




# DISCONTINUOUS STATIONARY SOLUTIONS TO CERTAIN REACTION-DIFFUSION SYSTEMS

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*Dedicated to Eiji Yanagida on the occasion of his birthday*

ABSTRACT. Systems consisting of a single ordinary differential equation coupled with one reaction-diffusion equation in a bounded domain and with the Neumann boundary conditions are studied in the case of particular nonlinearities from the Brusselator model, the Gray-Scott model, the Oregonator model and a certain predator-prey model. It is shown that the considered systems have the both smooth and discontinuous stationary solutions, however, only discontinuous ones can be stable.

## 1. INTRODUCTION

We discuss stationary solutions to certain particular systems consisting of a single ordinary differential equation coupled with one reaction-diffusion equation:

$$(1.1) \quad \begin{aligned} u_t &= f(u, v), & x \in \overline{\Omega}, & \quad t > 0, \\ v_t &= \gamma \Delta v + g(u, v), & x \in \Omega, & \quad t > 0, \end{aligned}$$

with unknown functions  $u = u(x, t)$  and  $v = v(x, t)$ . System (1.1) is considered in a bounded, open domain  $\Omega \subset \mathbb{R}^N$  with the  $C^2$ -boundary  $\partial\Omega$ . Moreover, the reaction-diffusion equation in (1.1) is supplemented with the Neumann boundary condition

$$(1.2) \quad \partial_\nu v = 0 \quad \text{for } x \in \partial\Omega, \quad t > 0,$$

where  $\partial_\nu = \nu \cdot \nabla$  with  $\nu$  denoting the unit outer normal vector to  $\partial\Omega$ . The letter  $\gamma > 0$  is a constant diffusion coefficient. In the following, we denote by  $\Delta_\nu$  the Laplacian operator together with the Neumann boundary condition (1.2).

Particular versions of such a system have been studied, *e.g.*, in the works [8, 11–13, 16–18, 20–25, 30, 31, 35] where the following two types of stationary solutions  $(U(x), V(x))$  were identified:

- (1) *Regular stationary solutions*, where  $U(x) = k(V(x))$  for one  $C^2$ -function  $k$  and all  $x \in \overline{\Omega}$ .
- (2) *Jump-discontinuous stationary solutions* with  $U(x)$  obtained using different branches of solutions to the equation  $f(U, V) = 0$  with respect to  $U$ .

Regular stationary solutions to general problem (1.1)-(1.2) have been investigated recently in the paper [2] showing that they all are unstable. The second work [3] is devoted to the second type stationary solutions with jump-discontinuities and

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provides sufficient conditions for their existence and stability. The theory developed in both papers [2, 3] can be applied to several reaction-diffusion-ODE models in spatially homogeneous environments arising from applications as *e.g.* studied in [1, 10–13, 17–19, 24, 25, 28].

In this work, we explain how to apply general results from the papers [2, 3] to systems of type (1.1) with particular well-known nonlinearities which appear in Turing models. Recall that a system of reaction-diffusion equations

$$(1.3) \quad u_t = \varepsilon \Delta u + f(u, v), \quad v_t = \gamma \Delta v + g(u, v)$$

(in a bounded domain and with the Neumann boundary condition) exhibits a Turing instability if it has a constant stationary solution which is stable if  $\varepsilon = \gamma = 0$  and unstable for  $\varepsilon > 0$  and  $\gamma > 0$ . In his work [32], Turing showed that this kind of instability may occur in some particular reaction-diffusion systems with  $\varepsilon > 0$  sufficiently small. Due to different nonzero diffusion rates, the system gives rise spontaneously to stationary pattern formation with a characteristic length scale from an initial configuration (see *e.g.* [34] and the references therein).

In this work, we apply the theory from the papers [2, 3] to study nonconstant stationary solutions of some reaction-diffusion models with the Turing instability in the diffusion degenerated case, namely, when  $\varepsilon = 0$  and  $\gamma > 0$ . In Section 3, we consider the diffusion-degenerated Gray-Scott model

$$u_t = u^2 v - \alpha u, \quad v_t = \gamma \Delta_\nu v - u^2 v + \beta(1 - v)$$

and the diffusion-degenerated Brusselator system

$$u_t = \alpha + u^2 v - (\beta + 1)u, \quad v_t = \gamma \Delta_\nu v + \beta u - u^2 v.$$

By applying results from [2, 3], we obtain these two systems (considered in bounded domains) have stationary solutions of both types: regular and discontinuous. However, their all nonconstant stationary solutions are unstable.

Next, in Section 4, we discuss two other models: the diffusion-degenerated Oregonator

$$u_t = u - u^2 + \alpha v \frac{\beta - u}{\beta + u}, \quad v_t = \gamma \Delta_\nu v + u - v$$

and a certain kind of predator-prey system

$$u_t = u \left( \frac{u^2}{u^3 + 1} - v \right), \quad v_t = \gamma \Delta_\nu v + v(\alpha u - v - \beta).$$

In the following, we show how to use theory from the work [3] to construct *stable discontinuous stationary solutions* of these two systems.

## 2. GENERAL RESULTS

Here, we review briefly results from [2, 3] formulating them in a form which is suitable to deal with a system consisting of one ODE coupled with one reaction-diffusion equation.

We deal with a weak solution  $(U, V) = (U(x), V(x))$  to the boundary value problem

$$(2.1) \quad \begin{aligned} f(U, V) &= 0, & x &\in \overline{\Omega}, \\ \gamma \Delta_\nu V + g(U, V) &= 0, & x &\in \Omega, \end{aligned}$$

with arbitrary  $C^2$ -functions  $f$  and  $g$ , with constant  $\gamma > 0$ , and in an open bounded domain  $\Omega \subseteq \mathbb{R}^N$  with a  $C^2$ -boundary. Recall (cf. [2, 3]) that a weak solution has the following properties:  $U \in L^\infty(\Omega)$ ,  $V \in W^{1,2}(\Omega)$ , first equation in (2.1) holds true almost everywhere and the second equation is satisfied in the usual weak sense in the Sobolev space  $W^{1,2}(\Omega)$ .

