$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}$$

Eliminating the constants a, b from above three equation we get

$$F(x, y, z, p, q) = 0$$

Example.

1. Form the PDE by eliminating the constants a and b from $z = (x+a)(y+b)$ —— (i)

$$\rightarrow \quad p = \frac{\partial z}{\partial x} = y+b \quad \text{—— (ii)}$$

$$q = \frac{\partial z}{\partial y} = x+a \quad \text{—— (iii)}$$

Put the value of (ii) and (iii) in (i), we get,

Required PDE $\Rightarrow z = qp$

2. $z = ax^n + by^n$ —— (i)

$$\rightarrow p = \frac{\partial z}{\partial x} = anx^{n-1} \Rightarrow a = \frac{p}{nx^{n-1}} \quad \text{—— (ii)}$$

$$q = \frac{\partial z}{\partial y} = bny^{n-1} \Rightarrow b = \frac{q}{ny^{n-1}} \quad \text{—— (iii)}$$

Put the value of (ii) and (iii) in (i), we get,

$$z = \frac{p}{nx^{n-1}} \times x^n + \frac{q}{ny^{n-1}} \times y^n$$

$$z = \frac{px\cancel{x^n}}{n\cancel{x^n}} + \frac{q\cancel{y^n}y}{n\cancel{y^n}}$$

$$\Rightarrow z = \frac{px+qy}{n} \Rightarrow zn = px+qy$$

3.

$$\rightarrow \frac{\partial z}{\partial x} = p = 2x(y^2+b^2) \Rightarrow y^2+b^2 = \frac{p}{2x} \quad \text{--- (ii)}$$

$$\frac{\partial z}{\partial y} = q = 2y(x^2+a^2) \Rightarrow x^2+a^2 = \frac{q}{2y} \quad \text{--- (iii)}$$

put the value of (ii) and (iii) in (i), we get:

$$z = \frac{q}{2y} \times \frac{p}{2x}$$

$$z = \frac{pq}{4xy}$$

$$\Rightarrow 4xyz = pq.$$

4. $z = ax + by + \sqrt{a^2+b^2}$ --- (i)

$$\Rightarrow \frac{\partial z}{\partial x} = p = a \quad \text{--- (ii)}$$

$$\frac{\partial z}{\partial y} = q = b \quad \text{--- (iii)}$$

put the value of $\underset{(ii)}{a}$ and $\underset{(iii)}{b}$ in (i), we get:

$$z = px + qy + \sqrt{p^2+q^2}$$

5. $\log(az-1) = x + ay + b$, where a and b are constants.

$\rightarrow$ partial differential w.r.t x

$$\frac{1}{az-1}\frac{\partial}{\partial z}(az-1) = 1+0+0$$

$$\Rightarrow \frac{ap}{az-1} = 1 \quad \text{--- (i)} \Rightarrow a =$$

d.p. w.r.t y

$$\frac{1}{az-1}\frac{\partial}{\partial z}(az-1) = a \Rightarrow \frac{1}{az-1}\cdot aq = a \quad \text{--- (ii)}$$

7. $x^2+y^2+(z-c)^2=r^2$ [c, r constants]

$\Rightarrow \frac{\partial z}{\partial x} = 2x+0+2(z-c)\frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial y} = 2y+0+2(z-c)\frac{\partial z}{\partial y} = 0$

$x+(z-c)p=0$

$y+(z-c)q=0$

$z-c = \frac{-x}{p}$ $\qquad$ $z-c = \frac{-y}{q}$

$\cancel{-}\frac{x}{p} = \cancel{-}\frac{y}{q}$

$\Rightarrow \boxed{xq = py}$

~~Solve~~

Solution of PDE by direct integration.

1. Solve $\frac{\partial z}{\partial x} = 0$

→ Here z is independent of x.

Integrating w.r.t x

$$z = f(y)$$

2. Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$

$$\rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0$$

Integrating w.r.t x

$$\frac{\partial z}{\partial y} = f_1(y)$$

Now integrating w.r.t y

$$z = \int f_1(y)\, dy + f_2(x)$$

$$= g_1(y) + f_2(x)$$

3. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + c$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{x}{y} + c$$

Integrating w.r.t x

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + cx + f_1(y)$$

Now integrating w.r.t y

$$z = \frac{x^2}{2}\log y + cxy + \int f_1(y)\, dy + f_2(x)$$

$$= \frac{x^2}{2}\log y + cxy + g_1(y) + f_2(x).$$

4. Solve $\frac{\partial^2 z}{\partial x^2} = \cos x$

→ Integrating w.r.t x

$$\frac{\partial z}{\partial x} = \sin x + f_1(y)$$

Again, integrating w.r.t x

$$z = -\cos x + x f_1(y) + f_2(y)$$

order PDE → PDE → ODE

28/09/20

PDE :

$\frac{dy}{dx} = y$

$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$

Order of a Differential Equation :

$\frac{dy}{dx} = y$

$\therefore$ order $= 1$, degree $= 1$

$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + x$

$\therefore$ order $= 2$, degree $= 1$

1) Eliminating Arbitary constant
2) Eliminating Arbitary function

$z = x^2 y$

$\frac{dz}{dx} = 2x$

$\frac{\partial z}{\partial x} = 2xy$

$\frac{\partial z}{\partial y} = x^2$

Eliminating Arbitary constant

(1) $z = (x+a)(y+b)$ — (i), where a and b are const.

$\frac{\partial z}{\partial x} = y+b$ — (ii), $\quad \frac{\partial z}{\partial y} = x+a$

$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = Q$

$p = y+b \qquad Q = x+a$

$z = pq$

(2) Find the PDE of all sphere whose radius is unity and centre lies on xy plane.

$\rightarrow \quad x^2+y^2+z^2 = 1$ eqn of unit sphere | $x^2+y^2 = 1$ eqn of unit circle.

// $(x-a)^2 + (y-b)^2 + z^2 = 1$

$\Rightarrow z^2 = 1 - (x-a)^2 - (y-b)^2$

$\frac{\partial z}{\partial x} = 2(x-a) + 2zp = 0 \; ; \quad \Rightarrow 2(y-b) + 2zq = 0$

$$(x-a)^2+(y-b)^2+z^2=1$$

$$\Rightarrow z^2+\left(\frac{-xzp}{x}\right)^2+\left(-\frac{xzq}{x}\right)^2=1$$

$$\Rightarrow z^2+z^2p^2+z^2q^2=1$$

$$\Rightarrow z^2(p^2+q^2+1)=1$$

$$(x-a)^2+(y-b)^2=5z^2$$

$$\Rightarrow z=\sqrt{\frac{(x-a)^2+(y-b)^2}{5}}$$

30/09/20

By eliminating Arbitrary functions :

$z=f(w)$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial w}\times\frac{\partial w}{\partial x}$$

$$f'(x+y)=\frac{\partial f}{\partial(x+y)}$$

1. $z=f(x+y)$

$$p=\frac{\partial z}{\partial x}=f'(x+y)\frac{\partial}{\partial x}(x+y)$$

$$Q=\frac{\partial z}{\partial y}=f'(x+y)\frac{\partial}{\partial y}(x+y)$$

$p=f'$, $Q=f'$

$$\boxed{P=Q}$$

2. $z=f(x^2+y^2)$

$$P=\frac{\partial z}{\partial x}=f'(x^2+y^2)\frac{\partial}{\partial x}(x^2+y^2)$$

$$P=f'\,2x$$

$$Q=\frac{\partial z}{\partial y}=f'(x^2+y^2)\frac{\partial}{\partial y}(x^2+y^2)$$

$$Q=f'\cdot 2y$$

$$f'=\frac{P}{2x}\ ;\ f'=\frac{Q}{2y}$$

$$\frac{P}{\cancel{2}x}=\frac{Q}{\cancel{2}y}$$

$$\Rightarrow\frac{P}{x}=\frac{Q}{y}$$

$$\Rightarrow\boxed{Py=Qx}$$

3. $z = x+y + f(xy)$

$p = \frac{\partial z}{\partial x} = 1+0+f'(xy)\frac{\partial}{\partial x}(xy)$

$q = \frac{\partial z}{\partial y} = 1+0+f'(xy)\frac{\partial}{\partial y}(xy)$

$p = 1+f'y \quad ; \quad q = 1+f'x$

$f' = \frac{p-1}{y} \qquad f' = \frac{q-1}{x}$

$\frac{p-1}{y} = \frac{q-1}{x}$

$\Rightarrow px - x = qy - y$

$\Rightarrow -(x-y) = -(px-qy) \Rightarrow x-y = px-qy$

4. $z = ax + by + \sqrt{a^2+b^2}$ —— (1)

$p = a+0+0 \quad ; \quad q = b+0+0$

$p = a \qquad q = b$

Putting the values of a and b in (1)

$z = px + qy + \sqrt{p^2+q^2}$

5. $z = f(x+y)$

$p = \frac{\partial z}{\partial x} = f'(x+y)\frac{\partial}{\partial x}(x+y)$

$q = \frac{\partial z}{\partial y} = f'(x+y)\frac{\partial}{\partial y}(x+y)$

$p = f' \quad ; \quad q = f'$

$\boxed{p = q}$

6. $z = f_1(x) + f_2(y)$

$\frac{\partial z}{\partial x} = f_1'(x) + 0 \qquad \frac{\partial z}{\partial y} = f_2'(y)$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 0$

$\frac{\partial^2 z}{\partial y \partial x} = 0$ (Req. eqn).

7. $z = y f_1(x) + f_2(y)$

$\frac{\partial z}{\partial x} = y f_1'(x)$ $\qquad$ $\frac{\partial z}{\partial y} = f_1(x) + f_2'(y)$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_1'(x)$

$\frac{\partial z}{\partial x} = y \frac{\partial^2 z}{\partial x \partial y}$

8. $z = f_1(x+2y) + f_2(x+2y)$

$\frac{\partial z}{\partial x} = f_1' + f_2'$ $\qquad$ $\frac{\partial^2 z}{\partial x^2} = f_1'' + f_2''$

$\frac{\partial z}{\partial y} = 2f_1' + 2f_2'$ $\qquad$ $\frac{\partial^2 z}{\partial y^2} = 4f_1'' + 4f_2''$

$= 4(f_1'' + f_2'')$

$\therefore \frac{\partial^2 z}{\partial y^2} = 4 \frac{\partial^2 z}{\partial x^2}$

9. $z = f_1(x+2y)$

$\frac{\partial z}{\partial x} = f_1'(x+2y)$ $\qquad$ $\frac{\partial z}{\partial y} = 2f_1'(x+2y)$

$\therefore \frac{\partial^2 z}{\partial y^2} = 4f_1''$

Solution of PDE by direct integration:-

// $\frac{\partial z}{\partial x} = 0$

z = constant

$z = f(y)$

$\frac{\partial y}{\partial x} = 0$

$\int dy = \int x\,dx$

$y = \frac{x^2}{2} + c$

$\frac{dy}{dx} = 0$

$y = A$

$z = y^2$

$\frac{\partial z}{\partial x} = 0$

// $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

$\dfrac{\partial z}{\partial x} = 0$

$z = f(y)$

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = 0$

$\dfrac{\partial z}{\partial y} = f(y)$

$z = \int f(y)\, dy + f_2(x)$

$z = g(y) + f_2(x)$

// $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + \text{const.}$

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{x}{y} + \text{const.}$

$= \int \dfrac{x}{y} + c$

$\dfrac{\partial z}{\partial y} = \dfrac{x^2}{2y} + cx + f_1(y)$

$z = \dfrac{x^2}{2} \log y + cxy + \int f_1(y)\, dy + f_2(x)$

$z = \dfrac{x^2}{2} \log y + cxy + g_1(y) + f_2(x)$

// $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \cos x$

→ Integrating w.r.t x

$\dfrac{\partial z}{\partial x} = \sin x + f_1(y)$

Integrating w.r.t x

$z = -\cos x + x f_1(y) + f_2(y)$.

