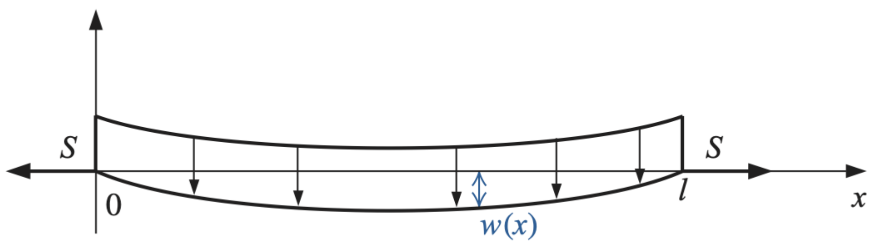
Project Description

We will experiment with solving linear systems using Gaussian Elimination (PA=LU) in a more realistic application and will observe some of the problems that might come with it.

Application A common problem in civil engineering concerns the deflection of a beam of rectangular cross-section subject to uniform loading while the ends of the beam are supported so that they undergo no deflection, like in a bridge.



Suppose that L, q, E, S, and I represent, respectively, the length of the beam, the intensity of the uniform load, the modulus of elasticity, the stress at the endpoints, and the central moment of inertia. The differential equation approximating the physical situation is of the form

$$\frac{d^2w}{dx^2}(x) = \frac{S}{EI}w(x) + \frac{qx}{2EI}(x - L) \tag{1}$$

where $w(x)$ is the deflection a distance x from the left end of the beam. Since no deflection occurs at the ends of the beam, there are two boundary conditions

$$w(0) = 0 \text{ and } w(L) = 0. \tag{2}$$

When the beam is of uniform thickness, the product EI is constant. In this case, the exact solution is easily obtained. When the thickness is not uniform, the moment of inertia I is a function of x, and approximation techniques are required.

The above system (1)-(2) is an example of a second-order linear boundary value problem (BVP). Discretization converts the differential equation model into a system of linear equations. The smaller the discretization size, the larger is the resulting system of equations. Here we will consider one of many discretization methods, namely the finite difference method. Techniques for discretizing derivatives are found in Section 5.1 of Sauer, where it will be shown that a reasonable approximation for the second derivative is

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} \tag{3}$$

for a small increment h. This formula is known as three-point centered Finite Difference for the second derivative. The discretization error of this approximation is proportional to h^2 (see equation (5.8) in Sauer). Our strategy will be to consider the beam as the union of many segments of length h, and to apply the discretized version of the differential equation on each segment.

For a positive integer n, set $h = L/n$. Consider the evenly spaced grid $0 = x_0 < x_1 < \cdots < x_n < x_n = L$, where $h = x_i - x_{i-1}$ for $i = 1, \ldots, n$. Replacing the differential equation (1) with the difference approximation (3) to get the system of linear equations for the displacements $w_i = w(x_i)$, $i = 0, \dots, n$ yields

$$\begin{aligned} \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} &= \frac{S}{EI} w_i + \frac{q x_i}{2EI}(x_i - L), \quad i = 1, \dots, n-1 \\ w_0 = w_n &= 0 \end{aligned} \tag{4}$$

Eliminating w_0 and w_n and some rearrangements lead to the linear system $A\mathbf{w} = \mathbf{b}$, where A is the $(n-1) \times (n-1)$ tridiagonal matrix

$$A = \begin{bmatrix} 2 + \frac{S}{EI}h^2 & -1 & & & \\ -1 & 2 + \frac{S}{EI}h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 + \frac{S}{EI}h^2 & -1 \\ & & & -1 & 2 + \frac{S}{EI}h^2 \end{bmatrix}, \tag{5}$$

$\mathbf{b}$ is the $(n-1)$-vector

$$\mathbf{b} = \frac{q}{2EI} h^2 \begin{bmatrix} x_1(L - x_1) \\ x_2(L - x_2) \\ \vdots \\ x_{n-1}(L - x_{n-1}) \end{bmatrix}, \tag{6}$$

and $\mathbf{w}$ is the $(n-1)$-vector of unknowns

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \end{bmatrix}.$$

Finally, we are ready to model the "clamped-clamped" beam (the ends of the beam are supported). Suppose the beam is a W10-type steel I-beam with the following characteristics: length $L = 120$ in, intensity of uniform load $q = 100$ lb/ft (i.e. $q = 100/12$ **lb/in**), modulus of elasticity $E = 3.0 \times 10^7$ lb/in^2, stress at ends $S = 1000$ lb, and central moment of inertia $I = 625$ in^4.

The boundary value problem (1)-(2) actually has a known analytical solution:

$$w(x) = c_1 e^{ax} + c_2 e^{-ax} + bx(x - L) + c, \tag{7}$$

where $c = -\frac{qEI}{S^2}$, $a = \sqrt{\frac{S}{EI}}$, $b = -\frac{q}{2S}$, $c_1 = c\,\frac{(1 - e^{-aL})}{(e^{-aL} - e^{aL})}$, and $c_2 = c\,\frac{(e^{aL} - 1)}{(e^{-aL} - e^{aL})}$.

Goals

1. Compare the approximate (finite difference) solution with the true solution when we increase the number of segments n.

2. Study the error at the middle of the beam and verify that it is proportional to h^2, as claimed above.

3. Verify that the condition number of the matrix A in (5) is proportional to h^{-2}.

To achieve our goals, we will run several levels of refinement with number of subintervals/segment $n = 2^{k+1}$, for $k = 1, \ldots, 20$ with length $h = L/n$. For each level, we will do the following:

1. Generate the matrix A and the vector $\mathbf{b}$ in (5).

2. Solve the system using Gaussian Elimination (PA=LU).

3. Calculate and store the error (true - approximate solution) at the middle of the beam.

4. For the two coarsest levels ($k = 1$ and $k = 2$ **only**), plot on **the same grid** the **solution** of the linear system and the **true solution**. (one figure per level)

5. For **all levels** (values of k), plot the **error in the solution** along the beam (x axis).

6. Calculate and store the condition number of the matrix A.