

Lecture 2

Camera Models



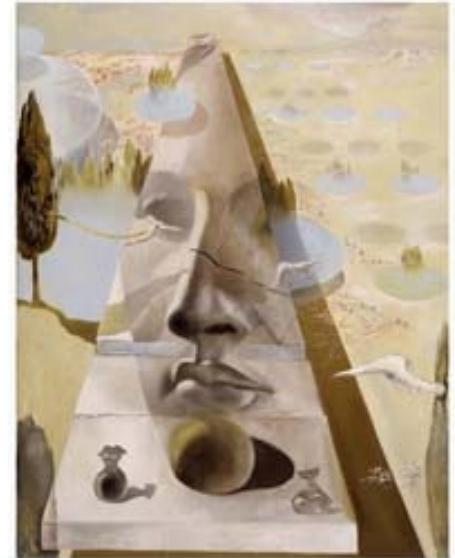
1891

Professor Silvio Savarese

Computational Vision and Geometry Lab

Lecture 2

Camera Models

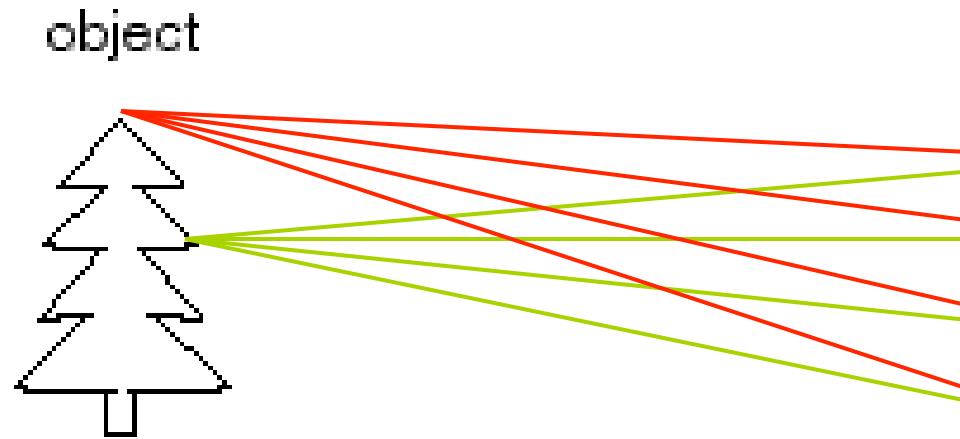


- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras

Reading: **[FP]** Chapter 1, “Geometric Camera Models”
[HZ] Chapter 6 “Camera Models”

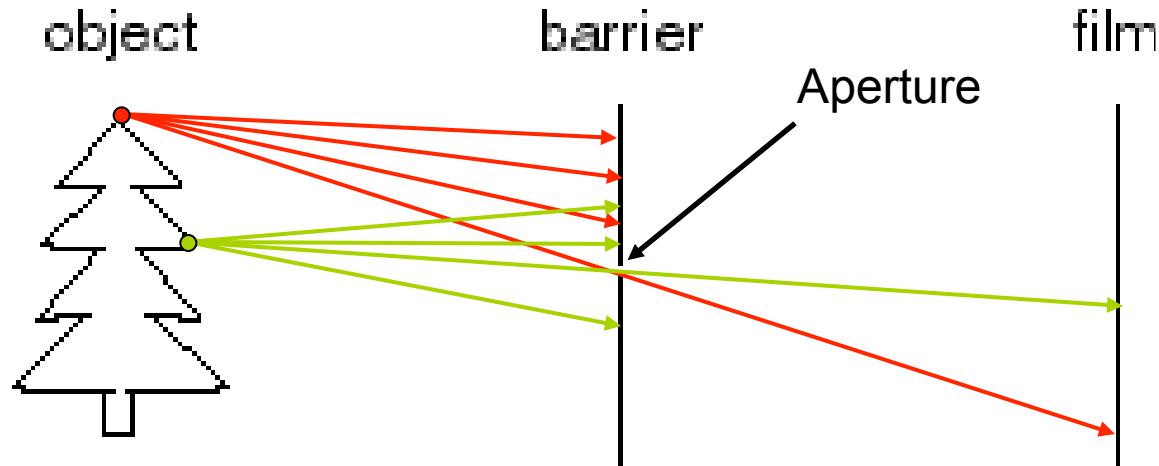
Some slides in this lecture are courtesy to Profs. J. Ponce, S. Seitz, F-F Li

How do we see the world?



- Let's design a camera
 - Idea 1: put a piece of film in front of an object
 - Do we get a reasonable image?

Pinhole camera

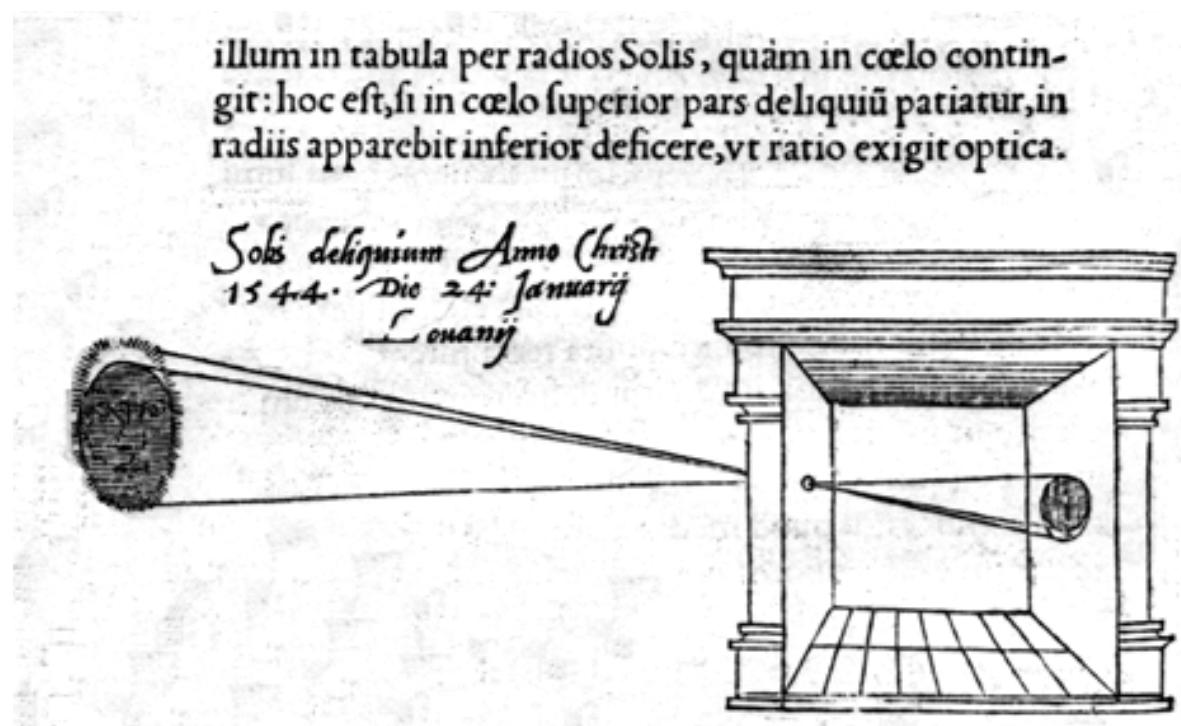


- Add a barrier to block off most of the rays
 - This reduces blurring
 - The opening known as the **aperture**

Some history...

Milestones:

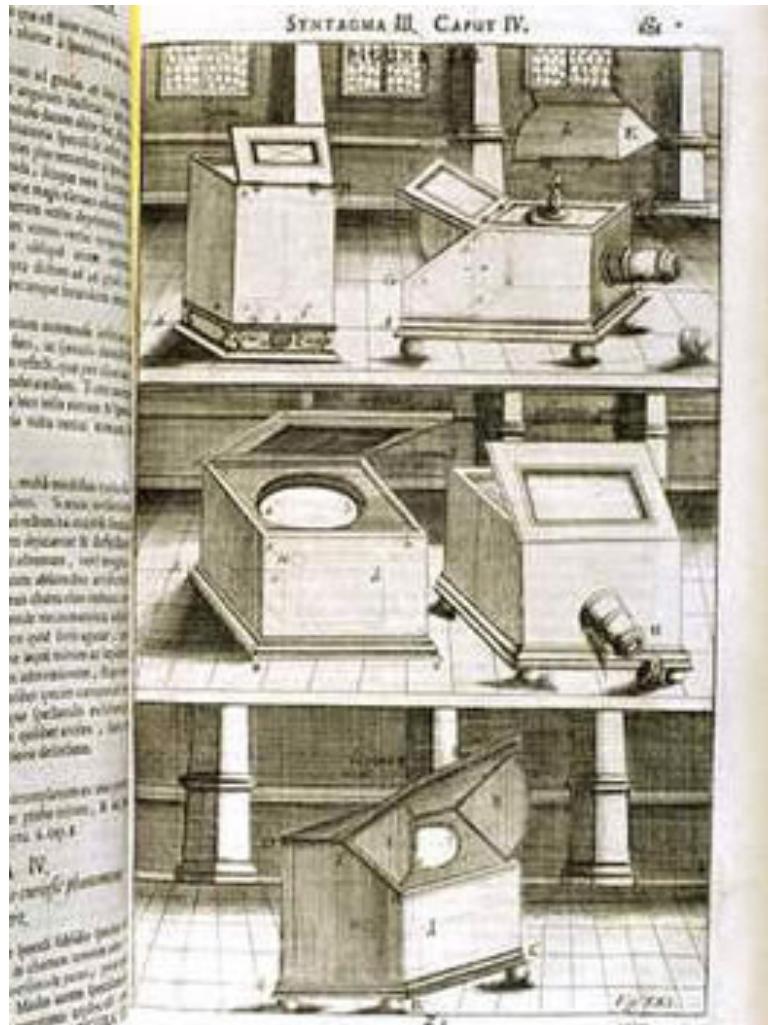
- Leonardo da Vinci (1452-1519):
first record of camera *obscura* (1502)



Some history...

Milestones:

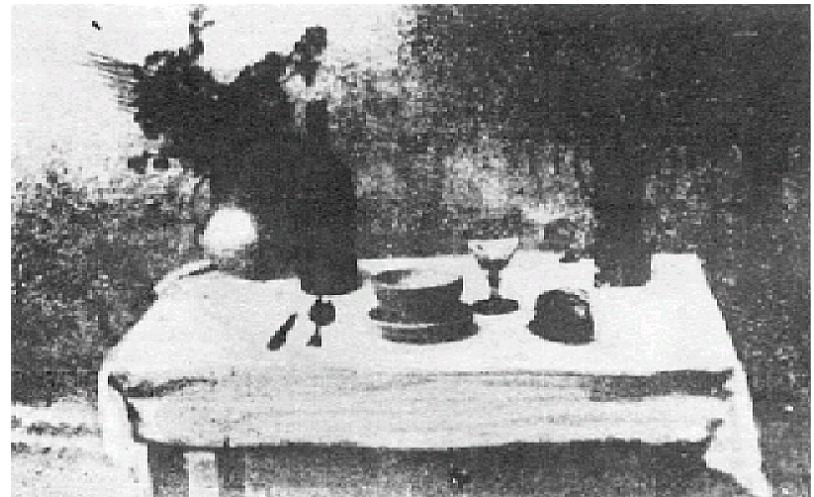
- Leonardo da Vinci (1452-1519): first record of camera *obscura*
- Johann Zahn (1685): first portable camera



Some history...

Milestones:

- Leonardo da Vinci (1452-1519): first record of camera *obscura*
- Johann Zahn (1685): first portable camera
- Joseph Nicéphore Niépce (1822): first photo - birth of photography

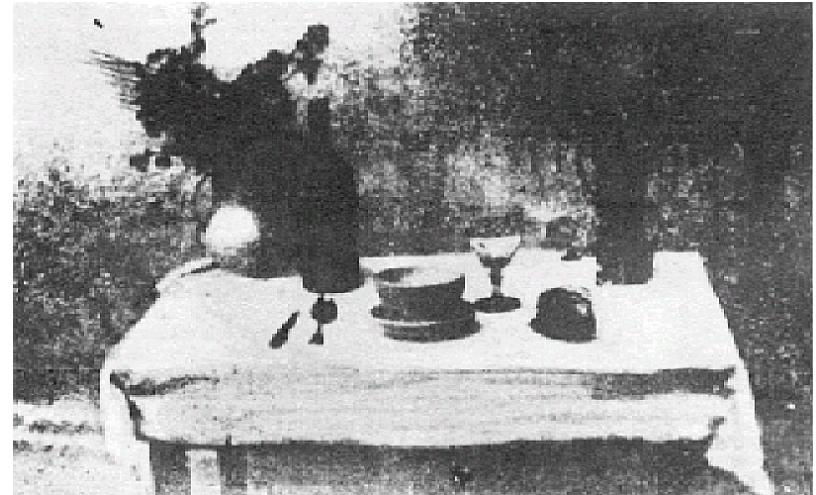


Photography (Niépce, "La Table Servie," 1822)

Some history...

Milestones:

- Leonardo da Vinci (1452-1519): first record of camera *obscura*
- Johann Zahn (1685): first portable camera
- Joseph Nicéphore Niépce (1822): first photo - birth of photography
 - Daguerreotypes (1839)
 - Photographic Film (Eastman, 1889)
 - Cinema (Lumière Brothers, 1895)
 - Color Photography (Lumière Brothers, 1908)



Photography (Niépce, “La Table Servie,” 1822)

Let's also not forget...