We require that problem (2.1) has a constant solution, namely, a constant vector

$$(\bar{U}, \bar{V}) \in \mathbb{R}^2 \quad \text{such that} \quad f(\bar{U}, \bar{V}) = 0 \quad \text{and} \quad g(\bar{U}, \bar{V}) = 0.$$

In this work, we use the following notation

$$(2.2) \quad a_0 = f_u(\bar{U}, \bar{V}), \quad b_0 = f_v(\bar{U}, \bar{V}), \quad c_0 = g_u(\bar{U}, \bar{V}), \quad d_0 = g_v(\bar{U}, \bar{V})$$

and we always impose the following assumption.

**Assumption 2.1.** *We assume that  $a_0 = f_u(\bar{U}, \bar{V}) \neq 0$ .*

**2.1. Regular stationary solutions.** A solution  $(U(x), V(x))$  of system (2.1) is called regular if there exists a  $C^2$ -function  $k : \mathbb{R} \rightarrow \mathbb{R}$  such that  $U(x) = k(V(x))$  for all  $x \in \Omega$  and

$$f(U(x), V(x)) = f(k(V(x)), V(x)) = 0 \quad \text{for all } x \in \Omega,$$

where  $V = V(x)$  is a solution of the elliptic Neumann problem

$$\gamma \Delta_\nu V + h(V) = 0 \quad \text{for } x \in \Omega$$

with  $h(V) = g(k(V), V)$ . Now, we recall a result from work [2] on the existence of regular stationary solutions in the case of one ODE coupled with one reaction-diffusion equation.

**Proposition 2.2** ([3, Prop. 2.5]). *Let  $N \leq 6$  and let  $(\bar{U}, \bar{V}) \in \mathbb{R}^2$  be a constant solution of problem (2.1) satisfying Assumption 2.1. If*

$$(2.3) \quad \frac{1}{a_0} \det \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \gamma \mu_k > 0,$$

*for some  $\mu_k$  eigenvalues of  $-\Delta_\nu$ , then there exists a sequence of real numbers  $d_\ell \rightarrow 1$  such that the following “perturbed” problem*

$$\begin{aligned} f(U, V) &= 0, & x \in \bar{\Omega}, \\ d_\ell \Delta_\nu V + (1 - d_\ell)(V - \bar{V}) + g(U, V) &= 0, & x \in \Omega \end{aligned}$$

*has a non-constant regular solution.*

All regular stationary solutions are unstable, except the degenerate case when  $f_u(U(x), V(x)) \leq 0$  for all  $x \in \bar{\Omega}$  and  $f_u(U(x_0), V(x_0)) = 0$  for some  $x_0 \in \bar{\Omega}$ , see the following theorem.

**Theorem 2.3** ([2, Thm. 2.7 and Thm. 2.8]). *Let  $(U, V) = (U(x), V(x))$  be a non-constant regular stationary solution of problem (1.1)-(1.2) such that*

- *either  $f_u(U(x_0), V(x_0)) > 0$  for some  $x_0 \in \bar{\Omega}$ ,*
- *or  $f_u(U(x), V(x)) < 0$  for all  $x \in \bar{\Omega}$  and the domain  $\Omega$  is convex.*

*Then  $(U, V)$  is nonlinearly unstable.*

**2.2. Discontinuous solutions.** Now, we impose additional assumptions on a constant stationary solution using notation (2.2).

**Assumption 2.4.** *The constant solution  $(\bar{U}, \bar{V}) \in \mathbb{R}^2$  of problem (2.1) satisfies*

$$\frac{1}{a_0} \det \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \neq \gamma \mu_k,$$

for each eigenvalue  $\mu_k$  of  $-\Delta_\nu$ .

Notice that if  $a_0 = f_u(\bar{U}, \bar{V}) \neq 0$ , by the Implicit mapping theorem, there exist an open neighborhood  $\mathcal{V} \subseteq \mathbb{R}$  of  $\bar{V}$  and a function  $k \in C^2(\mathcal{V}, \mathbb{R})$  such that

$$k(\bar{V}) = \bar{U} \quad \text{and} \quad f(k(w), w) = 0 \quad \text{for all } w \in \mathcal{V}.$$

In the following assumption, the equation  $f(U, V) = 0$  is required to have *two different branches of solutions* with respect to  $U$  on a common domain  $\mathcal{V}$ .

**Assumption 2.5.** *We assume that there exists an open set  $\mathcal{V} \subseteq \mathbb{R}$  and  $k_1, k_2 \in C^2(\mathcal{V}, \mathbb{R})$  such that*

- $\bar{V} \in \mathcal{V}$ ,
- $\bar{U} = k_1(\bar{V})$ ,  $k_1(\bar{V}) \neq k_2(\bar{V})$ ,
- $f(k_1(w), w) = f(k_2(w), w) = 0$  for all  $w \in \mathcal{V}$ .

Such two branches  $k_1, k_2$  allow us to construct discontinuous solution of problem (2.1).

