Principal Component Analysis

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Outline

1. Principal Component Analysis (PCA)
2. Empirical Principal Component Analysis
3. Applications of Principal Component Analysis

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Motivation

Let $X = (X_1, X_2, \ldots, X_n)^T$ be a n-dimensional random vector

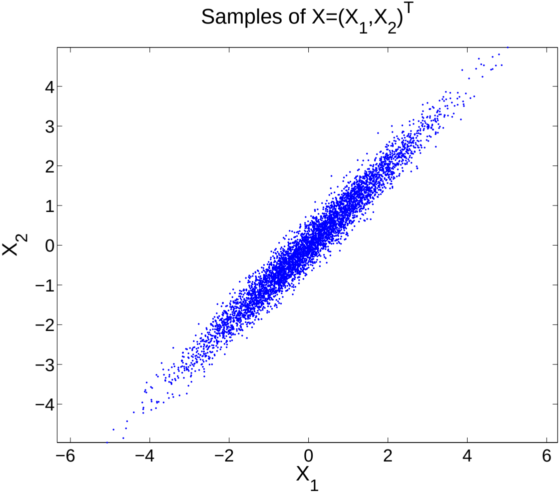
- X_i denotes the change of i-th risk factor over a specified time horizon
- These risk factors could represent
  - security prices
  - security returns
  - zero rates at different maturities
- $X_1, X_2, \ldots, X_n$ are generally correlated: $\text{Cov}(X_i, X_j) \neq 0$

Question: can we find a matrix $A \in \mathbb{R}^{n\times n}$, such that the components of random vector $Y = AX$ are uncorrelated with each other?

$$\text{Cov}(Y_i, Y_j) = 0, \ \forall i \neq j$$

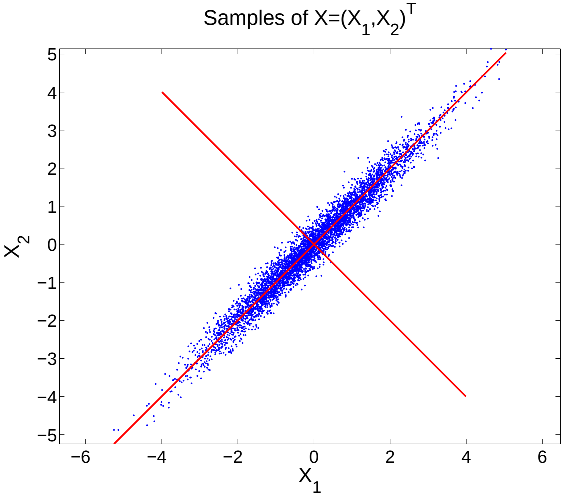
Motivation (continued)

Example of a random vector $X = (X_1, X_2)^T$ with $\mathrm{Cov}(X_1, X_2) > 0$



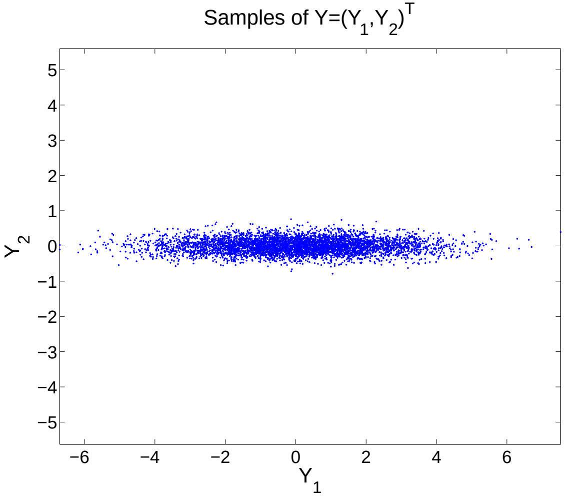
Motivation (continued)

Find two projection directions



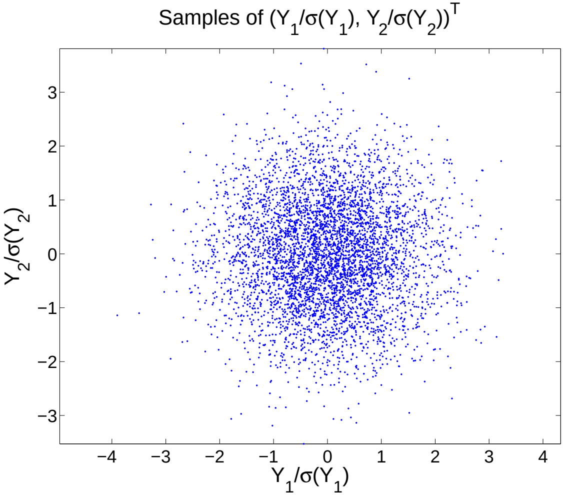
Motivation (continued)