Motzu
(468-376 BC)

Oldest existent
book on geometry
in China

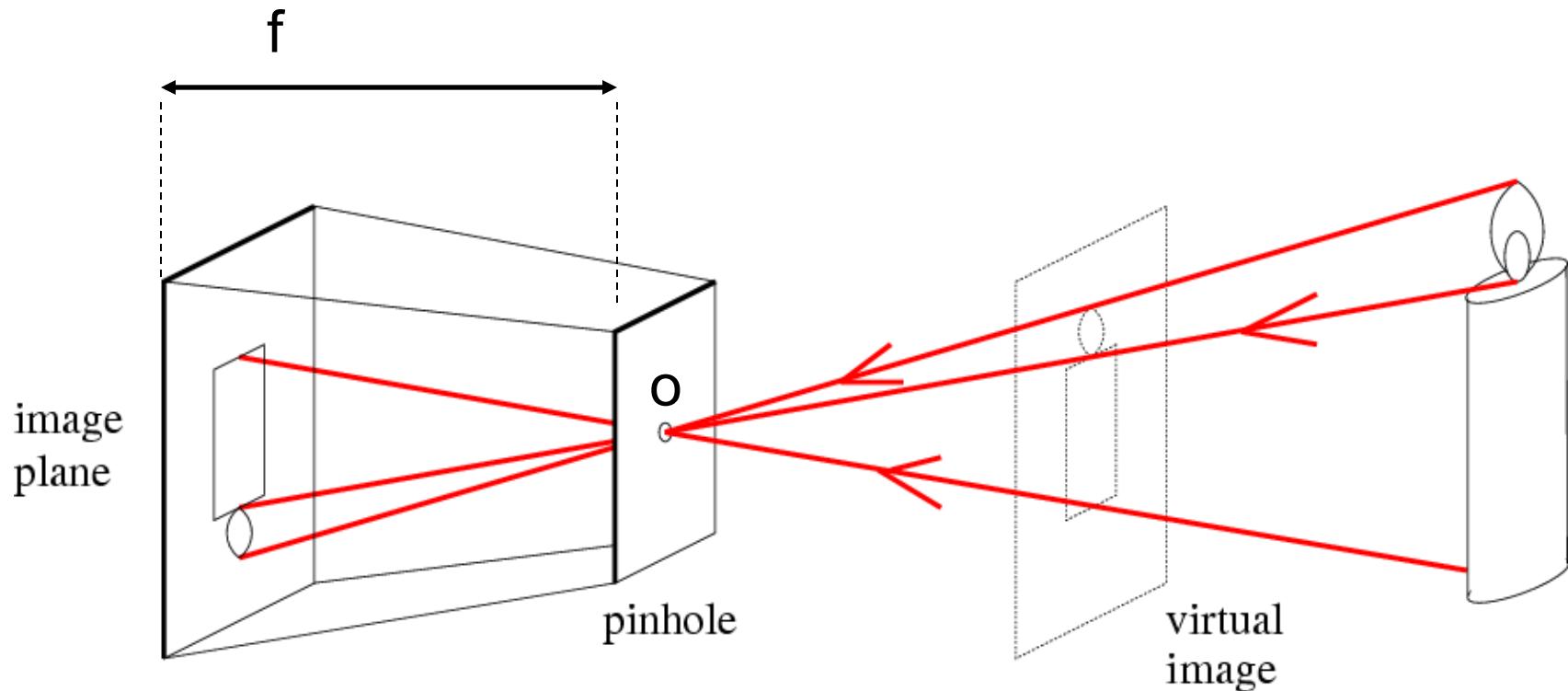


Aristotle
(384-322 BC)
Also: Plato, Euclid



Al-Kindi (c. 801–873)
Ibn al-Haitham
(965–1040)

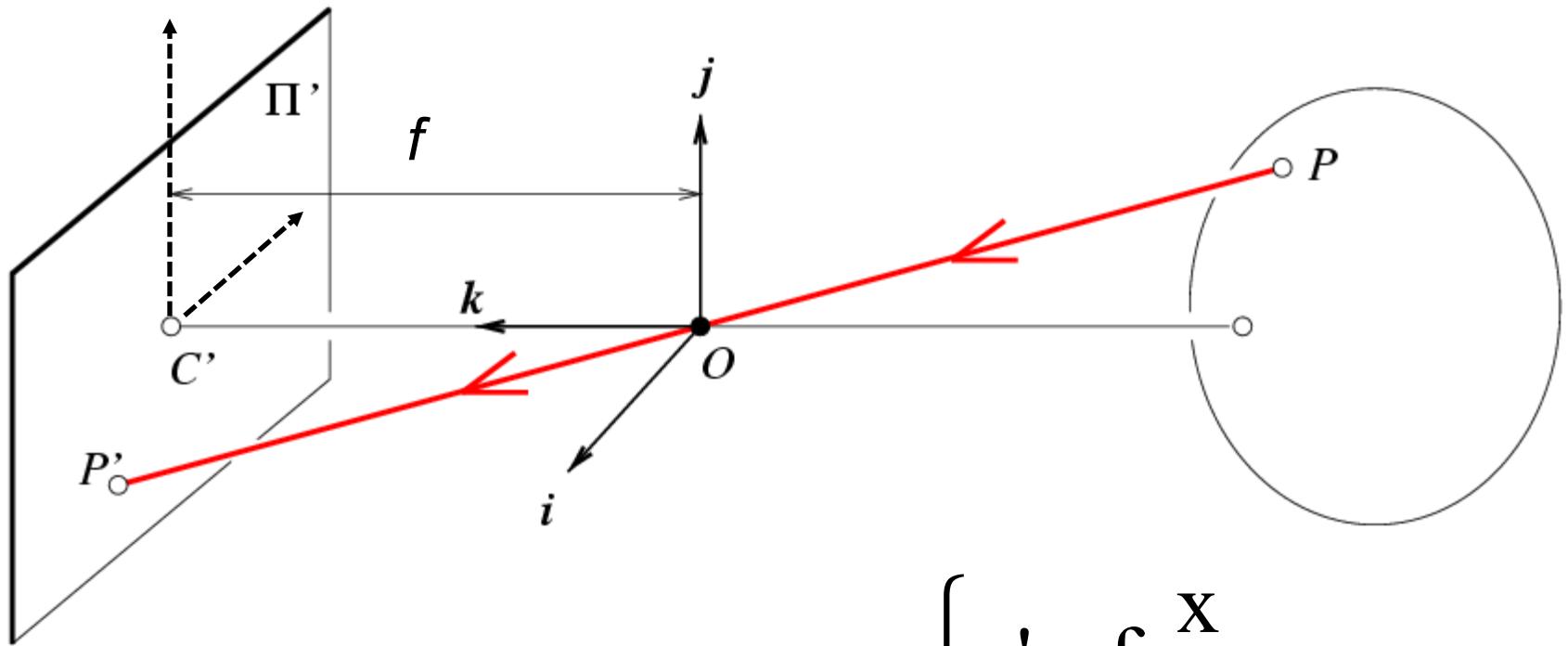
Pinhole camera



f = focal length

o = aperture = pinhole = center of the camera

Pinhole camera

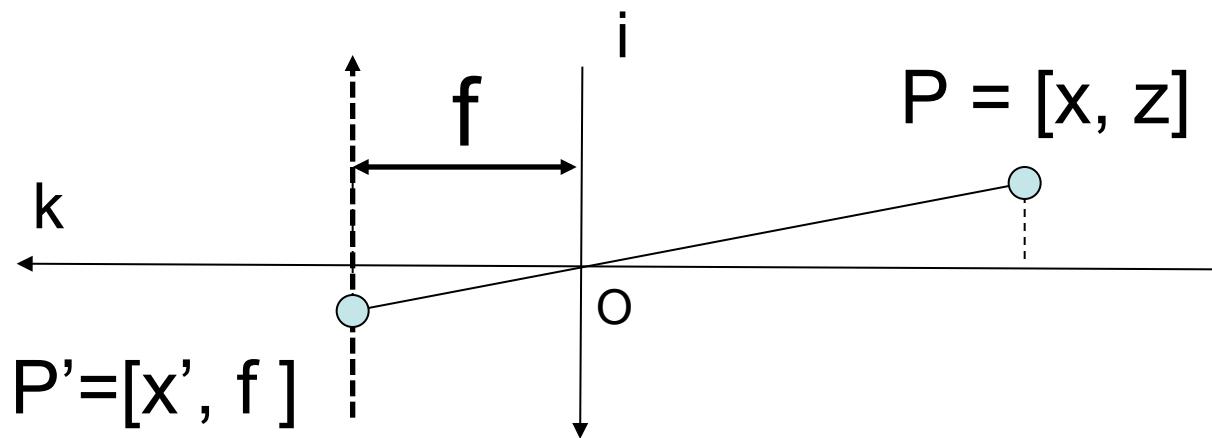
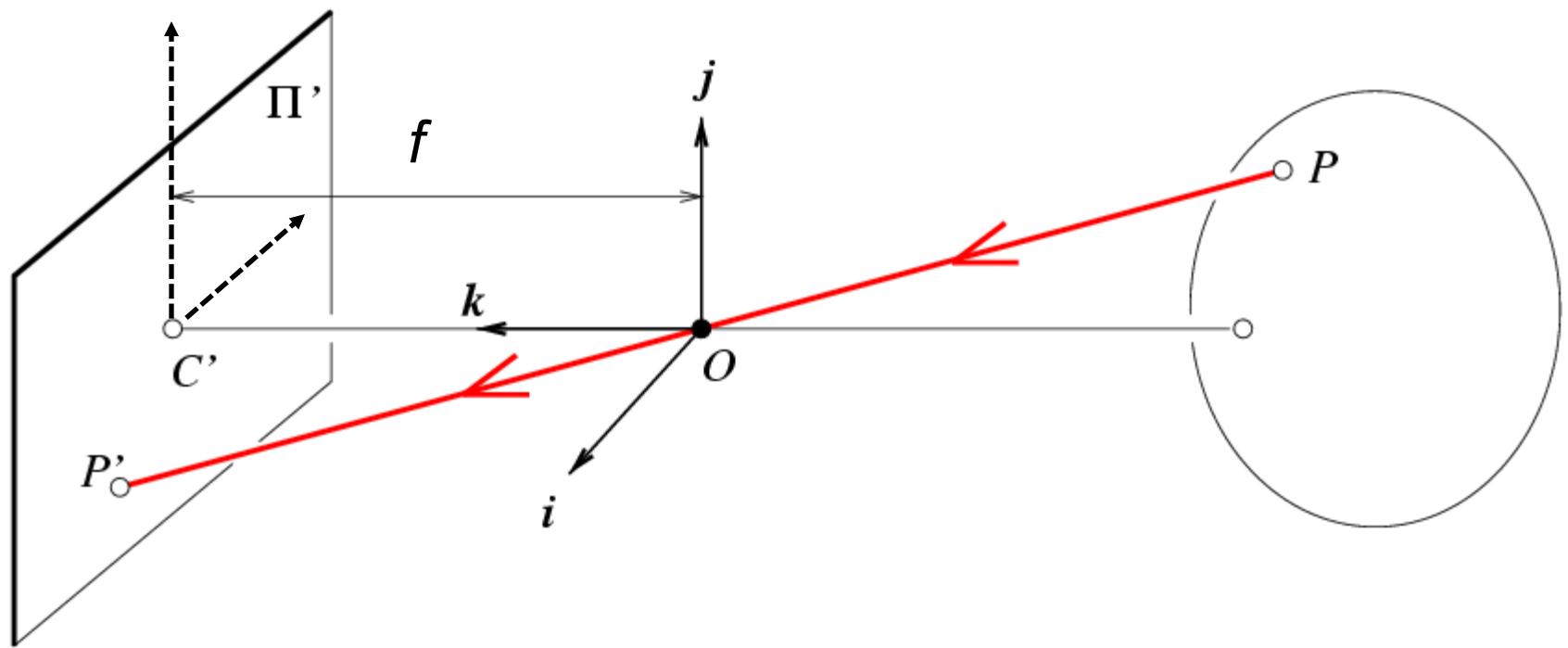


$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\left\{ \begin{array}{l} x' = f \frac{x}{z} \\ y' = f \frac{y}{z} \end{array} \right. \quad [\text{Eq. 1}]$$

Derived using similar triangles

Pinhole camera

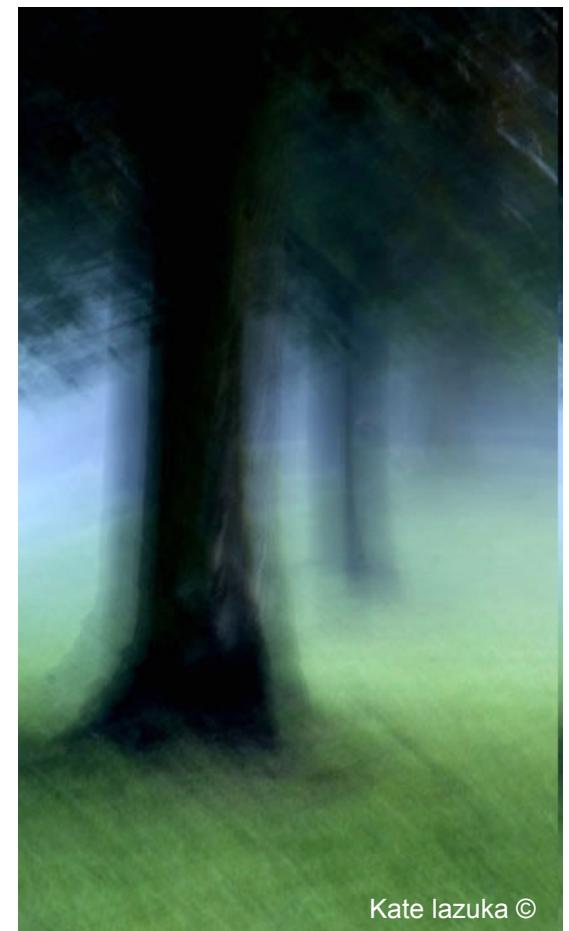
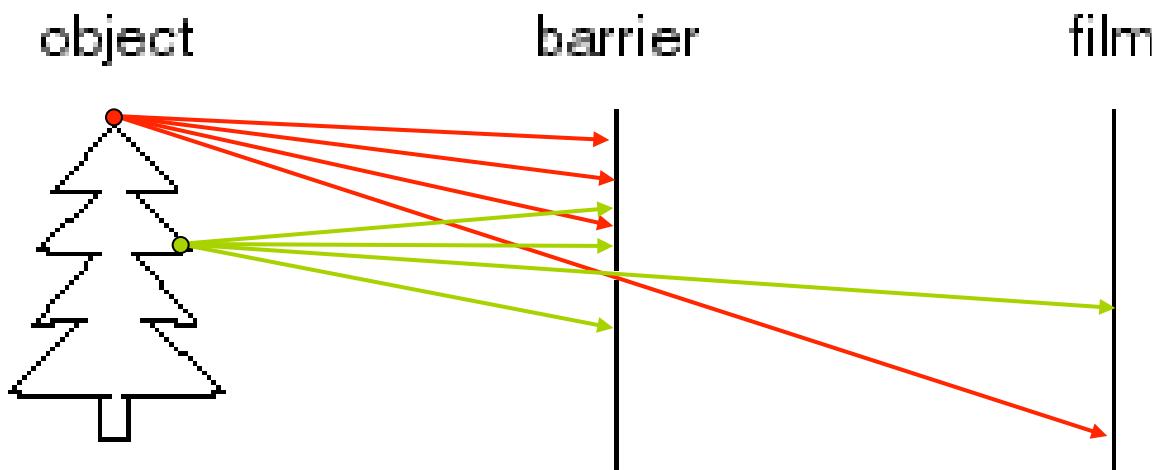


[Eq. 2]

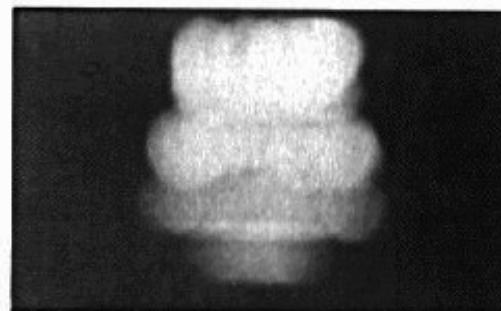
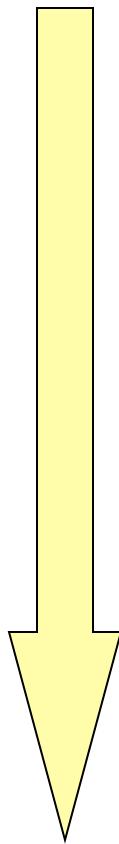
$$\frac{x'}{f} = \frac{x}{z}$$

Pinhole camera

Is the size of the aperture important?