**Theorem 2.6** (Existence of discontinuous stationary solutions [3, Thm. 2.6]). *Assume that*

- *problem (2.1) has a constant solution  $(\bar{U}, \bar{V}) \in \mathbb{R}^2$  satisfying Assumptions 2.1 and 2.4,*
- *Assumption 2.5 holds true,*
- *for an arbitrary open set  $\Omega_1 \subset \Omega$  we put  $\Omega_2 = \Omega \setminus \bar{\Omega}_1$  and the sets  $\Omega_1$  and  $\Omega_2$  have nonzero Lebesgue measures.*

*There is  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there exists  $\delta > 0$  such that if  $|\Omega_2| < \delta$  then problem (2.1) has a weak solution with the following properties:*

- $(U, V) \in L^\infty(\Omega) \times W_\nu^{2,p}(\Omega)$  for each  $p \in [2, \infty)$ ,
- $V = V(x)$  is a weak solution to problem

$$\gamma \Delta_\nu V + g(U, V) = 0 \quad \text{for } x \in \Omega,$$

where

$$U(x) = \begin{cases} k_1(V(x)), & x \in \Omega_1, \\ k_2(V(x)), & x \in \Omega_2, \end{cases}$$

satisfies  $f(U(x), V(x)) = 0$  for almost all  $x \in \bar{\Omega}$ .

- *the solution  $(U, V)$  stays around the points  $(\bar{U}, \bar{V})$  and  $(k_2(\bar{V}), \bar{V})$  in the following sense*

$$(2.4) \quad \|V - \bar{V}\|_{L^\infty(\Omega)} < \varepsilon \quad \text{and} \quad \|U - \bar{U}\|_{L^\infty(\Omega_1)} + \|U - k_2(\bar{V})\|_{L^\infty(\Omega_2)} < C\varepsilon$$

for a constant  $C = C(f, g)$ .

Next, we discuss a stability of discontinuous stationary solutions by the linearization procedure. Here, we say that  $(U, V)$  is *linearly stable (unstable)* if the zero solution to the following system

$$(2.5) \quad \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \Delta_\nu \psi \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv \mathcal{L}_p \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

with the bounded (and possibly  $x$ -dependent) coefficients

$$a = f_u(U, V), \quad b = f_v(U, V), \quad c = g_u(U, V), \quad d = g_v(U, V),$$

is stable (unstable).

**Theorem 2.7** (Instability of stationary solutions [2, 23]). *Let  $(U, V)$  be an arbitrary weak stationary solution to problem (1.1)-(1.2). If*

$$(2.6) \quad f_u(U(x), V(x)) > 0 \quad \text{for } x \text{ from a set of a positive measure}$$

*then  $(U, V)$  is linearly unstable in  $L^p(\Omega)^2$  for each  $p \in (1, \infty)$ .*

The assumption (2.6) is called the *autocatalysis condition* in the work [23]. It appears in models with the Turing instability and it leads to an instability of the both: constant and nonconstant stationary solutions. Theorem 2.7 is an immediate consequence of the results in [23, Thm. 4.5] and [2, Thm. 4.6], where the spectrum of the operator  $\mathcal{L}_p$  defined in (2.5) and acting on the space  $L^p(\Omega)^2$  is described.

Next, we discuss a stability of discontinuous stationary solutions and we require from the constant solution  $(\bar{U}, \bar{V}) \in \mathbb{R}^2$  used in Theorem 2.6 to have the following additional properties.

**Assumption 2.8.** *For the numbers defined in (2.2), assume that the following inequalities hold true*

$$a_0 < 0, \quad d_0 < 0 \quad \text{and the matrix} \quad \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$$

*has both eigenvalues with strictly negative real parts.*

**Theorem 2.9** (Stability of discontinuous stationary solutions [3]). *Let the assumptions of Theorem 2.6 hold true. If, moreover,*

- *Assumption 2.8 is valid,*
- *$\varepsilon > 0$  and  $|\Omega_2| > 0$  are small enough in Theorem 2.6,*
- *the following inequalities are satisfied*

$$f_u(k_2(\bar{V}), \bar{V}) < 0 \quad \text{and} \quad g_v(k_2(\bar{V}), \bar{V}) < 0.$$

*then the stationary solution  $(U, V)$  constructed in Theorem 2.6 is exponentially stable in  $L^\infty(\Omega)$ .*

Theorem 2.9 can be obtained from results in [3] in the following way. First, we apply [3, Prop. 4.2] to linear system (2.5) to show that the stationary solution is linearly exponentially stable in  $L^p(\Omega)$ . Here, our system consists of one ODE coupled with one PDE, thus Assumption 2.8 and the condition  $d_0 < 0$  imply that [3, Assumptions 2.8 and 2.9] are satisfied by [3, Rem. 2.16]. Next, the nonlinear stability in  $L^\infty(\Omega)$  is a direct consequence of [3, Thm. 2.13] which proof does not require from  $\gamma > 0$  to be large enough.

*Remark 2.10.* Other discontinuous stationary solutions can be constructed under the following more general version of Assumption 2.5, where we postulate an existence of a set  $\mathcal{V} \subseteq \mathbb{R}$  and different branches  $k_1, \dots, k_J \in C(\mathcal{V}, \mathbb{R})$  of solutions to the equation  $f(U, V) = 0$ . Then for an arbitrary decomposition

$$\Omega \subseteq \overline{\bigcup_{i \in \{1, \dots, J\}} \Omega_i}, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j \quad \text{and} \quad |\Omega_i| < \delta \text{ for } i \in \{2, \dots, J\},$$

we can construct a discontinuous stationary solution of the form

$$U(x) = \begin{cases} k_1(V(x)), & x \in \Omega_1, \\ \vdots \\ k_J(V(x)), & x \in \Omega_J, \end{cases}$$

using the same reasoning as in Theorem 2.6. If

$$f_u(k_i(\bar{V}), \bar{V}) < 0 \quad \text{and} \quad g_v(k_i(\bar{V}), \bar{V}) < 0 \quad \text{for each } i \in \{1, \dots, J\}$$

then this solution is linearly and nonlinearly stable.

