62.2023

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62.Consider a birth-death process on the state space $\{0, 1, 2, 3\}$. The birth rates are given by $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 0$. The death rates are given by $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 1$. If $[\pi_0, \pi_1, \pi_2, \pi_3]$ is the unique stationary distribution, then $\pi_0+2\pi_1+3\pi_2+4\pi_3$ (rounded off to two decimal places) equals

Solution: : Given, a birth-death process on the state space $\{0, 1, 2, 3\}$ with The birth rates $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 0$ and the death rates $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 1$ defined over a stationary distribution $[\pi_0, \pi_1, \pi_2, \pi_3]$.

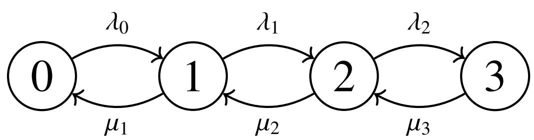

Fig. 0. Markov Chain for Birth-Death Process

$$\text{Let, } \pi = \begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\end{bmatrix} \tag{1}$$

The steady-state balanced solution of the birth and death process is given by

$$\begin{gathered}\lambda_0\pi_0 = \mu_1\pi_1 \quad (\text{for } i = 0)\\ (\lambda_i + \mu_i)\pi_i = \mu_{i+1}\pi_{i+1} + \lambda_{i-1}\pi_{i-1}(i = 1, 2, \ldots)\end{gathered} \tag{2}$$

$\therefore$ Transition Matrix P for the given states is given by

$$\begin{gathered}P = \begin{bmatrix}-\lambda_0 & \mu_1 & 0 & 0\\ \lambda_0 & -(\lambda_1+\mu_1) & \mu_2 & 0\\ 0 & \lambda_1 & -(\lambda_2+\mu_2) & \mu_3\\ 0 & 0 & \lambda_2 & -\mu_3\end{bmatrix}\\ \implies P = \begin{bmatrix}-1 & 1 & 0 & 0\\ 1 & -2 & 1 & 0\\ 0 & 1 & -3 & 1\\ 0 & 0 & 2 & -1\end{bmatrix}\end{gathered} \tag{3}$$

Stationary distribution π satisfies,

$$\begin{gathered}P\pi = 0\\ \Sigma_{i=0}^{3}\pi_i = 1\end{gathered} \tag{4}$$

Converting P to echelon form we get,

$$\begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & -2 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{5}$$

From the above we get

$$\begin{aligned}\pi_3 &= 2\pi_2\\ \pi_2 &= \pi_1\\ \pi_1 &= \pi_0\\ &\text{from,}\\ \Sigma_{i=0}^{3}\pi_i &= 1\\ \pi_0 &= \frac{1}{5}\\ \therefore \pi &= \begin{bmatrix}\frac{1}{5}\\ \frac{1}{5}\\ \frac{1}{5}\\ \frac{2}{5}\end{bmatrix}\end{aligned} \tag{6}$$

Using the results obtained in equations from (5), we get

$$\therefore \pi_0 + 2\pi_1 + 3\pi_2 + 4\pi_3 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \pi = 2.80 \tag{7}$$