Shrinking
aperture
size



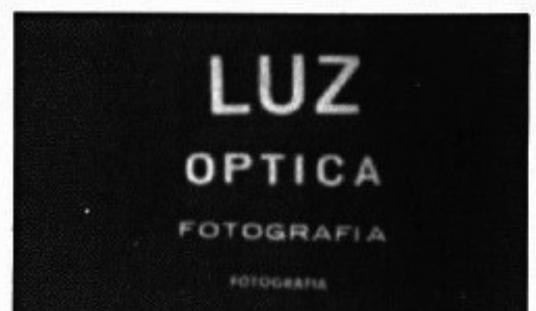
2 mm



1 mm



0.6mm



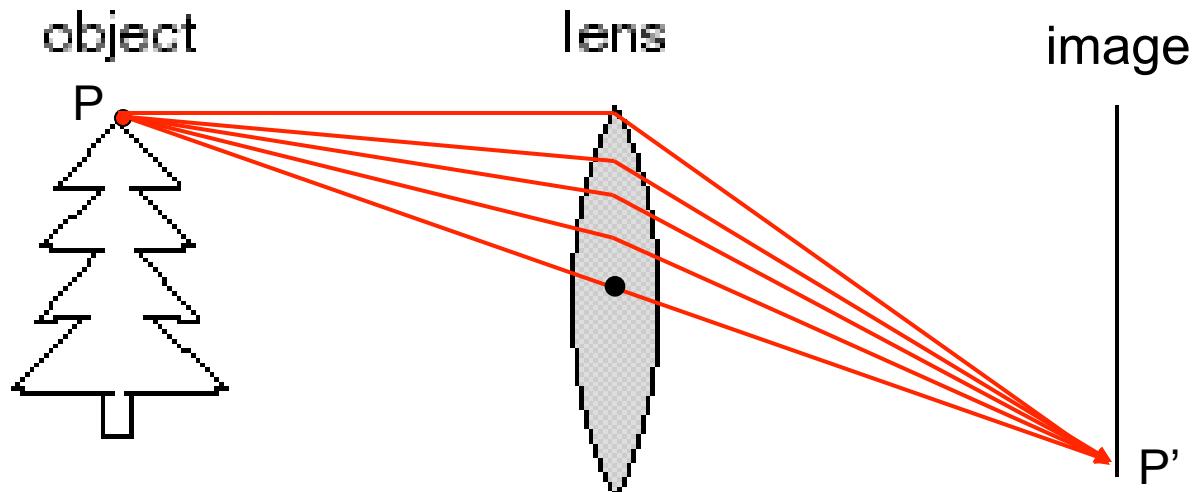
0.35 mm

-What happens if the aperture is too small?

-Less light passes through

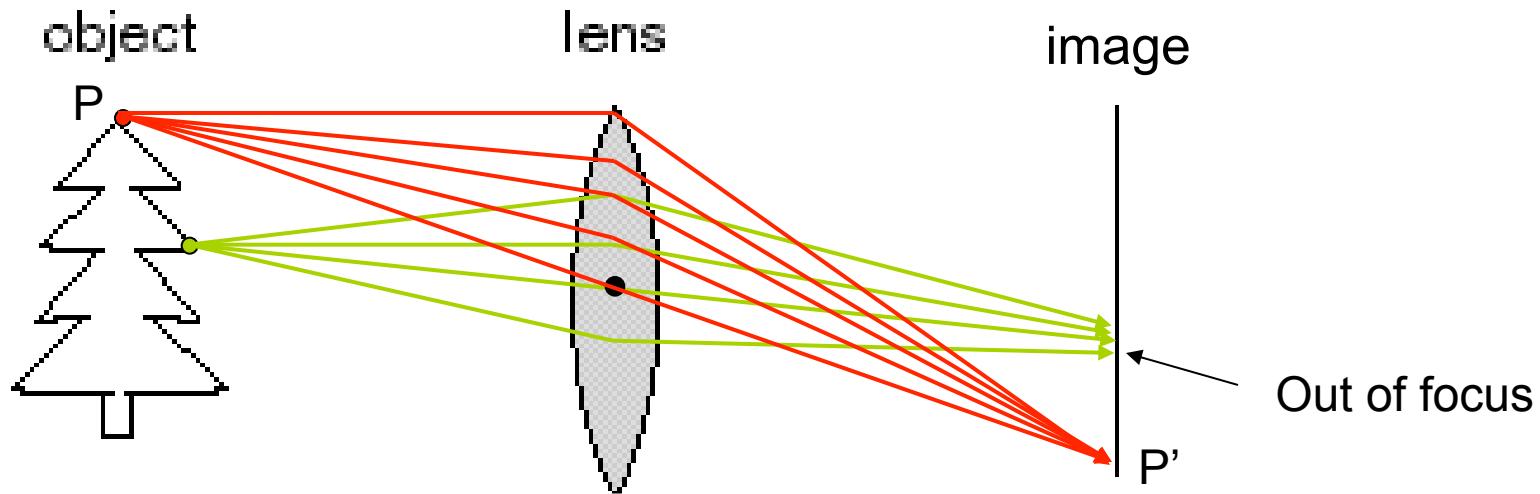
Adding lenses!

Cameras & Lenses



- A lens focuses light onto the film

Cameras & Lenses



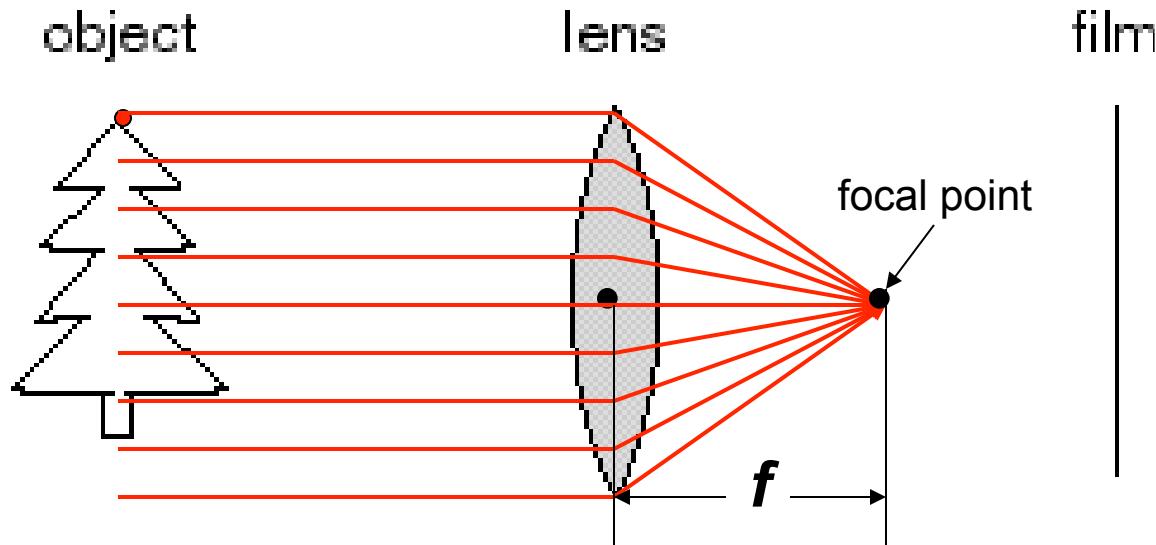
- A lens focuses light onto the film
 - There is a specific distance at which objects are “in focus”
 - Related to the concept of depth of field

Cameras & Lenses



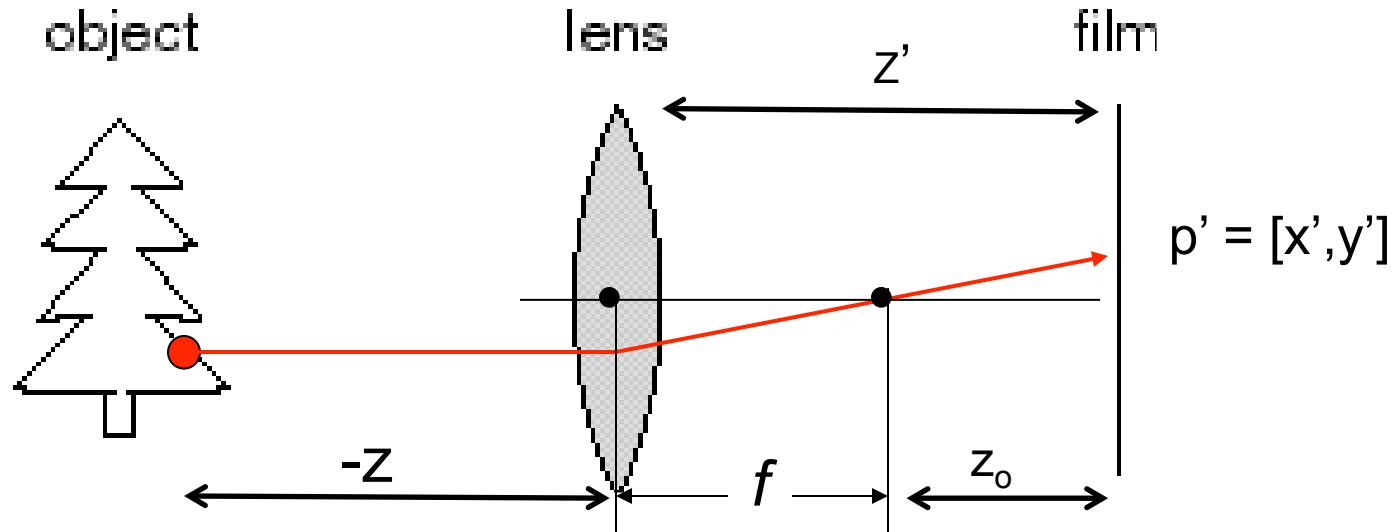
- A lens focuses light onto the film
 - There is a specific distance at which objects are “in focus”
 - Related to the concept of depth of field

Cameras & Lenses



- A lens focuses light onto the film
 - All rays parallel to the optical (or principal) axis converge to one point (the *focal point*) on a plane located at the *focal length* f from the center of the lens.
 - Rays passing through the center are not deviated

Paraxial refraction model



[Eq. 4]

From Snell's law:

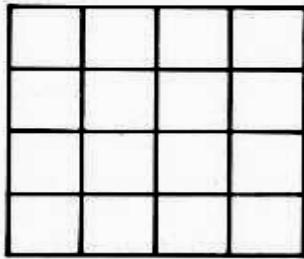
[Eq. 3]

$$\begin{cases} x' = z' \frac{x}{z} \\ y' = z' \frac{y}{z} \end{cases}$$

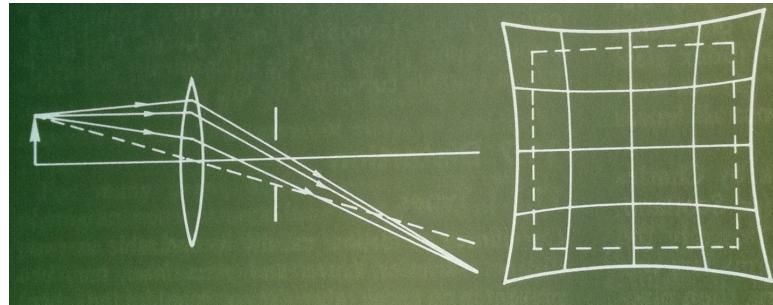
$$z' = f + z_0$$
$$f = \frac{R}{2(n - 1)}$$

Issues with lenses: Radial Distortion

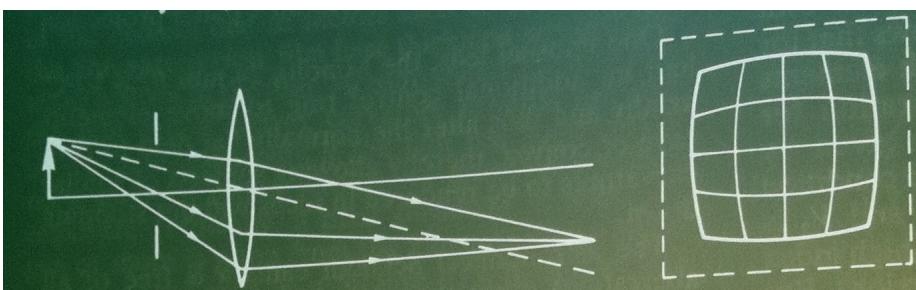
- Deviations are most noticeable for rays that pass through the edge of the lens



No distortion



Pin cushion



Barrel (fisheye lens)

