QUESTION : 12.13.6.9

ROLL NO:EE22BTECH11027
NAME: KATARI SIRI VARSHINI

12.13.6.9.An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be atleast 4 successes.

Solution: : The repetition in conduction of experiment constitute the bernoulli trails. Let X denote the number of times the experiment succeeds in all the 6 attempts.
Let p be the probability for the experiment to succeed and q for the failure.
Here, it is given that probability of success is twice that of the failure, so

$$\begin{aligned} p &= 2q \\ p+q &= 1 \\ 3q &= 1 \\ q &= \frac{1}{3} \\ p &= \frac{2}{3} \end{aligned} \tag{1}$$

Now, let's consider a single trial as a bernuolli random variable $X_i = 1$ represents success and $X_i = 0$ represents failure. Therefore we have, probability of $X_i = 1$ as $\frac{2}{3}$, probability of $X_i = 0$ as $\frac{1}{3}$
Since we have n=6 trials, the random variable X representing the number of successes in 6 trials follows a binomial distribution. The cumulative distribution function (CDF) of X is given by

$$F_X(k) = P_X(X \le k) = \Sigma_{k=0}^{n} \binom{n}{k} q^{n-k} p^k \tag{2}$$

Here,

$$n = 6, p = \frac{2}{3}, q = \frac{1}{3}, k = 0, 1, 2...6$$

We need to find the probability for the experiment to succeed to atleast 4 times, which means that we want probability for experiment to succeed for 4, 5, or 6 times i.e. $P_X(X \ge 4)$

$$P_X(X \ge 4) = P_X(4) + P_X(5) + P_X(6) \tag{3}$$

Using equation 2 we get,

$$\begin{aligned} P_X(X \ge 4) &= 1 - P_X(X \le 3) \\ &= 1 - F_X(3) \\ &= 1 - \binom{6}{0}\left(\frac{1}{3}\right)^{6-0}\left(\frac{2}{3}\right)^{0} + \binom{6}{1}\left(\frac{1}{3}\right)^{6-1}\left(\frac{2}{3}\right)^{1} \\ &+ \binom{6}{2}\left(\frac{1}{3}\right)^{6-2}\left(\frac{2}{3}\right)^{2} + \binom{6}{3}\left(\frac{1}{3}\right)^{6-3}\left(\frac{2}{3}\right)^{3} \\ &= 1 - \binom{6}{0}\left(\frac{1}{3}\right)^{6}\left(\frac{2}{3}\right)^{0} + \binom{6}{1}\left(\frac{1}{3}\right)^{5}\left(\frac{2}{3}\right)^{1} \\ &+ \binom{6}{2}\left(\frac{1}{3}\right)^{4}\left(\frac{2}{3}\right)^{2} + \binom{6}{3}\left(\frac{1}{3}\right)^{3}\left(\frac{2}{3}\right)^{3} \\ &= 1 - \frac{6!}{6! \times 0!}\left(\frac{1}{3^6}\right) + \frac{6!}{5! \times 1!}\left(\frac{2}{3^6}\right) \\ &+ \frac{6!}{4! \times 2!}\left(\frac{4}{3^6}\right) + \frac{6!}{3! \times 3!}\left(\frac{8}{3^6}\right) \\ &= 1 - \frac{1}{3^6} + \frac{12}{3^6} + \frac{60}{3^6} + \frac{160}{3^6} \\ &= 1 - \frac{233}{3^6} = \frac{496}{729} \end{aligned} \tag{4}$$

Therefore the probability that in the next six trials, there will be atleast 4 successes is $\frac{496}{729}$.