// $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

$\Rightarrow \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 z}{\partial x \partial y}\right) = \cos(2x+3y)$

I. w.r.t x

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\sin(2x+3y)}{2} + f_1(y)$

I. w.r.t. x

$$\frac{\partial z}{\partial y} = \frac{1}{2} - \frac{\cos(2x+3y)}{2} + xf_1(y) + f_2(y)$$

$$= \frac{-\cos(2x+3y)}{4} + xf_1(y) + f_2(y).$$

I. w.r.t y

$$z = -\sin(2x+3y) \times \frac{1}{4} \times \frac{1}{3} + xf_1'(y) + g'(y) + h(x)$$

$$z = -\frac{1}{12}\sin(2x+3y) + xf_1'(y) + g'(y) + h(x)$$

Q// $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2y$

$\rightarrow$ I. w.r.t x

$$\frac{\partial z}{\partial y} = \frac{x^3}{3}y + f_1(y)$$

I. w.r.t. y

$$z = \frac{x^3}{3}\cdot\frac{y^2}{2} + f_1(y) + f_2(x) \rightarrow (1)$$

put $z = x^2$ and $y = 0$ in (1)

$$x^2 = 0 + f_1(0) + f_2(x)$$

From (1)

$$z = \frac{x^3y^2}{6} + f_1(y) + x^2 - f_1(0) \quad - (11)$$

$$z(1,y) = \cos y$$

$$\cos y = \frac{y^2}{6} + f_1(y) + 1 - f_1(0)$$

$$\Rightarrow f_1(y) = \cos y - \frac{y^2}{6} - 1 + f_1(0)$$

16/10/20

Lagrange method for solving 1st order PDE → PDE → ODE

$Pp + Qq = R$

$p = \frac{\partial z}{\partial x}$ $\quad z = z(x,y)$

$q = \frac{\partial z}{\partial y}$

Rule:

1. write down the auxillary eqn

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

2. Solve the above auxillary eqns.

Let the solns be

$u = c_1$ and $v = c_2$

3. soln is

$f(u,v) = 0$ OR, $u = f(v)$ or $v = f(u)$

Ex

$yq - xp = z$

auxillary equation

$$\underbrace{\frac{dx}{-x} = \frac{dy}{y}} = \frac{dz}{z}$$

$xy = c_1$, $\frac{y}{z} = c_2$

$$\frac{dx}{-x} = \frac{dy}{y}$$

$\Rightarrow -\log x = \log y + \log c$

$\Rightarrow \cancel{xy = c_1}$

$\Rightarrow \cancel{-(\log x + \log y) =}$

$\Rightarrow -\log x = \log(yc)$

$\Rightarrow \log(x^{-1}) = \log(yc)$

soln :- $f(xy, y/z) = 0$ $\Rightarrow \frac{1}{x} = yc$

$\Rightarrow xy = \frac{1}{c} = c_1$

$xy = f(y/z)$

Method of multipliers

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

where l, m, n may be constants or functions of x, y, z.

we will choose l, m, n such a way that

$$lP + mQ + nR = 0$$

$$\frac{l\,dx + m\,dy + n\,dz}{0} = \frac{dx}{P}$$

Thus we get,

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$ $\longrightarrow$ $x\,dx + y\,dy + z\,dz = 0$

$d(x^2/2 + y^2/2 + z^2/2) = 0$

by solving this we will get $u = c_1 \longleftarrow d(x) = 0$

simi, we will choose another set of multipliers to get second solution as $v = c_2$

Req. soln $f(u, v) = 0$

Ex $x(y-z)p + y(z-x)q = z(x-y)$

$$\Rightarrow \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{d(x+y+z)}{0}$$

$l=1,\ m=1,\ n=1$

$$= \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)}$$

$$= \frac{dx+dy+dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{xy}+\cancel{zx}-\cancel{zy}}$$

$$d(x+y+z)=0$$

$$x+y+z = c_1$$

$$\frac{\frac{dx}{x}}{(y-z)} = \frac{dy/y}{(z-x)} = \frac{dz/z}{(x-y)}$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y-z+z-x+x-y}$$ [multiply 1 with num. and deno]

$$= \frac{dx}{x}+\frac{dy}{dy}+\frac{dz}{z} = 0$$

$$= \ln(x)+\ln(y)+\ln(z) = 0$$

$$\cancel{= d[\ln(xyz)] = 0}$$

$$\Rightarrow \ln(xyz) = \ln c_2$$

$$xyz = c_2$$

$$f(x+y+z,\ xyz) = 0$$

Ex

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

$\rightarrow$ $l = x \quad m = y \quad n = z$

$$= \cancel{\frac{dx+dy+dz}{x(z^2-y^2)+y(x^2-z^2)+z(y^2-x^2)}}$$

$$= \frac{x\,dx + x\,dy + z\,dz}{\cancel{x^2z^2} - \cancel{x^2y^2} + \cancel{y^2x^2} - \cancel{y^2z^2} + \cancel{z^2y^2} - \cancel{z^2x^2}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

~~$(x+y+z) = c_1$~~

$\Rightarrow x^2 + y^2 + z^2 = c_1$

$$\frac{dx/x}{z^2-y^2} = \frac{dy/y}{x^2-z^2} = \frac{dz/z}{y^2-x^2}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\cancel{z^2} - \cancel{y^2} + \cancel{x^2} - \cancel{z^2} + \cancel{y^2} - \cancel{x^2}}$$

[multiply 1 with num. and deno.]

$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

$\Rightarrow \ln(x) + \ln(y) + \ln(z) = 0$

$\Rightarrow \ln(xyz) = \ln c_2$

$\Rightarrow xyz = c_2$

$f(x^2+y^2+z^2,\ xyz) = 0$

2. $(y+z)p + (z+x)q = x+y$

$$\Rightarrow \frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{x+y}$$

$$= \frac{dx+dy+dz}{2(x+y+z)}$$

$l = 1 \quad m = -1, n = 0$

$l = 0 \quad m = 1, n - 1$

$$= \frac{d(x+y+z)}{2(x+y+z)}$$

$l = 1 \quad m = 1 \quad n = 1$

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{d(x-y)}{y-x} = \frac{d(y-z)}{z-y} = \frac{d(x+y+z)}{2(x+y+z)}$$

1 and 3

$$-\log(x-y) = \frac{1}{2}\log(x+y+z) + \log c_1$$

$$\Rightarrow -\log(x-y) = \log\left\{c_1\sqrt{x+y+z}\right\}$$

$$\Rightarrow \frac{1}{x-y} = c_1\sqrt{x+y+z}$$

$$\Rightarrow (x-y)\sqrt{x+y+z} = c_2 \quad \because \left[c_2 = \frac{1}{c_1}\right]$$

1 and 2

$$-\log(x-y) = -\log(y-z) - \log c_3$$

$$\Rightarrow \log\left(\frac{1}{x-y}\right) = \log\left\{\frac{1}{c_3(y-z)}\right\}$$

$$\Rightarrow \frac{x-y}{y-z} = c_4$$

$$f\left((x-y)\sqrt{x+y+z},\ \frac{x-y}{y-z}\right) = 0$$

3. $(y^2+z^2+x^2)p - 2xyq + 2xz = 0$

$$\Rightarrow \frac{dx}{(y^2+z^2+x^2)} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$$\cancel{= \frac{dx + (-dy) + dz}{y^2+z^2+x^2 - 2xy - 2xz}}$$

$$\frac{dy}{-2xy} = \frac{dz}{-2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow y/z = c_1$$

$$\frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 + x^3 - 2xy^2 - 2xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dx}{-xy^2 - xz^2 + x^3} = \frac{x\,dx + y\,dy + z\,dx}{-x(x^2+y^2+z^2)}$$

$$= \frac{2x\,dx + 2y\,dy + 2z\,dz}{-2x(x^2+y^2+z^2)}$$

$$= \frac{d(x^2+y^2+z^2)}{-2x(x^2+y^2+z^2)}$$

$$\frac{dy}{-\cancel{2x}y} = \frac{d(x^2+y^2+z^2)}{-\cancel{2x}(x^2+y^2+z^2)}$$

$$\Rightarrow \log(x^2+y^2+z^2) = \log y + \log c_2$$

$$\Rightarrow \frac{x^2+y^2+z^2}{y} = c_2$$

$$f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{y}\right) = 0$$

4. $yzp + xzq = xy$

$$\Rightarrow \frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

$$\cancel{\frac{dy}{xz} = \frac{dz}{xy}}$$

$$\frac{dx}{yz} = \frac{dz}{xy}$$

$$\Rightarrow \frac{dx}{z} = \frac{dz}{x}$$

$$\Rightarrow x\,dx = z\,dz$$

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + \frac{c}{2} \Rightarrow x^2 - z^2 = c_1$$

$$\frac{dy}{xz} = \frac{dz}{xy} \qquad \text{or} \qquad \frac{dx}{yz} = \frac{dy}{xz}$$

$$\Rightarrow \frac{dy}{z} = \frac{dz}{y} \qquad \Rightarrow x^2 - y^2 = c_5$$

$$\Rightarrow y\,dy = z\,dz$$

$$\cancel{\Rightarrow \frac{y^2}{2} = \frac{z^2}{2}}$$

$$\cancel{\Rightarrow y^2 - z^2 = c_3} \Rightarrow$$

$$\Rightarrow y^2 - z^2 = c_3$$

$$f(x^2 - z^2, x^2 - y^2) = 0$$

21/10/20

Solution of non-linear PDE

Type - 1: Eqn of the form $f(p, q) = 0$ → $pq = 1$

$pq - 1 = 0$

$f(p, q) = 0$

$x^2 p^2 + q^2 = 1$

$\sqrt{p} + \sqrt{q} = 1$

$p^{2/3} + q^{2/3} = 1$

Method:

Let the req. soln be

$$z = ax + by + c$$

$$\frac{dz}{dx} = a, \quad \frac{dz}{dy} = b$$

$$\Rightarrow p = a, \quad q = b$$

$\cancel{pq = 1}$

$$f(p, q) = 0 \Rightarrow f(a, b) = 0$$

$$\Downarrow$$

$$\cancel{g(b) =} \; b = g(a)$$

$$z = ax + by + c$$

$$= ax + g(a) \cdot y + c$$

Ex

$pq = 1$ $\qquad f(p, q) = 0$

$z = ax + by + c$

$p = a, \; q = b$

$$pq = 1 \Rightarrow ab = 1$$

$$\Rightarrow b = \frac{1}{a}$$

$$\therefore \boxed{z = ax + \frac{1}{a} \cdot y + c}$$

Ex $\sqrt{p}+\sqrt{q}=1$, $f(p,q)=0$

$z=ax+by+c$

$\frac{dz}{dx}=a$, $\frac{dz}{dy}=b$

$p=a$, $q=b$

$\sqrt{p}+\sqrt{q}=1 \Rightarrow \sqrt{a}+\sqrt{b}=1$

$\Rightarrow \sqrt{b}=1-\sqrt{a}$

$\Rightarrow b=(1-\sqrt{a})^2$

$\therefore \boxed{z=ax+(1-\sqrt{a})^2 \cdot y+c}$

Ex

$p+q+pq=0$

$\rightarrow z=ax+by+c$

$\frac{dz}{dx}=a$, $\frac{dz}{dy}=b$

$p+q+pq=0 \Rightarrow a+b+ab=0$

$\Rightarrow a+b(1+a)=0$

$\Rightarrow b=\frac{-a}{1+a}$

$\boxed{z=ax+\left(\frac{-a}{1+a}\right)\cdot y+c}$

Ex Special PDE

$z = px + qy + \sqrt{pq} \Rightarrow$ ~~$z = px + qy$~~

$\Rightarrow z = ax + by + \sqrt{ab}$ ✓

$z = px + qy + \frac{pq}{p+q}$

$z = ax + by + \frac{ab}{a+b}$ ✓

$p^2 - q^2 = 1$
non linear

Type - II

equation of the type :-

$z = px + qy + f(p, q)$ $\quad p = a$, $q = b$

$\Rightarrow z = ax + by + f(a, b)$

General PDE → $f(x, y, z, p, q) = 0$

Type III

Equation of the type :-

$f(z, p, q) = 0$ $\quad p(1+q) = qz$
non-linear

$f(z, p, q) = 0$

$p = \frac{dz}{dx}$ $\quad pq, \; p^2, \; q^2, \; q^3$ → Non-linear

Let z be a function of u

$u = x + ay$

$\frac{du}{dx} = 1, \quad \frac{du}{dy} = a$

$p = \frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx} = \frac{dz}{du}$

$q = \frac{dz}{dy} = \frac{dz}{du} \frac{du}{dy} = a \frac{dz}{du}$

$p(1+q) = qz$

$u = x + ay$

$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$

$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$

$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du} z$

$1 + a\frac{dz}{du} = az$

$\Rightarrow a\frac{dz}{du} = az - 1$

$\Rightarrow du = \frac{a\,dz}{az-1}$

$u = \log(az-1) - \log c$

$x + ay = \log c(az-1)$

<u>H|W</u> ~~$p(1+q^2) = q(z-a)$~~

$p(1+q^2) = q\,\underline{\underline{(z-c)}}$

$\rightarrow u = x + ay$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a$

~~$p = \frac{\partial z}{}$~~ $p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$

$\frac{dz}{du}\left(1 + a^2\left(\frac{dz}{du}\right)^2\right) = a\frac{dz}{du}(z-c)$

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$$* \quad p(1+q^2) = q(z-c)$$

$$= f(z,p,q) = 0$$

$$u = x + ay \quad \text{——} \ (1)$$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du} \ , \quad q = a\frac{dz}{du}$$

$$\cancel{\frac{dz}{du}}\left[1 + a^2\left(\frac{dz}{du}\right)^2\right] = a\cancel{\frac{dz}{du}}(z-c)$$

$$\cancel{= \frac{dz}{du} + a^2\left(\frac{dz}{du}\right)^3 = az\frac{dz}{du} - ac\frac{dz}{du}}$$

$$\cancel{= \frac{dz}{du}}\left[1\right. \quad \Rightarrow 1 + a^2\left(\frac{dz}{du}\right)^2 = a(z-c)$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{a(z-c) - 1}{a^2}$$

$$\Rightarrow \frac{dz}{du} = \frac{1}{a}\sqrt{a(z-c)-1}$$

$$\Rightarrow \int \frac{a\,dz}{\sqrt{a(z-c)-1}} = \int du + c_1$$

$$\Rightarrow 2\sqrt{a(z-c)-1} = u + c_1$$

$$= x + ay + c_1$$

[using (1)]

Type IV

$$f_1(x,p) = f_2(y,q)$$

Method: let $f_1(x,p) = f_2(y,q) = a$

$f_1(x,p) = a \Rightarrow p = f_1(x)$

$f_2(y,q) = a \Rightarrow q = f_2(y)$

Ex

$$p - x^2 = q + y^2$$

$\Rightarrow f_1(x,p) = f_2(y,q) = a$

$p - x^2 = a \Rightarrow p = a + x^2$

$q + y^2 = a \Rightarrow q = a - y^2$

$dz = p\,dx + q\,dy$

$\int dz = \int (a+x^2)\,dx + \int (a-y^2)\,dy$

$\Rightarrow z = ax + \frac{x^3}{3} + ay + \frac{y^3}{3} + c$

// **Charpit's Method:**

$$f(x,y,z,p,q) = 0$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z}$$

Rule:

1. write the given problem as $f(x,y,z,p,q) = 0$

Ex $(p^2+q^2)y = qz$

$\Rightarrow (p^2+q^2)y - qz = 0$

find f_p, f_q, f_x, f_y, f_z

$\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

... the charpit's equations and put the value ...

① Linear:-

$xy^2 \cdot z \quad \log(xy)\cdot z$

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} + c(x,y)\cdot z = d(x,y)$$

② semilinear:

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = c(x,y,z)$$

xy^2z^2

$\log(xyz)$

③ Quasilinear:

$$a(x,y,z)\frac{\partial z}{\partial x} + b(x,y,z)\frac{\partial z}{\partial y} = c(x,y,z)$$

(coefficient is z)

$$x^2zp + 2yq = 5xz^2$$

$$x^2zp^2 + 2yq^2 = 5xz^2$$

$$x^2p + 2yq = 5xz$$

$$x^2p + 2yq = 5xz^2 \rightarrow \text{semilinear}$$

Lagrange's method

semiLinear, QuasiLinear, Linear

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

$$p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$u(x,y,z) = c_1 \;;\; u_2(x,y,z) = c_2$$

General solⁿ:

$$\boxed{f(u_1,u_2) = 0 \quad \text{or} \quad u_1 = f(u_2)}$$

$$xp + yq = z$$

Auxiliary eqⁿ

$P = x$

$Q = y$

$R = z$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} + c$$

$$\Rightarrow \log x = \log y + c_1$$

$$x = y c_1$$

$$c_1 = \frac{x}{y}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\Rightarrow \log y = \log z + \log c_2$$

$$\Rightarrow y = z c_2$$

$$c_2 = \frac{y}{z}$$

$$\boxed{\frac{x}{y} = \frac{y}{z} \Rightarrow y^2 = xz}$$

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

OR

$$\frac{x}{y} = f\left(\frac{y}{z}\right)$$

$$y^2 p - xyq = x(z - 2y)$$

$P = y^2$

$Q = -xy$

$R = x(z - 2y)$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\int \frac{dx}{y^2} = \int \frac{dy}{-xy}$$

$$\rightarrow \int \frac{dx}{y} = \int -\frac{dy}{x} \Rightarrow \int x\,dx = -\int y\,dy$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$\Rightarrow x^2 + y^2 = c_1$$

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$\Rightarrow \frac{dy}{-y} = \frac{dz}{z-2y}$$

$$\Rightarrow z\,dy - 2y\,dy = -y\,dz$$

$$\Rightarrow z\,dy + y\,dz = 2y\,dy$$

$$\Rightarrow d(yz) = d(y^2)$$

$$\Rightarrow yz = y^2 + c_2$$

$$\Rightarrow yz - y^2 = c_2$$

$$f(x^2+y^2,\ yz-y^2)=0.$$

$$x^2+y^2=f_2(yz-y^2).$$

// $z(x+y)p+z(x-y)q=x^2+y^2$

$$\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2}$$

$$\Rightarrow \frac{dx}{\cancel{z}(x+y)}=\frac{dy}{\cancel{z}(x-y)}$$

$$\rightarrow \boxed{\frac{dy}{dx}=\frac{x-y}{x+y}}$$

$$y=vx$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1-v}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v}{1+v}-v$$

$$=\frac{1-v-v-v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-2v-v^2}{1+v}$$

$$\Rightarrow \frac{(1+v)\,dv}{1-2v-v^2}=\frac{dx}{x}$$

$$\Rightarrow \frac{-2(1+v)\,dv}{-2(1-2v-v^2)}=\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\log(1-2v-v^2)=\log x+\log c$$

// $z=y^2+2f\left(\frac{1}{x}+\log y\right)$

$$\frac{\partial z}{\partial x}=0+2f'\left(\frac{1}{x}+\log y\right)\frac{\partial}{\partial x}\left(\frac{1}{x}+\log y\right)$$

$$=2f'\left(\frac{1}{x}+\log y\right)\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow p=\frac{2f'\left(\frac{1}{x}+\log y\right)}{-x^2}$$

$$\frac{\partial z}{\partial y}=2y+2f'\left(\frac{1}{x}+\log y\right)\frac{\partial}{\partial y}\left(\frac{1}{x}+\log y\right)$$

$$q=2y+2f'\left(\frac{1}{x}+\log y\right)\cdot\frac{1}{y}$$

$-x^2 p = 2f'\left(\frac{1}{x} + \log y\right)$ | $yq - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right)$

$\Rightarrow -x^2 p = yq - 2y^2$

$\Rightarrow \boxed{x^2 p + yq = 2y^2}$


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4. Choose two proper fractions from Charpit's auxiliary equations and find p and q.

5. put the values of p and q in $dz = p\,dx + q\,dy$ and solve.

Ex solve $3p^2 = q$

$f(p,q) = 0$

$3p^2 - q = 0$ —— (1)

$f = 3p^2 - q$

$f_x = 0,\ f_y = 0,\ f_z = 0,\ f_p = 6p,\ f_q = -1$

$$\frac{dx}{6p} = \frac{dy}{-1} = \frac{dz}{6p^2 - q} = \frac{-dp}{0} = \frac{-dq}{0}$$

$dp = 0 \Rightarrow p = a$

put $p = a$ in (1), we get:

$3a^2 - q = 0$

$\Rightarrow q = 3a^2$

$\int dz = \int a\,dx + \int 3a^2\,dy$

$\Rightarrow z = ax + 3a^2 y + c$

2.