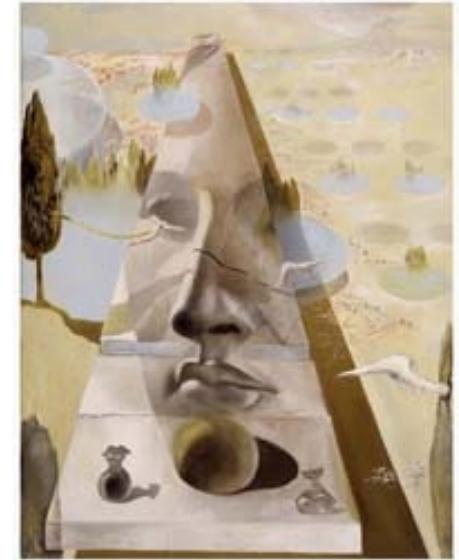


Image magnification decreases with distance from the optical axis

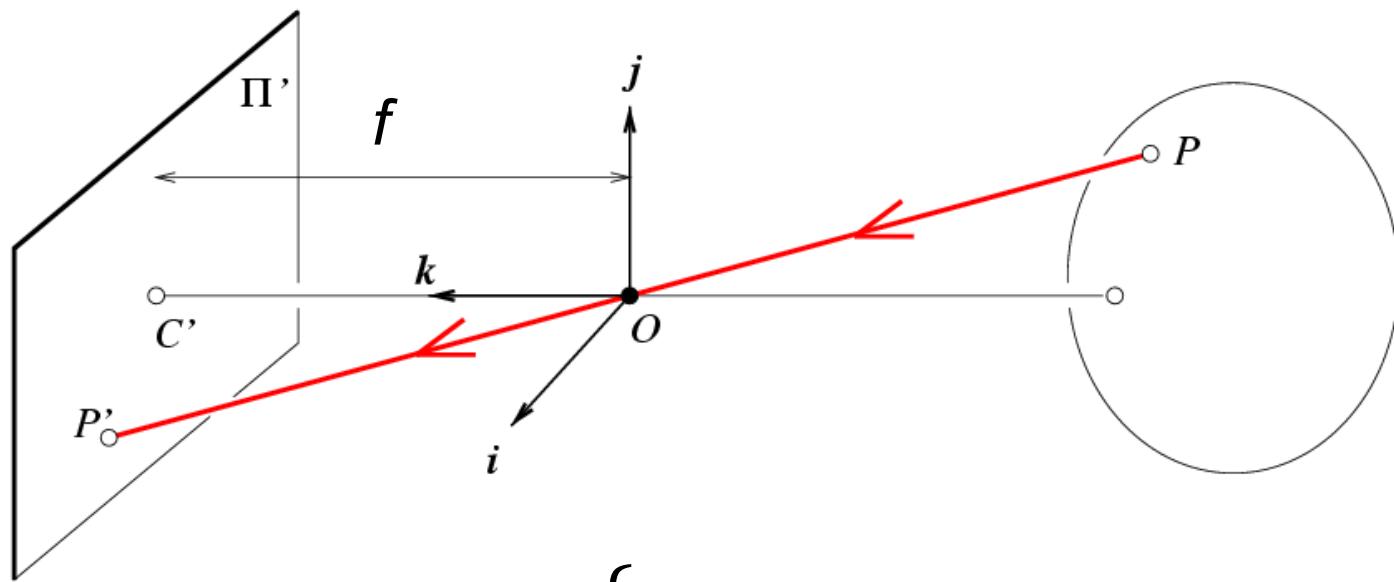
Lecture 2

Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
 - Intrinsic
 - Extrinsic



Pinhole camera



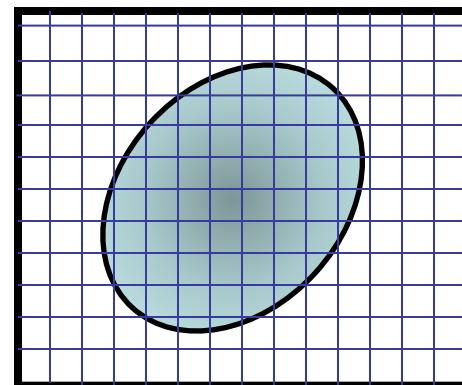
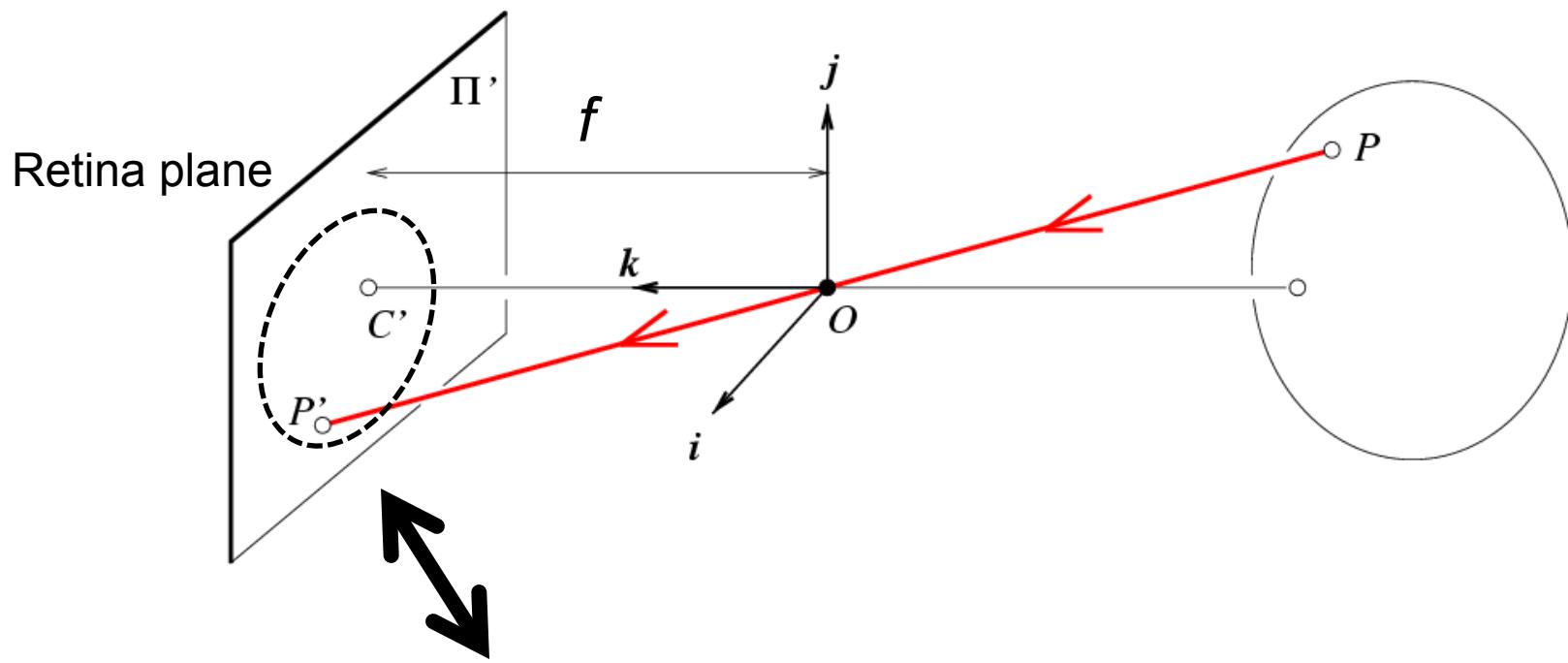
$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \left\{ \begin{array}{l} x' = f \frac{x}{z} \\ y' = f \frac{y}{z} \end{array} \right. \quad \Re^3 \xrightarrow{E} \Re^2$$

[Eq. 1]

f = focal length

O = center of the camera

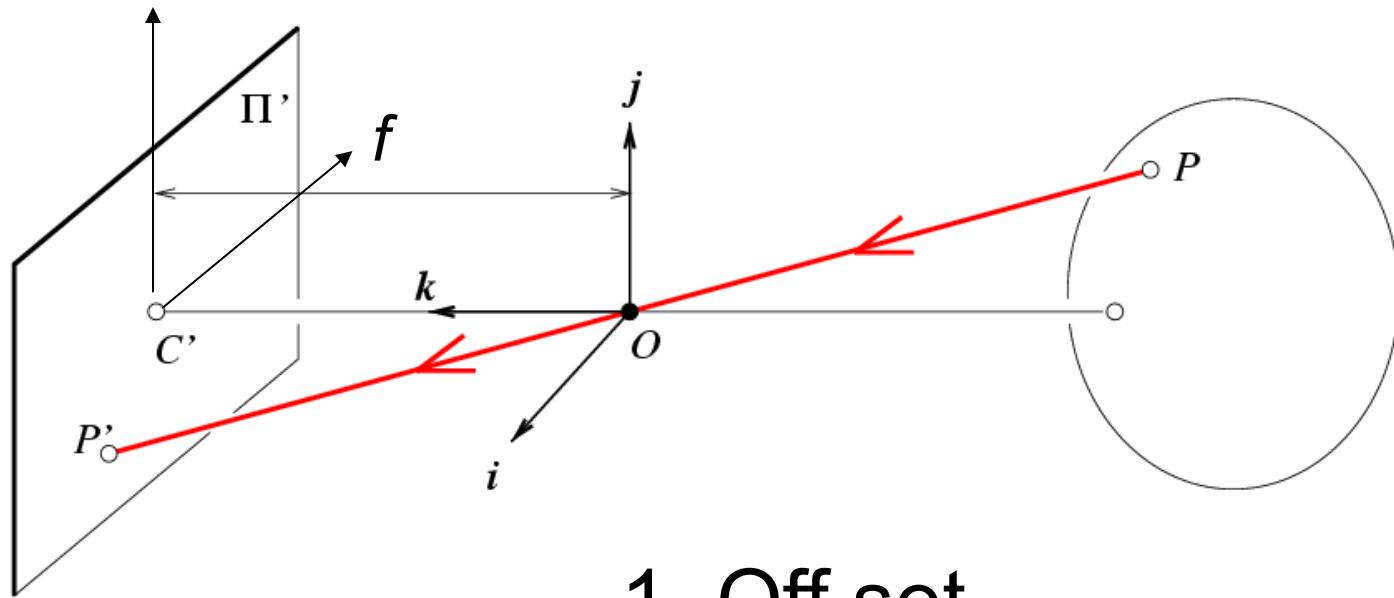
From retina plane to images



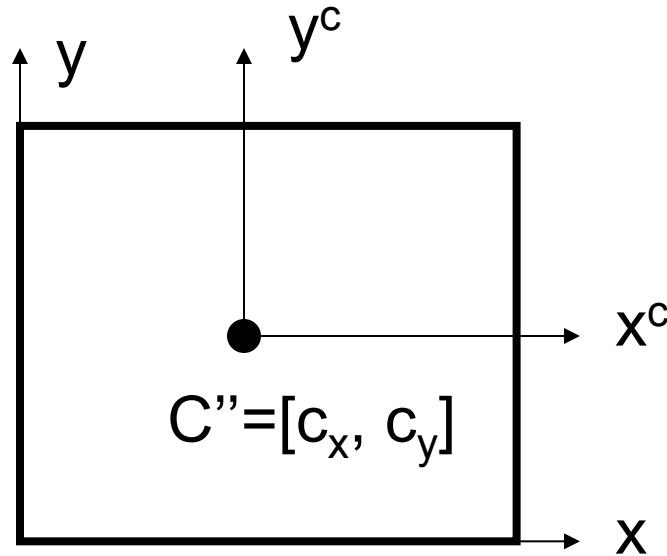
Digital image

Pixels, bottom-left coordinate systems

Coordinate systems



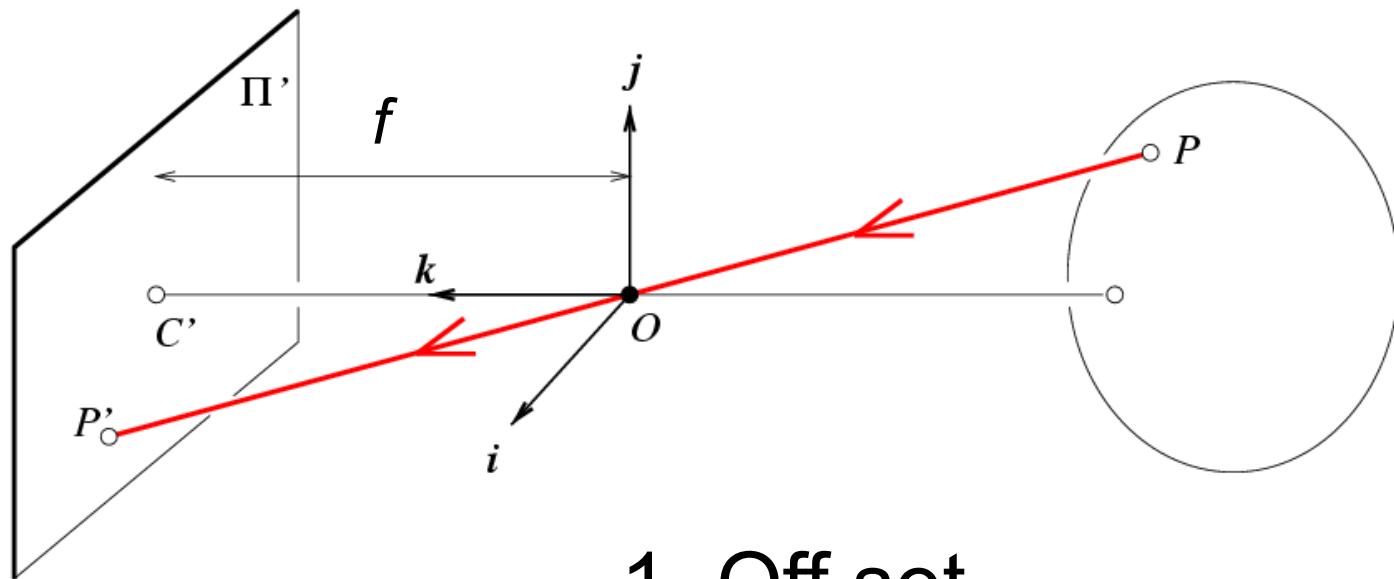
1. Off set



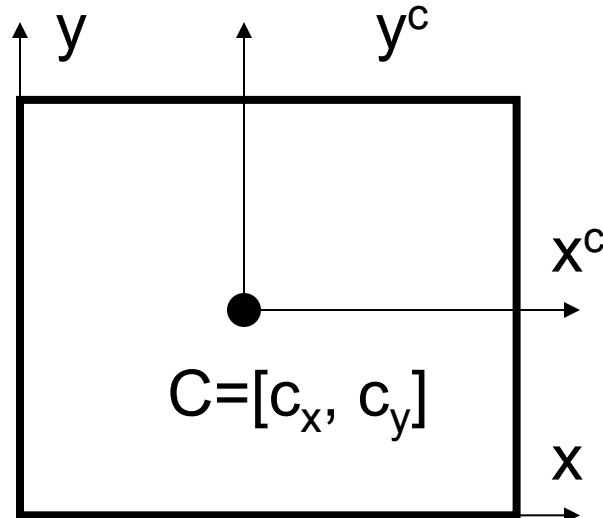
$$(x, y, z) \rightarrow \left(f \frac{x}{z} + c_x, f \frac{y}{z} + c_y \right)$$

[Eq. 5]

Converting to pixels



1. Off set
2. From metric to pixels

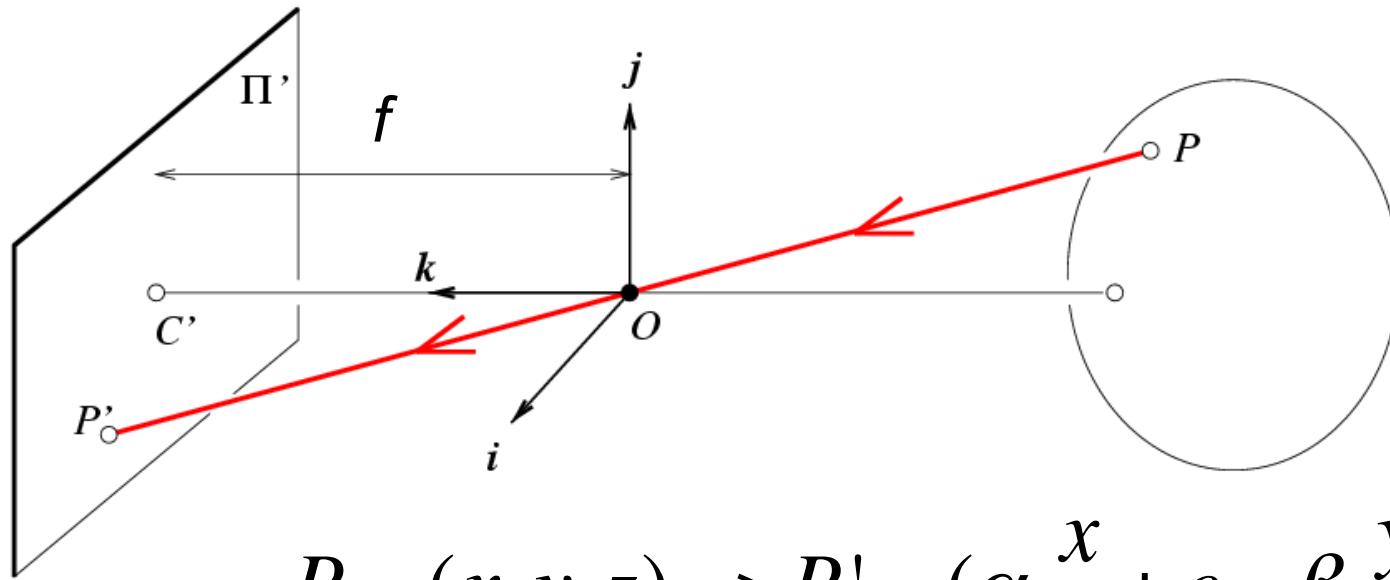


$$(x, y, z) \rightarrow \left(\frac{f}{\alpha} \frac{x}{z} + c_x, \frac{f}{\beta} \frac{y}{z} + c_y \right) \quad [\text{Eq. 6}]$$

Units: k, l : pixel/m
 f : m

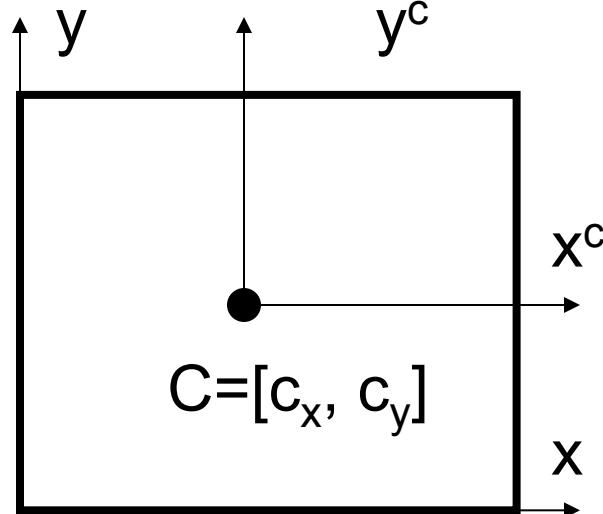
Non-square pixels
 α, β : pixel

Is this projective transformation linear?



$$P = (x, y, z) \rightarrow P' = (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)$$

[Eq. 7]



- Is this a linear transformation?
No — division by z is nonlinear
- Can we express it in a matrix form?