### 3. UNSTABLE STATIONARY SOLUTIONS

We apply results from Section 2 to particular reaction-diffusion-ODE systems. The following two models with classical nonlinearities have stationary solutions  $(U, V)$ , the both regular and discontinuous. For each stationary solution the autocatalysis condition (2.6) is satisfied, hence such solutions are unstable.

**3.1. Gray-Scott type model.** First, we deal with the reaction-diffusion-ODE model with the Gray-Scott nonlinearities

$$(3.1) \quad \begin{aligned} u_t &= u^2 v - \alpha u, & x \in \bar{\Omega}, \quad t > 0, \\ v_t &= \gamma \Delta_\nu v - u^2 v + \beta(1 - v), & x \in \Omega, \quad t > 0, \end{aligned}$$

with arbitrary constants  $\alpha > 0$ ,  $\beta > 0$  and the diffusion coefficient  $\gamma > 0$ . This is the diffusion degenerate case of the classical Gray-Scott model introduced in [9] and studied in *e.g.* [4, 15] and in the references therein.

Notice that problem (3.1) has the constant stationary solution  $(\bar{U}_1, \bar{V}_1) = (0, 1)$ . Two other constant solutions  $(\bar{U}_2, \bar{V}_2)$ ,  $(\bar{U}_3, \bar{V}_3)$  satisfy the relations

$$(3.2) \quad \bar{U} = \frac{\alpha}{\bar{V}} \quad \text{and} \quad 0 = \bar{V}^2 - \bar{V} + \frac{\alpha^2}{\beta}$$

and they exist if  $\alpha^2/\beta < 1/4$ .

**Theorem 3.1.** *For a certain choice of coefficients  $\alpha, \beta > 0$  and for a discrete sequence of diffusion coefficients  $\gamma > 0$  problem (3.1) has a regular stationary solution. All regular stationary solutions (not only those obtained via Proposition 2.2) are nonlinearly unstable.*

*Proof.* We choose  $\alpha, \beta > 0$  such that  $\alpha^2/\beta < 1/4$  and consider a solution  $\bar{V}_3$  of the quadratic equation in (3.2) satisfying  $\bar{V}_3 < 1/2$ . We apply Proposition 2.2 with the constant stationary solution  $(\bar{U}, \bar{V}) = (\bar{U}_3, \bar{V}_3)$ , where  $\bar{U}_3 = \alpha/\bar{V}_3$ . Assumption 2.1

holds true, because  $a_0 = f_u(\bar{U}_3, \bar{V}_3) = \alpha > 0$ . Moreover, for the numbers  $a_0, b_0, c_0, d_0$  as in (2.2), we have

$$\det \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{vmatrix} \alpha & (\alpha/\bar{V}_3)^2 \\ -2\alpha & -(\alpha/\bar{V}_3)^2 - \beta \end{vmatrix} = \alpha ((\alpha/\bar{V}_3)^2 - \beta) > 0,$$

where the last inequality follows from an explicit formula for  $\bar{V}_3$ . Thus, we may choose  $\gamma > 0$  to satisfy equation (2.3) for some eigenvalue  $\mu_k > 0$ .

Next, we show that every stationary solution satisfies  $f_u(U(x), V(x)) \neq 0$  for each  $x \in \Omega$ . Indeed, notice that either  $U(x) = 0$  for all  $x \in \bar{\Omega}$  or  $U(x) = \alpha/V(x)$  for all  $x \in \bar{\Omega}$ . Thus, since,  $f_u(U, V) = 2UV - \alpha$  we have either  $f_u(U(x), V(x)) = \alpha$  or  $f_u(U(x), V(x)) = -\alpha$ .

Hence, by Theorem 2.3, we conclude that all regular solutions to problem (3.1) are unstable.  $\square$

**Theorem 3.2.** *For arbitrary  $\alpha > 0$  and  $\beta > 0$  and for each diffusion coefficient  $\gamma > 0$ , except of a discrete set, there exists a family of discontinuous stationary solutions to problem (3.1). All discontinuous stationary solutions (not only those obtained via Theorem 2.6) are linearly unstable in  $L^p(\Omega)^2$  for each  $p \in (1, \infty)$ .*

*Proof.* First, we apply Theorem 2.6 to construct discontinuous stationary solutions to problem (3.1). Assumption 2.4 is satisfied for all stationary points  $(\bar{U}_1, \bar{V}_1)$ ,  $(\bar{U}_2, \bar{V}_2)$ ,  $(\bar{U}_3, \bar{V}_3)$  and for all diffusion coefficients  $\gamma > 0$ , possibly except of a discrete set. To check Assumption 2.5, we notice that the equation  $U(UV - \alpha) = 0$  has the following two branches of solutions

$$U = k_1(V) = 0 \quad \text{and} \quad U = k_2(V) = \frac{\alpha}{V} \quad \text{for all } V \neq 0.$$

Thus, for an arbitrary open set  $\Omega_1 \subset \Omega$ , with  $\Omega_2 = \Omega \setminus \bar{\Omega}_1$  and such that the measure  $|\Omega_2| > 0$  is sufficiently small, there exist a family of discontinuous stationary solutions around the points either  $(\bar{U}_1, \bar{V}_1)$  or  $(\bar{U}_2, \bar{V}_2)$  or  $(\bar{U}_3, \bar{V}_3)$  as stated in Theorem 2.6.