The projected random vector $Y = (Y_1, Y_2)^T$ has uncorrelated components: $\mathrm{Cov}(Y_1, Y_2) = 0$



Motivation (continued)

The normalized random vector $\left(\frac{Y_1}{\sigma(Y_1)}, \frac{Y_2}{\sigma(Y_2)}\right)^T$



Normalization

- We first normalize each component of $X = (X_1, X_2, \ldots, X_n)^T$:
  - Remove the mean of X_i: $X_i \leftarrow X_i - E[X_i]$
  - Normalize the variance of X_i: $X_i \leftarrow \frac{X_i}{\sqrt{\mathrm{Var}(X_i)}}$
  - The normalized random vector is a linear transform of X
- After normalization, $\mu = E[X] = 0$ and $\mathrm{Var}(X_i) = 1$ for all $i = 1, 2, \ldots, n$.
- Normalizing the variance is not necessary but is a common practice.

Formulate the Problem

Let $X = (X_1, X_2, \ldots, X_n)^T$ be a random vector with $E[X] = 0$ (Var(X_i) is not necessarily equal to 1)

- Let $\Sigma := E[XX^T]$ be the variance-covariance matrix of X
- Let $Y = AX$, where $A \in \mathbb{R}^{n\times n}$ is the matrix to be determined:

$$\mathrm{Cov}(Y) = A\Sigma A^T$$

- Can we find A such that $A\Sigma A^T$ is a diagonal matrix?

The key method is the **spectral decomposition** of symmetric matrix.

Spectral Decomposition

Spectral decomposition: any real-valued symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$ can be decomposed into

$$\Sigma = \Gamma\Lambda\Gamma^T, \text{ where}$$

- Λ is a diagonal matrix $\text{diag}\{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_1, \ldots, \lambda_n$ are the **eigen values** of Σ
  - Without loss of generality the eigen values can be ordered such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$
  - If Σ is positive semi-definite, then $\lambda_i \geq 0$, for all i
- Γ is an orthogonal matrix with the i-th column $\gamma_i \in \mathbb{R}^n$ being the i-th standardized **eigen vector** of Σ
  - $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$
  - γ_i is the i-th eigen vector: $\Sigma\gamma_i = \lambda_i\gamma_i$
  - γ_i is standardized: $\gamma_i^T\gamma_i = 1$
  - Γ is an orthogonal matrix: $\Gamma\Gamma^T = \Gamma^T\Gamma = I_n$

Principal Components Analysis (PCA)

Let $\Sigma = \Gamma\Lambda\Gamma^T$ be the spectral decomposition of Σ and define

$$Y = \Gamma^T X$$

- The covariance matrix of Y is

$$\mathrm{Cov}(Y) = \Gamma^T \Sigma (\Gamma^T)^T = \Gamma^T \Gamma \Lambda \Gamma^T (\Gamma^T)^T = \Lambda$$

- Components of Y are uncorrelated and Y is called the **principal components** of X
- The i-th component of Y is $Y_i = \gamma_i^T X$:

$$Y = \Gamma^T X = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T X = \begin{pmatrix} \gamma_1^T \\ \gamma_2^T \\ \cdots \\ \gamma_n^T \end{pmatrix} X = \begin{pmatrix} \gamma_1^T X \\ \gamma_2^T X \\ \cdots \\ \gamma_n^T X \end{pmatrix}.$$

- $\mathrm{Var}(Y_i) = \lambda_i$
- The matrix Γ^T is called the matrix of **factor loadings**.