Homogeneous coordinates

E → H

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image
coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene
coordinates

- Converting back *from* homogeneous coordinates

H → E

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Projective transformation in the homogenous coordinate system

$$P_h^{-1} = \begin{bmatrix} \alpha & x + c_x z \\ \beta & y + c_y z \\ z \end{bmatrix} = \boxed{\begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \boxed{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}} \quad P_h$$

[Eq.8]

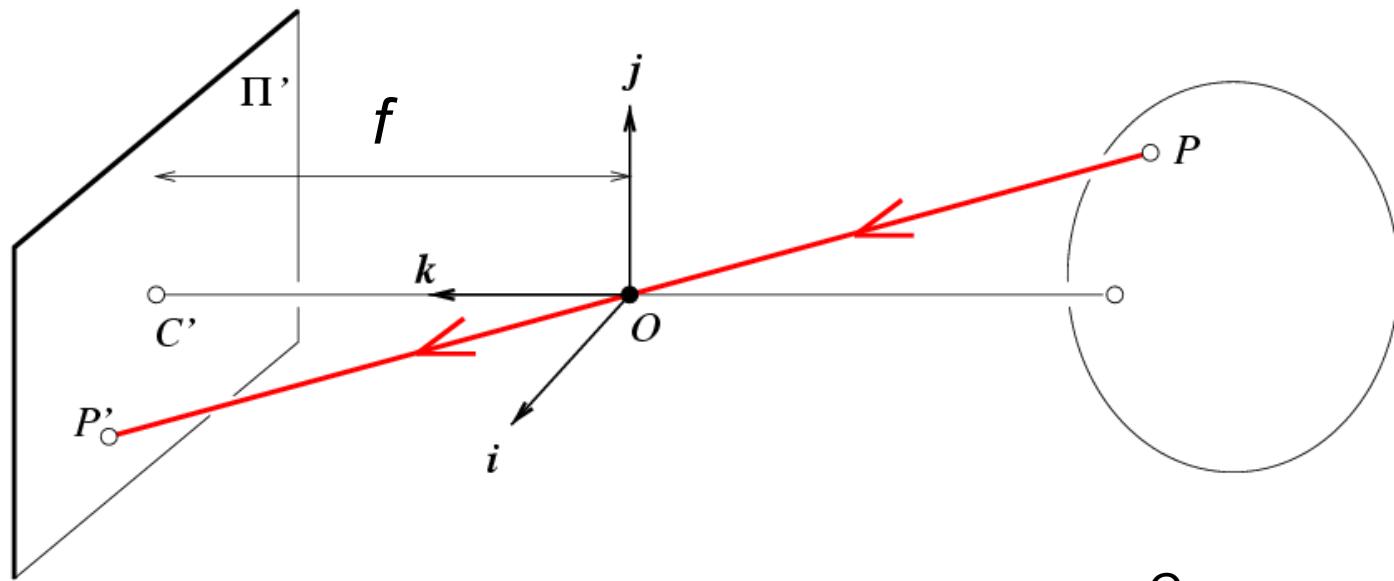
Homogenous

Euclidian

$$P_h \rightarrow P = (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)$$

$$M = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Camera Matrix



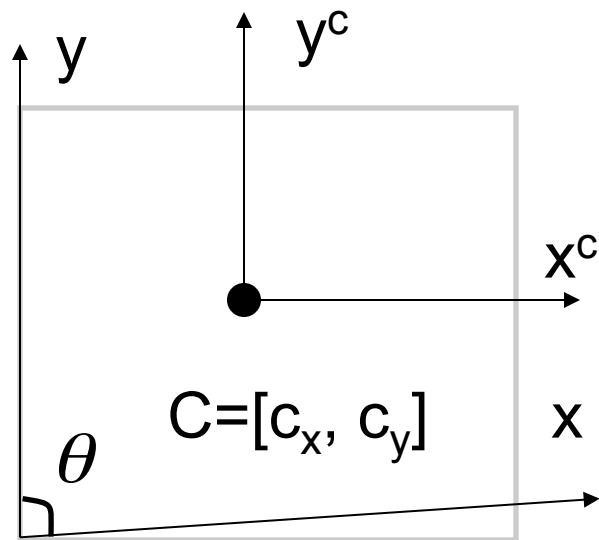
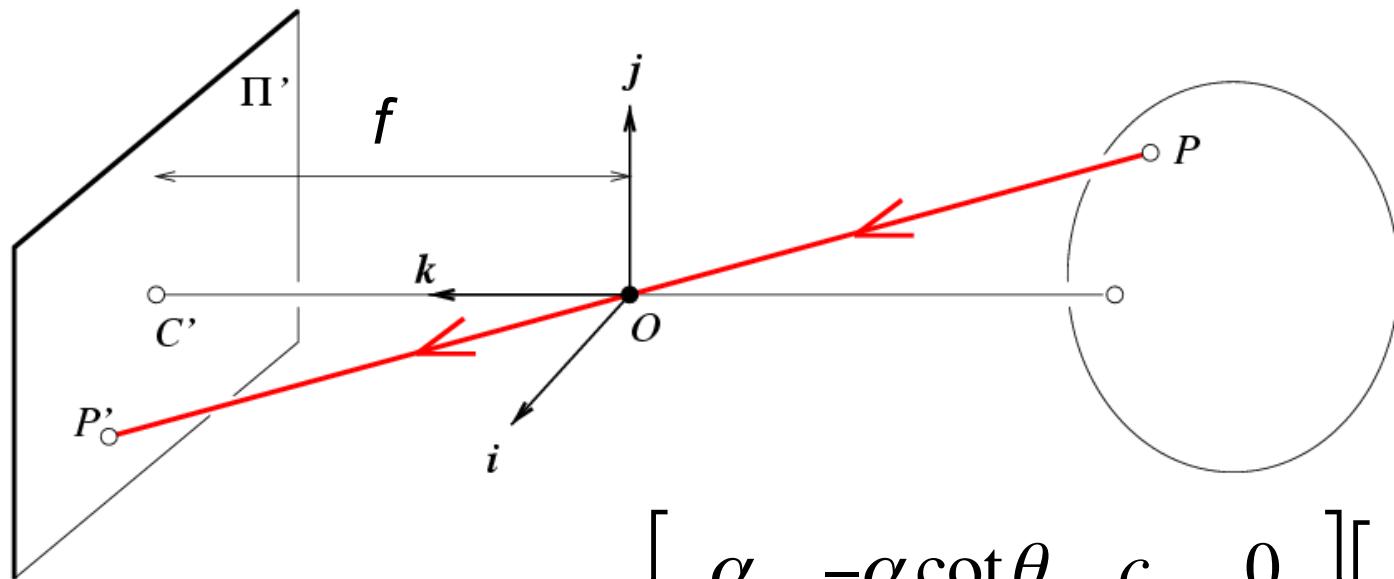
[Eq.9]

$$\begin{aligned}P' &= M P \\&= K \begin{bmatrix} I & 0 \end{bmatrix} P\end{aligned}$$

Camera matrix K

$$P' = \begin{bmatrix} \alpha & 0 & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Camera Skewness



$$P' = \begin{bmatrix} \alpha & -\alpha \cot \theta & c_x & 0 \\ 0 & \frac{\beta}{\sin \theta} & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How many degrees does K have?
5 degrees of freedom!

Canonical Projective Transformation

$$P' = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_M \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad [\text{Eq.10}]$$
$$P' = M P$$
$$\mathfrak{R}^4 \xrightarrow{H} \mathfrak{R}^3$$

$$P_i' = \begin{bmatrix} \frac{x}{z} \\ \frac{y}{z} \end{bmatrix}$$

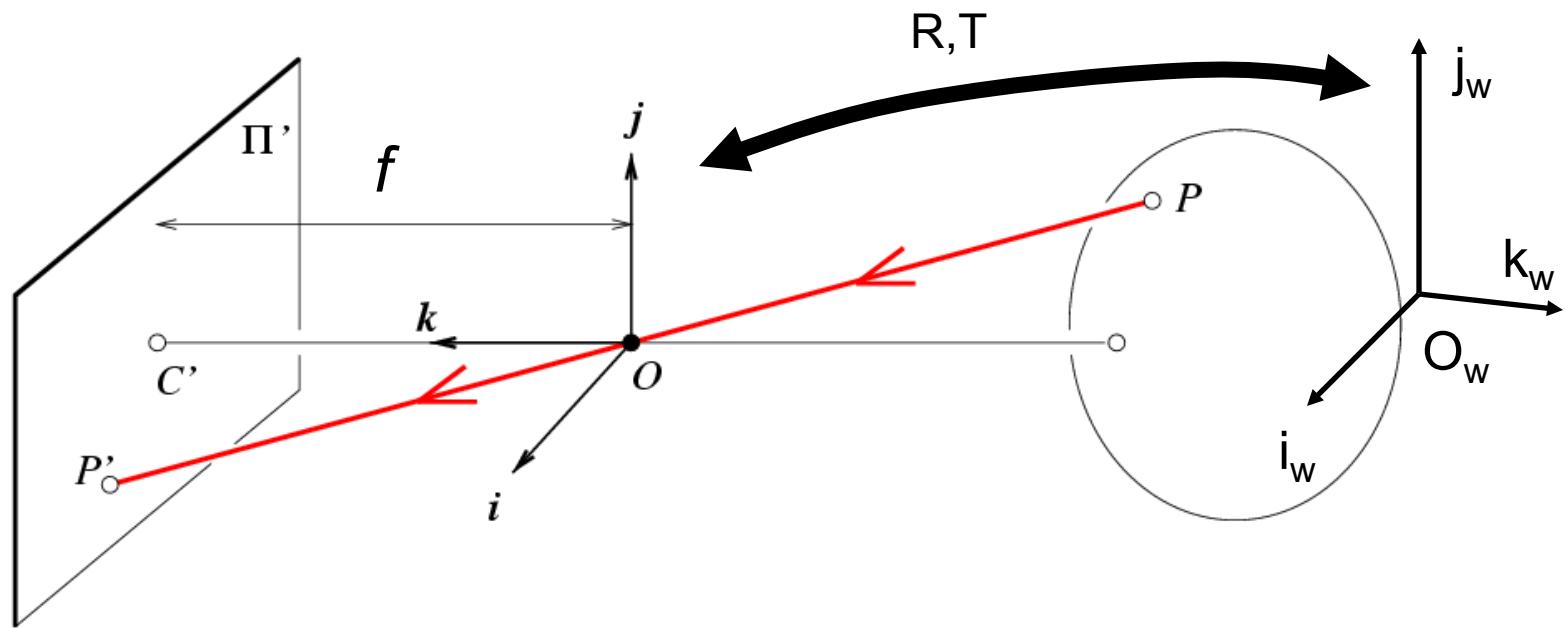
Lecture 2

Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
 - Intrinsic
 - Extrinsic
- Other camera models



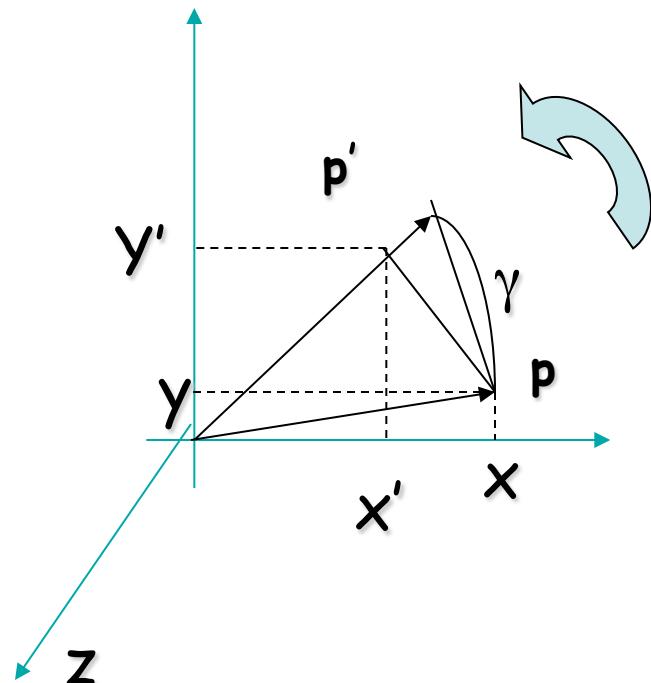
World reference system



- The mapping so far is defined within the camera reference system
- What if an object is represented in the world reference system?
- Need to introduce an additional mapping from world ref system to camera ref system

3D Rotation of Points

Rotation around the coordinate axes,
counter-clockwise:



