62.2023

ROLL NO:EE22BTECH11027
NAME: KATARI SIRI VARSHINI

62.Consider a birth-death process on the state space $\{0,1,2,3\}$. The birth rates are given by $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 0$. The death rates are given by $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 1$. If $[\pi_0, \pi_1, \pi_2, \pi_3]$ is the unique stationary distribution, then $\pi_0+2\pi_1+3\pi_2+4\pi_3$ (rounded off to two decimal places) equals

Solution: : Given, a birth-death process on the state space $\{0,1,2,3\}$ with The birth rates $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 0$ and the death rates $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 1$ defined over a stationary distribution $[\pi_0, \pi_1, \pi_2, \pi_3]$.

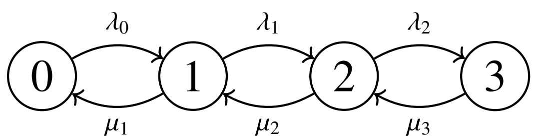

Fig. 0. Markov Chain for Birth-Death Process

$$\text{Let, } \pi = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} \tag{1}$$

The steady-state balanced solution of the birth and death process is given by

$$\begin{aligned} \lambda_0\pi_0 &= \mu_1\pi_1 \quad (\text{for } i = 0) \\ (\lambda_i + \mu_i)\pi_i &= \mu_{i+1}\pi_{i+1} + \lambda_{i-1}\pi_{i-1} (i = 1, 2, \ldots) \end{aligned} \tag{2}$$

$\therefore$ Transition Matrix P for the given states is given by

$$\begin{aligned} P &= \begin{bmatrix} -\lambda_0 & \mu_1 & 0 & 0 \\ \lambda_0 & -(\lambda_1+\mu_1) & \mu_2 & 0 \\ 0 & \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 \\ 0 & 0 & \lambda_2 & -\mu_3 \end{bmatrix} \\ \implies P &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \end{aligned} \tag{3}$$

Stationary distribution π satisfies $\pi P= 0$ and $\Sigma_{i=0}^{3}\pi_i= 1$. Converting P to echelon form we get,

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

From the above we get

$$\begin{aligned} \pi_1 &= \pi_0 \\ \pi_2 &= \pi_0 \\ \pi_3 &= 2\pi_0 \end{aligned} \tag{5}$$

Using the results obtained in equations from (5) and using the property of stationary distribution we get

$$\begin{aligned} \pi_0 &= \frac{1}{5} \\ \therefore \pi_0 + 2\pi_1 + 3\pi_2 + 4\pi_3 &= (1 + 2 + 3 + 8)\pi_0 \\ &= 2.80 \end{aligned} \tag{6}$$