Each discontinuous stationary solution  $(U, V)$  satisfies  $k_2(V(x)) = \alpha/V(x)$  for  $x$  from a set of a positive measure. Thus,  $f_u(k_2(V(x)), V(x)) = \alpha > 0$  and this solution is linearly unstable in  $L^p(\Omega)^2$  for each  $p \in (1, \infty)$  by Theorem 2.7.  $\square$

**3.2. Brusselator type model.** Next, we deal with the reaction-diffusion-ODE model with the nonlinearity as in the Brusselator system

$$(3.3) \quad \begin{aligned} u_t &= \alpha + u^2v - (\beta + 1)u, & x \in \bar{\Omega}, & t > 0, \\ v_t &= \gamma \Delta_\nu v + \beta u - u^2v, & x \in \Omega, & t > 0, \end{aligned}$$

with arbitrary constants  $\alpha > 0$ ,  $\beta > 0$  and the diffusion coefficient  $\gamma > 0$ . This is the diffusion-degenerate case of the classical Brusselator model which was introduced in [29] as a model for an autocatalytic oscillating chemical reaction. It has been shown (*e.g.* in [5, 14, 26] and in references therein) that the Brusselator system with the both nonzero diffusion coefficients (as in (1.3)) exhibits the Turing instability.

Notice that a constant stationary solution  $(\bar{U}, \bar{V}) \in \mathbb{R}^2$  of problem (3.3) satisfies the equations

$$0 = \alpha + \bar{U}^2\bar{V} - (\beta + 1)\bar{U}, \quad 0 = \beta\bar{U} - \bar{U}^2\bar{V}$$

with the only solution

$$(\bar{U}_1, \bar{V}_1) = (\alpha, \beta/\alpha).$$

**Theorem 3.3.** *For each  $\alpha > 0$  and  $\beta > 1$  and for a discrete sequence of the diffusion coefficients  $\gamma > 0$  problem (3.3) has a regular stationary solution. All regular stationary solutions (not only those obtained via Proposition 2.2) are nonlinearly unstable.*

*Proof.* We use the notation from (2.2) with the constant stationary solution  $(\bar{U}_1, \bar{V}_1)$  and we notice that

$$a_0 = \beta - 1 \quad \text{and} \quad \det \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \alpha^2.$$

Thus, for  $\beta > 1$ , the proof of the existence of regular stationary solutions, may be completed in the analogous way as the proof of Theorem 3.1. All regular stationary solutions are unstable by Theorem 2.3.  $\square$

**Theorem 3.4.** *For some  $\alpha > 0$ ,  $\beta > 0$ ,  $\beta \neq 1$ , and for each diffusion coefficient  $\gamma > 0$  except of a discrete set, there exists a family of discontinuous stationary solutions to problem (3.3). All discontinuous stationary solutions (not only those obtained via Theorem 2.6) are linearly unstable in  $L^p(\Omega)^2$  for each  $p \in (1, \infty)$ .*

*Proof.* We choose the coefficient  $\alpha > 0$ ,  $\beta > 0$  and  $\gamma > 0$  in such a way that Assumptions 2.1 and 2.4 are satisfied (notice that  $a_0 = \beta - 1 \neq 0$ ). The equation

$$f(U, V) = \alpha + U^2V - (\beta + 1)U = 0$$

has two different branches of solutions given by the explicit formulas

$$U = k_1(V) = \frac{\beta + 1 + \sqrt{(\beta + 1)^2 - 4\alpha V}}{2V}, \quad U = k_2(V) = \frac{\beta + 1 - \sqrt{(\beta + 1)^2 - 4\alpha V}}{2V}$$

for  $V < (\beta + 1)^2/(4\alpha)$ . Note that

$$\bar{V}_1 = \beta/\alpha < (\beta + 1)^2/(4\alpha) \quad \text{and} \quad \bar{U}_1 = k_1(\bar{V}_1).$$

Thus, for an open set  $\Omega_1 \subset \Omega$  with  $\Omega_2 = \Omega \setminus \bar{\Omega}_2$  and such that the measure  $|\Omega_2| > 0$  is sufficiently small, there exist a family of discontinuous stationary solutions around points  $(\bar{U}_1, \bar{V}_1)$  and  $(k_2(\bar{V}_1), \bar{V}_1)$  as stated in Theorem 2.6.

Each discontinuous stationary solution  $(U, V)$  satisfies  $U(x) = k_1(V(x))$  for  $x$  from a set of positive measure. Thus,

$$f_u(k_1(V(x)), V(x)) = \sqrt{(\beta + 1)^2 - 4\alpha V(x)} > 0$$

and this solution is linearly unstable in  $L^p(\Omega)^2$  for each  $p \in (1, \infty)$  by Theorem 2.7.  $\square$

#### 4. STABLE DISCONTINUOUS STATIONARY SOLUTIONS

Now, we discuss reaction-diffusion-ODE models with stable discontinuous stationary solutions.

**4.1. Oregonator type model.** The following reaction-diffusion-ODE model

$$(4.1) \quad \begin{aligned} u_t &= u - u^2 + \alpha v \frac{\beta - u}{\beta + u}, & x \in \bar{\Omega}, \quad t > 0, \\ v_t &= \gamma \Delta_\nu v + u - v, & x \in \Omega, \quad t > 0, \end{aligned}$$

contains the nonlinearities as in the classical Oregonator system with the parameters  $\alpha > 0$ ,  $\beta > 0$  and the diffusion coefficient  $\gamma > 0$ . A general version of this model was introduced in [6, 7, 33] to describe the FKN-Belousov reaction. It is known that



system (4.1) may exhibit the Turing instability for a certain choice of parameters and diffusion coefficients, see *e.g.* [27, 36] and the references therein.