$p^2 - y^2 q = y^2 - x^2$

$\Rightarrow p^2 - y^2 q - y^2 + x^2 = 0$ —— (1)

$f_p = 2p,\ f_q = -y^2,\ f_x = 2x,\ f_y = -2yq - 2y$

$f_z = 0$

$$\frac{dx}{2p} = \frac{dy}{-y^2} = \frac{dz}{2p^2 - qy^2} = \frac{-dp}{2x + 0} = \frac{-dq}{-2yq - 2y + 0}$$

$$\frac{dx}{\cancel{2}p} = \frac{-dp}{\cancel{2}x} \Rightarrow x\,dx + p\,dp = 0$$

$\Rightarrow x^2 + p^2 = c^2$

$P = \sqrt{c^2 - x^2}$

Put the value of p in ①, we get

$(\sqrt{c^2 - x^2})^2 - y^2 q - y^2 + x^2 = 0$

$\Rightarrow c^2 - \cancel{x^2} - y^2 q - y^2 + \cancel{x^2} = 0$

$\Rightarrow q = \dfrac{c^2 - y^2}{y^2}$

$q = c^2 y^{-2} - 1$

$\int dz = \int (\sqrt{c^2 - x^2}) dx + (c^2 y^{-2} - 1) dy$

$\cancel{\Rightarrow z = \frac{c}{2} \sqrt{c^2 - x^2}}$

$\Rightarrow z = \dfrac{x\sqrt{c^2 - x^2}}{2} + \dfrac{c^2}{2} \sin^{-1}\left(\dfrac{x}{c}\right)$

$+ \dfrac{c^2 y^{-2+1}}{-2+1} - y + c_1$

$\Rightarrow z = \dfrac{x\sqrt{c^2 - x^2}}{2} + \dfrac{c^2}{2} \sin^{-1}\left(\dfrac{x}{c}\right) + c^2\left(\dfrac{-1}{y}\right) - y + c_1$

H/W

$$(p^2+q^2)y = qz$$

$$\Rightarrow (p^2+q^2)y - qz = 0 \quad \text{--- (1)}$$

$$f = p^2y + q^2y - qz$$

$$f_p = 2py, \quad f_q = 2qy - z, \quad f_x = 0, \quad f_y = p^2+q^2$$

$$f_z = -q$$

$$\frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{2p^2y + q(2qy-z)} = \frac{dp}{0+(-pq)} = \frac{-dq}{p^2+\cancel{q^2}-\cancel{q^2}}$$

$$\frac{dp}{\cancel{p}q} = \frac{-dq}{\cancel{p^2}\ p}$$

$$\Rightarrow \frac{dp}{q} = \frac{dq}{p}$$

$$\Rightarrow p\,dp + q\,dq = 0$$

$$\Rightarrow p^2 + q^2 = c^2$$

$$\Rightarrow p = \sqrt{c^2 - q^2}$$

~~α~~

$$\left(\left(\sqrt{c^2-q^2}\right)^2 + q^2\right)y - qz = 0$$

$$\Rightarrow (c^2 - \cancel{q^2} + \cancel{q^2})y - qz = 0$$

$$\Rightarrow c^2y - qz = 0$$

$$\Rightarrow q = \frac{c^2y}{z}$$

$$\therefore p = \sqrt{c^2 - \frac{c^4y^2}{z^2}}$$

$$= \sqrt{\frac{c^2z^2 - c^4y^2}{z^2}}$$

$$p = \sqrt{\frac{c^2(z^2 - c^2y^2)}{z^2}}$$

$$= \frac{c}{z}\sqrt{z^2 - c^2y^2}$$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{c}{z}\sqrt{z^2 - c^2y^2}\,dx + \frac{c^2y}{z}\,dy$$

$$\Rightarrow z\,dz = c\sqrt{z^2 - c^2y^2}\,dx + c^2y\,dy$$

$$\Rightarrow \frac{z\,dz - c^2y\,dy}{\sqrt{z^2 - c^2y^2}} = c\,dx$$

$$\Rightarrow \frac{1}{2}\frac{d(z^2 - c^2y^2)}{\sqrt{z^2 - c^2y^2}} = c\,dx$$

$$\Rightarrow \sqrt{z^2 - c^2y^2} = cx + d$$

$$\Rightarrow z^2 - c^2y^2 = (cx + d)^2$$

where c and d are constants.

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Ex

$\cancel{\frac{xy^2}{y^2}}$ $\cancel{\frac{x^3}{y^2}}$

$$x(x^2+y^2+z^2) = cy^2$$

$$\Rightarrow f(x,y,z) = \frac{x(x^2+y^2+z^2)}{y^2} = c$$

$$\cancel{\frac{\partial f}{\partial x} = \frac{x^3\times 0 - y^2\cdot 3x^2}{y^4}} \quad , \quad \cancel{\frac{\partial f}{\partial y} = 0}$$

$$\cancel{= \frac{-3x^2y^2}{y^4}}$$

$$\cancel{= \frac{-3x^2}{y^2}}$$

$$\cancel{\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{xz^2}{y^2}\right)}$$

$$\cancel{= 2zx}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3}{y^2} + x + \frac{xz^2}{y^2}\right)$$

$$= \frac{3x^2}{y^2} + 1 + \frac{z^2}{y^2} \cancel{\frac{z^2\times 0}{y^2}} \Rightarrow \frac{3x^2+y^2+z^2}{y^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3}{y^2} + x + \frac{xz^2}{y^2}\right)$$

$$= -x^3 2y^{-3} + 0 + \left(-2y^{-3}xz^2\right)$$

$$= -\frac{2x^3}{y^3} - \frac{2xz^2}{y^3} \Rightarrow \frac{-2x(x^2+z^2)}{y^3}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x^3}{y^2} + x + \frac{xz^2}{y^2}\right)$$

$$= 0 + 0 + \frac{2zx}{y^2}$$

$$= \frac{2zx}{y^2}$$

$$\Rightarrow \frac{dx}{\frac{3x^2+y^2+z^2}{y^2}} = \frac{dy}{\frac{-2x(x^2+z^2)}{y^3}} = \frac{dz}{\frac{2zx}{y^2}}$$

$$\Rightarrow \frac{dx}{y(3x^2+y^2+z^2)} = \frac{dy}{-2x(x^2+z^2)} = \frac{dz}{2zxy}$$

$$\cancel{\frac{dx}{3x^2+y^2+z^2} = \frac{dz}{2zx}}$$

$$\cancel{xy(x^2+z^2+y^2) \quad x^3y - xyz^2 + xy^3}$$

$$\cancel{\frac{ydy+zdz}{-2x^3y} \quad \frac{xdx+ydy}{3x^3y+xy^3+xyz^2} \quad -2x^3y - 2xyz^2}$$

$$\cancel{\Rightarrow 2zx\,dx = (3x^2+y^2+z^2)\,dz}$$

$$\cancel{\Rightarrow zx^2 = 3x^2z + y^2z + \frac{z^3}{3}}$$

$$\Rightarrow \frac{x\,dx}{xy(3x^2+y^2+z^2)} = \frac{y\,dy}{-2xy(x^2+z^2)} = \frac{z\,dz}{2z^2xy}$$

$$\frac{x\,dx + y\,dy + z\,dz}{3x^3y + xy^3 + xz^2y - 2x^3y - \cancel{2xyz^2} + \cancel{2xz^2y}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^3y + xy^3 + xz^2y}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{xy(x^2+z^2+y^2)}$$

$$= \frac{1}{2}\frac{d(x^2+y^2+z^2)}{xy(x^2+y^2+z^2)} = \frac{z\,dz}{2z^2xy}$$

$$\Rightarrow \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)} = \frac{dz}{z}$$

$$\Rightarrow \ln(x^2+y^2+z^2) = \ln z + \ln c_1$$

$$\Rightarrow x^2+y^2+z^2 = z c_1$$

$$\Rightarrow \frac{x^2+y^2+z^2}{z} = c_1$$

$$\frac{4x\,dx + 2y\,dy}{12x^3y + 4xy^3 + 4\cancel{xy}z^2 - 4x^3y - 4\cancel{xy}z^2}$$

$$= \frac{d(2x^2+y^2)}{2\cancel{4}xy(2x^2+y^2)} = \frac{dz}{\cancel{2}xyz}$$

$$\Rightarrow \cancel{\text{too } \ln(2x^2+y^2)}.$$

$$\Rightarrow \frac{1}{2}\ln(2x^2+y^2) = \ln z + \tfrac{1}{2}\ln c_2$$

$$\Rightarrow \ln(2x^2+y^2) = \ln z^2 + \ln c_2$$

$$\Rightarrow 2x^2+y^2 = z^2 c_2$$

$$\Rightarrow \frac{2x^2+y^2}{z^2} = c_2$$

$$\phi\left(\frac{x^2+y^2+z^2}{z}, \frac{2x^2+y^2}{z^2}\right) = 0$$

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1. $z^2(p^2z^2+q^2)=1$

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{-dp}{f_x+pf_z}=\frac{-dq}{f_y+qf_z}$$

$$f=z^2(p^2z^2+q^2)-1$$

$$f_x=0\ ;\ f_y=0\ ;\ f_z=\frac{\partial}{\partial z}\left(z^4p^2+z^2q^2-1\right)$$

$$=4z^3p^2+2zq^2$$

$$f_p=2pz^4\ ;\ f_q=2qz^2$$

$$\frac{dx}{2pz^4}=\frac{dy}{2qz^2}=\frac{dz}{2p^2z^4+2q^2z^2}=\frac{-dp}{4z^3p^3+2zpq^2}$$

$$=\frac{-dq}{4z^3p^2q+2zq^3}$$

$$\frac{+dp}{4z^3p^3+2zpq^2}=\frac{+dq}{4z^3p^2q+2zq^3}$$

$$\Rightarrow (4z^3p^2q+2zq^3)\,dp=(4z^3p^3+2zpq^2)\,dq$$

$$\Rightarrow \frac{4z^3p^3q}{3}+2zq^3p=4z^3p^3q+\frac{2zpq^3}{3}$$

$$\Rightarrow 4z^3p^3q+6zq^3p=12z^3p^3q+2zpq^3$$

$$\Rightarrow \cancel{4}zq^3p=\overset{2}{\cancel{8}}z^3p^3q$$

$$\Rightarrow \cancel{z}\,\overset{q^2}{\cancel{q^3}}\,\cancel{p}=2\,\overset{z^2}{\cancel{z^3}}\,\overset{p^2}{\cancel{p^3}}\,\cancel{q}$$

$$\Rightarrow q^2=2z^2p^2 \qquad q=\sqrt{2z^2p^2}$$

$$\frac{dp}{p(4z^3p^2+2zq^2)} = \frac{dq}{q(4z^3p^2+2zq^2)}$$

$$\Rightarrow p = aq$$

Lagrange - quasilinear

$$z^2(p^2z^2+q^2) = 1$$

$$\cancel{\Rightarrow z^2(a^2q^2z^2+q^2) = 1}$$

$$\cancel{\Rightarrow z^2a^2q^2z^4+z^2q^2 = 1}$$

$$\Rightarrow p^2z^2+q^2 = \frac{1}{z^2}$$

$$\Rightarrow q^2(a^2z^2+1) = \frac{1}{z^2}$$

$$\Rightarrow q = \cancel{\frac{1}{z}\sqrt{a^2z^2+1}} \quad \frac{1}{z\sqrt{a^2z^2+1}}$$

$$p = aq$$
$$= \cancel{a\frac{1}{z}\sqrt{a^2z^2+1}} = \frac{a}{z\sqrt{a^2z^2+1}}$$

$$dz = p\,dx + q\,dy$$
$$= \frac{a}{z\sqrt{a^2z^2+1}}\,dx + \frac{1}{z\sqrt{a^2z^2+1}}\,dy$$

$$\Rightarrow z\sqrt{a^2z^2+1}\,dz = a\,dx + dy$$

$$\Rightarrow a^2z^2+1 = t \qquad \Rightarrow \frac{1}{2a^2}\sqrt{t}\,dt = a\,dx + dy$$

$$\Rightarrow 2a^2z\,dz = dt \qquad \Rightarrow \frac{2}{3}\frac{t^{3/2}}{2a^2} = ax+y+b$$