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

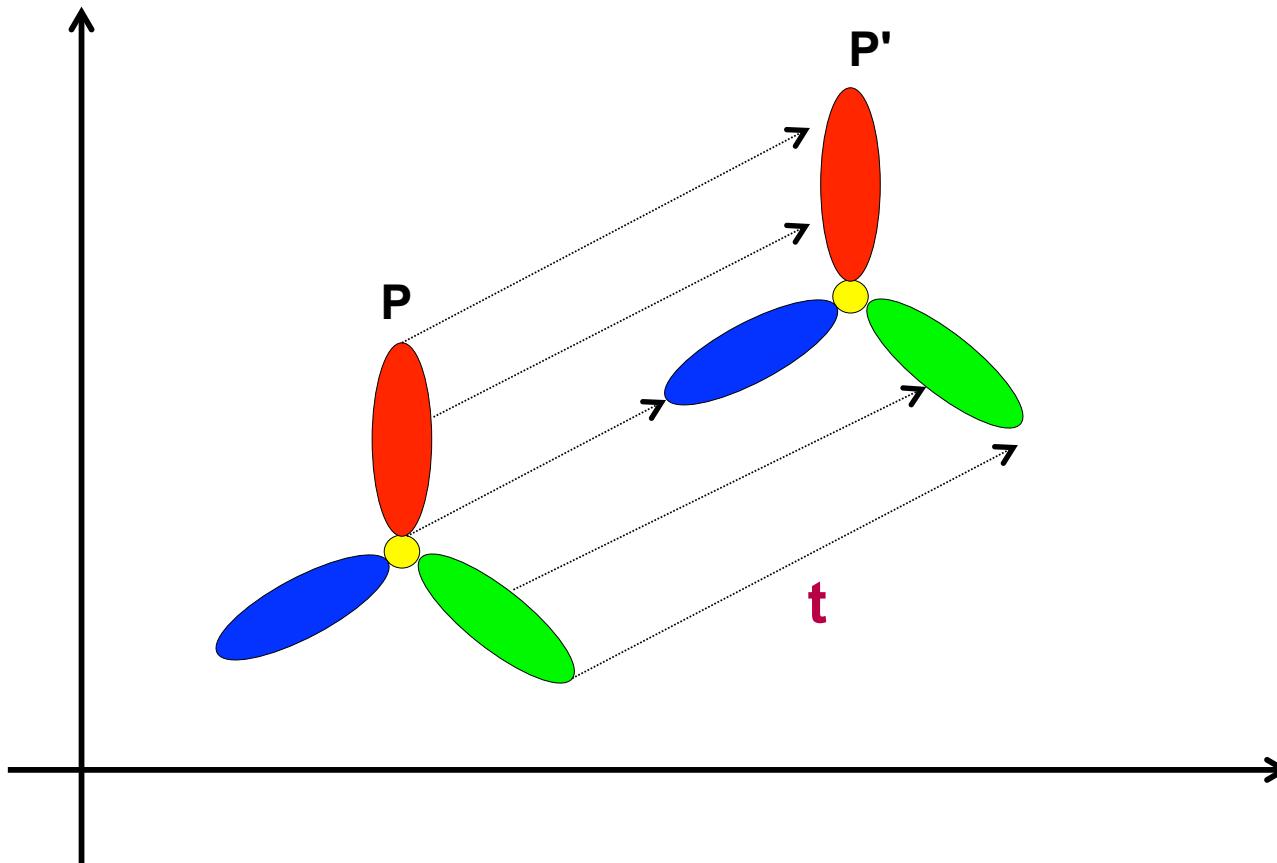
$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

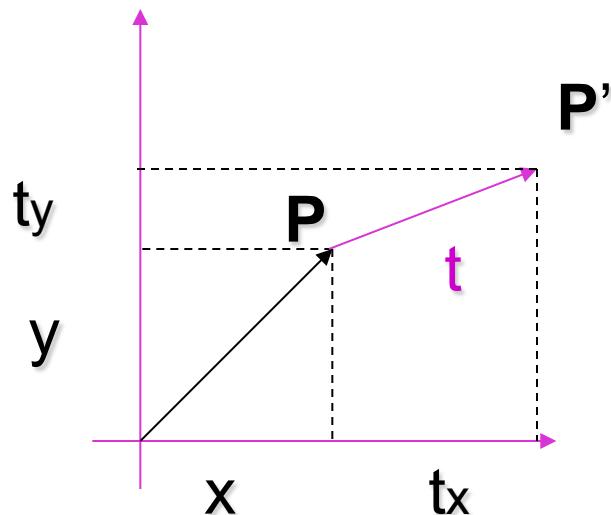
A rotation matrix in 3D has 3 degree of freedom

Please refer to CA session
on transformations for
more details

2D Translation



2D Translation Equation

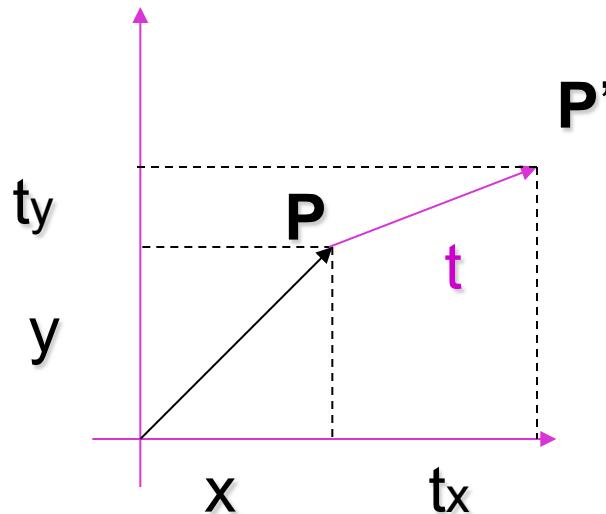


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (x + t_x, y + t_y)$$

2D Translation using Homogeneous Coordinates

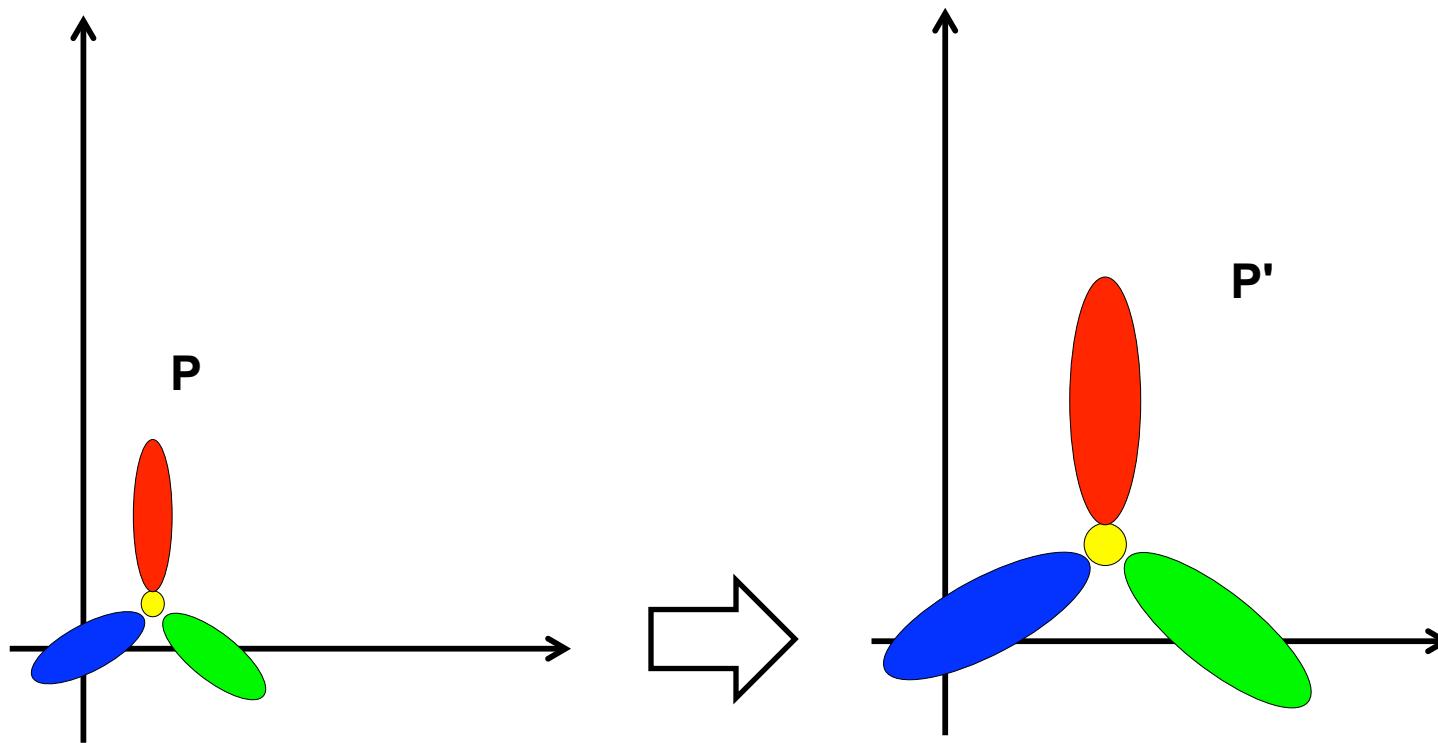


$$P = (x, y) \rightarrow (x, y, 1)$$

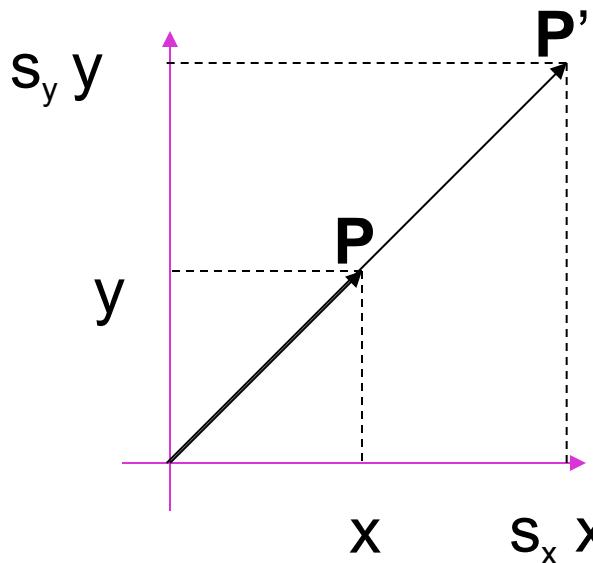
$$P' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} I & t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling



Scaling Equation

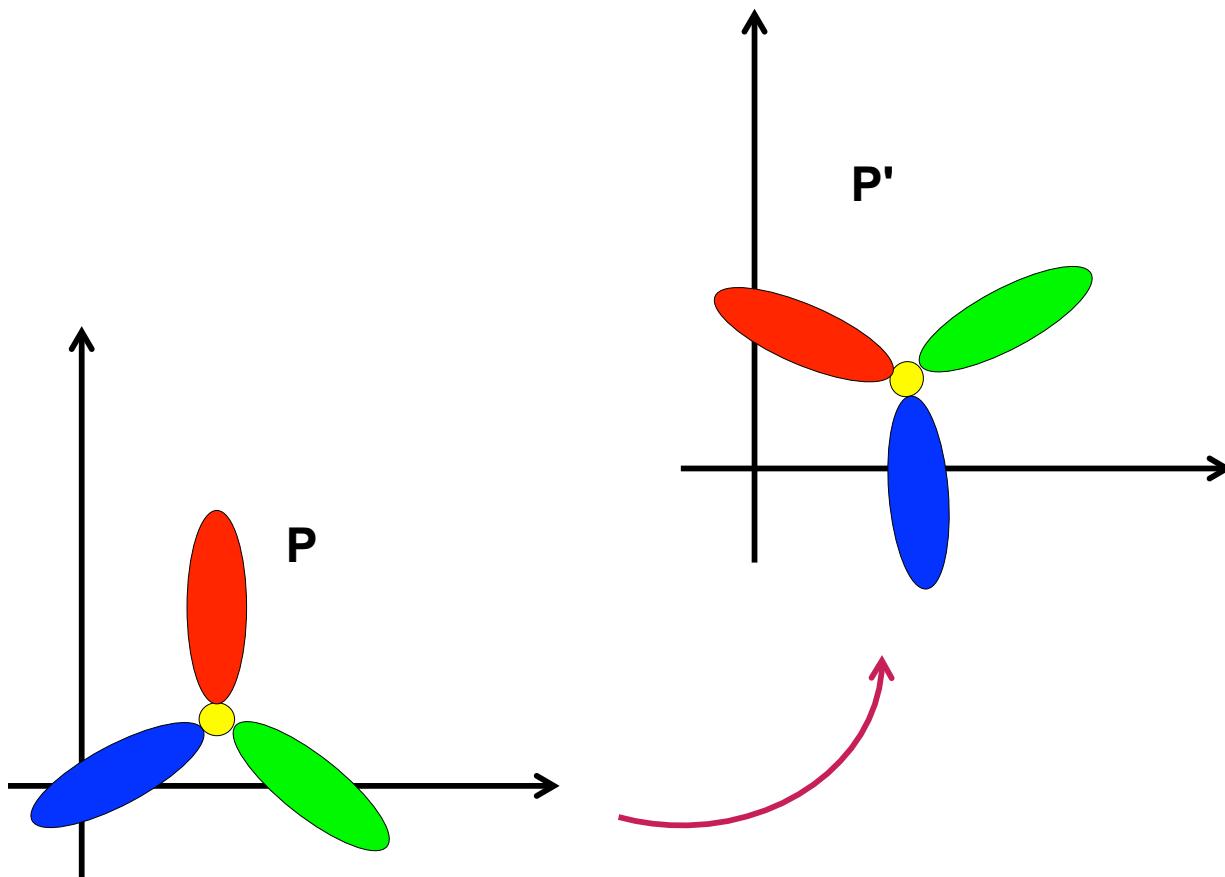


$$P = (x, y) \rightarrow P' = (s_x x, s_y y)$$

$$P = (x, y) \rightarrow (x, y, 1)$$

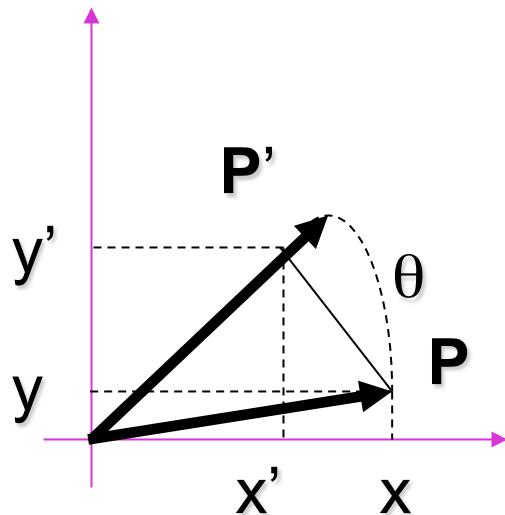
$$P' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{s}' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{S} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

- Counter-clockwise rotation by an angle θ



$$x' = \cos \theta x - \sin \theta y$$

$$y' = \cos \theta y + \sin \theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \mathbf{P}$$

