{35}{\pi}}\frac{(x+iy)^2 z(143z^4 - 110z^2r^2 + 15r^4)}{r^7}, \end{aligned} \tag{129}$$

$$
\begin{aligned}
Y_{73} &= -\frac{3}{64}\sqrt{\frac{35}{2\pi}}e^{3i\varphi}\sin^3\theta(143\cos^4\theta - 66\cos^2\theta + 3) \\
&= -\frac{3}{64}\sqrt{\frac{35}{2\pi}}\frac{(x+iy)^3(143z^4 - 66z^2r^2 + 3r^4)}{r^7}, \qquad (130) \\
Y_{74} &= \frac{3}{32}\sqrt{\frac{385}{2\pi}}e^{4i\varphi}\sin^4\theta(13\cos^3\theta - 3\cos\theta) = \frac{3}{32}\sqrt{\frac{385}{2\pi}}\frac{(x+iy)^4 z(13z^2 - 3r^2)}{r^7}, \qquad (131) \\
Y_{75} &= -\frac{3}{64}\sqrt{\frac{385}{2\pi}}e^{5i\varphi}\sin^5\theta(13\cos^2\theta - 1) = -\frac{3}{64}\sqrt{\frac{385}{2\pi}}\frac{(x+iy)^5(13z^2 - r^2)}{r^6}, \qquad (132) \\
Y_{76} &= \frac{3}{64}\sqrt{\frac{5005}{\pi}}e^{6i\varphi}\sin^6\theta\cos\theta = \frac{3}{64}\sqrt{\frac{5005}{\pi}}\frac{(x+iy)^6 z}{r^7}, \qquad (133) \\
Y_{77} &= -\frac{3}{64}\sqrt{\frac{715}{2\pi}}e^{7i\varphi}\sin^7\theta = -\frac{3}{64}\sqrt{\frac{715}{2\pi}}\frac{(x+iy)^7}{r^7}. \qquad (134)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c_{n7-7} &= \frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2 + r_i^2 - 2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^7 \\
&= \frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r} - \frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x-iy)^7 \\
&= \frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x + \frac{x_i}{1+\alpha_{k7}} - i\left(y + \frac{y_i}{1+\alpha_{k7}}\right)\right)^7 \\
&= \frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i - iy_i)^7}{(1+\alpha_{k7})^7} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{715}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i - iy_i)^7, \qquad (135)
\end{aligned}
$$

$$
\begin{aligned}
c_{n7-6} &= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^6 z \\
&= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x-iy)^6 z \\
&= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(z+\frac{z_i}{1+\alpha_{k7}}\right) \\
&\times\left(x+\frac{x_i}{1+\alpha_{k7}}-i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^6 \\
&= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i-iy_i)^6 z_i}{(1+\alpha_{k7})^7} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{5005}\pi\beta_{nk}}{64(1+\alpha_{n7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i-iy_i)^6 z_i, \qquad (136)
\end{aligned}
$$

$$
\begin{aligned}
c_{n7-5} &= \frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^5(13z^2-r^2) \\
&= \frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x-iy)^5(13z^2-r^2) \\
&= \frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}-i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^5 \\
&\times\left[13\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right] \\
&= \frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i-iy_i)^5(13z_i^2-r_i^2)}{(1+\alpha_{k7})^7} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{385}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i-iy_i)^5(13z_i^2-r_i^2), \qquad (137)
\end{aligned}
$$

$$
\begin{aligned}
c_{n7-4} &= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^4 z(13z^2-3r^2)\\
&= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x-iy)^4 z(13z^2-3r^2)\\
&= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}-i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^4\\
&\times\left(z+\frac{z_i}{1+\alpha_{k7}}\right)\left[13\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right]\\
&= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i-iy_i)^4 z_i(13z_i^2-3r_i^2)}{(1+\alpha_{k7})^7}\\
&= \sum_{k=1}^{3}\beta_{nk}\frac{3\sqrt{385}\pi\beta_{nk}}{32\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i-iy_i)^4 z_i(13z_i^2-3r_i^2),
\end{aligned}
\tag{138}
$$

$$
\begin{aligned}
c_{n7-3} &= \frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^3(143z^4-66z^2r^2+3r^4)\\
&= \frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x-iy)^3(143z^4-66z^2r^2+3r^4)\\
&= \frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}-i\left(y+\frac{y_i}{1+\alpha_{n7}}\right)\right)^3\\
&\times\left[143\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^4-66\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right.\\
&\left.+3\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^2\right]\\
&= \frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i-iy_i)^3(143z_i^4-66z_i^2r_i^2+3r_i^4)}{(1+\alpha_{k7})^7}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i-iy_i)^3(143z_i^4-66z_i^2r_i^2+3r_i^4),
\end{aligned}
\tag{139}
$$

$$
\begin{aligned}
c_{n7-2} &= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2 z(143z^4-110z^2r^2+15r^4) \\
&= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x-iy)^2 z(143z^4-110z^2r^2+15r^4) \\
&= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}-i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^2\left(z+\frac{z_i}{1+\alpha_{k7}}\right) \\
&\times\left[143\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^4-110\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right. \\
&\left.+15\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^2\right] \\
&= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i-iy_i)^2 z_i(143z_i^4-110z_i^2r_i^2+15r_i^4)}{(1+\alpha_{k7})^7} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{64(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i-iy_i)^2 z_i(143z_i^4-110z_i^2r_i^2+15r_i^4),
\end{aligned}
\tag{140}
$$

$$
\begin{aligned}
c_{n7-1} &= \frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)(429z^6-495z^4r^2+135z^2r^4-5r^6) \\
&= \frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x-iy)(429z^6-495z^4r^2+135z^2r^4-5r^6) \\
&= \frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}-i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right) \\
&\times\left[429\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^6-495\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right. \\
&+135\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^2 \\
&\left.-5\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^3\right] \\
&= \frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i-iy_i)(429z_i^6-495z_i^4r_i^2+135z_i^2r_i^4-5r_i^6)}{(1+\alpha_{k7})^7} \\
&= \sum_{k=1}^{3}\frac{\sqrt{105}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i-iy_i)(429z_i^6-495z_i^4r_i^2+135z_i^2r_i^4-5r_i^6),
\end{aligned}
\tag{141}
$$

$$
\begin{aligned}
c_{n70} &= \frac{1}{32}\sqrt{\frac{15}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})} z(429z^6-693z^4r^2+315z^2r^4-35r^6) \\
&= \frac{1}{32}\sqrt{\frac{15}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2} z(429z^6-693z^4r^2+315z^2r^4-35r^6) \\
&= \frac{1}{32}\sqrt{\frac{15}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(z+\frac{z_i}{1+\alpha_{k7}}\right) \\
&\times\left[429\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^6-693\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right. \\
&+315\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^2 \\
&\left.-35\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^3\right] \\
&= \frac{1}{32}\sqrt{\frac{15}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N} e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{z_i(429z_i^6-693z_i^4r_i^2+315z_i^2r_i^4-35r_i^6)}{(1+\alpha_{k7})^7} \\
&= \sum_{k=1}^{3}\frac{\sqrt{15}\pi\beta_{nk}}{32(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N} e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2} z_i(429z_i^6-693z_i^4r_i^2+315z_i^2r_i^4-35r_i^6),
\end{aligned}
\tag{142}
$$