$$\Rightarrow \frac{2}{3}\frac{(a^2z^2+1)^{3/2}}{2a^2} = ax+y+b$$

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Canonical forms

1. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$

$\rightarrow$

$A = 3 \quad B = 10 \quad C = 3$

$\Delta = B^2 - 4AC$

$\Delta = 100 - 4 \times 3 \times 3$

$\Delta = 100 - 36$

$\Delta = 64 > 0$

>0 case

$3\lambda^2 + 10\lambda + 3 = 0$

$\Rightarrow 3\lambda^2 + 9\lambda + \lambda + 3 = 0$

$\Rightarrow 3\lambda(\lambda + 3) + (\lambda + 3) = 0$

$\Rightarrow (3\lambda + 1)(\lambda + 3) = 0$

$\lambda_1 = -1/3 \quad \lambda_2 = -3$

$\frac{dy}{dx} - \frac{1}{3} = 0 \qquad \frac{dy}{dx} - 3 = 0$

$\Rightarrow y - \frac{1}{3}x = c_1 \quad , \quad \Rightarrow y - 3x = c_2$

$\xi = y - \frac{1}{3}x \quad , \quad \eta = y - 3x$

$\xi_x = -\frac{1}{3} \qquad \eta_x = -3$

$\xi_y = 1 \qquad \eta_y = 1$

$$u_{xx} = u_{\xi\xi}\left(-\frac{1}{3}\right)^2 + 2u_{\xi\eta}\left(-\frac{1}{3}\right)(-3) + u_{\eta\eta}(-3)^2$$

$$+ u_{\xi}\xi_{xx} + u_{\eta}\eta_{xx}$$

$$\cancel{= \frac{1}{9}u_{\xi\xi} + 2u_{\xi\eta} + 9u_{\eta\eta}}$$

$$u_{xx} = u_{\xi\xi}\frac{1}{9} + 2u_{\xi\eta} + 9u_{\eta\eta}$$

$$u_{xy} = -u_{\xi\xi}\frac{1}{3} + u_{\xi\eta}\left(-\frac{1}{3} - 3\right) + u_{\eta\eta}(-3)$$

$$= -\frac{1}{3}u_{\xi\xi} - \frac{10}{3}u_{\xi\eta} - 3u_{\eta\eta}$$

$$\frac{-1-9}{3}$$

$$\frac{1}{3} + 3 = \frac{10}{3}$$

$$\frac{1+9}{3} = \frac{10}{3}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

$$\Rightarrow 3\left(\frac{1}{9}u_{\xi\xi} + 2u_{\xi\eta} + 9u_{\eta\eta}\right)$$

$$+ 10\left(-\frac{1}{3}u_{\xi\xi} - \frac{10}{3}u_{\xi\eta} - 3u_{\eta\eta}\right)$$

$$+ 3\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) = 0$$

$$\frac{18 - 100 + 6}{3}$$

$$-\frac{100}{3}$$

$$\frac{1}{3} + 3 \quad \frac{1+9}{3}$$

$$27 + 12 = 39, \quad 39 - 10 = 29$$

$$30 - 112 = 28$$

$$\Rightarrow \frac{1}{3}u_{\xi\xi} + 6u_{\xi\eta} + 27u_{\eta\eta} - \frac{10}{3}u_{\xi\xi}$$

$$-\frac{100}{3}u_{\xi\eta} - 30u_{\eta\eta} + 3u_{\xi\xi} + 6u_{\xi\eta} + 12u_{\eta\eta}$$

$$\cancel{= -124 u_{\xi\eta} + 294 u_{\eta\eta} +}$$

$$\cancel{= 84 u_{\xi\eta} + 294 u_{\eta\eta}}$$

$$\Rightarrow -\frac{64}{3} u_{\xi\eta} = 0$$

$$\Rightarrow u_{\xi\eta} = 0$$

$\frac{\begin{matrix}100 \\ 36\end{matrix}}{64}$

$12 - \frac{100}{3}$

$\frac{36 - 100}{3}$

$\frac{64}{3}$

$$\cancel{2.\ 4xx + 44xy + 44yy = 0}$$

$$\cancel{A = 1 \quad B = 4 \quad C = 0}$$

$$\cancel{\Delta \Rightarrow B^2 - 4AC}$$

$$\cancel{\Delta \Rightarrow 16 - 0}$$

$$\cancel{\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda = -2}$$

$$\cancel{\frac{dy}{dx} - 2}$$

Canonical forms or normal forms

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = (u_x)_x = (u_x)_\xi \xi_x + (u_x)_\eta \eta_x$$

$$= (u_\xi \xi_x + u_\eta \eta_x)_\xi \xi_x + (u_\xi \xi_x + u_\eta \eta_x)_\eta \eta_x$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{yy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$\bar{A}(\xi_x,\xi_y)u_{\xi\xi} + \bar{B}(\xi_x,\xi_y,\eta_x,\eta_y)u_{\xi\eta} + \bar{C}(\eta_x,\eta_y)u_{\eta\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$$

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$\bar{B}^2 - 4\bar{A}\bar{C} = (\xi_x \eta_y - \xi_y \eta_x)^2 \, (B^2 - 4AC)$$

$\xi = x + y$ $\quad \eta = c\xi$

$\eta = x - y$

Example:

1. $u_{xx} = x^2 u_{yy}$

$\rightarrow A = 1, \; B = 0, \; C = -x^2$

$\Delta = 4x^2$

$x^2 > 0 \longrightarrow$ case

$A\alpha^2 + B\alpha + C = 0$

$\Rightarrow \alpha^2 - x^2 = 0$

$\alpha = \pm x$

$\lambda_1 = x, \; \lambda_2 = -x$

$\frac{dy}{dx} \pm \lambda_i = 0$

$\frac{dy}{dx} + x = 0 \Rightarrow y + \frac{x^2}{2} = c_1$

$\frac{dy}{dx} - x = 0 \Rightarrow y - \frac{x^2}{2} = c_2$

$\xi = y + \frac{x^2}{2}, \; \eta = y - \frac{x^2}{2}$

$\xi_x = x \quad \xi_y = 1$

$\eta_x = -x, \; \eta_y = 1$

$u_{xx} = x^2 u_{yy}$

$\Rightarrow \cancel{u_{\xi\xi} x^2} - 2u_{\xi\eta} x^2 + \cancel{u_{\eta\eta} x^2}$
$+ u_\xi - u_\eta$
$= x^2 (\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}})$

$\Rightarrow -4u_{\xi\eta} x^2 + u_\xi - u_\eta = 0$

$$u_{\xi\eta} = \frac{u_\xi - u_\eta}{4x^2} = \frac{u_\xi - u_\eta}{4(\xi - \eta)}$$

Case:

$B^2-4AC<0$

$A\alpha^2+B\alpha+C=0$

λ_1, λ_2

(*) $u_{xx}+x^2u_{yy}=0, \quad x\neq 0$

$B^2-4AC = -4x^2<0$

$\alpha^2+x^2=0 \Rightarrow \alpha^2=-x^2$

$\alpha = \pm ix$

$\frac{dy}{dx}+ix=0$

$\Rightarrow y+\frac{ix^2}{2}=c_1$

$\alpha = y+\frac{ix^2}{2}$

$y-ix^2=c_2$

$\beta = y-\frac{ix^2}{2}$

$\xi = \frac{1}{2}(\alpha+\beta)$

$= y$

$\eta = \frac{1}{2i}(\alpha-\beta)$

$= \frac{1}{2i}\left(\cancel{y}+\frac{ix^2}{2}-\cancel{y}+\right.$

$= \frac{1}{\cancel{2i}}\times\frac{\cancel{2i}x^2}{2}$

$= \frac{x^2}{2}$

$\xi_x=0 \quad \xi_y=1$

$\eta_x=x \quad \eta_y=\cancel{1}\,0$

$u_{xx}=u_{\eta\eta}x^2+u_\eta$

$u_{yy}=u_{\xi\xi}$ *

$\Rightarrow u_{\eta\eta}x^2+u_\eta+x^2u_{\xi\xi}=0$

$\Rightarrow x^2(u_{\eta\eta}+u_{\xi\xi})+u_\eta=0$

$\Rightarrow u_{\xi\xi}+u_{\eta\eta}=0$

2. $yu_{xx} + u_{yy} = 0$ $y > 0$

0 -

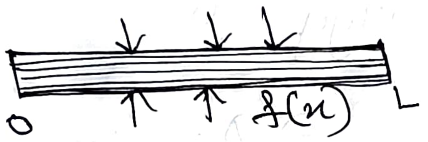
α^4 -

3. $u_t = \alpha^2 u_{xx}$, $0 \le x \le L$, $0 < t < \infty$

Boundary cond^n (separation of variable)

$u(0,t) = 0$, $u(L,t) = 0$ $u(x,t)$

I.C $u(x,0) = f(x)$, $0 \le x \le L$
(initial cond^n)



$u_t = \alpha^2 u_{xx}$

$u(x,t) = X(x) \cdot T(t)$

$u_t = X(x)\, T'(t)$ $T' = \frac{d}{dt} T$

$u_{xx} = X'' T$ $X'' - cX = 0$

$XT' = \alpha^2 X'' T$ $T' - c\alpha^2 T = 0$

$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = c$

$X'' - cX = 0$

$X(0) = 0$

$X(L) = 0$

$u(0,t) = 0$
$X(0) \cdot T(t) = 0$
$X(0) = 0$
$u(L,t) = 0$
$X(L) \cdot T(t) = 0$
$X(L) = 0$

$c = (\lambda^2) > 0$

$X'' - \lambda^2 X = 0$ | $m^2 - \lambda^2 = 0$

$\lambda, -\lambda$

$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0 \Rightarrow c_1 (e^{2\lambda L} - 1) = 0$

$c_1 = 0$

$c_1 = 0 \Rightarrow c_2 = 0$

$\cancel{\forall t}$ $X(x) = 0$

$u(x,t) = 0$ ×

$c = 0$ $X'' = 0$

$X' = c_3$

$X = c_3 x + c_4$

$c_3 = c_4 = 0$

$X(0) = 0$

$X(L) = 0$

$u(x,t) = 0$

$X(x) = 0$

×

Case: parabolic

$u_{xx} + 4u_{xy} + 4u_{yy} = 0$

$\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda = -2$

$\frac{dy}{dx} - 2 = 0$

$\Rightarrow y - 2x = c_1$

$\xi_x = -2$

$\xi_y = 1$

$\eta_x = 0$

$\eta_y = 1$

$u_{\eta\eta} = f(\cdot)$

or

$u_{\xi\xi} = f(\cdot)$

$\xi = y - 2x$

$\eta = y$

$u_{xx} = 4u_{\xi\xi} +$

$u_{xy} = -2u_{\xi\xi} - 2u_{\xi\eta}$

~~u_y~~ ~~u_x~~

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$\Rightarrow$ putting the values

$u_{xx} + 4u_{xy} + 4u_{yy} = 0$

$\Rightarrow 4u_{\xi\xi} + 4(-2u_{\xi\xi} - 2u_{\xi\eta}) + 4(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$

$= \cancel{4u_{\xi\xi}} - \cancel{8u_{\xi\xi}} - \cancel{8u_{\xi\eta}} + \cancel{4u_{\xi\xi}} + \cancel{8u_{\xi\eta}} + \cancel{4u_{\eta\eta}}\ 4u_{\eta\eta}$

$\Rightarrow 4u_{\eta\eta} = 0$

$u_{\eta\eta} = 0$ $\quad \therefore$ parabolic in nature.

continuation.

