Ex 1. 设 P 是数域, $A \in P^{m\times s}, B \in P^{s\times n}$, 记

$$W = \{B\alpha | \alpha \in P^n, AB\alpha = 0\}$$

试证明: W 是 P 的子空间且 $\dim W = R(B) - R(AB)$.[1]

证明. $\mathbf{B}\vec{x} = \vec{0}$ 之解空间 $\mathbf{V_1}$ 维数 $\dim V_1 = n - R(B)$, $\mathbf{AB}\vec{x} = \vec{0}$ 之解空间 $\mathbf{V_2}$ 维数 $\dim V_2 = n - R(AB)$.
易知 $V_1 \subseteq V_2$. 对 V_1 进行基扩充.

$$V_1 = \mathrm{L}(\vec{\gamma_1}, \cdots, \gamma_{n-\vec{R}(B)})$$

$$V_2 = \mathrm{L}(\vec{\gamma_1}, \cdots, \gamma_{n-\vec{R}(B)}, \cdots, \gamma_{n-\vec{R}(AB)})$$

对 $\alpha \in V_2$, 有 $B\alpha \in W$. 对于 $B\vec{\gamma}$, 有 $B\vec{\gamma_k} = \vec{0}, k = 1, \cdots, n - R(B)$, 且 $B\vec{\gamma_k} \neq \vec{0}, k = n - R(B) + 1, \cdots, n - R(AB)$. 下证 $B\vec{\gamma_k}$ 线性无关.
若 $B\sum_{i=n-R(B)+1}^{n-R(AB)} k_i\vec{\gamma_i} = \vec{0}$, 因为 $\vec{\gamma_i}, {}_{i=n-R(B)+1,\cdots,n-R(AB)}$ 线性无关

$$\begin{aligned}
&\Rightarrow \sum_{i=n-R(B)+1}^{n-R(AB)} k_i\vec{\gamma_i} \neq \vec{0} \\
&\Rightarrow \sum_{i=n-R(B)+1}^{n-R(AB)} k_i\vec{\gamma_i}\text{为}B\vec{\alpha} = 0\text{的非零解} \\
&\Rightarrow \sum_{i=n-R(B)+1}^{n-R(AB)} k_i\vec{\gamma_i} \in V_1 \\
&\Rightarrow V_2 \subseteq V_1
\end{aligned}$$

矛盾.
由是 $\dim W = (n - R(AB)) - (n - R(B)) = R(B) - R(AB)$ □

[1] 该题利用基扩充来做.