$$
\begin{aligned}
c_{n71} &= -\frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)(429z^6-495z^4r^2+135z^2r^4-5r^6) \\
&= -\frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x+iy)(429z^6-495z^4r^2+135z^2r^4-5r^6) \\
&= -\frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}+i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right) \\
&\times\left[429\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^6-495\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right. \\
&+135\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^2 \\
&\left.-5\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^3\right] \\
&= -\frac{1}{64}\sqrt{\frac{105}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i+iy_i)(429z_i^6-495z_i^4r_i^2+135z_i^2r_i^4-5r_i^6)}{(1+\alpha_{k7})^7} \\
&= -\sum_{k=1}^{3}\frac{\sqrt{105}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i+iy_i)(429z_i^6-495z_i^4r_i^2+135z_i^2r_i^4-5r_i^6),
\end{aligned}
\tag{143}
$$

$$
\begin{aligned}
c_{n72} &= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2z(143z^4-110z^2r^2+15r^4) \\
&= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x+iy)^2z(143z^4-110z^2r^2+15r^4) \\
&= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}+i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^2\left(z+\frac{z_i}{1+\alpha_{k7}}\right) \\
&\times\left[143\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^4-110\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right. \\
&\left.+15\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^2\right] \\
&= \frac{3}{64}\sqrt{\frac{35}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i+iy_i)^2z_i(143z_i^4-110z_i^2r_i^2+15r_i^4)}{(1+\alpha_{k7})^7} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{64(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i+iy_i)^2z_i(143z_i^4-110z_i^2r_i^2+15r_i^4),
\end{aligned}
\tag{144}
$$

$$
\begin{aligned}
c_{n73} &= -\frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^3(143z^4-66z^2r^2+3r^4)\\
&= -\frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x+iy)^3(143z^4-66z^2r^2+3r^4)\\
&= -\frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}+i\left(y+\frac{y_i}{1+\alpha_{n7}}\right)\right)^3\\
&\times\left[143\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^4-66\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right.\\
&\left.+3\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)^2\right]\\
&= -\frac{3}{64}\sqrt{\frac{35}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i+iy_i)^3(143z_i^4-66z_i^2r_i^2+3r_i^4)}{(1+\alpha_{k7})^7}\\
&= -\sum_{k=1}^{3}\frac{3\sqrt{35}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i+iy_i)^3(143z_i^4-66z_i^2r_i^2+3r_i^4),
\end{aligned}
\tag{145}
$$

$$
\begin{aligned}
c_{n74} &= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^4z(13z^2-3r^2)\\
&= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x+iy)^4z(13z^2-3r^2)\\
&= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}+i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^4\\
&\times\left(z+\frac{z_i}{1+\alpha_{k7}}\right)\left[13\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right]\\
&= \frac{3}{32}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i+iy_i)^4z_i(13z_i^2-3r_i^2)}{(1+\alpha_{k7})^7}\\
&= \sum_{k=1}^{3}\beta_{nk}\frac{3\sqrt{385}\pi\beta_{nk}}{32\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i+iy_i)^4z_i(13z_i^2-3r_i^2),
\end{aligned}
\tag{146}
$$

$$
\begin{aligned}
c_{n75} &= -\frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^5(13z^2-r^2)\\
&= -\frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x+iy)^5(13z^2-r^2)\\
&= -\frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}+i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^5\\
&\times\left[13\left(z+\frac{z_i}{1+\alpha_{k7}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k7})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k7}}\right)\right]\\
&= -\frac{3}{64}\sqrt{\frac{385}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i+iy_i)^5(13z_i^2-r_i^2)}{(1+\alpha_{k7})^7}\\
&= -\sum_{k=1}^{3}\frac{3\sqrt{385}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i+iy_i)^5(13z_i^2-r_i^2),
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
c_{n76} &= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^6 z\\
&= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x+iy)^6 z\\
&= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(z+\frac{z_i}{1+\alpha_{k7}}\right)\\
&\times\left(x+\frac{x_i}{1+\alpha_{k7}}+i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^6\\
&= \frac{3}{64}\sqrt{\frac{5005}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i+iy_i)^6 z_i}{(1+\alpha_{k7})^7}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{5005}\pi\beta_{nk}}{64(1+\alpha_{n7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i+iy_i)^6 z_i,
\end{aligned}
\tag{148}
$$

$$
\begin{aligned}
c_{n77} &= -\frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k7})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^7 \\
&= -\frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k7})(\mathbf{r}-\frac{1}{1+\alpha_{k7}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x+iy)^7 \\
&= -\frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\left(x+\frac{x_i}{1+\alpha_{k7}}+i\left(y+\frac{y_i}{1+\alpha_{k7}}\right)\right)^7 \\
&= -\frac{3}{64}\sqrt{\frac{715}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}\int dV e^{-(1+\alpha_{k7})r^2}\frac{(x_i+iy_i)^7}{(1+\alpha_{k7})^7} \\
&= -\sum_{k=1}^{3}\frac{3\sqrt{715}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k7})^{\frac{17}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}(x_i+iy_i)^7, \qquad (149)
\end{aligned}
$$

The power spectrum for $\ell = 7$ is

$$
\begin{aligned}
\mathbf{P}_{nn'7}(\mathbf{r}_i) &= \sum_m c_{n7m}c^*_{n'7m} \\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{15\pi^2\beta_{nk}\beta^*_{n'k'}}{1024(1+\alpha_{k7})^{\frac{17}{2}}(1+\alpha_{k'7})^{\frac{17}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{k7}}{1+\alpha_{k7}}r_i^2}e^{-\frac{\alpha_{k'7}}{1+\alpha_{k'7}}r_j^2}\Big[\frac{429}{4}\mathrm{Re}[X_{ij}^7] \\
&+\frac{3003}{2}z_iz_j\mathrm{Re}[X_{ij}^6]+\frac{231}{4}(13z_i^2-r_i^2)(13z_j^2-r_j^2)\mathrm{Re}[X_{ij}^5] \\
&+231z_iz_j(13z_i^2-3r_i^2)(13z_j^2-3r_j^2)\mathrm{Re}[X_{ij}^4] \\
&+\frac{21}{4}(143z_i^4-66z_i^2r_i^2+3r_i^4)(143z_j^4-66z_j^2r_j^2+3r_j^4)\mathrm{Re}[X_{ij}^3] \\
&+\frac{21}{2}z_iz_j(143z_i^4-110z_i^2r_i^2+15r_i^4)(143z_j^4-110z_j^2r_j^2+15r_j^4)\mathrm{Re}[X_{ij}^2] \\
&+\frac{7}{4}(429z_i^6-495z_i^4r_i^2+135z_i^2r_i^4-5r_i^6)(429z_j^6-495z_j^4r_j^2+135z_j^2r_j^4-5r_j^6)\mathrm{Re}[X_{ij}] \\
&+z_iz_j(429z_i^6-693z_i^4r_i^2+315z_i^2r_i^4-35r_i^6)(429z_j^6-693z_j^4r_j^2+315z_j^2r_j^4-35r_j^6)\Big]. \qquad (150)
\end{aligned}
$$