$c < 0 \quad c(= -\lambda^2) < 0 \quad , \lambda > 0$

$X'' - cX = 0$ $\quad \rightarrow m^2 + \lambda^2 = 0,$

$m = \pm i\lambda$

$\Rightarrow X'' + \lambda^2 X = 0$ $\quad \Rightarrow X(x) = C_5 \cos\lambda x + C_6 \sin\lambda x$

$$\boxed{\begin{array}{c} d^2y + a^2y = 0 \\ y = e^{mx} \\ m^2 + a^2 = 0 \end{array}}$$

$X(0) = 0 \quad \Rightarrow C_5 = 0$

$X(x) = C_6 \sin\lambda x$

$X(L) = 0$

$C_6 \sin\lambda L = 0$

$\sin\lambda L = 0$

$\lambda L = n\pi \quad , n = 0, \pm 1, \pm 2, \ldots$

$\lambda = \frac{n\pi}{L} \quad , n = 1, 2, 3, \ldots$

$X(x) = a_n \sin\frac{n\pi}{L}x$ $\quad\quad C_6 \sin\lambda x$

$n = 1, 2, 3, \ldots$ $\quad\quad = C_6 \sin\frac{n\pi}{L}x$

$T' - c\alpha^2 T = 0$ $\quad\quad T = b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$

$\frac{T'}{T} = -\lambda^2\alpha^2$

$\ln T = -\lambda^2\alpha^2 t + c_7$ $\quad\quad u_n(x,t) = X_n(x)\, T_n(t)$

$$\begin{cases} u_t = 16 u_{xx}, & 0 < x < 1,\ t > 0 \\ u(0,t) = 0, \quad u(1,t) = 0, & t > 0 \\ u(x,0) = (1-x)x, & 0 < x < 1. \\ \quad = x - x^2 \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

I.C $\quad u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

H/w $\quad f(x) = 20$

$f(x) = 6 \sin \frac{\pi n}{L} x$

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wave equation.

$$u_{tt} = \alpha^2 u_{xx}$$

$B^2 - 4AC$ $\Rightarrow \alpha^2 u_{xx} - u_{tt} = 0$ General form of pde

$= 0 + 4\alpha^2 > 0$ Hyperbolic $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x$

$+ Eu_y + Fu = G$

Seperation of variables method

$$u(x,t) = X(x)\, T(t)$$

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial}{\partial t}\left[X(x)\, T(t)\right]$$

$$= X(x) \frac{\partial}{\partial t}\left[T(t)\right]$$

$$= X(x)\, T'(t)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(u_t) = \frac{\partial}{\partial t}\left[X(x)\, T'(t)\right]$$

$$= X(x) \frac{\partial}{\partial t}\left[T'(t)\right]$$

$$= X(x)\, T''(t)$$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial}{\partial x}\left[X(x) T(t)\right]$$

$$= T(t) \frac{\partial}{\partial x}\left[X(x)\right]$$

$$= T(t) \cdot X'(x)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}\left[T(t) \cdot X'(x)\right]$$

$$= T(t) \cdot \frac{\partial}{\partial x}\left[X'(x)\right]$$

$$= T(t) \cdot X''(x)$$

Substituting these values, we get,

$$X(x) \cdot T''(t) = \alpha^2 \left[T(t) X''(x) \right]$$

$$\Rightarrow \frac{T''(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda^2$$

$$\frac{X''(x)}{X(x)} = \lambda^2$$

$$\Rightarrow X''(x) - \lambda^2 X(x) = 0$$

$$\Rightarrow X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

A·E $m^2 - \lambda^2 = 0$

$$\Rightarrow m = \pm \lambda$$

Case (1): $\lambda^2 > 0$

$$\frac{T''(t)}{\alpha^2 T(t)} = \lambda^2$$

$$\Rightarrow T''(t) - \lambda^2 \alpha^2 T(t) = 0$$

$$T(t) = c_3 e^{\lambda \alpha t} + c_4 e^{-\lambda \alpha t}$$

$$u(x,t) = X(x)\, T(t)$$

$$= \left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right)\left(c_3 e^{\lambda \alpha t} + c_4 e^{-\lambda \alpha t}\right)$$

Case II when constant $= 0$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{\alpha^2 T(t)} = 0$$

$$\Rightarrow X''(x) = 0$$

$$\Rightarrow X(x) = c_5 x + c_6$$

$$\frac{d^2X}{dx^2} = 0$$

integrating

$$\frac{dX}{dx} = A$$

integrating

$$X(x) = Ax + B$$

$$T''(t) = 0$$

$$\Rightarrow T(t) = c_7 t + c_8$$

$$u(x,t) = (c_5 x + c_6)(c_7 t + c_8)$$

Case (3): when the constant is -ve

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{\alpha^2 T(t)} = -\lambda^2$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$A.E = m^2 + \lambda^2 = 0$$

$$m = \pm \lambda i$$

$$X(x) = c_9 \cos\lambda x + c_{10} \sin\lambda x$$

$$T''(t) + \alpha^2\lambda^2 T(t) = 0$$

$$AE = m_1^2 + \alpha^2\lambda^2 = 0$$

$$\Rightarrow m_1 = \pm \alpha\lambda i$$

$$T(t) = c_{11}\cos(\alpha\lambda t) + c_{12}\sin(\alpha\lambda t)$$

$$u(x,t) = X(x)\,T(t)$$

$$= (c_9\cos\lambda x + c_{10}\sin\lambda x)\left(c_{11}\cos(\alpha\lambda t) + c_{12}\sin(\alpha\lambda t)\right)$$

Vector Algebra

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What is a vector

vector by $\vec{a}$

magnitude $= |\vec{a}|$

let a unit vector be $\hat{a}$

$|\hat{a}| = 1$

suppose you have

$\vec{a} = 3\hat{i} + 4\hat{j}$

$\hat{i}$ = unit vector along x-axis

$\hat{j}$ = unit vector along y-axis

$\hat{k}$ = unit vector along z-axis.

Find out an unit vector in the direction of $\vec{a}$.

$\vec{a} = 3\hat{i} + 4\hat{j}$

→ unit vector along $\vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i} + 4\hat{j} + 0\hat{k}}{\sqrt{3^2 + 4^2 + 0^2}}$

$= \frac{3\hat{i} + 4\hat{j} + 0\hat{k}}{5}$

let a vector be $x\hat{i} + y\hat{j} + z\hat{k}$, then magnitude of $\vec{b} = |\vec{b}|$

$= \sqrt{x^2 + y^2 + z^2}$

Addition of vectors

let, $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{A} + \vec{B} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

Linear combination of vectors :-

A vector $\vec{r}$ is said to be the linear combination of vectors $\vec{a}, \vec{b}, \vec{c}$ if there exists some scalars α, β, γ such that

$\vec{r} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$

Linearly Dependent vectors:

$\vec{a}$, $\vec{b}$ and $\vec{c}$ vectors are said to be linearly dependent vectors if there exist scalars α, β, γ (not all zero)

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$$

Linearly Independent vectors:

$\vec{a}$, $\vec{b}$ and $\vec{c}$ vectors are said to be linearly independent vectors if there exist scalars α, β and γ (all zero) such that

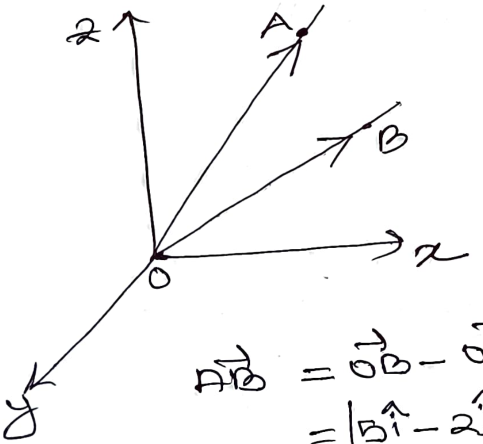
$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$$

$$\alpha = \beta = \gamma = 0$$

Q If the position vectors of A and B with respect to be $(\hat{i} + 3\hat{j} - 7\hat{k})$ and $(5\hat{i} - 2\hat{j} + 4\hat{k})$ respectively. Then find $\vec{AB}$. Also find the unit vector parallel to $\vec{AB}$.

→ $\vec{OA} = (\hat{i} + 3\hat{j} - 7\hat{k})$

$\vec{OB} = (5\hat{i} - 2\hat{j} + 4\hat{k})$



$\vec{AB} = \vec{OB} - \vec{OA}$

$= (5\hat{i} - 2\hat{j} + 4\hat{k}) - (\hat{i} + 3\hat{j} - 7\hat{k})$

$= (4\hat{i} - 5\hat{j} + 11\hat{k})$

Magnitude $= \sqrt{162} = 9\sqrt{2}$.

Vectors

Q show that the vectors $(2,4,10)$ and $(3,6,15)$ are linearly dependent

→ Let $\vec{a} = 2\hat{i} + 4\hat{j} + 10\hat{k}$

and $\vec{b} = 3\hat{i} + 6\hat{j} + 15\hat{k}$

Let us consider two scalars α and β such that

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

implies $\alpha(2\hat{i} + 4\hat{j} + 10\hat{k}) + \beta(3\hat{i} + 6\hat{j} + 15\hat{k})$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$2\alpha + 3\beta = 0$

$4\alpha + 6\beta = 0$

$10\alpha + 15\beta = 0$

$\beta = \frac{-2}{3}\alpha$

If $\alpha = 1$

then $\beta = -\frac{2}{3}$

If $\alpha = 3$

$\beta = -2$

Thus, we are getting non zero scalars.

The given vectors are linearly dependent.

Q show that the vectors $(2,1,2)$ and $(8,4,8)$ are linearly dependent.

→ Let $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$

and $\vec{b} = \cancel{2\hat{i} + 6\hat{j}}\ 8\hat{i} + 4\hat{j} + 8\hat{k}$

Let us consider two scalars α and β such that

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

implies $\alpha(2\hat{i} + \hat{j} + 2\hat{k}) + \beta(8\hat{i} + 4\hat{j} + 8\hat{k})$

$$= 0\hat{j} + 0\hat{j} + 0\hat{k}$$

Same as above

Q Show that the vectors (1, 2, 3) and (4, −2, 7) are linearly independent.

→ Let α, β be 2 scalars and $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\vec{b} = 4\hat{i} - 2\hat{j} + 7\hat{k}$

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

$$\Rightarrow \alpha(\hat{i} + 2\hat{j} + 3\hat{k}) + \beta(4\hat{i} - 2\hat{j} + 7\hat{k}) = \vec{0}$$

$$\left.\begin{aligned} \alpha + 4\beta = 0 &\Rightarrow \alpha = -4\beta \\ 2\alpha - 2\beta = 0 &\Rightarrow \alpha = \beta \\ 3\alpha + 7\beta = 0 &\Rightarrow \alpha = -\tfrac{7}{3}\beta \end{aligned}\right\} \text{The only possibility is } \alpha = \beta = 0$$

Thus, the vectors are L.I.