Principal Components Analysis (continued)

Let $Y = \Gamma^T X$ be the principal components of X. Then

$$X = \Gamma Y = (\gamma_1, \gamma_2, \ldots, \gamma_n) \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix} = Y_1\gamma_1 + Y_2\gamma_2 + \ldots + Y_n\gamma_n$$

- X is a random linear combination of $\gamma_1, \gamma_2, \ldots, \gamma_n$; the combination coefficients are random and are uncorrelated

Explain Variability of X by Its Principal Components

Let $Y = \Gamma^T X$ be the principal components of X

- Total variance of principal components:

$$\sum_{i=1}^{n} \mathrm{Var}(Y_i) = \sum_{i=1}^{n} \lambda_i = \mathrm{trace}(\Lambda) = \mathrm{trace}(\Gamma^T \Sigma \Gamma)$$

$$= \mathrm{trace}(\Gamma\Gamma^T \Sigma) = \mathrm{trace}(\Sigma) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

- The proportion of variability explained by the i-th principal component

$$\frac{\mathrm{Var}(Y_i)}{\sum_{i=1}^{n} \mathrm{Var}(X_i)} = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$$

- The proportion of variability explained by the first k principal components

$$\frac{\sum_{i=1}^{k} \mathrm{Var}(Y_i)}{\sum_{i=1}^{n} \mathrm{Var}(X_i)} = \frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{n} \lambda_i}$$

Explain Variability of X by Its Principal Components

Let $Y = \Gamma^T X$ be the principal components of X

- The first component $Y_1 = \gamma_1^T X$ satisfies

$$\mathrm{Var}(\gamma_1^T X) = \max\{\mathrm{Var}(a^T X) : a^T a = 1\}$$

- γ_1 is the projection direction along which the variability of X is the largest
- The second component $Y_2 = \gamma_2^T X$ satisfies

$$\mathrm{Var}(\gamma_2^T X) = \max\{\mathrm{Var}(a^T X) : a^T a = 1, a^T \gamma_1 = 0\}$$

where $a^T \gamma_1 = 0$ means that a is orthogonal to γ_1, or equivalently, $a^T X$ is orthogonal to Y_1 (i.e., $E(a^T X \cdot Y_1) = 0$).

- The successive principal components $Y_i = \gamma_i^T X$ satisfy the optimization problem

$$\mathrm{Var}(\gamma_i^T X) = \max\{\mathrm{Var}(a^T X) : a^T a = 1, a^T \gamma_j = 0, j = 1, 2, \ldots, i-1\}.$$

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Empirical PCA

- In practice, the true variance-covariance matrix is unknown and may only be estimated from historical data.
- Suppose we observe $X^t = (X_1^t, X_2^t, \ldots, X_n^t)^T$, $t = 1, 2, \ldots, T$: X^t is the t-th observation (e.g, the sample observed at time t)
- It is important to make sure that $\{X^t, t = 1, 2, \ldots, T\}$ is (weakly) stationary
  - $E(X^1) = E(X^2) = \cdots = E(X^T)$
  - $\mathrm{Cov}(X^1, X^1) = \mathrm{Cov}(X^2, X^2) = \cdots = \mathrm{Cov}(X^T, X^T)$
- If X denotes asset returns or yields, $\{X^t, t = 1, 2, \ldots, T\}$ is usually assumed to be stationary
- If X denotes asset prices, $\{X^t, t = 1, 2, \ldots, T\}$ usually have a trend and may not be assumed to be stationary

Empirical PCA (continued)

Demean the sample

- Let $\hat{\mu}_i := \frac{1}{T}\sum_{t=1}^{T} X_i^t$ be the sample mean of X_i, $i = 1, 2, \ldots, n$
- Let $Z_i^t := X_i^t - \hat{\mu}_i,\ i = 1, 2, \ldots, n, t = 1, \ldots, T$

$$Z^t = \begin{pmatrix} Z_1^t \\ Z_2^t \\ \cdots \\ Z_n^t \end{pmatrix} = X^t - \hat{\mu},\ t = 1, \ldots, T$$

Empirical PCA (continued)

- Compute the sample covariance matrix

$$\Sigma = \frac{1}{T}\sum_{t=1}^{T} Z^t (Z^t)^T$$

- Spectral decomposition of $\Sigma = \Gamma\Lambda\Gamma^T$
- Compute principal components

$$Y^t = \Gamma^T Z^t = \Gamma^T \cdot (X^t - \hat{\mu})$$

The original data X^t can be recovered from principal components by

$$X^t = \Gamma Y^t + \hat{\mu}.$$

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Zero-coupon Bond Yields

- Continuously compounded zero-coupon bond yield y_t:

$$D(0,t) = \frac{1}{e^{y_t \cdot t}} = e^{-y_t \cdot t}$$

- zero-coupon bond yield y_t is also called **zero rate** or **spot rate**.
- y_t is a function of t: it is called the **zero-coupon bond yield curve**.
- The zero-coupon bond yield curve is also called the **term structure of interest rates**.