System (4.1) has the zero constant solution  $(\bar{U}_1, \bar{V}_1) = (0, 0)$  and two other constant solutions have the form

$$(\bar{U}_2, \bar{V}_2) = (\bar{U}_2, \bar{U}_2) \quad (\bar{U}_3, \bar{V}_3) = (\bar{U}_3, \bar{U}_3),$$

where the numbers  $\bar{U}_2$  and  $\bar{U}_3$  satisfy the quadratic equation

$$\bar{U}^2 + \bar{U}(\beta + \alpha - 1) - \beta(\alpha + 1) = 0.$$

Fig. 4.1 presents a location of the nullclines defined by the following equation for  $\beta > 1$  and for sufficiently small  $\alpha > 0$ :

$$(4.2) \quad f(U, V) = U - U^2 + \alpha V \frac{\beta - U}{\beta + U} = 0 \quad \text{and} \quad g(U, V) = U - V = 0.$$

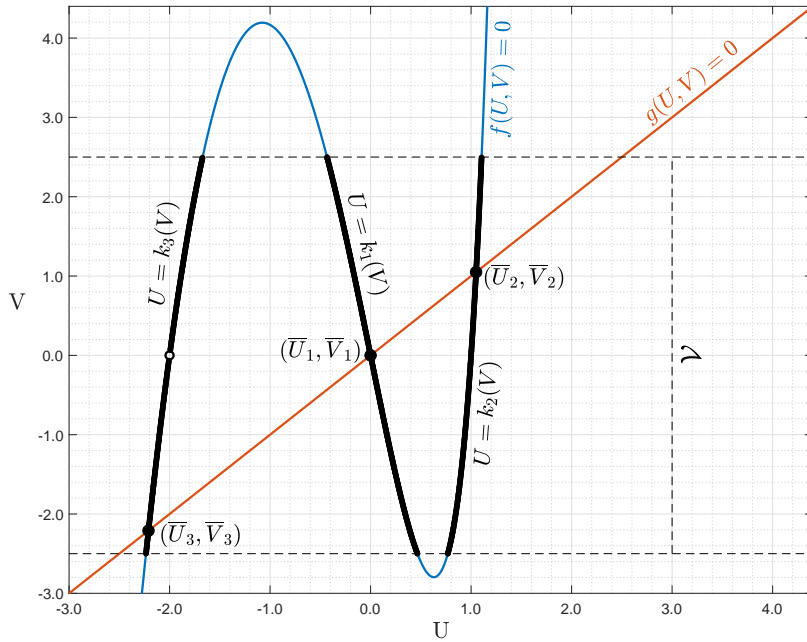


FIGURE 4.1. The nullclines (4.2) for the Oregonator model with  $\beta = 2$  and sufficiently small  $\alpha > 0$ . The point  $(-2, 0)$  is excluded because it does not satisfy the equation  $f(U, V) = 0$ .

**Theorem 4.1.** *For each  $\beta > 1$  and sufficiently small  $\alpha > 0$  there exist a family of discontinuous stationary solutions to problem (4.1).*

*Proof.* Fix  $\beta > 1$  and choose sufficiently small  $\alpha > 0$  to have the nullclines as in Fig. 4.1. In such a case, we have an open subset  $\mathcal{V} \subset \mathbb{R}$  satisfying  $V_1, V_2, V_3 \in \mathcal{V}$  and three branches  $k_1, k_2, k_3$  defined on  $\mathcal{V}$ .

Here, we check assumptions of Theorem 2.6 for the stationary points either  $(\bar{U}_1, \bar{V}_1)$  or  $(\bar{U}_2, \bar{V}_2)$  or  $(\bar{U}_3, \bar{V}_3)$  in the cases shown in Fig. 4.1. First, we notice that for a branch  $k \in \{k_1, k_2, k_3\}$ , we have

$$(4.3) \quad \frac{d}{dV} f(k(V), V) = f_u(k(V), V)k'(V) + f_v(k(V), V) = 0.$$

Since

$$f_v(U, V) = \frac{\alpha(\beta - U)}{\beta + U},$$

choosing the constant stationary solution  $(\bar{U}_i, \bar{V}_i) \in \mathbb{R}^2$  satisfying  $\bar{U} \neq \beta$  and  $k'(\bar{V}) \neq 0$  we obtain  $a_0 = f_u(\bar{U}, \bar{V}) \neq 0$  as required in Assumption 2.1. This is the case of all constant stationary solutions  $(\bar{U}_1, \bar{V}_1)$ ,  $(\bar{U}_2, \bar{V}_2)$  and  $(\bar{U}_3, \bar{V}_3)$  shown in Fig. 4.1.

To apply Theorem 2.6, we need a constant stationary solution  $(\bar{U}_i, \bar{V}_i)$ , an open set  $\mathcal{V} \subset \mathbb{R}$  with  $V_i \in \mathcal{V}$  and two branches of solutions  $k_i, k_j \in C(\mathcal{V}, \mathbb{R})$  to equation  $f(U, V) = 0$ . On Fig 4.1, we may choose each constant stationary solution with corresponding branches, namely, the point  $(\bar{U}_1, \bar{V}_1)$  with the branches  $k_1$  and  $k_2$ , the point  $(\bar{U}_2, \bar{V}_2)$  with the branches  $k_2$  and either  $k_1$  or  $k_3$  as well as the point  $(\bar{U}_3, \bar{V}_3)$  with the branches  $k_3$  and either  $k_1$  or  $k_2$ . We do not choose the point  $(\bar{U}_1, \bar{V}_1)$  with the branch  $k_3$  in order to avoid the singular point  $(-\beta, 0)$ .

Thus, for  $\gamma > 0$  satisfying Assumption 2.4, an arbitrary open set  $\Omega_1 \subset \Omega$  with  $\Omega_2 = \Omega \setminus \bar{\Omega}_1$  and  $|\Omega_2| > 0$  sufficiently small, there exist a family of discontinuous stationary solutions around the points  $(\bar{U}, \bar{V}) = (k_i(\bar{V}), \bar{V})$  and  $(k_j(\bar{V}), \bar{V})$  as stated in Theorem 2.6.