- $\ell = 8$

The form of spherical harmonics for $\ell = 8$ are

$$Y_{8-8} = \frac{3}{256}\sqrt{\frac{12155}{2\pi}}e^{-8i\varphi}\sin^8\theta = \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\frac{(x-iy)^8}{r^8}, \tag{151}$$

$$Y_{8-7} = \frac{3}{64}\sqrt{\frac{12155}{2\pi}}e^{-7i\varphi}\sin^7\theta\cos\theta = \frac{3}{64}\sqrt{\frac{12155}{2\pi}}\frac{(x-iy)^7 z}{r^8} \tag{152}$$

$$Y_{8-6} = \frac{1}{128}\sqrt{\frac{7293}{\pi}}e^{-6i\varphi}\sin^6\theta(15\cos^2\theta-1) = \frac{1}{128}\sqrt{\frac{7293}{\pi}}\frac{(x-iy)^6(15z^2-r^2)}{r^8}, \tag{153}$$

$$Y_{8-5} = \frac{3}{64}\sqrt{\frac{17017}{2\pi}}e^{-5i\varphi}\sin^5\theta(5\cos^3\theta-\cos\theta) = \frac{3}{64}\sqrt{\frac{17017}{2\pi}}\frac{(x-iy)^5 z(5z^2-r^2)}{r^8}, \tag{154}$$

$$\begin{aligned} Y_{8-4} &= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}e^{-4i\varphi}\sin^4\theta(65\cos^4\theta-26\cos^2\theta+1) \\ &= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\frac{(x-iy)^4(65z^4-26z^2r^2+r^4)}{r^8}, \end{aligned} \tag{155}$$

$$\begin{aligned} Y_{8-3} &= \frac{1}{64}\sqrt{\frac{19635}{2\pi}}e^{-3i\varphi}\sin^3\theta(39\cos^5\theta-26\cos^3\theta+3\cos\theta) \\ &= \frac{1}{64}\sqrt{\frac{19635}{2\pi}}\frac{(x-iy)^3 z(39z^4-26z^2r^2+3r^4)}{r^8}, \end{aligned} \tag{156}$$

$$\begin{aligned} Y_{8-2} &= \frac{3}{128}\sqrt{\frac{595}{\pi}}e^{-2i\varphi}\sin^2\theta(143\cos^6\theta-143\cos^4\theta+33\cos^2\theta-1) \\ &= \frac{3}{128}\sqrt{\frac{595}{\pi}}\frac{(x-iy)^2(143z^6-143z^4r^2+33z^2r^4-r^6)}{r^8}, \end{aligned} \tag{157}$$

$$\begin{aligned} Y_{8-1} &= \frac{3}{64}\sqrt{\frac{17}{2\pi}}e^{-i\varphi}\sin\theta(715\cos^7\theta-1001\cos^5\theta+385\cos^3\theta-35\cos\theta) \\ &= \frac{3}{64}\sqrt{\frac{17}{2\pi}}\frac{(x-iy)z(715z^6-1001z^4r^2+385z^2r^4-35r^6)}{r^8}, \end{aligned} \tag{158}$$

$$\begin{aligned} Y_{80} &= \frac{1}{256}\sqrt{\frac{17}{\pi}}(6435\cos^8\theta-12012\cos^6\theta+6930\cos^4\theta-1260\cos^2\theta+35) \\ &= \frac{1}{256}\sqrt{\frac{17}{\pi}}\frac{(6435z^8-12012z^6r^2+6930z^4r^4-1260z^2r^6+35r^8)}{r^8}, \end{aligned} \tag{159}$$

$$\begin{aligned}
Y_{81} &= -\frac{3}{64}\sqrt{\frac{17}{2\pi}}e^{i\varphi}\sin\theta(715\cos^7\theta - 1001\cos^5\theta + 385\cos^3\theta - 35\cos\theta) \\
&= -\frac{3}{64}\sqrt{\frac{17}{2\pi}}\frac{(x+iy)z(715z^6 - 1001z^4r^2 + 385z^2r^4 - 35r^6)}{r^8}, && (160) \\
Y_{82} &= \frac{3}{128}\sqrt{\frac{595}{\pi}}e^{2i\varphi}\sin^2\theta(143\cos^6\theta - 143\cos^4\theta + 33\cos^2\theta - 1) \\
&= \frac{3}{128}\sqrt{\frac{595}{\pi}}\frac{(x+iy)^2(143z^6 - 143z^4r^2 + 33z^2r^4 - r^6)}{r^8}, && (161) \\
Y_{83} &= -\frac{1}{64}\sqrt{\frac{19635}{2\pi}}e^{3i\varphi}\sin^3\theta(39\cos^5\theta - 26\cos^3\theta + 3\cos\theta) \\
&= -\frac{1}{64}\sqrt{\frac{19635}{2\pi}}\frac{(x+iy)^3z(39z^4 - 26z^2r^2 + 3r^4)}{r^8}, && (162) \\
Y_{84} &= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}e^{4i\varphi}\sin^4\theta(65\cos^4\theta - 26\cos^2\theta + 1) \\
&= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\frac{(x+iy)^4(65z^4 - 26z^2r^2 + r^4)}{r^8}, && (163) \\
Y_{85} &= -\frac{3}{64}\sqrt{\frac{17017}{2\pi}}e^{5i\varphi}\sin^5\theta(5\cos^3\theta - \cos\theta) = \frac{3}{64}\sqrt{\frac{17017}{2\pi}}\frac{(x+iy)^5z(5z^2 - r^2)}{r^8}, && (164) \\
Y_{86} &= \frac{1}{128}\sqrt{\frac{7293}{\pi}}e^{6i\varphi}\sin^6\theta(15\cos^2\theta - 1) = \frac{1}{128}\sqrt{\frac{7293}{\pi}}\frac{(x+iy)^6(15z^2 - r^2)}{r^8}, && (165) \\
Y_{87} &= -\frac{3}{64}\sqrt{\frac{12155}{2\pi}}e^{7i\varphi}\sin^7\theta\cos\theta = \frac{3}{64}\sqrt{\frac{12155}{2\pi}}\frac{(x+iy)^7z}{r^8}, && (166) \\
Y_{88} &= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}e^{8i\varphi}\sin^8\theta = \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\frac{(x+iy)^8}{r^8}. && (167)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
c_{n8-8} &= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^8 \\
&= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)^8 \\
&= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^8 \\
&= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2} \\
&\times\left(x^8+y^8-28(x^6y^2+x^2y^6)+70x^4y^4+\frac{(x_i-iy_i)^8}{(1+\alpha_{k8})^8}\right) \\
&= \sum_{k=1}^{3}\frac{3\sqrt{12155}\pi\beta_{nk}}{256\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)^8, \qquad (168)
\end{aligned}
$$

$$
\begin{aligned}
c_{n8-7} &= \frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^7 z \\
&= \frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)^7 z \\
&= \frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(z+\frac{z_i}{1+\alpha_{k8}}\right) \\
&\times\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^7 \\
&= \frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i-iy_i)^7 z_i}{(1+\alpha_{k8})^8} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{12155}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)^7 z_i, \qquad (169)
\end{aligned}
$$

$$
\begin{aligned}
c_{n8-6} &= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^6(15z^2-r^2)\\
&= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)^6(15z^2-r^2)\\
&= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^6\\
&\times\left[15\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right]\\
&= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i-iy_i)^6(15z_i^2-r_i^2)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{\sqrt{7293}\pi\beta_{nk}}{128(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)^6(15z_i^2-r_i^2), \qquad (170)
\end{aligned}
$$

$$
\begin{aligned}
c_{n8-5} &= \frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^5 z(5z^2-r^2)\\
&= \frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)^5 z(5z^2-r^2)\\
&= \frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^5\\
&\times\left(z+\frac{z_i}{1+\alpha_{k8}}\right)\left[5\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right]\\
&= \frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i-iy_i)^5 z_i(5z_i^2-r_i^2)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{17017}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)^5 z_i(5z_i^2-r_i^2), \qquad (171)
\end{aligned}
$$