Zero-coupon Bond Yields (continued)

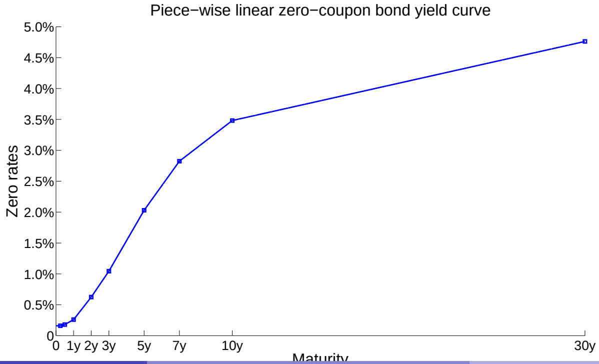
- The price of a coupon bond with face value F, coupon payments C_i at time T_i, $i = 1, 2, \ldots, m$ and maturity $T = T_m$:

$$\begin{aligned} P &= \sum_{i=1}^{m-1} D(0, T_i) \cdot C_i + D(0, T_m) \cdot (C_m + F) \\ &= \sum_{i=1}^{m-1} e^{-y_{T_i} \cdot T_i} \cdot C_i + e^{-y_{T_m} \cdot T_m} \cdot (C_m + F) \end{aligned} \tag{1}$$

Piece-wise Linear Zero-coupon Bond Yield Curve

- Assumptions (models) of the zero-coupon bond yield curve
  - Piece-wise linear assumption: y_t is constant on $[0, 0.25]$ and y_t is linear on each interval between the time points 0.25, 0.5, 1, 2, 3, 5, 7, 10, and 30

$$y_t = y_{T_{i-1}} + \frac{y_{T_i} - y_{T_{i-1}}}{T_i - T_{i-1}}(t - T_{i-1}), \forall t \in (T_{i-1}, T_i], i = 2, \ldots, 8$$



PCA of Change of Zero-coupon Curve

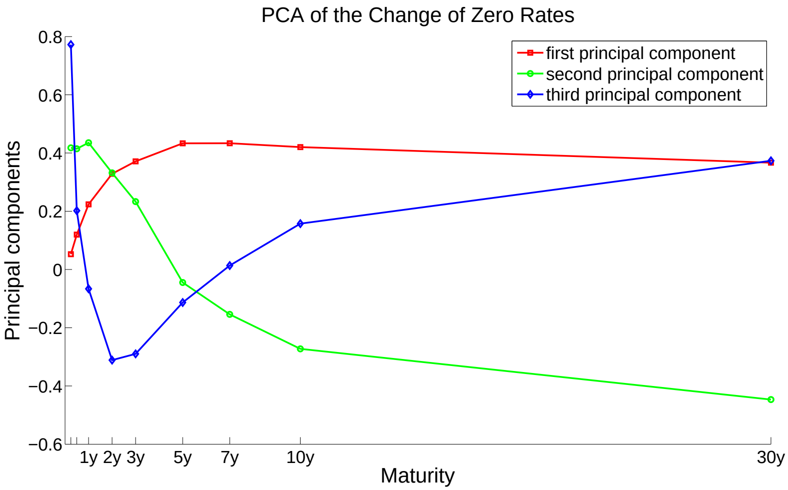
- Historical data of discount factors: “s0023_disc_factors_hist.xls"
- Compute the zero rates from the discount factors:

$$Z_i^t := -\frac{\log D(t, t+T_i)}{T_i}, i = 1, 2, \ldots, 9, t = 1, 2, \ldots, S$$

- $Z^t = (Z_1^t, Z_2^t, \ldots, Z_9^t)^T$ is the zero-coupon yield curve observed on day t, $t = 1, 2, \ldots, S$
- Compute change of zero rates: $X^t := Z^t - Z^{t-1}$
- PCA in Python: sklearn.decomposition.PCA in the package scikit-learn

PCA of Change of Zero-coupon Curve (continued)

