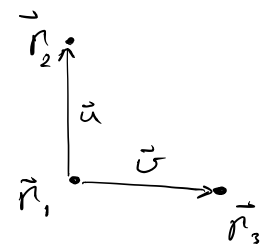
We get 3 lines: L_1, L_2, L_3 where

$$L_1(t_1) = \vec{a}_1 + t_1 \vec{b}_1$$
$$L_2(t_2) = \vec{a}_2 + t_2 \vec{b}_2$$
$$L_3(t_3) = \vec{a}_3 + t_3 \vec{b}_3$$

a_i and b_i constants

Want to find points $\vec{r}_1, \vec{r}_2, \vec{r}_3$ on L_1, L_2, L_3 respectively such that $|\vec{u}| = |\vec{v}| = 1$ U $\vec{u} \cdot \vec{v} = 0$

where
$$\vec{u} = \vec{r}_2 - \vec{r}_1$$
$$\vec{v} = \vec{r}_3 - \vec{r}_1$$



$$\Rightarrow \quad \vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\begin{cases} u_x^2 + u_y^2 + u_z^2 = 1 \\ v_x^2 + v_y^2 + v_z^2 = 1 \\ u_x v_x + u_y v_y + u_z v_z = 0 \end{cases} \Rightarrow \text{3 equations to determine 3 unknowns}$$

$$\vec{u} = L_2(t_2') - L_1(t_1')$$
$$\vec{v} = L_3(t_3') - L_1(t_1')$$

6 eq $\rightarrow$

$$\vec{u} = (\vec{a}_2 + t_2' \vec{b}_2) - (\vec{a}_1 + t_1' \vec{b}_1)$$
$$= \vec{a}_2 - \vec{a}_1 + t_2' \vec{b}_2 - t_1' \vec{b}_1$$
$$\vec{v} = \vec{a}_3 - \vec{a}_1 + t_3' \vec{b}_3 - t_1 \vec{b}_1$$

Reducing to 3 eq, 3 var system:

$$|u| = \left((a_{2x} + t_2 b_{2x}) - (a_{1x} + t_1 b_{1x})\right)^2$$
$$+ \left((a_{2y} + t_2 b_{2y}) - (a_{1y} + t_1 b_{1y})\right)^2$$
$$+ \left((a_{2z} + t_2 b_{2z}) - (a_{2z} + t_1 b_{1z})\right)^2 = 1$$

$$|v| = \left((a_{3x} + t_3 b_{3x}) - (a_{1x} + t_1 b_{1x})\right)^2 + \dots = 1$$

$\Rightarrow$ we know lines go through origin, ie $\vec{a}_n = \vec{0}$

so $\vec{u} = t_2 \vec{b}_2 - t_1 \vec{b}_1$ etc and

$$|\vec{u}| = \left(t_2 b_{2x} - t_1 b_{1x}\right)^2 + \left(t_2 b_{2y} - t_1 b_{1y}\right)^2 + \left(t_2 b_{2z} - t_1 b_{1z}\right)^2$$

$$= \left(t_2 b_{2x}\right)^2 - 2 t_2 b_{2x} t_1 b_{1x} + \left(t_1 b_{1x}\right)^2 +$$

$$\left(t_2 b_{2y}\right)^2 - 2 t_2 b_{2y} t_1 b_{1y} + \left(t_1 b_{1y}\right)^2 +$$

$$\left(t_2 b_{2z}\right)^2 - 2 t_2 b_{2z} t_1 b_{1z} + \left(t_1 b_{1z}\right)^2 = 1$$

$$|\vec{v}| = \left(t_3 b_{3x} - t_1 b_{1x}\right)^2 + \left(t_3 b_{3y} - t_1 b_{1y}\right)^2 + \left(t_3 b_{3z} - t_1 b_{1z}\right)^2 = 1$$

Solve for t_2:

$$t_2^2 \left(b_{2x}^2 + b_{2y}^2 + b_{2z}^2\right) - 2 t_2 \left(b_{2x} t_1 b_{1x} + b_{2y} t_1 b_{1y} + b_{2z} t_1 b_{2z}\right)$$
$$+ \left(t_1 b_{1x}\right)^2 + \left(t_1 b_{1y}\right)^2 + \left(t_1 b_{1z}\right)^2 - 1 = 0$$

Second order polynomial $\quad m t_2^2 + n t_2 + \sigma = 0$

$$t_2 = \frac{-n \pm \sqrt{n^2 - 4 m \sigma}}{2m}$$

where
$$m = b_{2x}^2 + b_{2y}^2 + b_{2z}^2 \qquad = |\vec{b}_2|^2$$
$$n = -2\left(b_{2x} t_1 b_{1x} + b_{2y} t_1 b_{1y} + b_{2z} t_1 t_{2z}\right) \qquad = -2 t_1 \left(\vec{b}_2 \cdot \vec{b}_1\right)$$
$$\sigma = \left(t_1 b_{1x}\right)^2 + \left(t_1 b_{1y}\right)^2 + \left(t_1 b_{1z}\right)^2 - 1 \qquad = t_1^2 |\vec{b}_1|^2 - 1$$