By Remark 2.10, we can also construct discontinuous stationary solutions with all three branches  $k_1, k_2, k_3$  and with an arbitrary partition  $\Omega_1, \Omega_2, \Omega_3$  of  $\Omega$ .  $\square$

**Theorem 4.2.** *For each  $\beta > 1$  and for sufficiently small  $\alpha > 0$  there exist discontinuous stationary solutions constructed around the points  $(\bar{U}_2, \bar{V}_2)$  and  $(k_3(\bar{V}_2), \bar{V}_2)$  which are stable.*

*Proof.* As in the proof of Theorem 4.1, we have an open subset  $\mathcal{V} \subset \mathbb{R}$  satisfying  $V_1, V_2, V_3 \in \mathcal{V}$  and three branches  $k_1, k_2, k_3$  defined on  $\mathcal{V}$ , see Fig. 4.1. Let  $(U, V)$  be a discontinuous stationary solution constructed around points  $(\bar{U}_2, \bar{V}_2)$  and  $(k_3(\bar{V}_2), \bar{V}_2)$  via Theorem 2.6.

First, we check Assumption 2.8. Since  $|\bar{U}_2| < \beta$  and since  $k_2(V)$  is increasing in the neighbourhood of  $\bar{V}_2$  (see Fig. 4.1) we have  $a_0 < 0$  by equation (4.3). In order to show that the matrix  $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$  has both eigenvalues with negative real parts, it suffices to show that

$$a_0 + d_0 < 0 \quad \text{and} \quad a_0 d_0 - b_0 c_0 > 0.$$

Since  $c_0 = 1$  and  $d_0 = -1$ , first inequality is immediately satisfied. Moreover, since  $k'_2(\bar{V}_2) < 1$ , by identity (4.3), we have

$$a_0 d_0 - b_0 c_0 = -a_0(1 - k'_2(\bar{V}_2)) > 0.$$

Finally,  $|k_3(\bar{V}_2)| < \beta$  (see Fig. 4.1) and the function  $k_3(V)$  is increasing in a neighbourhood of  $\bar{V}_2$  which implies that  $f_u(k_3(\bar{V}_2), \bar{V}_2) < 0$ .

By Theorem 2.9, the discontinuous stationary solution  $(U, V)$  is stable provided  $\varepsilon > 0$  and  $|\Omega_2| > 0$  are small enough.  $\square$

*Remark 4.3.* On the other hand, all discontinuous stationary solutions constructed in Theorem 4.1 around the points  $(\bar{U}_1, \bar{V}_1)$  and  $(\bar{U}_3, \bar{V}_3)$  are unstable. Indeed, analogously as in the proof of Theorem 4.1, we analyse nullclines (4.2) presented in Fig. 4.1. Following the calculations from equation (4.3) we obtain, that the number  $f_u(k(\bar{V}), \bar{V})$  is positive if either

$$|\bar{U}| < \beta \quad \text{and} \quad k'(\bar{V}) < 0 \quad \text{or} \quad |\bar{U}| > \beta \quad \text{and} \quad k'(\bar{V}) > 0.$$

Since the constant solutions given by explicit formulas satisfy  $\bar{U}_1 = 0$ ,  $\bar{U}_3 < -\beta$  and since the function  $k(V)$  is decreasing in a neighbourhood of  $\bar{U}_1$  and increasing in a neighbourhood of  $\bar{U}_3$  (see Fig. 4.1) we obtain that every solution constructed around points  $(\bar{U}_1, \bar{V}_1)$  and  $(\bar{U}_3, \bar{V}_3)$  is linearly unstable by Theorem 2.7.

**4.2. Predator-prey model.** Finally, we discuss stationary solutions to the following model of predator-prey interactions which is a particular case of a system considered in [25]:

$$(4.4) \quad \begin{aligned} u_t &= u \left( \frac{u^2}{u^3 + 1} - v \right), & x \in \bar{\Omega}, \quad t > 0 \\ v_t &= \gamma \Delta_\nu v + v(\alpha u - v - \beta) & x \in \Omega, \quad t > 0. \end{aligned}$$

which  $\alpha > 0$  and  $\beta > 0$ . Here, the couple  $(\bar{U}_2, \bar{V}_2) = (0, 0)$  is the constant stationary solution and the second constant stationary solution  $(\bar{U}_1, \bar{V}_1)$  satisfies the system

$$(4.5) \quad f(U, V) = \frac{U^2}{U^3 + 1} - V = 0 \quad \text{and} \quad g(U, V) = \alpha U - V - \beta = 0.$$

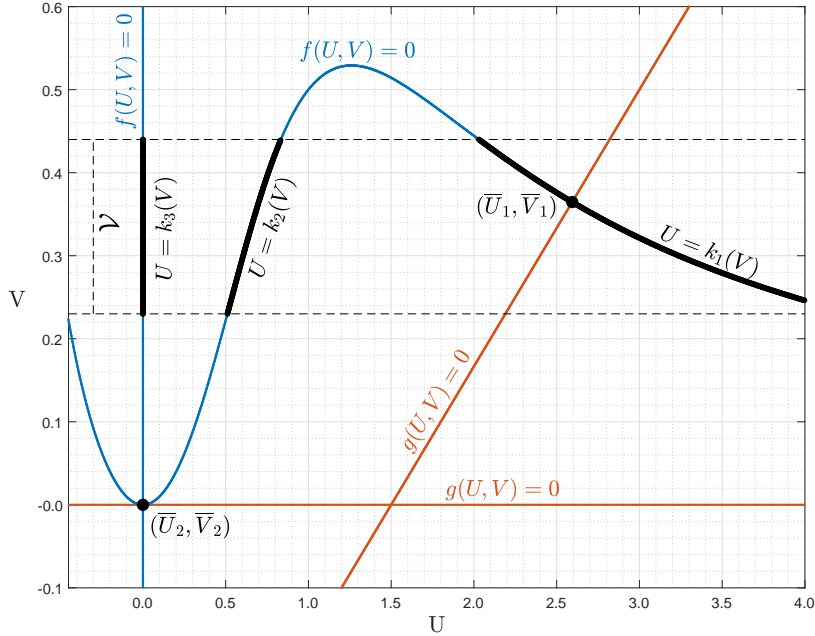


FIGURE 4.2. The nullclines (4.5) for the predator-prey model with  $\bar{U}_1 > U_m$ .