The graph of γ_1, γ_2, and γ_3



PCA of Change of Zero-coupon Curve (continued)

- The first principal component γ_1 can usually be interpreted as (approximate) parallel shift in the zero curve
- The second component represents a flattening or steepening of the zero curve
- The third component represents the curvature of the curve
- The first three principal components explain 95% of the total variability:

$$\frac{\lambda_1}{\sum_{i=1}^{9}\lambda_i} = 0.76, \frac{\lambda_1+\lambda_2}{\sum_{i=1}^{9}\lambda_i} = 0.89, \frac{\sum_{i=1}^{3}\lambda_i}{\sum_{i=1}^{9}\lambda_i} = 0.95$$

Using PCA to Building Factor Models

- Recall that $X = \Gamma Y + \mu$
- Partition the matrix Γ into $\Gamma = [\Gamma_1, \Gamma_2]$, where

$$\Gamma_1 = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{R}^{n \times k}, \Gamma_2 = (\gamma_{k+1}, \ldots, \gamma_n) \in \mathbb{R}^{n \times (n-k)}$$

- $X = (\gamma_1 Y_1 + \cdots + \gamma_k Y_k) + (\gamma_{k+1} Y_{k+1} + \cdots + \gamma_n Y_n) + \mu$
- If first k principal components explain sufficiently large amount of total variability, then $\varepsilon := \gamma_{k+1} Y_{k+1} + \cdots + \gamma_n Y_n$ is small in magnitude and can be viewed as noise, and

$$X = \gamma_1 Y_1 + \cdots + \gamma_k Y_k + \epsilon + \mu$$

This is a k-factor model for X.

Using PCA to Building Factor Models (continued)

ε can be ignored to obtain an exact k-factor model

$$X = \gamma_1 Y_1 + \cdots + \gamma_k Y_k + \mu$$

- The joint distribution of the first k principal components can be estimated by using data $Y_i^1, Y_i^2, \ldots, Y_i^S$, $i = 1, 2, \ldots, k$
  - $(Y_1, Y_2, \ldots, Y_k)^T$ can be assumed to have a $N_k(0, \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_k\})$ distribution
  - $(Y_1, Y_2, \ldots, Y_k)^T$ can be assumed to have a multivariate-t distribution
- This reduces the number of risk factors from n to $k \ll n$.

Portfolio Immunization

- A fund manager at a pension fund or a life insurance company needs to pay out a stream of liability ℓ_i at time T_i, $i = 1, 2, \ldots, n$, to retirement policy holders.
- The current zero-coupon yield curve is $\{y_t, t \geq 0\}$.
- The present value $L(y)$ of the liability stream is $L(y) = \sum_{i=1}^{n} e^{-y_{T_i} \cdot T_i} \cdot \ell_i$.
- The fund manager is facing interest rate risk: when interest rates change from y to $y + X$, $L(y)$ changes to

$$L(y + X) = \sum_{i=1}^{n} e^{-(y_{T_i} + X_{T_i}) \cdot T_i} \cdot \ell_i$$

Portfolio Immunization (continued)

- The fund manager is managing an investment portfolio which will generate cash to pay off the liability
- The portfolio is composed of x_i units of bond i, $i = 1, 2, \ldots, n$.
- The currnt value of the portfolio: $P(y) = \sum_{i=1}^{n} x_i \cdot P_i(y)$
- When interest rates change from y to $y + X$, the value of the portfolio changes to $P(y + X) = \sum_{i=1}^{n} x_i \cdot P_i(y + X)$

Portfolio Immunization (continued)

We want to construct a portfolio with x_i units of bond P_i such that the portfolio value matches the liability value no matter how the zero curve changes:

$$L(y+X) \approx P(y+X), \text{ for small random change } X$$

Portfolio Immunization

Portfolio of bonds can be immunized under the k-factor model of the change of zero-coupon yield curve

- Let $y = (y_1, y_2, \ldots, y_n)^T$ be the current zero-coupon yield curve, where y_i is the zero rate of maturity T_i, $i = 1, 2, \ldots, n$.
- Bond portfolio value as a function of zero-coupon yield curve

$$P(y) = P(y_1, y_2, \ldots, y_n) = \sum_{j=1}^{m} e^{-y_{t_j} \cdot t_j} C_j$$

- Note that in general y_{t_j} is a function of y.
- The k-factor model of the change of zero curve obtained by PCA

$$X = \gamma_1 Y_1 + \cdots + \gamma_k Y_k + \mu$$

- Change of portfolio value

$$P(y + X) - P(y) = P(\gamma_1 Y_1 + \gamma_2 Y_2 + \cdots + \gamma_k Y_k + \mu + y) - P(y)$$