How many degrees of freedom? 1

$$\mathbf{P}' \rightarrow \left[\begin{array}{ccc|c} \cos \theta & -\sin \theta & 0 & x \\ \sin \theta & \cos \theta & 0 & y \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Scale + Rotation + Translation

$$\mathbf{P}' \rightarrow \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

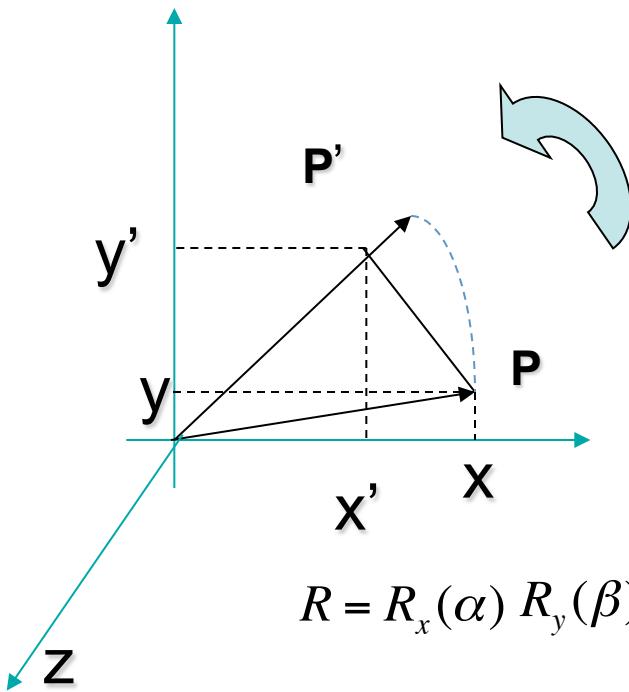
$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If $s_x = s_y$, this is a similarity transformation

$$= \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \mathbf{R} & \mathbf{S} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3D Rotation of Points

Rotation around the coordinate axes,
counter-clockwise:



$$R = R_x(\alpha) R_y(\beta) R_z(\gamma)$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

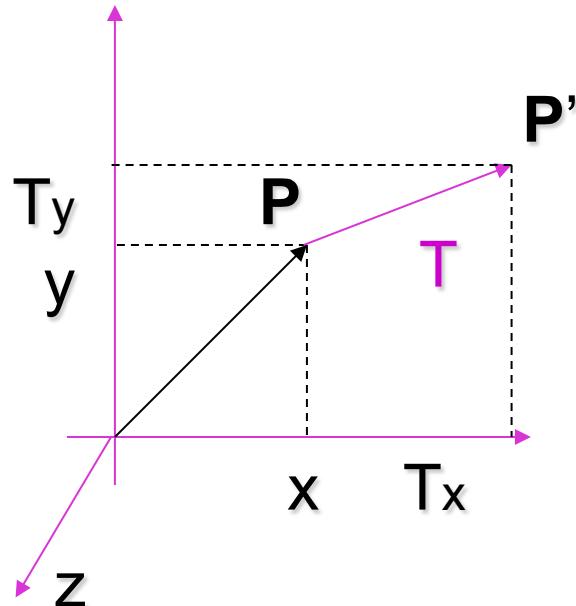
$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A rotation matrix in 3D has 3 degrees of freedom

3D Translation of Points



$$T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

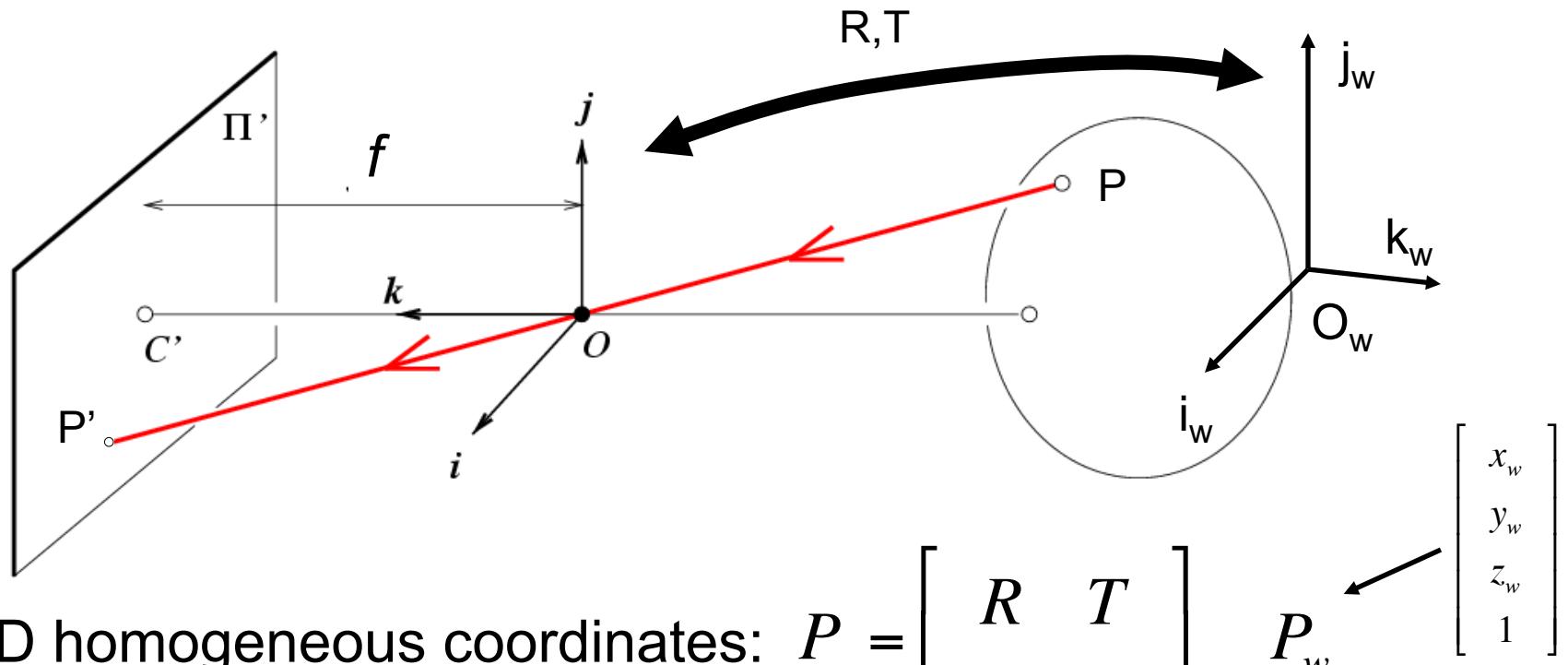
A translation vector in 3D has 3 degrees of freedom

3D Translation and Rotation

$$R = R_x(\alpha) \ R_y(\beta) \ R_z(\gamma) \quad T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$P' \rightarrow \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

World reference system



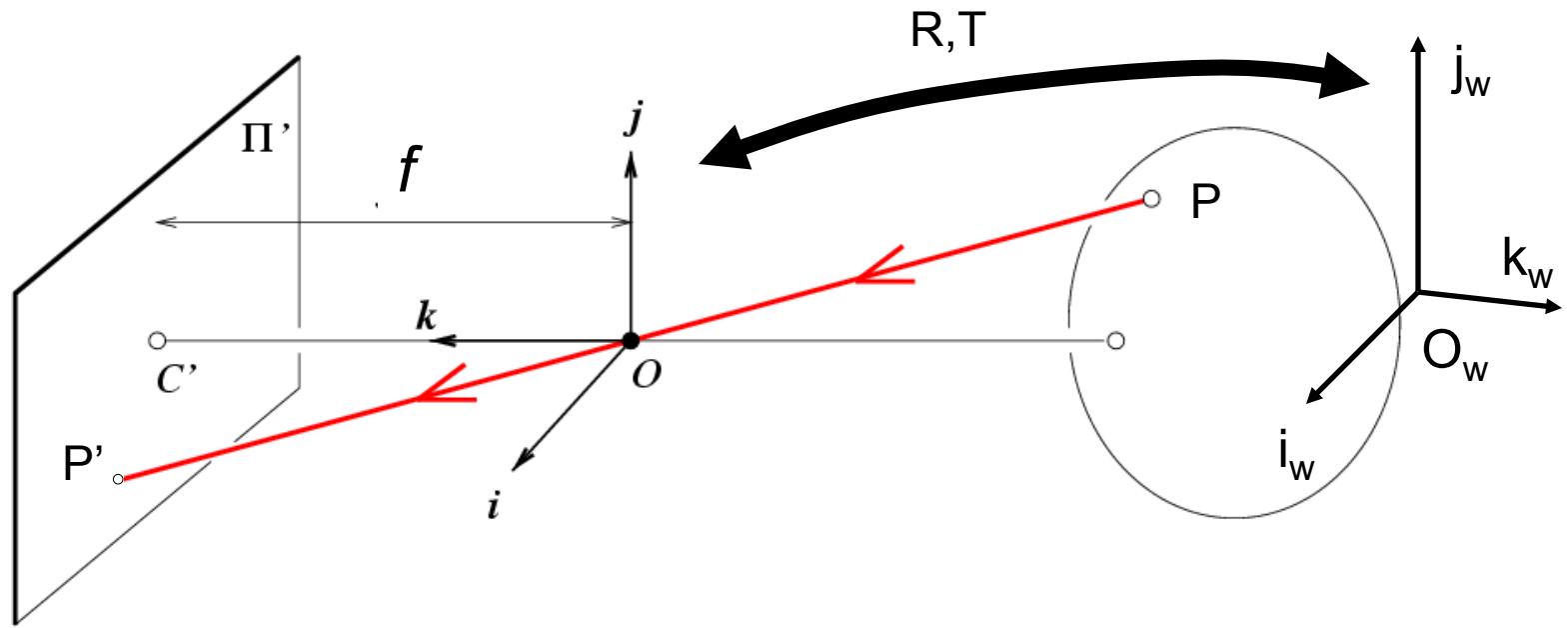
In 4D homogeneous coordinates: $P = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} P_w$

Internal parameters External parameters

$$P' = K \begin{bmatrix} I & 0 \end{bmatrix} P = K \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} P_w = K \boxed{\begin{bmatrix} R & T \end{bmatrix}} P_w$$

M [Eq.11]

The projective transformation

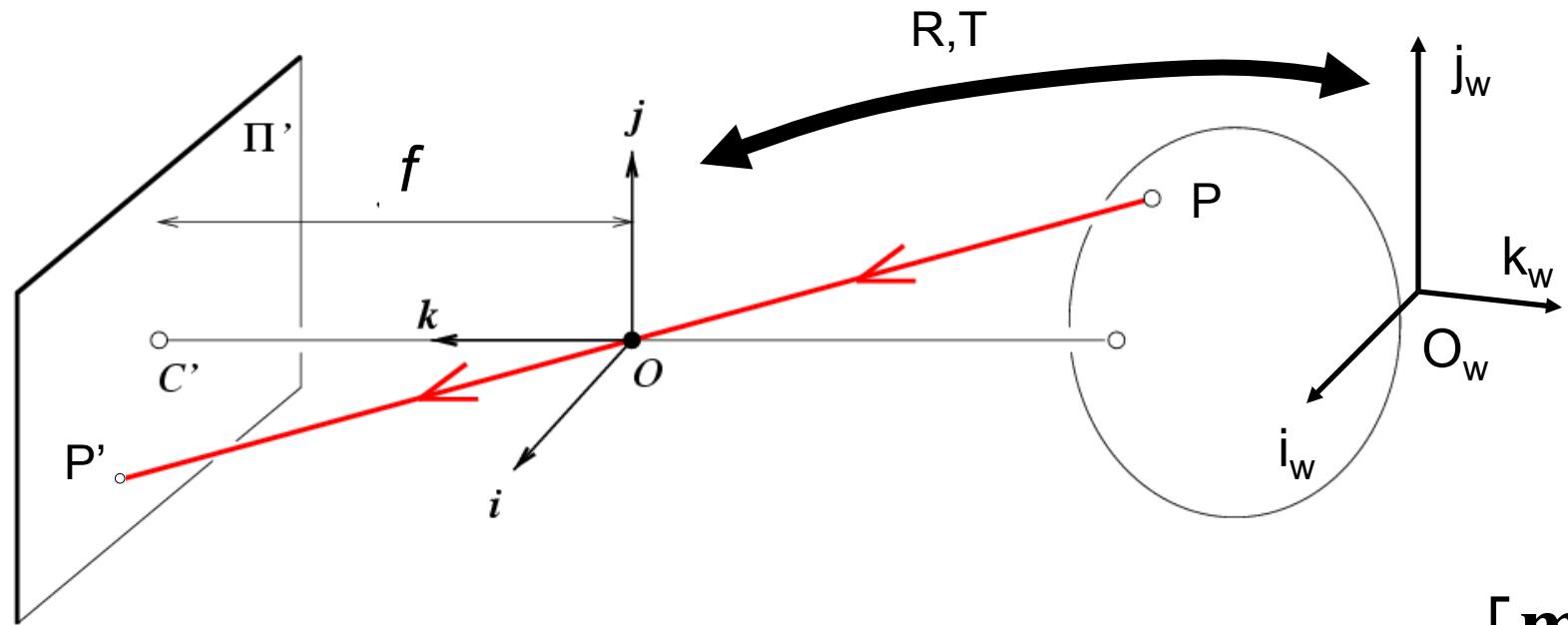


$$P'_{3 \times 1} = M_{3 \times 4} P_w = K_{3 \times 3} \begin{bmatrix} R & T \end{bmatrix}_{3 \times 4} P_w^{4 \times 1}$$

How many degrees of freedom?

$$5 + 3 + 3 = 11!$$

The projective transformation



$$\begin{aligned}
 P'_{3 \times 1} &= M P_w = K_{3 \times 3} \begin{bmatrix} R & T \end{bmatrix}_{3 \times 4} P_w_{4 \times 1} & M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} P_w = \begin{bmatrix} \mathbf{m}_1 P_w \\ \mathbf{m}_2 P_w \\ \mathbf{m}_3 P_w \end{bmatrix} & \mathbf{E} \rightarrow \left(\frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w} \right) \quad [\text{Eq.12}]
 \end{aligned}$$

Theorem (Faugeras, 1993)

$$M = K \begin{bmatrix} R & T \end{bmatrix} = \begin{bmatrix} KR & KT \end{bmatrix} = [A \quad b]$$

[Eq.13]

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$$

- A necessary and sufficient condition for \mathcal{M} to be a perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for \mathcal{M} to be a zero-skew perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$ and