$$
\begin{aligned}
c_{n8-4} &= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^4(65z^4-26z^2r^2+r^4)\\
&= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)^4(65z^4-26z^2r^2+r^4)\\
&= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^4\\
&\times\left[65\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4-26\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right.\\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\right]\\
&= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i-iy_i)^4(65z_i^4-26z_i^2r_i^2+r_i^4)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{1309}\pi\beta_{nk}}{128\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)^4(65z_i^4-26z_i^2r_i^2+r_i^4),
\end{aligned}
\tag{172}
$$

$$
\begin{aligned}
c_{n8-3} &= \frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^3z(39z^4-26z^2r^2+3r^4)\\
&= \frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)^3z(39z^4-26z^2r^2+3r^4)\\
&= \frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^3\\
&\times\left(z+\frac{z_i}{1+\alpha_{k8}}\right)\left[39\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4-26\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right.\\
&\left.+3\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\right]\\
&= \frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i-iy_i)^3z_i(39z_i^4-26z_i^2r_i^2+3r_i^4)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{\sqrt{19635}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)^3z_i(39z_i^4-26z_i^2r_i^2+3r_i^4),
\end{aligned}
\tag{173}
$$

$$
\begin{aligned}
c_{n8-2} &= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2(143z^6-143z^4r^2+33z^2r^4-r^6)\\
&= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)^2z(143z^6-143z^4r^2+33z^2r^4-r^6)\\
&= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^2\\
&\times\left[143\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^6-143\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right.\\
&+33\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\\
&\left.-\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^3\right]\\
&= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i-iy_i)^2(143z_i^6-143z_i^4r_i^2+33z_i^2r_i^4-r_i^6)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{595}\pi\beta_{nk}}{128(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)^2(143z_i^6-143z_i^4r_i^2+33z_i^2r_i^4-r_i^6),
\end{aligned}
\tag{174}
$$

$$
\begin{aligned}
c_{n8-1} &= \frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)z(715z^6-1001z^4r^2+385z^2r^4-35r^6) \\
&= \frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x-iy)z \\
&\times(715z^6-1001z^4r^2+385z^2r^4-35r^6) \\
&= \frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)\left(z+\frac{z_i}{1+\alpha_{k8}}\right) \\
&\times\left[715\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^6-1001\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right. \\
&+385\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2 \\
&\left.-35\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^3\right] \\
&= \frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i-iy_i)z_i(715z_i^6-1001z_i^4r_i^2+385z_i^2r_i^4-35r_i^6)}{(1+\alpha_{k8})^8} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{17}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i-iy_i)z_i(715z_i^6-1001z_i^4r_i^2+385z_i^2r_i^4-35r_i^6),
\end{aligned}
\tag{175}
$$

$$
\begin{aligned}
c_{n80} &= \frac{1}{256}\sqrt{\frac{17}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(6435z^8-12012z^6r^2+6930z^4r^4-1260z^2r^6+35r^8)\\
&= \frac{1}{256}\sqrt{\frac{17}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\\
&\times(6435z^8-12012z^6r^2+6930z^4r^4-1260z^2r^6+35r^8)\\
&= \frac{1}{256}\sqrt{\frac{17}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}-i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)\left(z+\frac{z_i}{1+\alpha_{k8}}\right)\\
&\times\Bigg[6435\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^8-12012\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^6\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\\
&+6930\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\\
&-1260\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^3\\
&+35\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^4\Bigg]\\
&= \frac{1}{256}\sqrt{\frac{17}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(6435z_i^8-12012z_i^6r_i^2+6930z_i^4r_i^4-1260z_i^2r_i^6+35r_i^8)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{\sqrt{17}\pi\beta_{nk}}{256(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(6435z_i^8-12012z_i^6r_i^2+6930z_i^4r_i^4-1260z_i^2r_i^6+35r_i^8),
\end{aligned}
\tag{176}
$$

$$
\begin{aligned}
c_{n81} &= -\frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)z(715z^6-1001z^4r^2+385z^2r^4-35r^6)\\
&= -\frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)z\\
&\times(715z^6-1001z^4r^2+385z^2r^4-35r^6)\\
&= -\frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)\left(z+\frac{z_i}{1+\alpha_{k8}}\right)\\
&\times\left[715\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^6-1001\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right.\\
&+385\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\\
&\left.-35\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^3\right]\\
&= -\frac{3}{64}\sqrt{\frac{17}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i+iy_i)z_i(715z_i^6-1001z_i^4r_i^2+385z_i^2r_i^4-35r_i^6)}{(1+\alpha_{k8})^8}\\
&= -\sum_{k=1}^{3}\frac{3\sqrt{17}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)z_i(715z_i^6-1001z_i^4r_i^2+385z_i^2r_i^4-35r_i^6),
\end{aligned}
\tag{177}
$$

$$
\begin{aligned}
c_{n82} &= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2(143z^6-143z^4r^2+33z^2r^4-r^6)\\
&= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)^2 z(143z^6-143z^4r^2+33z^2r^4-r^6)\\
&= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^2\\
&\times\left[143\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^6-143\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right.\\
&+33\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\\
&\left.-\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^3\right]\\
&= \frac{3}{128}\sqrt{\frac{595}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i+iy_i)^2(143z_i^6-143z_i^4r_i^2+33z_i^2r_i^4-r_i^6)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{595}\pi\beta_{nk}}{128(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)^2(143z_i^6-143z_i^4r_i^2+33z_i^2r_i^4-r_i^6),
\end{aligned}
\tag{178}
$$

$$
\begin{aligned}
c_{n83} &= -\frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^3 z(39z^4-26z^2r^2+3r^4)\\
&= -\frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)^3 z(39z^4-26z^2r^2+3r^4)\\
&= -\frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^3\\
&\times\left(z+\frac{z_i}{1+\alpha_{k8}}\right)\left[39\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4-26\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right.\\
&\left.+3\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\right]\\
&= \frac{1}{64}\sqrt{\frac{19635}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i+iy_i)^3 z_i(39z_i^4-26z_i^2r_i^2+3r_i^4)}{(1+\alpha_{k8})^8}\\
&= -\sum_{k=1}^{3}\frac{\sqrt{19635}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)^3 z_i(39z_i^4-26z_i^2r_i^2+3r_i^4),
\end{aligned}
\tag{179}
$$

$$
\begin{aligned}
c_{n84} &= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^4(65z^4-26z^2r^2+r^4) \\
&= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)^4(65z^4-26z^2r^2+r^4) \\
&= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^4 \\
&\times\left[65\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^4-26\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right. \\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)^2\right] \\
&= \frac{3}{128}\sqrt{\frac{1309}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i+iy_i)^4(65z_i^4-26z_i^2r_i^2+r_i^4)}{(1+\alpha_{k8})^8} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{1309}\pi\beta_{nk}}{128\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)^4(65z_i^4-26z_i^2r_i^2+r_i^4),
\end{aligned}
\tag{180}
$$

$$
\begin{aligned}
c_{n85} &= -\frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^5z(5z^2-r^2) \\
&= -\frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)^5z(5z^2-r^2) \\
&= -\frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^5 \\
&\times\left(z+\frac{z_i}{1+\alpha_{k8}}\right)\left[5\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right] \\
&= -\frac{3}{64}\sqrt{\frac{17017}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\frac{(x_i+iy_i)^5z_i(5z_i^2-r_i^2)}{(1+\alpha_{k8})^8} \\
&= -\sum_{k=1}^{3}\frac{3\sqrt{17017}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)^5z_i(5z_i^2-r_i^2),
\end{aligned}
\tag{181}
$$

