change of the integration variable. The weights and the nodes are computed such that (3.8) is exactly satisfied for polynomials of degree $2n-1$ or less. For instance, if $n=2$, denote $f_i(x)=x^{i-1}$, $i=1,\dots,4$. The weights and nodes satisfy

$$w_1 f_1(x_1)+w_2 f_1(x_2)=\int_{-1}^{1} f_1(x)\,dx,$$

$$w_1 f_2(x_1)+w_2 f_2(x_2)=\int_{-1}^{1} f_2(x)\,dx,$$

$$w_1 f_3(x_1)+w_2 f_3(x_2)=\int_{-1}^{1} f_3(x)\,dx,$$

$$w_1 f_4(x_1)+w_2 f_4(x_2)=\int_{-1}^{1} f_4(x)\,dx.$$

This is a system of four equations with four unknowns. The solutions are $w_1=w_2=1$ and $x_2=-x_1=0.578$. For larger values of n, the computation is similar. By increasing the number of nodes n, the precision increases. Notice that the nodes are not necessarily equally spaced. The weights and the value of the nodes are published in the literature for commonly used values of n.

Approximating an Autoregressive Process with a Markov Chain
In this discussion we follow Tauchen (1986) and Tauchen and Hussey (1991) and show how to approximate an autoregressive process of order one by a first-order Markov process. This way we can simplify the computation of expected values in the value function iteration framework.

To return to the value function in the cake-eating problem, we need to calculate the expected value given ε:

$$V(W,\varepsilon)=\max[\varepsilon u(W),E_{\varepsilon'|\varepsilon}V(\rho W,\varepsilon')].$$

The calculation of an integral at each iteration is cumbersome. So we discretize the process ε_t, into N points ε^i, $i=1,\dots,N$. Now we can replace the expected value by

$$V(W,\varepsilon^i)=\max\left[\varepsilon u(W),\sum_{j=1}^{N}\pi_{i,j}V(\rho W,\varepsilon^j)\right],\qquad i=1,\dots,N.$$

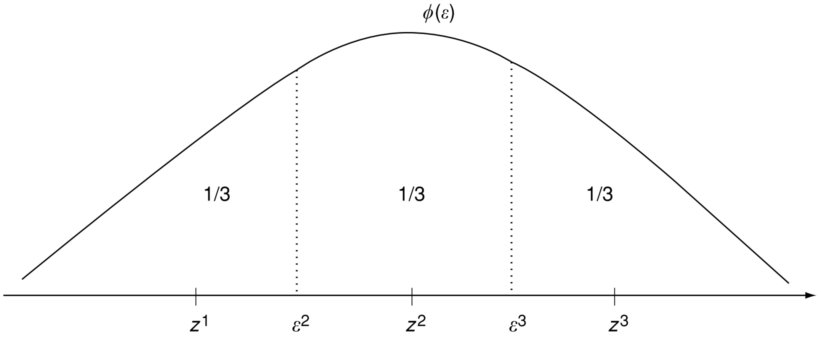

Figure 3.10
Example of discretization, $N = 3$

As in the quadrature method, the method involves finding nodes ε^j and weights $\pi_{i,j}$. As we will see below, the ε^i and the $\pi_{i,j}$ can be computed prior to the iterations.

Suppose that ε_t follows an AR(1) process, with an unconditional mean μ and an autocorrelation ρ:

$$\varepsilon_t = \mu(1-\rho) + \rho\varepsilon_{t-1} + u_t, \tag{3.9}$$

where u_t is a normally distributed shock with variance σ^2. To discretize this process, we need to determine three different objects. First, we need to discretize the process ε_t into N intervals. Second, we need to compute the conditional mean of ε_t within each intervals, which we denote by z^i, $i, \dots, N$. Third, we need to compute the probability of transition between any of these intervals, $\pi_{i,j}$. Figure 3.10 shows the plot of the distribution of ε and the cut-off points ε^i as well as the conditional means z^i.

We start by discretizing the real line into N intervals, defined by the limits $\varepsilon^1, \dots, \varepsilon^{N+1}$. As the process ε_t is unbounded, $\varepsilon^1 = -\infty$ and $\varepsilon^{N+1} = +\infty$. The intervals are constructed such that ε_t has an equal probability of $1/N$ of falling into them. Given the normality assumption, the cut-off points $\{\varepsilon^i\}_{i=1}^{N+1}$ are defined as

$$\Phi\left(\frac{\varepsilon^{i+1}-\mu}{\sigma_\varepsilon}\right) - \Phi\left(\frac{\varepsilon^{i}-\mu}{\sigma_\varepsilon}\right) = \frac{1}{N}, \qquad i = 1, \dots, N, \tag{3.10}$$

where $\Phi(\ )$ is the cumulative of the normal density and σ_ε is the standard deviation of ε equal to $\sigma/\sqrt{(1-\rho)}$. Working recursively, we get

$$\varepsilon^i = \sigma_\varepsilon \Phi^{-1}\left(\frac{i-1}{N}\right) + \mu.$$