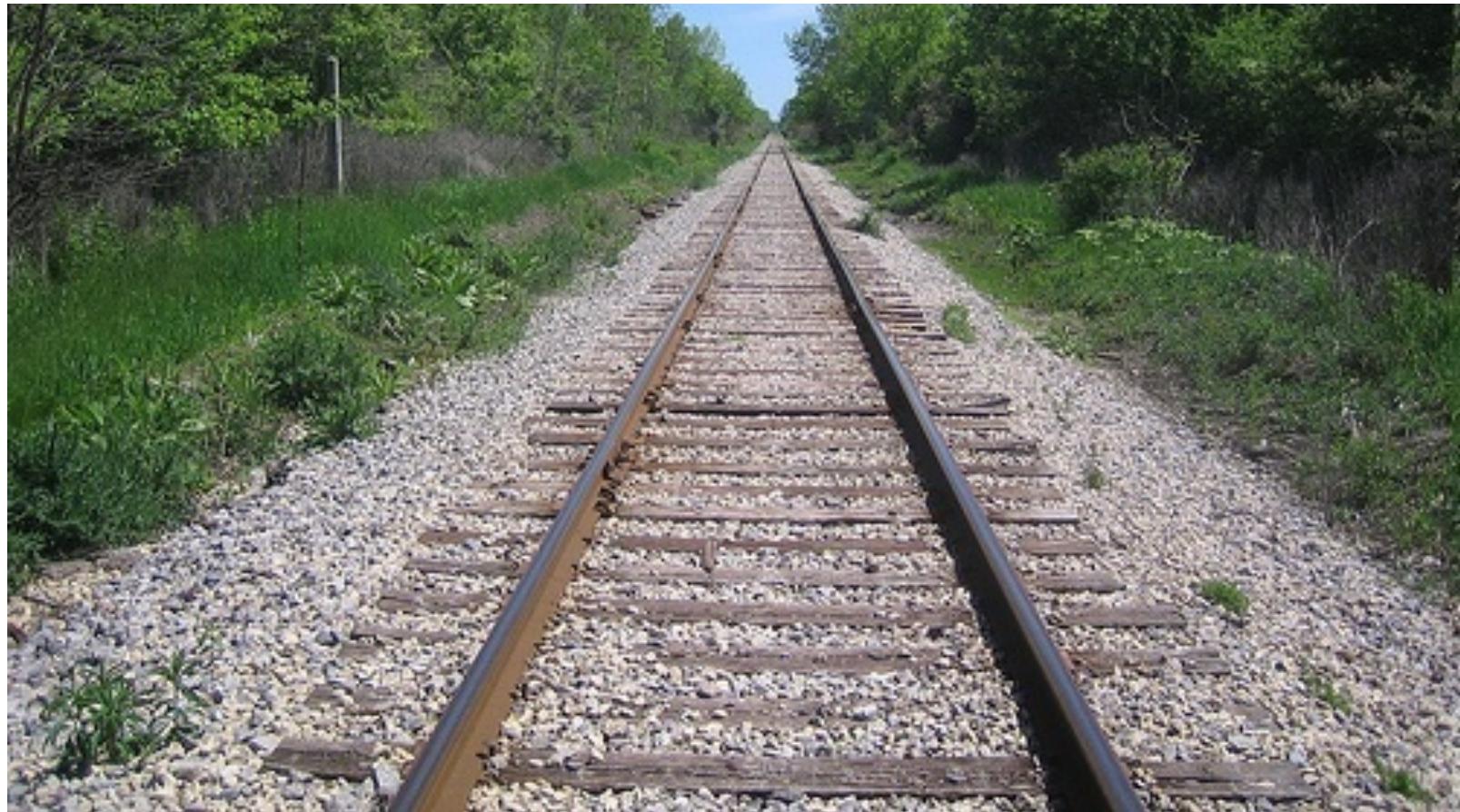
$$(\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = 0.$$

- A necessary and sufficient condition for \mathcal{M} to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\text{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = 0, \\ (\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_1 \times \mathbf{a}_3) = (\mathbf{a}_2 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3). \end{cases}$$

Properties of projective transformations

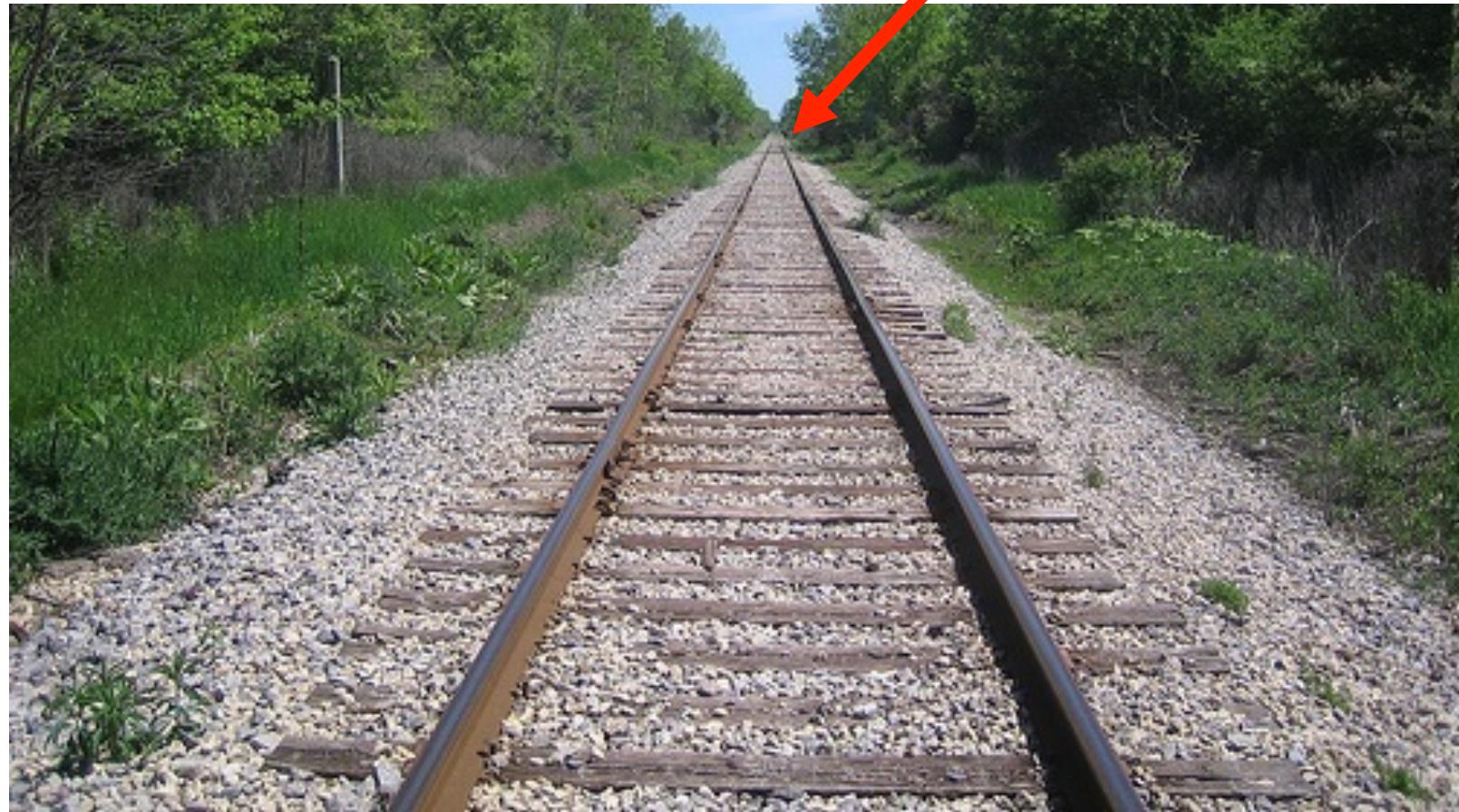
- Points project to points
- Lines project to lines
- Distant objects look smaller



Properties of Projection

- Angles are not preserved
- Parallel lines meet!

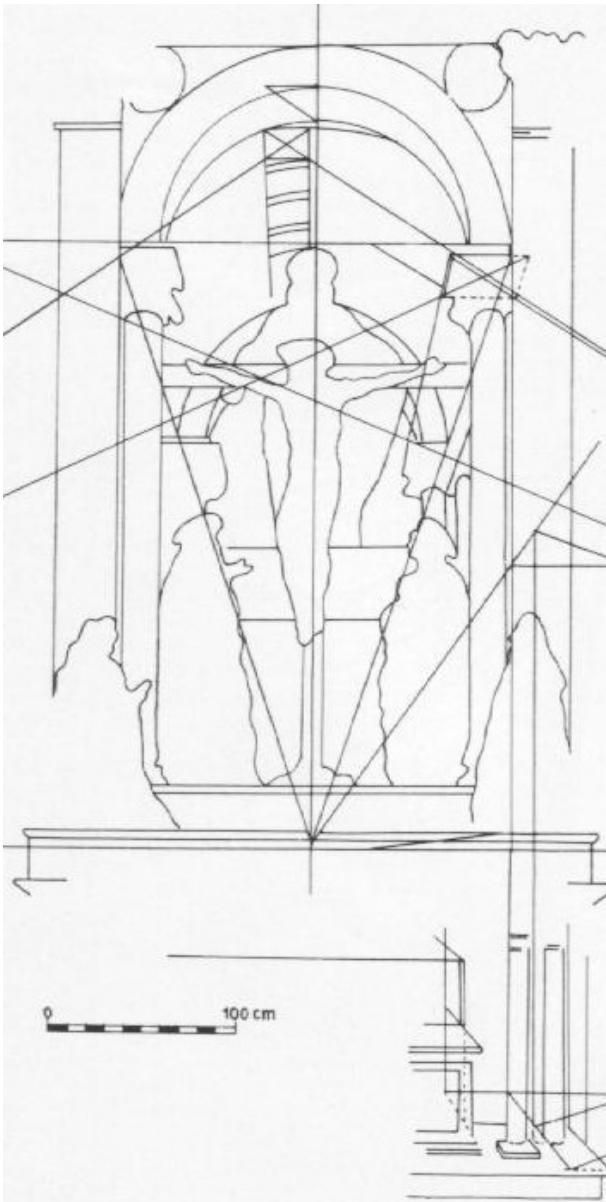
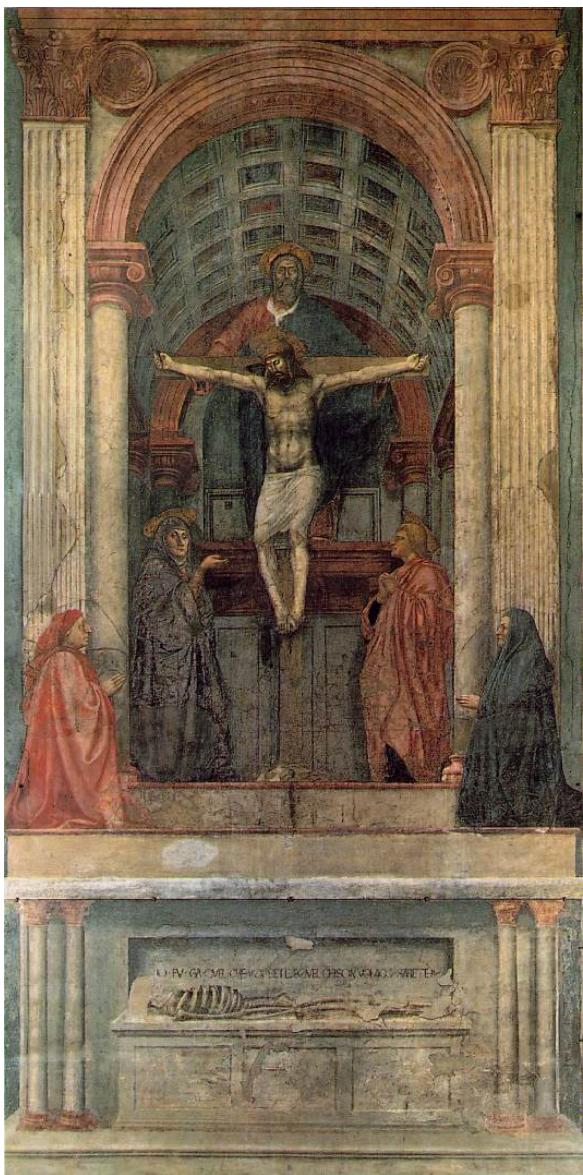
Parallel lines in the world intersect in the image at a “vanishing point”



Horizon line (vanishing line)



One-point perspective



- Masaccio, *Trinity*, Santa Maria Novella, Florence, 1425-28

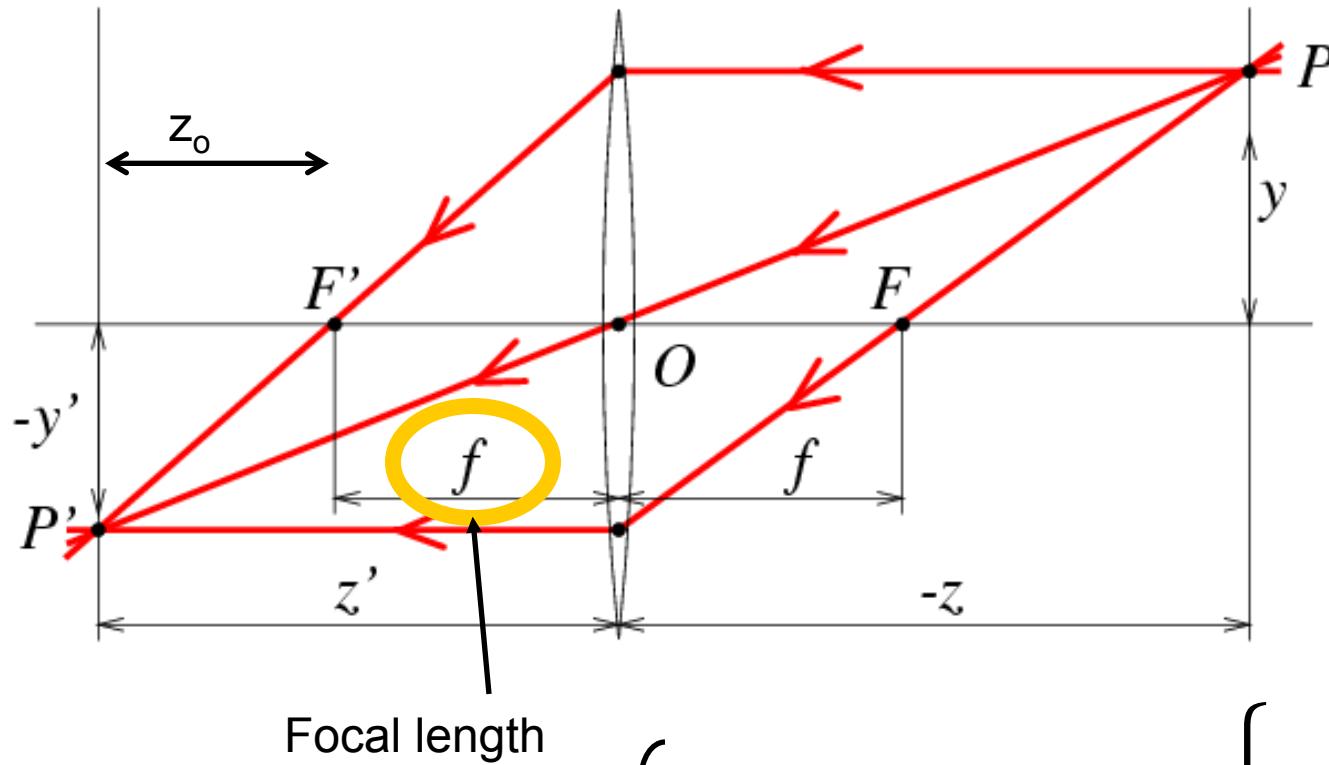
Next lecture

- How to calibrate a camera?

Supplemental material

Thin Lenses

[FP] sec 1.1, page 8.



$$z' = f + z_0$$

$$f = \frac{R}{2(n-1)}$$

Snell's law:
 $n_1 \sin \alpha_1 = n_2 \sin \alpha_2$

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$



$$\left\{ \begin{array}{l} \text{Small angles: } \\ n_1 \alpha_1 \approx n_2 \alpha_2 \\ n_1 = n \text{ (lens)} \\ n_1 = 1 \text{ (air)} \end{array} \right.$$



$$\left\{ \begin{array}{l} x' = z' \frac{x}{z} \\ y' = z' \frac{y}{z} \end{array} \right.$$

Horizon line (vanishing line)