$$
\begin{aligned}
c_{n86} &= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^6(15z^2-r^2)\\
&= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)^6(15z^2-r^2)\\
&= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dVe^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^6\\
&\times\left[15\left(z+\frac{z_i}{1+\alpha_{k8}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k8})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k8}}\right)\right]\\
&= \frac{1}{128}\sqrt{\frac{7293}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dVe^{-(1+\alpha_{k8})r^2}\frac{(x_i+iy_i)^6(15z_i^2-r_i^2)}{(1+\alpha_{k8})^8}\\
&= \sum_{k=1}^{3}\frac{\sqrt{7293}\pi\beta_{nk}}{128(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)^6(15z_i^2-r_i^2), \qquad (182)
\end{aligned}
$$

$$
\begin{aligned}
c_{n8-7} &= -\frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^7z\\
&= -\frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)^7z\\
&= -\frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dVe^{-(1+\alpha_{k8})r^2}\left(z+\frac{z_i}{1+\alpha_{k8}}\right)\\
&\times\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^7\\
&= -\frac{3}{64}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dVe^{-(1+\alpha_{k8})r^2}\frac{(x_i+iy_i)^7z_i}{(1+\alpha_{k8})^8}\\
&= -\sum_{k=1}^{3}\frac{3\sqrt{12155}\pi\beta_{nk}}{64\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)^7z_i, \qquad (183)
\end{aligned}
$$

$$
\begin{aligned}
c_{n88} &= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k8})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^8 \\
&= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k8})(\mathbf{r}-\frac{1}{1+\alpha_{k8}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x+iy)^8 \\
&= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2}\left(x+\frac{x_i}{1+\alpha_{k8}}+i\left(y+\frac{y_i}{1+\alpha_{k8}}\right)\right)^8 \\
&= \frac{3}{256}\sqrt{\frac{12155}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}\int dV e^{-(1+\alpha_{k8})r^2} \\
&\times\left(x^8+y^8-28(x^6y^2+x^2y^6)+70x^4y^4+\frac{(x_i+iy_i)^8}{(1+\alpha_{k8})^8}\right) \\
&= \sum_{k=1}^{3}\frac{3\sqrt{12155}\pi\beta_{nk}}{256\sqrt{2}(1+\alpha_{k8})^{\frac{19}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}(x_i+iy_i)^8, \qquad (184)
\end{aligned}
$$

The power spectrum for $\ell = 8$ is

$$
\begin{aligned}
\mathbf{P}_{nn'8}(\mathbf{r}_i) &= \sum_m c_{n8m}c^*_{n'8m} \\
&= \sum_{k=1}^{3}\sum_{k'=1}^{3}\frac{17\pi^2\beta_{nk}\beta^*_{n'k'}}{65536(1+\alpha_{k8})^{\frac{19}{2}}(1+\alpha_{k'8})^{\frac{19}{2}}}\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{\alpha_{k8}}{1+\alpha_{k8}}r_i^2}e^{-\frac{\alpha_{k'8}}{1+\alpha_{k'8}}r_j^2} \\
&\times\big[6435\mathrm{Re}[X_{ij}^8]+102960z_iz_j\mathrm{Re}[X_{ij}^7] \\
&+3432(15z_i^2-r_i^2)(15z_j^2-r_j^2)\mathrm{Re}[X_{ij}^6]+144144z_iz_j(5z_i^2-r_i^2)(5z_j^2-r_j^2)\mathrm{Re}[X_{ij}^5] \\
&+2772(65z_i^4-26z_i^2r_i^2+r_i^4)(65z_j^4-26z_j^2r_j^2+r_j^4)\mathrm{Re}[X_{ij}^4] \\
&+18480z_iz_j(39z_i^4-26z_i^2r_i^2+3r_i^4)(39z_j^4-26z_j^2r_j^2+3r_j^4)\mathrm{Re}[X_{ij}^3] \\
&+2520(143z_i^6-143z_i^4r_i^2+33z_i^2r_i^4-r_i^6)(143z_j^6-143z_j^4r_j^2+33z_j^2r_j^4-r_j^6)\mathrm{Re}[X_{ij}^2] \\
&+144z_iz_j(715z_i^6-1001z_i^4r_i^2+385z_i^2r_i^4-35r_i^6)(715z_j^6-1001z_j^4r_j^2+385z_j^2r_j^4-35r_j^6)\mathrm{Re}[X_{ij}] \\
&+(6435z_i^8-12012z_i^6r_i^2+6930z_i^4r_i^4-1260z_i^2r_i^6+35r_i^8) \\
&\times(6435z_j^8-12012z_j^6r_j^2+6930z_j^4r_j^4-1260z_j^2r_j^6+35r_j^8)\big]. \qquad (185)
\end{aligned}
$$

- $\ell = 9$

The form of spherical harmonics for $\ell = 9$ are

$$Y_{9-9} = \frac{1}{512}\sqrt{\frac{230945}{\pi}}e^{-9i\varphi}\sin^9\theta = \frac{1}{512}\sqrt{\frac{230945}{\pi}}\frac{(x-iy)^9}{r^9}, \tag{186}$$

$$Y_{9-8} = \frac{3}{256}\sqrt{\frac{230945}{2\pi}}e^{-8i\varphi}\sin^8\theta\cos\theta = \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\frac{(x-iy)^8 z}{r^9}, \tag{187}$$

$$Y_{9-7} = \frac{3}{512}\sqrt{\frac{13585}{\pi}}e^{-7i\varphi}\sin^7\theta(17\cos^2\theta - 1) = \frac{3}{512}\sqrt{\frac{13585}{\pi}}\frac{(x-iy)^7(17z^2 - r^2)}{r^9}, \tag{188}$$

$$Y_{9-6} = \frac{1}{128}\sqrt{\frac{40755}{\pi}}e^{-6i\varphi}\sin^6\theta(17\cos^3\theta - 3\cos\theta) = \frac{1}{128}\sqrt{\frac{40755}{\pi}}\frac{(x-iy)^6 z(17z^2 - 3r^2)}{r^9}, \tag{189}$$

$$Y_{9-5} = \frac{3}{256}\sqrt{\frac{2717}{\pi}}e^{-5i\varphi}\sin^5\theta(85\cos^4\theta - 30\cos^2\theta + 1) = \frac{3}{256}\sqrt{\frac{2717}{\pi}}\frac{(x-iy)^5(85z^4 - 30z^2r^2 + r^4)}{r^9}, \tag{190}$$

$$\begin{aligned} Y_{9-4} &= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}e^{-4i\varphi}\sin^4\theta(17\cos^5\theta - 10\cos^3\theta + \cos\theta) \\ &= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\frac{(x-iy)^4 z(17z^4 - 10z^2r^2 + r^4)}{r^9}, \end{aligned} \tag{191}$$

$$\begin{aligned} Y_{9-3} &= \frac{1}{256}\sqrt{\frac{21945}{\pi}}e^{-3i\varphi}\sin^3\theta(221\cos^6\theta - 195\cos^4\theta + 39\cos^2\theta - 1) \\ &= \frac{1}{256}\sqrt{\frac{21945}{\pi}}\frac{(x-iy)^3(221z^6 - 195z^4r^2 + 39r^4z^2 - r^6)}{r^9}, \end{aligned} \tag{192}$$