Now that we have defined the intervals, we want to find the average value of ε within a given interval. We denote this value by z^i, which is computed as the mean of ε_t conditional on $\varepsilon_t \in [\varepsilon^i, \varepsilon^{i+1}]$:

$$z^i = E(\varepsilon_t | \varepsilon_t \in [\varepsilon^i, \varepsilon^{i+1}]) = \sigma_\varepsilon \frac{\phi((\varepsilon^i - \mu)/\sigma_\varepsilon) - \phi((\varepsilon^{i+1} - \mu)/\sigma_\varepsilon)}{\Phi((\varepsilon^{i+1} - \mu)/\sigma_\varepsilon) - \Phi((\varepsilon^i - \mu)/\sigma_\varepsilon)} + \mu.$$

From (3.10), we know that the expression simplifies to

$$z^i = N\sigma_\varepsilon \left(\phi\left(\frac{\varepsilon^i - \mu}{\sigma_\varepsilon}\right) - \phi\left(\frac{\varepsilon^{i+1} - \mu}{\sigma_\varepsilon}\right)\right) + \mu.$$

Next we define the transition probability as

$$\pi_{i,j} = P(\varepsilon_t \in [\varepsilon^j, \varepsilon^{j+1}] \,|\, \varepsilon_{t-1} \in [\varepsilon^i, \varepsilon^{i+1}])$$

$$\pi_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon}} \int_{\varepsilon^i}^{\varepsilon^{i+1}} e^{-(u-\mu)^2/(2\sigma_\varepsilon^2)} \left[\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho u}{\sigma}\right) \right.$$
$$\left. - \Phi\left(\frac{\varepsilon^{j} - \mu(1-\rho) - \rho u}{\sigma}\right)\right] du.$$

The computation of $\pi_{i,j}$ requires the computation of a nontrivial integral. This can be done numerically. Note that if $\rho = 0$, meaning ε is an iid process, the expression above is simply

$$\pi_{i,j} = \frac{1}{N}.$$

We can now define a Markov process z_t that will mimic an autoregressive process of order one, as defined in (3.9). z_t takes its values in $\{z^i\}_{i=1}^N$ and the transition between period t and $t+1$ is defined as

$$P(z_t = z^j | z_{t-1} = z^i) = \pi_{i,j}.$$

By increasing N, the discretization becomes finer and the Markov process gets closer to the real autoregressive process.

Example For N = 3, $\rho = 0.5$, $\mu = 0$, and $\sigma = 1$, we have

$$z^1 = -1.26, \quad z^2 = 0, \quad z^3 = 1.26,$$

and

$$\pi = \begin{bmatrix} 0.55 & 0.31 & 0.14 \\ 0.31 & 0.38 & 0.31 \\ 0.14 & 0.31 & 0.55 \end{bmatrix}.$$

3.5.3 How to Simulate the Model

Once the value function is computed, the estimation or the evaluation of the model often requires the simulation of the behavior of the agent through time. If the model is stochastic, the first step is to generate a series for the shocks, for $t = 1, \dots, T$. Then we go from period to period and use the policy function to find out the optimal choice for this period. We also update the state variable and proceed to next period.

How to Program a Markov Process

The Markov process is characterized by grid points, $\{z^i\}$ and by a transition matrix π, with elements $\pi_{ij} = \text{Prob}(y_t = z^j / y_{t-1} = z^i)$. We start in period 1. The process z_t is initialized at, say, z^i. Next, we have to assign a value for z_2. To this end, using the random generator of the computer, we draw a uniform variable u in $[0, 1]$. The state in period 2, j, is defined as

```
t=1
oldind=1                        * Variable to keep track of
                                  state in period t-1 *

y[t]=z[oldind]                  * Initialize first period *
  do until t>T                  * Loop over all time periods *
    u=uniform(0,1)              * Generate a uniform random
                                  variable *
    sum=0                       * Will contain the cumulative
                                  sum of pi *
    ind=1                       * Index over all possible
                                  values for process *
      do until u<=sum           * Loop to find out the state
                                  in period t *
        sum=sum+pi[oldind,ind]  * Cumulative sum of pi *
        ind=ind+1
      endo
  y[t]=z[ind]                   * State in period t *
  oldind=ind                    * Keep track of lagged state *
  t=t+1
  endo
```

Figure 3.11
Simulation of a Markov process

$$\sum_{l=1}^{j} \pi_{i,l} < u \leq \sum_{l=1}^{j+1} \pi_{i,l},$$

or $j = 1$ if $u < \pi_{i,1}$. The values for the periods ahead are constructed in a similar way. Figure 3.11 presents a computer code that will construct iteratively the values for T periods.

How to Simulate the Model

For the model we need to initialize all stochastic processes that are the exogenous shock and the state variables. The state variables can be initialized to their long-run values or to some other value. Often the model is simulated over a long number of periods and the first periods are discarded to get rid of initial condition problems.

The value of the state variables and the shock in period 1 are used to determine the choice variable in period 1. In the case of the continuous stochastic cake-eating problem of section 3.2, we would construct $c_1 = c(X_1)$. Next we can generate the values of the state variable in period 2, $X_2 = R(X_1 - c_1) + y_2$, where y_2 is calculated using the method described in section 3.5.3 above. This procedure would be repeated over T periods to successively construct all the values for the choice variables and the state variables.