The nullclines defined by equations (4.5) are sketched in Fig. 4.2. Notice that the function  $U^2/(U^3 + 1)$  attains a maximum for  $U > 0$  at some point which we denoted by  $U_m$ .

**Theorem 4.4.** *Assume that  $\alpha > 0$  and  $\beta > 0$  are such that the constant stationary solution  $(\bar{U}_1, \bar{V}_1)$  satisfies  $\bar{U}_1 \neq U_m$ . Then there exist a family of discontinuous stationary solutions to problem (4.4).*

*Proof.* It is sufficient to apply Theorem 2.6 in the same way as in the proof of Theorem 4.1 with the stationary point  $(\bar{U}_1, \bar{V}_1)$  with the branch  $k_1$  and the branches either  $k_2$  or  $k_3$  (or the both, see Remark 2.10).  $\square$

**Theorem 4.5.** *Assume that  $\alpha > 0$  and  $\beta > 0$  are such that the constant stationary solution  $(\bar{U}_1, \bar{V}_1)$  satisfies  $\bar{U}_1 > U_m$ . Then there exist stable discontinuous stationary solutions to problem (4.4).*

*Proof.* Following the reasoning from Theorem 4.2 we obtain that the number  $f_u(\bar{U}_1, \bar{V}_1)$  is negative because the function  $k_1(V)$  is decreasing in a neighbourhood of  $\bar{V}_1$  (see Fig. 4.2). Moreover, since  $b_0 < 0$ ,  $c_0 > 0$  and  $d_0 < 0$  (with the notation from (2.2)) we obtain that

$$a_0 + d_0 < 0 \quad \text{and} \quad a_0 d_0 - b_0 c_d > 0$$

and Assumption 2.8 holds true. By a direct calculation, we obtain that

$$f_u(0, \bar{V}_1) = -\bar{V}_1 < 0 \quad \text{and} \quad g_v(0, \bar{V}_1) = -2\bar{V}_1 - \beta < 0.$$

Thus, the solution is stable by Theorem 2.9.  $\square$

**4.3. Numerical illustrations of discontinuous stationary solutions.** We conclude this work by numerical simulations of solutions to problem (4.4) with  $\alpha$  and  $\beta$  as in Fig. 4.1. To obtain Fig. 4.3, we used the explicit finite difference Euler method and we choose the initial conditions  $(u_0, v_0)$  in the following way. First, we decompose the domain  $\Omega = [0, 1]^2 = \Omega_1 \cup \Omega_2$  with the *EY*-shape set  $\Omega_2$  and  $|\Omega_2| > 0$  sufficiently small. Next, we set

$$(4.6) \quad u_0 = \bar{U}_1 \text{ and } v_0 = \bar{V}_1 \text{ on } \Omega_1 \quad \text{as well as} \quad u_0 = 0 \text{ and } v_0 = \bar{V}_1 \text{ on } \Omega_2.$$

The graph of the corresponding solution in Fig. 4.3 was obtained for large values of  $t > 0$  and, by the stability results from this work, this is a good approximation of discontinuous stationary solution. Here, we observe a graphical illustration of inequalities (2.4), namely,  $U$  stays close to  $\bar{U}_1 = k_1(\bar{V}_1)$  on the set  $\Omega_1$  and close to  $k_3(\bar{V}_1) = 0$  on the set  $\Omega_2$ .

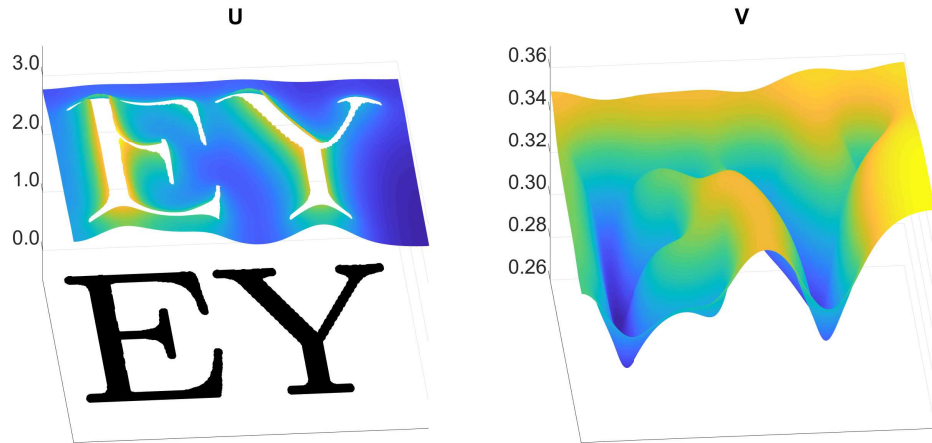


FIGURE 4.3. The solution  $u, v$  to problem (4.4) for sufficiently large  $t > 0$  and with the *EY*-shape initial condition satisfying equations (4.6). The solution  $u$  is equal to 0 on the *EY*-shape set  $\Omega_2$  (black) and stays close to  $\bar{U}_1$  on a set  $\Omega_1$  (blue-green-yellow). The solution  $v$  is an *EY*-perturbation of the constant solution  $\bar{V}_2$ .

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