$$\begin{aligned} Y_{9-2} &= \frac{3}{128}\sqrt{\frac{1045}{\pi}}e^{-2i\varphi}\sin^2\theta(221\cos^7\theta - 273\cos^5\theta + 91\cos^3\theta - 7\cos\theta) \\ &= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\frac{(x-iy)^2 z(221z^6 - 273z^4r^2 + 91z^2r^4 - 7r^6)}{r^9}, \end{aligned} \tag{193}$$

$$\begin{aligned} Y_{9-1} &= \frac{3}{256}\sqrt{\frac{95}{2\pi}}e^{-i\varphi}\sin\theta(2431\cos^8\theta - 4004\cos^6\theta + 2002\cos^4\theta - 308\cos^2\theta + 7) \\ &= \frac{3}{256}\sqrt{\frac{95}{2\pi}}\frac{(x-iy)(2431z^8 - 4004z^6r^2 + 2002z^4r^4 - 308z^2r^6 + 7r^8)}{r^9}, \end{aligned} \tag{194}$$

$$\begin{aligned} Y_{90} &= \frac{1}{256}\sqrt{\frac{19}{\pi}}(12155\cos^9\theta - 25740\cos^7\theta + 18018\cos^5\theta - 4620\cos^3\theta + 315\cos\theta) \\ &= \frac{1}{256}\sqrt{\frac{19}{\pi}}\frac{z(12155z^8 - 25740z^6r^2 + 18018z^4r^4 - 4620z^2r^6 + 315r^8)}{r^9}, \end{aligned} \tag{195}$$

$$Y_{9-1} = -\frac{3}{256}\sqrt{\frac{95}{2\pi}}e^{i\varphi}\sin\theta(2431\cos^8\theta - 4004\cos^6\theta + 2002\cos^4\theta - 308\cos^2\theta + 7)$$
$$= -\frac{3}{256}\sqrt{\frac{95}{2\pi}}\frac{(x+iy)(2431z^8 - 4004z^6r^2 + 2002z^4r^4 - 308z^2r^6 + 7r^8)}{r^9}, \tag{196}$$

$$Y_{9-2} = \frac{3}{128}\sqrt{\frac{1045}{\pi}}e^{2i\varphi}\sin^2\theta(221\cos^7\theta - 273\cos^5\theta + 91\cos^3\theta - 7\cos\theta)$$
$$= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\frac{(x+iy)^2z(221z^6 - 273z^4r^2 + 91z^2r^4 - 7r^6)}{r^9}, \tag{197}$$

$$Y_{9-3} = -\frac{1}{256}\sqrt{\frac{21945}{\pi}}e^{3i\varphi}\sin^3\theta(221\cos^6\theta - 195\cos^4\theta + 39\cos^2\theta - 1)$$
$$= -\frac{1}{256}\sqrt{\frac{21945}{\pi}}\frac{(x+iy)^3(221z^6 - 195z^4r^2 + 39r^4z^2 - r^6)}{r^9}, \tag{198}$$

$$Y_{9-4} = \frac{3}{128}\sqrt{\frac{95095}{2\pi}}e^{4i\varphi}\sin^4\theta(17\cos^5\theta - 10\cos^3\theta + \cos\theta)$$
$$= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\frac{(x+iy)^4z(17z^4 - 10z^2r^2 + r^4)}{r^9}, \tag{199}$$

$$Y_{9-5} = -\frac{3}{256}\sqrt{\frac{2717}{\pi}}e^{5i\varphi}\sin^5\theta(85\cos^4\theta - 30\cos^2\theta + 1) = -\frac{3}{256}\sqrt{\frac{2717}{\pi}}\frac{(x+iy)^5(85z^4 - 30z^2r^2 + r^4)}{r^9}, \tag{200}$$

$$Y_{9-6} = \frac{1}{128}\sqrt{\frac{40755}{\pi}}e^{6i\varphi}\sin^6\theta(17\cos^3\theta - 3\cos\theta) = \frac{1}{128}\sqrt{\frac{40755}{\pi}}\frac{(x+iy)^6z(17z^2 - 3r^2)}{r^9}, \tag{201}$$

$$Y_{9-7} = -\frac{3}{512}\sqrt{\frac{13585}{\pi}}e^{7i\varphi}\sin^7\theta(17\cos^2\theta - 1) = -\frac{3}{512}\sqrt{\frac{13585}{\pi}}\frac{(x+iy)^7(17z^2 - r^2)}{r^9}, \tag{202}$$

$$Y_{9-8} = \frac{3}{256}\sqrt{\frac{230945}{2\pi}}e^{8i\varphi}\sin^8\theta\cos\theta = \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\frac{(x+iy)^8z}{r^9}, \tag{203}$$

$$Y_{9-9} = -\frac{1}{512}\sqrt{\frac{230945}{\pi}}e^{9i\varphi}\sin^9\theta = -\frac{1}{512}\sqrt{\frac{230945}{\pi}}\frac{(x+iy)^9}{r^9}. \tag{204}$$

Therefore,

$$
\begin{aligned}
c_{n9-9} &= \frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^9 \\
&= \frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^9 \\
&= \frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^9 \\
&= \frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i-iy_i)^9}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{\sqrt{230945}\pi\beta_{nk}}{512(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^9, \qquad (205)
\end{aligned}
$$

$$
\begin{aligned}
c_{n9-8} &= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^8 z \\
&= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^8 z \\
&= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(z+\frac{z_i}{1+\alpha_{k9}}\right) \\
&\times\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^8 \\
&= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i-iy_i)^8 z_i}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{230945}\pi\beta_{nk}}{256\sqrt{2}(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^8 z_i, \qquad (206)
\end{aligned}
$$

$$
\begin{aligned}
c_{n9-7} &= \frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^7(17z^2-r^2)\\
&= \frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^7(17z^2-r^2)\\
&= \frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^7\\
&\times\left[17\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right]\\
&= \frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\frac{(x_i-iy_i)^7(17z_i^2-r_i^2)}{(1+\alpha_{k9})^9}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{13585}\pi\beta_{nk}}{512(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^7(17z_i^2-r_i^2), \qquad (207)
\end{aligned}
$$

$$
\begin{aligned}
c_{n9-6} &= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^6z(17z^2-3r^2)\\
&= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^6z(17z^2-3r^2)\\
&= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^6\\
&\times\left(z+\frac{z_i}{1+\alpha_{k9}}\right)\left[17\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right]\\
&= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\frac{(x_i-iy_i)^6z_i(17z_i^2-3r_i^2)}{(1+\alpha_{k9})^9}\\
&= \sum_{k=1}^{3}\frac{\sqrt{40755}\pi\beta_{nk}}{128(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^6z_i(17z_i^2-3r_i^2), \qquad (208)
\end{aligned}
$$

$$
\begin{aligned}
c_{n9-5} &= \frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^5(85z^4-30z^2r^2+r^4) \\
&= \frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^5(85z^4-30z^2r^2+r^4) \\
&= \frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^5 \\
&\times\left[85\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4-30\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2\right] \\
&= \frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\frac{(x_i-iy_i)^5(85z_i^4-30z_i^2r_i^2+r_i^4)}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{2717}\pi\beta_{nk}}{256(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^5(85z_i^4-30z_i^2r_i^2+r_i^4),
\end{aligned}
\tag{209}
$$

$$
\begin{aligned}
c_{n9-4} &= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^4z(17z^4-10z^2r^2+r^4) \\
&= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^4z(17z^4-10z^2r^2+r^4) \\
&= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^4 \\
&\times\left(z+\frac{z_i}{1+\alpha_{k9}}\right)\left[17\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4-10\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k}}\right)^2\right] \\
&= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{n9})r^2}\frac{(x_i-iy_i)^4z_i(17z_i^4-10z_i^2r_i^2+r_i^4)}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{95095}\pi\beta_{nk}}{128\sqrt{2}(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^4z_i(17z_i^4-10z_i^2r_i^2+r_i^4),
\end{aligned}
\tag{210}
$$