Dot product / scalar product:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

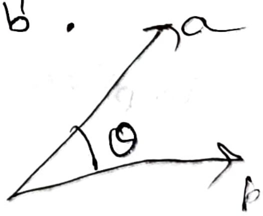
$\vec{b} = b_1\hat{i} + b\hat{j} + b_3\hat{k}$

$\vec{a}\cdot\vec{b} = (a_1\hat{j} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

or

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$, θ is the angle in betn $\vec{a}$ and $\vec{b}$.



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wave Eqn

Prob 1

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions

$u(0,t) = 0$

$u(l,t) = 0$

Initial conditions $u(x,0) = 0$

$u_t(x,0) = \lambda x(l-x)$

~~Thes~~ Three solutions depending on 3 cases.

case I

$$u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x}) \times (c_3 e^{\alpha \lambda t} + c_4 e^{-\alpha \lambda t})$$

case II

$$u(x,t) = (c_5 x + c_6)(c_7 t + c_8)$$

Case III

$$u(x,t) = (c_9 \cos \lambda x + c_{10} \sin \lambda x)(c_{11} \cos(\alpha \lambda t) + c_{12} \sin(\alpha \lambda t))$$

$$u(x,t) = [c_1 \cos(c\lambda t) + c_2 \sin(c\lambda t)][c_3 \cos \lambda x + c_4 \sin \lambda x]$$

For $u(0,t) = 0$

$$0 = [c_1 \cos(c\lambda t) + c_2 \sin(c\lambda t)] \times [c_3 \cos(\lambda \times 0) + c_4 \sin(\lambda \times 0)]$$

$$\Rightarrow c_3 [c_1 \cos(c\lambda t) + c_2 \sin(c\lambda t)] = 0$$

$\therefore c_3 = 0$

$$u(x,t) = c_4 \sin \lambda x \left[c_1 \cos(c\lambda t) + c_2 \sin(c\lambda t) \right]$$

Using second boundary condition,

$$u(l,t) = 0$$

$$c_4 \sin(\lambda l) \left[c_1 \cos(c\lambda t) + c_2 \sin(c\lambda t) \right] = 0$$

$$\therefore c_4 \sin(\lambda l) = 0$$

$$\sin(\lambda l) = 0 \quad \left[\text{If } c_4 = 0 \text{ then we shall get trivial sol}^n \right]$$

$$\sin(\lambda l) = \sin n\pi, \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \lambda l = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{l}$$

$$\therefore u(x,t) = c_4 \sin\left(\frac{n\pi}{l}\right)x \left[c_1 \cos\left\{ c\left(\frac{n\pi}{l}\right)t \right\} + c_2 \sin\left\{ c\left(\frac{n\pi}{l}\right)t \right\} \right]$$

Using the Initial conditions

$$u(x,0) = 0$$

$$c_4 \sin\left(\frac{n\pi}{l}\right)x \left[c_1 \cos 0 + \cancel{c_2 \sin 0} \right] = 0$$

$$\Rightarrow c_1 c_4 \sin\left(\frac{n\pi}{l}\right)x = 0$$

$$\therefore c_1 = 0$$

$$\therefore u(x,t) = c_2 c_4 \sin\left(\frac{n\pi}{l}\right)x \sin\left\{ c\left(\frac{n\pi}{l}\right)t \right\}$$

Using the second initial cond^n;

$$u_t(x,0) = \lambda x(l - x)$$

$$\text{Now, } u_t = \frac{c_2 c_4}{B}\left(\frac{cn\pi}{l}\right) \sin\left(\frac{n\pi}{l}\right)x \cos\left(c\left(\frac{n\pi}{l}\right)\right)t$$

$$= A \sin\left(\frac{n\pi}{l}\right)x \cos\left(\frac{cn\pi}{l}\right)t$$

$u_t(x,0) = A \sin\left(\frac{n\pi}{l}\right)x$

$\therefore A \sin\left(\frac{n\pi}{l}\right)x = \lambda x(l-x)$

General solⁿ:

$$u(x,t) = \sum_{n=0}^{\infty} \underbrace{c_2 c_4}_{B_n} \sin\left(\frac{n\pi}{l}\right)x \sin\left\{c\left(\frac{n\pi}{l}\right)t\right\}$$

we can find A from Fourier series

$\frac{d^2y}{dx^2} - y = 0$

A.E. $m^2 - 1 = 0$

$m = \pm 1$

$y = c_1 e^{x} + c_2 e^{-x}$

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$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = 0$

$u(l,t) = 0$

$u(x,0) = \mu x(l-x)$

$\frac{\partial u}{\partial t}(x,0) = 0$

$\rightarrow u(x,t) = (c_1 \cos c\lambda t + c_2 \sin c\lambda t)(c_3 \cos\lambda x + c_4 \sin\lambda x)$

For, $u(0,t) = \left[(c_1 \cos c\lambda t + c_2 \sin c\lambda t)(c_3 \cos\lambda\times 0 + c_4 \sin\lambda\times 0)\right] = 0$

$\Rightarrow c_3\left[c_1 \cos c\lambda t + c_2 \sin c\lambda t\right] = 0$

$\therefore c_3 = 0$

$\therefore u(x,t) = (c_1 \cos c\lambda t + c_2 \sin c\lambda t)(c_4 \sin\lambda x)$

$u(l,t) = 0$

$\Rightarrow c_4 \sin\lambda l \,(c_1 \cos c\lambda t + c_2 \sin c\lambda l) = 0$

$\sin\lambda l = 0 = \sin n\pi \quad , n = 0, 1, 2 \ldots$

$\lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$

$$u(x,t) = c_4 \sin\left(\frac{n\pi}{l}\right)x \left[c_1 \cos\left(\frac{cn\pi}{l}\right)t + c_2 \sin\left(\frac{cn\pi}{l}\right)\right.$$

$$\frac{\partial u}{\partial t} = c_4 \sin\left(\frac{n\pi}{l}\right)x \left[-\frac{c_1 cn\pi}{l} \sin\left(\frac{cn\pi}{l}\right)t + \frac{c_2 cn\pi}{l} \cos\left(\frac{cn\pi}{l}\right)\right]$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{c_4 cn\pi}{l} \sin\left(\frac{n\pi}{l}\right)x \left[-c_1 \sin\left(\frac{cn\pi}{l}\right)\times 0 + c_2 \cos\left(\frac{cn\pi}{l}\right)\times 0\right]$$

$$= \frac{c_4 cn\pi}{l} \sin\left(\frac{n\pi}{l}\right)x \cdot c_2$$

By condition.

$$\frac{c_4 c_2 cn\pi}{l} \sin\left(\frac{n\pi}{l}\right)x = 0$$

$$\therefore c_2 = 0$$

$$u(x,t) = c_4 \sin\left(\frac{n\pi}{l}\right)x \times \left(c_1 \cos\left(\frac{cn\pi}{l}\right)t\right)$$

By the principle of superposition,

$$u(x,t) = \sum_{n=0}^{\infty} c_1 c_4 \sin\left(\frac{n\pi}{l}\right)x \cos\left(\frac{cn\pi}{l}\right)t$$

$$u(x,0) = \sum_{n=0}^{\infty} c_1 c_4 \sin\left(\frac{n\pi}{l}\right)x = kx(l-x)$$

12/01/21

Scalar Triple product

Let there be three vectors:
$\vec{a}, \vec{b}$ and $\vec{c}$

$\vec{a} \cdot (\vec{b} \times \vec{c})$
$= [\vec{a}\ \vec{b}\ \vec{c}]$

$[\vec{b}\ \vec{c}\ \vec{a}]$
$= \vec{b} \cdot (\vec{c} \times \vec{a})$

This represents the ~~area~~ volume of a parallelopiped whose adjacent sides are represented by $\vec{a}, \vec{b}, \vec{c}$

Q compute the following scalar triple product

$$(\hat{i} - 2\hat{j} + 3\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{j} + \hat{k})$$

$$\cancel{\Rightarrow 2 - 2 - 0 = (2 - 2 + 3) \cdot (\hat{j} + \hat{k})}$$

$$\cancel{\Rightarrow 3(\hat{j} + \hat{k})}$$

$$\cancel{= 3\hat{j} + 3\hat{k}}$$

$$\cancel{\Rightarrow \left(2 + \hat{k} + \hat{j} - 2(\hat{j}\hat{i}) - 2(\hat{j}\hat{j}) + 2(\hat{j}\hat{k}) + 3\hat{k}\hat{i} + 3\hat{k}\hat{j} - 3\hat{k}\hat{k}\right)}$$

$$\cancel{(\hat{j} + \hat{k})}$$

$$\cancel{\Rightarrow 2 + \hat{k} + \hat{j} + 2\hat{k} - 2 +}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 1 & -1 \end{vmatrix} = \hat{i}(2-3) - \hat{j}(-1-6) + \hat{k}(1+4)$$

$$= -\hat{i} + 7\hat{j} + 5\hat{k}$$

$$(\hat{i} + 7\hat{j} + 5\hat{k}) \cdot (\hat{j} + \hat{k}) = 0 + 7 + 5 = 12$$

Q Find the volume of the parallelopiped whose edges are represented by

$$\underbrace{(\hat{i} + 2\hat{j} + 3\hat{k})}_{\vec{a}}, \underbrace{(3\hat{i} + 7\hat{j} - 4\hat{k})}_{\vec{b}}, \underbrace{(\hat{i} - 5\hat{j} + 3\hat{k})}_{\vec{c}}$$

$\rightarrow$ volume $= \vec{a} \cdot (\vec{b} \times \vec{c})$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix} = \hat{i}($$

$$= -91$$

Vector Triple product

$\vec{a} \times (\vec{b} \times \vec{c})$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{i}(b_2c_3 - c_2b_3) + \hat{j}(b_3c_1 - b_1c_3) + \hat{k}(b_1c_2 - b_2c_1)$$

$$\cancel{\vec{a}\times\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 & b_3c_1 & b_1c_2 \\ -c_2b_3 & -c_3b_1 & -b_2c_1 \end{vmatrix}}$$

$\neq 0$

Remember

$\vec{a} \times (\vec{b}\times\vec{c})$

$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$\vec{c} \times (\vec{a}\times\vec{b})$

$= (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

Note: If $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ the $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Q If $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\vec{c} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$

the prove that the scalar triple product is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$\rightarrow \vec{a} \cdot (\vec{b} \times \vec{c})$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \hat{i}(y_2 z_3 - z_2 y_3) + \hat{j}(z_2 x_3 - x_2 z_3) + \hat{k}(x_2 y_3 - y_2 x_3)$$

$$= \vec{a} \cdot \left[\hat{i}(y_2 z_3 - z_2 y_3) + \hat{j}(z_2 x_3 - x_2 z_3) + \hat{k}(x_2 y_3 - y_2 x_3)\right]$$

$$\Rightarrow (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) \cdot \quad u$$

$$\Rightarrow x_1 y_2 z_3 - x_1 z_2 y_3 + y_1 z_2 x_3 - y_1 x_2 z_3 + z_1 x_2 y_3 - z_1 y_2 x_3$$

Differentiation

If $\vec{a}$ and $\vec{b}$ are differentiable vector functions of a scalar t, then prove that