Similarly, for v we solve for t_3 to get

$$m' = |\vec{b}_3|^2$$
$$n' = -2 t_1 \left(\vec{b}_3 \cdot \vec{b}_1\right)$$
$$\sigma' = t_1^2 |\vec{b}_1|^2 - 1$$

$$\Rightarrow \quad t_2 = \frac{2t_1(\vec{b}_2\cdot\vec{b}_1) \pm \sqrt{4t_1^2(\vec{b}_2\cdot\vec{b}_1)^2 - 4t_1|\vec{b}_2|^2|\vec{b}_1|^2}}{2|b_2|^2}$$

$$\begin{cases} t_2(t_1) = \dfrac{t_1(\vec{b}_2\cdot\vec{b}_1) \pm \sqrt{t_1}\sqrt{(\vec{b}_2\cdot\vec{b}_1)^2 t_1 - |\vec{b}_2|^2|\vec{b}_1|^2}}{|b_2|^2} \\[2ex] t_3(t_1) = \dfrac{t_1(\vec{b}_3\cdot\vec{b}_1) \pm \sqrt{t_1}\sqrt{(\vec{b}_3\cdot\vec{b}_1)\, t_1 - |\vec{b}_3|^2|\vec{b}_1|^2}}{|b_3|^2} \end{cases}$$

Now, consider last eq in system:

$$\begin{aligned} \vec{u}\cdot\vec{v} = \; & (t_2 b_{2x} - t_1 b_{1x})(t_3 b_{3x} - t_1 b_{1x}) + \\ & (t_2 b_{2y} - t_1 b_{1y})(t_3 b_{3y} - t_1 b_{1y}) + \\ & (t_2 b_{2z} - t_1 b_{1z})(t_3 b_{3z} - t_1 b_{1z}) = 0 \\ = \; & t_2 b_{2x} t_3 b_{3x} - t_2 b_{2x} t_1 b_{1x} - t_3 b_{3x} t_1 b_{1x} + (t_1 b_{1x})^2 + \ldots \\ = \; & t_2 t_3(\vec{b}_2\cdot\vec{b}_3) - t_2 t_1(\vec{b}_2\cdot\vec{b}_1) - t_3 t_1(\vec{b}_3\cdot\vec{b}_1) + t_1^2|\vec{b}_1|^2 = 0 \end{aligned}$$

$\Rightarrow$ we can now sub in t_1 and t_2 to get one equation with one unknown.

Strategy for solving this:

$f(t_1) = 0$, solving options:

- solve for t_1 (ugly!)
- iterate on t_1
- Newtons method!

$$\vec{u} = t_2 \vec{b}_2 - t_1 \vec{b}_1 \Rightarrow \quad |\vec{u}| = 1$$

$$\Rightarrow |t_2 \vec{b}_2 - t_1 \vec{b}_1|$$

$$|\vec{u}| = (t_2 b_{2x} - t_1 b_{1x})^2 + (t_2 b_{2y} - t_1 b_{1y})^2 + (t_2 b_{2z} - t_1 b_{1z})$$

$$= (t_2 b_{2x})^2 - 2 t_2 b_{2x} t_1 b_{1x} + (t_1 b_{1x})^2 +$$

$$(t_2 b_{2y})^2 - 2 t_2 b_{2y} t_1 b_{1y} + (t_1 b_{1y})^2 +$$

$$(t_2 b_{2z})^2 - 2 t_2 b_{2z} t_1 b_{1z} + (t_1 b_{1z})^2 = 1$$

$$\Rightarrow t_2^2 \underline{|\vec{b}_2|^2} - \underline{2} t_1 t_2 \underline{(\vec{b}_1 \cdot \vec{b}_2)} + \underline{t_1^2 |\vec{b}_1|^2} - 1 = 0$$

$$a = |\vec{b}_1|^2$$

$$b = 2 t_1 (\vec{b}_1 \cdot \vec{b}_2)$$

$$c = t_1^2 |\vec{b}_1|^2 - 1$$

$$t_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This can be reduced to a single equation of one variable t_1 whose roots are solutions to the problem.
From t_1 we see that t_2 and t_3 follows, allowing us to determine points p_1, p_2 and p_3.

Next task : transform system so that p_1, p_2, p_3 falls on $(0,0,0)$, $(0,1,0)$, $(1,0,0)$ respectively.

Procedure :

1. Translate p_1 to $(0,0,0)$
2. Rotate p_2 to $(0,1,0)$ around (x, y, z)
3. Rotate p_3 to $(1,0,0)$ around y

System $\Rightarrow$ base station pose