$$
\begin{aligned}
c_{n9-3} &= \frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^3(221z^6-195z^4r^2+39r^4z^2-r^6) \\
&= \frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^3 \\
&\times(221z^6-195z^4r^2+39r^4z^2-r^6) \\
&= \frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{n9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^3 \\
&\times\left[221\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^6-195\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&+39\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2 \\
&\left.-\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^3\right] \\
&= \frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i-iy_i)^3(221z_i^6-195z_i^4r_i^2+39z_i^2r_i^4-r_i^6)}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{\sqrt{21945}\pi\beta_{nk}}{256(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^3(221z_i^6-195z_i^4r_i^2+39z_i^2r_i^4-r_i^6),
\end{aligned}
\tag{211}
$$

$$
\begin{aligned}
c_{n9-2} &= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy)^2 z(221z^6-273z^4r^2+91z^2r^4-7r^6)\\
&= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy)^2 z\\
&\times(221z^6-273z^4r^2+91z^2r^4-7r^6)\\
&= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^2\\
&\times\left(z+\frac{z_i}{1+\alpha_{k9}}\right)\left[221\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^6-273\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right.\\
&+91\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2\\
&\left.-7\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^3\right]\\
&= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i-iy_i)^2 z_i(221z_i^6-273z_i^4r_i^2+91z_i^2r_i^4-7r_i^6)}{(1+\alpha_{k9})^9}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{1045}\pi\beta_{nk}}{128(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)^2 z_i(221z_i^6-273z_i^4r_i^2+91z_i^2r_i^4-7r_i^6),
\end{aligned}
\tag{212}
$$

$$
\begin{aligned}
c_{n9-1} &= \frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x-iy) \\
&\times(2431z^8-4004z^6r^2+2002z^4r^4-308z^2r^6+7r^8) \\
&= \frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x-iy) \\
&\times(2431z^8-4004z^6r^2+2002z^4r^4-308z^2r^6+7r^8) \\
&= \frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}-i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right) \\
&\times\left[2431\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^8-4004\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^6\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&+2002\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2 \\
&-308\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^3 \\
&\left.+7\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^4\right] \\
&= \frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2} \\
&\times\frac{(x_i-iy_i)(2431z_i^8-4004z_i^6r_i^2+2002z_i^4r_i^4-308z_i^2r_i^6+7r_i^8)}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{95}\pi\beta_{nk}}{256\sqrt{2}(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i-iy_i)(2431z_i^8-4004z_i^6r_i^2+2002z_i^4r_i^4-308z_i^2r_i^6+7r_i^8),
\end{aligned}
\tag{213}
$$

$$
\begin{aligned}
c_{n90} &= \frac{1}{256}\sqrt{\frac{19}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})} \\
&\times z(12155z^8-25740z^6r^2+18018z^4r^4-4620z^2r^6+315r^8) \\
&= \frac{1}{256}\sqrt{\frac{19}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2} \\
&\times z(12155z^8-25740z^6r^2+18018z^4r^4-4620z^2r^6+315r^8) \\
&= \frac{1}{256}\sqrt{\frac{19}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(z+\frac{z_i}{1+\alpha_{k9}}\right) \\
&\times\left[12155\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^8-25740\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^6\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&+18018\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2 \\
&-4620\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^3 \\
&\left.+315\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^4\right] \\
&= \frac{1}{256}\sqrt{\frac{19}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2} \\
&\times\frac{z_i(12155z_i^8-25740z_i^6r_i^2+18018z_i^4r_i^4-4620z_i^2r_i^6+315r_i^8)}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{\sqrt{19}\pi\beta_{nk}}{256(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}z_i(12155z_i^8-25740z_i^6r_i^2+18018z_i^4r_i^4-4620z_i^2r_i^6+315r_i^8),
\end{aligned}
\tag{214}
$$

$$
\begin{aligned}
c_{n91} &= -\frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy) \\
&\times(2431z^8-4004z^6r^2+2002z^4r^4-308z^2r^6+7r^8) \\
&= -\frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy) \\
&\times(2431z^8-4004z^6r^2+2002z^4r^4-308z^2r^6+7r^8) \\
&= -\frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right) \\
&\times\Bigg[2431\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^8-4004\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^6\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right) \\
&+2002\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2 \\
&-308\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^3 \\
&+7\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^4\Bigg] \\
&= -\frac{3}{256}\sqrt{\frac{95}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2} \\
&\times\frac{(x_i+iy_i)(2431z_i^8-4004z_i^6r_i^2+2002z_i^4r_i^4-308z_i^2r_i^6+7r_i^8)}{(1+\alpha_{k9})^9} \\
&= -\sum_{k=1}^{3}\frac{3\sqrt{95}\pi\beta_{nk}}{256\sqrt{2}(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)(2431z_i^8-4004z_i^6r_i^2+2002z_i^4r_i^4-308z_i^2r_i^6+7r_i^8),
\end{aligned}
\tag{215}
$$

$$
\begin{aligned}
c_{n92} &= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^2 z(221z^6-273z^4r^2+91z^2r^4-7r^6)\\
&= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^2 z\\
&\times(221z^6-273z^4r^2+91z^2r^4-7r^6)\\
&= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^2\\
&\times\left(z+\frac{z_i}{1+\alpha_{k9}}\right)\Bigg[221\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^6-273\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\\
&+91\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2\\
&-7\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^3\Bigg]\\
&= \frac{3}{128}\sqrt{\frac{1045}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i+iy_i)^2z_i(221z_i^6-273z_i^4r_i^2+91z_i^2r_i^4-7r_i^6)}{(1+\alpha_{k9})^9}\\
&= \sum_{k=1}^{3}\frac{3\sqrt{1045}\pi\beta_{nk}}{128(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^2z_i(221z_i^6-273z_i^4r_i^2+91z_i^2r_i^4-7r_i^6),
\end{aligned}
\tag{216}
$$

$$
\begin{aligned}
c_{n93} &= -\frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^3(221z^6-195z^4r^2+39r^4z^2-r^6) \\
&= -\frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^3 \\
&\times(221z^6-195z^4r^2+39r^4z^2-r^6) \\
&= -\frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{n9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^3 \\
&\times\left[221\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^6-195\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&+39\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2 \\
&\left.-\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^3\right] \\
&= -\frac{1}{256}\sqrt{\frac{21945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i+iy_i)^3(221z_i^6-195z_i^4r_i^2+39z_i^2r_i^4-r_i^6)}{(1+\alpha_{k9})^9} \\
&= -\sum_{k=1}^{3}\frac{\sqrt{21945}\pi\beta_{nk}}{256(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^3(221z_i^6-195z_i^4r_i^2+39z_i^2r_i^4-r_i^6),
\end{aligned}
\tag{217}
$$

$$
\begin{aligned}
c_{n94} &= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^4 z(17z^4-10z^2r^2+r^4) \\
&= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^4 z(17z^4-10z^2r^2+r^4) \\
&= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^4 \\
&\times\left(z+\frac{z_i}{1+\alpha_{k9}}\right)\left[17\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4-10\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_k}\right)^2\right] \\
&= \frac{3}{128}\sqrt{\frac{95095}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{n9})r^2}\frac{(x_i+iy_i)^4 z_i(17z_i^4-10z_i^2r_i^2+r_i^4)}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{95095}\pi\beta_{nk}}{128\sqrt{2}(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^4 z_i(17z_i^4-10z_i^2r_i^2+r_i^4),
\end{aligned}
\tag{218}
$$