(i) $\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$

(ii) $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{b}$

(iii) $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$

(i) let $\vec{u}(t) = \vec{a}(t) \pm \vec{b}(t)$

Then $\vec{u}(t+\Delta t) = \vec{a}(t+\Delta t) \pm \vec{b}(t+\Delta t)$

LHS
$$\frac{d}{dt}\left[\vec{a}(t) \pm \vec{b}(t)\right]$$
$$= \frac{d}{dt}\left[\vec{u}(t)\right]$$
$$= \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0}\left[\frac{\vec{a}(t+\Delta t) \pm \vec{b}(t+\Delta t) - \vec{a}(t) - \vec{b}(t)}{\Delta t}\right]$$
$$= \lim_{\Delta t \to 0}\left[\frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t}\right] + \lim_{\Delta t \to 0}\left[\frac{\vec{b}(t+\Delta t) - \vec{b}(t)}{\Delta t}\right]$$
$$= \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt} = \text{RHS}$$

(ii) Let $\vec{u}(t) = \vec{a}(t)\cdot\vec{b}(t)$

Then $\vec{u}(t+\Delta t) = \vec{a}(t+\Delta t)\cdot\vec{b}(t+\Delta t)$

LHS $= \frac{d}{dt}\left[\vec{a}(t)\cdot\vec{b}(t)\right]$

$= \frac{d}{dt}\left[\vec{u}(t)\right]$

$$= \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t) \cdot \vec{b}(t+\Delta t) - \vec{a}(t) \cdot \vec{b}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t) \cdot \vec{b}(t+\Delta t) - \vec{a}(t+\Delta t) \cdot \vec{b}(t) + \vec{a}(t+\Delta t) \cdot \vec{b}(t) - \vec{a}(t) \vec{b}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \vec{a}(t+\Delta t) \cdot \left[\frac{\vec{b}(t+\Delta t) - \vec{b}(t)}{\Delta t}\right] + \lim_{\Delta t \to 0} \left[\frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t}\right] \cdot \vec{b}(t)$$

$$= \lim_{\Delta t \to 0} \vec{a}(t+\Delta t) \cdot \lim_{\Delta t \to 0} \left[\frac{\vec{b}(t+\Delta t) - \vec{b}(t)}{\Delta t}\right] + \lim_{\Delta t} \left[\frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t}\right] \cdot \lim_{\Delta t \to 0} \vec{b}(t)$$

$$= \vec{a}(t) \cdot \frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt} \cdot \vec{b}(t)$$

$$= \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b}$$

$= RHS.$

(iii) Let $\vec{u}(t) = \vec{a}(t) \times \vec{b}(t)$

Then $\vec{u}(t+\Delta t) = \vec{a}(t+\Delta t) \times \vec{b}(t+\Delta t)$

LHS $= \frac{d}{dt}\left[\vec{a}(t) \times \vec{b}(t)\right]$

$= \frac{d}{dt}\left[\vec{u}(t)\right]$

$$= \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left[\frac{\vec{a}(t+\Delta t) \times \vec{b}(t+\Delta t) - \vec{a}(t) \times \vec{b}(t)}{\Delta t} \right]$$

$$= \lim_{\Delta t \to 0} \left[\frac{\vec{a}(t+\Delta t) \times \vec{b}(t+\Delta t) - \vec{a}(t+\Delta t) \times \vec{b}(t) + \vec{a}(t+\Delta t) \times \vec{b}(t) - \vec{a}(t) \times \vec{b}(t)}{\Delta t} \right]$$

$$= \lim_{\Delta t \to 0} \left[\vec{a}(t+\Delta t) \times \frac{\left(\vec{b}(t+\Delta t) - \vec{b}(t)\right)}{\Delta t} + \left[\frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t}\right] \times \vec{b}(t) \right]$$

$$= \lim_{\Delta t \to 0} \vec{a}(t+\Delta t) \times \lim_{\Delta t \to 0} \left[\frac{\vec{b}(t+\Delta t) - \vec{b}(t)}{\Delta t}\right] + \lim_{\Delta t \to 0} \left[\frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t}\right] \times \lim_{\Delta t \to 0} \vec{b}(t)$$

$$= \vec{a}(t) \times \frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt} \times \vec{b}(t)$$

$$= \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}.$$

$$= \underline{\underline{\text{RHS}}}$$

Q prove the necessary and sufficient condition for the vector function $\vec{a}(t)$.

i) to be constant is $\frac{d\vec{a}}{dt} = \vec{0}$, a null vector

ii) to have constant magnitude is $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$

→(i) let $\vec{a}(t)$ be a constant vector function

Then $\vec{a}(t+\Delta t) - \vec{a}(t) = 0$

$$= \lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t} = 0$$

$$\Rightarrow \frac{d\vec{a}(t)}{dt} = 0$$

Now, let us suppose that $\frac{d\vec{a}}{dt} = 0$

and $\vec{a}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

Then,

$$\frac{d\vec{a}}{dt} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

Again since $\frac{d\vec{a}}{dt} = 0$

so, $\frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k} = \vec{0}$

$$\Rightarrow \frac{df}{dt} = 0, \frac{dg}{dt} = 0, \frac{dh}{dt} = 0$$

f, g, h are constant scalar function

Thus, $\vec{a}(t)$ is a constant.

(ii) let $\vec{a}(t)$ is of constant magnitude

$\vec{a} \cdot \vec{a} = a^2$

Taking derivative on both sides w.r.t t

$$\frac{d}{dt}(\vec{a}.\vec{a}) = 0$$

$$\Rightarrow \vec{a} \cdot \frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{a} = 0$$

$$\Rightarrow \vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

let $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$

$$\Rightarrow \frac{1}{2}\left[\vec{a} \cdot \frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{a}\right] = 0$$

$$\Rightarrow \frac{1}{2}\left[\frac{d}{dt}(\vec{a} \cdot \vec{a})\right) = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{a} \cdot \vec{a}) = 0 \Rightarrow \vec{a} \cdot \vec{a} = \text{constant}$$

$\Rightarrow \vec{a}$ is constant magnitude.

Q prove that $\frac{d}{dt}\left(\vec{a}\times\frac{d\vec{b}}{dt}-\frac{d\vec{a}}{dt}\times\vec{b}\right)$

$$=\vec{a}\times\frac{d^2\vec{b}}{dt^2}-\frac{d^2\vec{a}}{dt^2}\times\vec{b}$$

Q If $\vec{r}=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$

find at $t=0$, the values of

$$\frac{d\vec{r}}{dt},\ \frac{d^2\vec{r}}{dt^2},\ \left|\frac{d\vec{r}}{dt}\right| \text{ and } \left|\frac{d^2\vec{r}}{dt^2}\right|$$

Gradient of a scalar function

$$\vec{\nabla}=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$$

Let ϕ be a scalar function Gradient ϕ

or Grad ϕ

$$\text{or } \nabla\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

= vector quantity.

$$\phi=x^3+y^3+z^3-3xyz$$

find $\nabla\phi$

$$\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial x}\left[x^3+y^3+z^3-3xyz\right]$$

$$=3x^2-3yz$$

$$\frac{\partial\phi}{\partial y}=3y^2-3xz$$

$$\frac{\partial\phi}{\partial z}=3z^2-3xy$$

$$\nabla\phi=(3x^2-3yz)\hat{i}+(3y^2-3xz)\hat{j}+(3z^2-3xy)$$

Divergence of a vector

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector function

Then

$$\text{Div}\,\vec{F}$$

$$\vec{\nabla}\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \text{scalar}$$

curl of a vector

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector function

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Q prove that $(\vec{r}\cdot\vec{\nabla})\phi = \vec{r}\cdot(\nabla\phi)$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$(\vec{r}\cdot\nabla)\phi$$

$$= \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z} \quad \text{--- (1)}$$

Q prove that $\text{div}(\text{curl}\,\vec{v}) = 0$

i.e., to prove $\nabla\cdot(\nabla\times\vec{v}) = 0$

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Now $$\nabla\cdot(\nabla\times\vec{v}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\right]$$

$$= \cancel{\frac{\partial^2 v_3}{\partial x\partial y}} - \cancel{\frac{\partial^2 v_2}{\partial x\partial z}} - \cancel{\frac{\partial^2 v_3}{\partial y\partial x}} + \cancel{\frac{\partial^2 v_1}{\partial y\partial z}} + \cancel{\frac{\partial^2 v_2}{\partial z\partial x}} - \cancel{\frac{\partial^2 v_1}{\partial z\partial y}} = 0$$

Q prove that $\text{curl}\,\text{curl}\,\vec{v} = \text{grad}\,\text{div}\,\vec{v} - \nabla^2\vec{v}$

LHS $= \nabla\times(\nabla\times\vec{v})$

$= (\nabla\cdot\vec{v})\nabla - (\nabla\cdot\nabla)\vec{v}$

$= (\text{Div}.\,\vec{v}) - \nabla^2\vec{v}$

$= \nabla(\text{Div}.\,\vec{v}) - \nabla^2\vec{v}$

$= \text{grad}(\text{Div}\,\vec{v}) - \nabla^2\vec{v}$

$= \text{RHS}$

Q prove that $\text{grad}\, r^m = m r^{m-2}\vec{r}$

~~$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$~~

$+\ r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$LHS = \text{grad } r^m$

$= \nabla(r^m)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\left(\frac{\partial}{\partial z}\right)\right)(r^m)$

$\therefore r^m = (x^2+y^2+z^2)^{m/2}$

$= \hat{i}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{m/2}\right] + \hat{j}\left[\frac{\partial}{\partial y}(x^2+y^2+z^2)^{m/2}\right]$

$+ \hat{k}\left[\frac{\partial}{\partial z}(x^2+y^2+z^2)^{m/2}\right]$

$= \frac{m}{2}(x^2+y^2+z^2)^{m/2-1}\cdot(2x)$

$= m(x^2+y^2+z^2)^{m/2-1}\cdot x$

$= m r^{m-1}\cdot x$

$\frac{\partial}{\partial y}(x^2+y^2+z^2)^{m/2}$

$= \cancel{m r^{m-1} y}\ m/2\,(x^2+y^2+z^2)^{m/2-1}\cdot 2y$

$= m r^{m-1}\cdot y$

$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{m/2} = m r^{m-1}\cdot z$

$= m r^{m-1}(x\hat{i} + y\hat{j} + z\hat{k})$

$= m r^{m-1}\,\vec{r} = RHS.$

Q prove that $\nabla \times \vec{r} = \vec{0}$

$\rightarrow$ $\cancel{\hat{i} \quad \hat{j} \quad \hat{k}}$

$$\cancel{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(x\hat{i} + y\hat{j} + z\hat{k}\right)}$$

$$= \cancel{0 + \frac{\partial}{\partial x} y(\hat{i}\times\hat{j}) + \frac{\partial}{\partial x} z(\hat{i}\times\hat{k})}$$

$$\cancel{+ 0 + \frac{\partial}{\partial y} x(\hat{j}\times\hat{i}) + \frac{\partial}{\partial y} z(\hat{j}\times\hat{k})}$$

$$\cancel{+ 0 + \frac{\partial}{\partial z} x(\hat{k}\times\hat{i}) + \frac{\partial}{\partial z} y(\hat{k}\times\hat{j})}$$

$$= \cancel{\frac{\partial}{\partial x} y\hat{k} - \frac{\partial}{\partial x} z\hat{j}}$$

$$\cancel{- \frac{\partial}{\partial y} x\hat{k} + \frac{\partial}{\partial y} z\hat{i}}$$

$\cancel{+}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0$$