- Risk factors are the principal components $Y_1, Y_2, \ldots, Y_k$

Portfolio Immunization (continued)

Change of portfolio value

$$
\begin{aligned}
P(y+X)-P(y) &= P(\gamma_1 Y_1 + \gamma_2 Y_2 + \cdots + \gamma_k Y_k + \mu + y) - P(y) \\
&\approx \left(\frac{\partial P(y)}{\partial y_1}, \frac{\partial P(y)}{\partial y_2}, \ldots, \frac{\partial P(y)}{\partial y_n}\right)(\gamma_1 Y_1 + \cdots + \gamma_k Y_k + \mu) \\
&= \nabla P(y)^T (\gamma_1 Y_1 + \gamma_2 Y_2 + \cdots + \gamma_k Y_k + \mu) \\
&= \nabla P(y)^T \gamma_1 Y_1 + \nabla P(y)^T \gamma_2 Y_2 + \cdots + \nabla P(y)^T \gamma_k Y_k + \nabla P(y)^T \mu
\end{aligned}
$$

Percentage change of portfolio value

$$
\frac{P(y+X)-P(y)}{P(y)} = \frac{\nabla P(y)^T \gamma_1}{P(y)} Y_1 + \cdots + \frac{\nabla P(y)^T \gamma_k}{P(y)} Y_k + \frac{\nabla P(y)^T \mu}{P(y)}
$$

- $\frac{\nabla P(y)^T \gamma_i}{P(y)}$ is the sensitivity of the percentage change of portfolio w.r.t. the i-th factor

Portfolio Immunization (continued)

Consider a bond portfolio with x_i units of bond i with price $P_i(y)$, $i = 1, 2, \ldots, N$

- Total portfolio value: $P(y) = \sum_{i=1}^{N} x_i P_i(y)$
- Change in the value of the bond portfolio:

$$P(y+X) - P(y) = \sum_{i=1}^{N} x_i (P_i(y+X) - P_i(y))$$

$$\approx \sum_{i=1}^{N} x_i \left(\nabla P_i(y)^T \gamma_1 Y_1 + \nabla P_i(y)^T \gamma_2 Y_2 + \cdots + \nabla P_i(y)^T \gamma_k Y_k \right)$$

$$+ \sum_{i=1}^{N} x_i \nabla P_i(y)^T \mu$$

Portfolio Immunization (continued)

- Percentage change of the portfolio value

$$
\begin{aligned}
&\frac{P(y+X)-P(y)}{P(y)} \\
&= \sum_{i=1}^{N} \frac{x_i P_i(y)}{P(y)} \frac{\nabla P_i(y)^T \gamma_1}{P_i(y)} Y_1 + \cdots + \sum_{i=1}^{N} \frac{x_i P_i(y)}{P(y)} \frac{\nabla P_i(y)^T \gamma_k}{P_i(y)} Y_k \\
&+ \sum_{i=1}^{N} x_i \frac{\nabla P_i(y)^T \mu}{P(y)}
\end{aligned}
$$

Portfolio Immunization (continued)

Let $L(y)$ be the present value of the future stream of liabilities. We want to construct a portfolio with x_i units of bond $P_i(y)$ such that the portfolio is immunized against the move of the zero curve

$$L(y+X) \approx P(y+X), \text{ for small random change } X$$

- Matching current present value:

$$L(y) = \sum_{i=1}^{N} x_i P_i(y)$$

- Matching sensitivities w.r.t. principal components

$$\frac{\nabla L(y)^T \gamma_j}{L(y)} = \sum_{i=1}^{N} \frac{x_i P_i(y)}{P(y)} \frac{\nabla P_i(y)^T \gamma_j}{P_i(y)}, j = 1, \ldots, k$$

$$\frac{\nabla L(y)^T \mu}{L(y)} = \sum_{i=1}^{N} \frac{x_i P_i(y)}{P(y)} \frac{\nabla P_i(y)^T \mu}{P_i(y)}$$