$$
\begin{aligned}
c_{n95} &= -\frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^5(85z^4-30z^2r^2+r^4) \\
&= -\frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^5(85z^4-30z^2r^2+r^4) \\
&= -\frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^5 \\
&\times\left[85\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^4-30\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right. \\
&\left.+\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)^2\right] \\
&= -\frac{3}{256}\sqrt{\frac{2717}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\frac{(x_i+iy_i)^5(85z_i^4-30z_i^2r_i^2+r_i^4)}{(1+\alpha_{k9})^9} \\
&= -\sum_{k=1}^{3}\frac{3\sqrt{2717}\pi\beta_{nk}}{256(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^5(85z_i^4-30z_i^2r_i^2+r_i^4),
\end{aligned}
\tag{219}
$$

$$
\begin{aligned}
c_{n96} &= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^6 z(17z^2-3r^2)\\
&= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^6 z(17z^2-3r^2)\\
&= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^6\\
&\times\left(z+\frac{z_i}{1+\alpha_{k9}}\right)\left[17\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2-3\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right]\\
&= \frac{1}{128}\sqrt{\frac{40755}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i+iy_i)^6 z_i(17z_i^2-3r_i^2)}{(1+\alpha_{k9})^9}\\
&= \sum_{k=1}^{3}\frac{\sqrt{40755}\pi\beta_{nk}}{128(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^6 z_i(17z_i^2-3r_i^2),
\end{aligned}
\tag{220}
$$

$$
\begin{aligned}
c_{n97} &= -\frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^7(17z^2-r^2)\\
&= -\frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dV e^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2} e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^7(17z^2-r^2)\\
&= -\frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^7\\
&\times\left[17\left(z+\frac{z_i}{1+\alpha_{k9}}\right)^2-\left(r^2+\frac{r_i^2}{(1+\alpha_{k9})^2}+2\frac{xx_i+yy_i+zz_i}{1+\alpha_{k9}}\right)\right]\\
&= -\frac{3}{512}\sqrt{\frac{13585}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dV e^{-(1+\alpha_{k9})r^2}\frac{(x_i+iy_i)^7(17z_i^2-r_i^2)}{(1+\alpha_{k9})^9}\\
&= -\sum_{k=1}^{3}\frac{3\sqrt{13585}\pi\beta_{nk}}{512(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^7(17z_i^2-r_i^2),
\end{aligned}
\tag{221}
$$

$$
\begin{aligned}
c_{n98} &= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^8 z \\
&= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^8 z \\
&= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(z+\frac{z_i}{1+\alpha_{k9}}\right) \\
&\times\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^8 \\
&= \frac{3}{256}\sqrt{\frac{230945}{2\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\frac{(x_i+iy_i)^8 z_i}{(1+\alpha_{k9})^9} \\
&= \sum_{k=1}^{3}\frac{3\sqrt{230945}\pi\beta_{nk}}{256\sqrt{2}(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^8 z_i, \qquad (222)
\end{aligned}
$$

$$
\begin{aligned}
c_{n99} &= -\frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-((1+\alpha_{k9})r^2+r_i^2-2\mathbf{r}\cdot\mathbf{r_i})}(x+iy)^9 \\
&= -\frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}\int dVe^{-(1+\alpha_{k9})(\mathbf{r}-\frac{1}{1+\alpha_{k9}}\mathbf{r_i})^2}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x+iy)^9 \\
&= -\frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\left(x+\frac{x_i}{1+\alpha_{k9}}+i\left(y+\frac{y_i}{1+\alpha_{k9}}\right)\right)^9 \\
&= -\frac{1}{512}\sqrt{\frac{230945}{\pi}}\sum_{k=1}^{3}\beta_{nk}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}\int dVe^{-(1+\alpha_{k9})r^2}\frac{(x_i+iy_i)^9}{(1+\alpha_{k9})^9} \\
&= -\sum_{k=1}^{3}\frac{\sqrt{230945}\pi\beta_{nk}}{512(1+\alpha_{k9})^{\frac{21}{2}}}\sum_{i=1}^{N}e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}}r_i^2}(x_i+iy_i)^9, \qquad (223)
\end{aligned}
$$

The power spectrum for $\ell = 9$ is

$$\begin{aligned}
\mathbf{P}_{nn'9}(\mathbf{r}_i) &= \sum_m c_{n9m} c^*_{n'9m} \\
&= \sum_{k=1}^{3} \sum_{k'=1}^{3} \frac{19\pi^2 \beta_{nk} \beta^*_{n'k'}}{65536(1+\alpha_{k9})^{\frac{21}{2}}(1+\alpha_{k'9})^{\frac{21}{2}}} \sum_{i=1}^{N} \sum_{j=1}^{N} e^{-\frac{\alpha_{k9}}{1+\alpha_{k9}} r_i^2} e^{-\frac{\alpha_{k'9}}{1+\alpha_{k'9}} r_j^2} \Bigg[\frac{12155}{2} \mathrm{Re}[X_{ij}^9] \\
&+ 109395 z_i z_j \mathrm{Re}[X_{ij}^8] + \frac{6435}{2}(17z_i^2 - r_i^2)(17z_j^2 - r_j^2)\mathrm{Re}[X_{ij}^7] \\
&+ 17160 z_i z_j (17z_i^2 - 3r_i^2)(17z_j^2 - 3r_j^2)\mathrm{Re}[X_{ij}^6] \\
&+ 2574(85z_i^4 - 30z_i^2 r_i^2 + r_i^4)(85z_j^4 - 30z_j^2 r_j^2 + r_j^4)\mathrm{Re}[X_{ij}^5] \\
&+ 180180 z_i z_j (17z_i^4 - 10z_i^2 r_i^2 + r_i^4)(17z_j^4 - 10z_j^2 r_j^2 + r_j^4)\mathrm{Re}[X_{ij}^4] \\
&+ 2310(221z_i^6 - 195z_i^4 r_i^2 + 39z_i^2 r_i^4 - r_i^6)(221z_j^6 - 195z_j^4 r_j^2 + 39z_j^2 r_j^4 - r_j^6)\mathrm{Re}[X_{ij}^3] \\
&+ 3960 z_i z_j (221z_i^6 - 273z_i^4 r_i^2 + 91z_i^2 r_i^4 - 7r_i^6)(221z_j^6 - 273z_j^4 r_j^2 + 91z_j^2 r_j^4 - 7r_j^6)\mathrm{Re}[X_{ij}^2] \\
&+ 45(2431z_i^8 - 4004z_i^6 r_i^2 + 2002z_i^4 r_i^4 - 308z_i^2 r_i^6 + 7r_i^8) \\
&\times (2431z_j^8 - 4004z_j^6 r_j^2 + 2002z_j^4 r_j^4 - 308z_j^2 r_j^6 + 7r_j^8)\mathrm{Re}[X_{ij}] \\
&+ z_i z_j (12155z_i^8 - 25740z_i^6 r_i^2 + 18018z_i^4 r_i^4 - 4620z_i^2 r_i^6 + 315r_i^8) \\
&\times (12155z_j^8 - 25740z_j^6 r_j^2 + 18018z_j^4 r_j^4 - 4620z_j^2 r_j^6 + 315r_j^8) \Bigg] . \qquad (224)
\end{aligned}$$