Spectral flow demonstrated with N unit cells - Changing K1 and K2 simultaneously; adding a K2 spring at the end of the system (needed to get edge mode)

restart;
with(*LinearAlgebra*) : *with*(*Physics*) : *with*(*Statistics*) : *with*(*plots*) :

Nominal values for mass m, stiffnesses K1 and K2, and the number of unit cells (comes from paper with Nehemiah and Sai)

$m := 0.1; k := 6000; N := 20;$

$$m := 0.1$$
$$k := 6000$$
$$N := 20 \tag{1.1}$$

Set this to 1 if the final K2 spring to ground should be added at the end of the chain. Otherwise, set to 0. The system of equations are only chiral when edge spring is present (see Theocharis's paper, Phys Rev. B, 024106, 2021) for acoustic system. Without the edge spring, we do not get the topological mode (unless use quantum system, at which point diagonalModifier should be set to zero).

$endSpring := 1;$

$$endSpring := 1 \tag{1.2}$$

Note: if I set diagonalModifier to zero, I get the QM tight-binding system where the diagonals of the "stiffness" matrix are zero. Otherwise, set to 1.

$diagonalModifier := 1;$

$$diagonalModifier := 1 \tag{1.3}$$

$M := Matrix(2\cdot N, 2\cdot N, [\ ]) : K := Matrix(2\cdot N, 2\cdot N, [\ ]) :$

Mass matrix for system

```
for i from 1 to 2·N do
   for j from 1 to 2·N do
         M(i, j) := m·KroneckerDelta[i, j];
   end do;
end do;
```

This helps in the stiffness rows to assign either K1 or K2 to the entry to the left/right of the diagonal depending on whether the row is even numbered or odd numbered

mod(1, 2);**mod**(2, 2);**mod**(3, 2);**mod**(4, 2);**mod**(5, 2);

$$1$$
$$0$$
$$1$$
$$0$$
$$1 \tag{1.4}$$

Here, we let K1 and K2 be functions of lambda

$K1 := k\cdot(1 - \text{lambda}); K2 := k\cdot(1 + \text{lambda});$

$$K1 := 6000 - 6000\ \lambda$$

$$K2 := 6000 + 6000\ \lambda \tag{1.5}$$

Stiffness matrix for system.

for i **from** 1 **to** $2 \cdot N$ **do**

  if $(i \neq 1$ **and** $i \neq 2 \cdot N)$ **then** $K(i, i) := diagonalModifier \cdot (K1 + K2); K(i, i-1) := -K1 \cdot$ **mod**$(i-1, 2) - K2 \cdot$ **mod**$(i, 2); K(i, i+1) := -K1 \cdot$ **mod**$(i, 2) - K2 \cdot$ **mod**$(i+1, 2)$; **end if**;

  if $(i = 1)$ **then** $K(1, 1) := diagonalModifier \cdot K1; K(1, 2) := -K1$; **end if**;

  if $(i = 2 \cdot N)$ **then** $K(2 \cdot N, 2 \cdot N) := diagonalModifier \cdot (K1 + endSpring \cdot K2); K(2 \cdot N, 2 \cdot N - 1) := -K1$; **end if**;

end do;

The eigenvalues from the generalized EVP, K*x=lambda*M*x, are found from omega^2=lambda. Note that the usage of lambda here is different from the spectral flow lambda used above

$eigs := x \rightarrow Eigenvalues(subs(\text{lambda} = x, K), M);$

$$eigs := x \mapsto Eigenvalues(subs(\lambda = x, K), M) \tag{1.6}$$

$\#eigs(0.1);$

Create a sequence for various values of lambda from -1 to 1. Do this by specifying the number of lambda values to use.

$numLambda := 300;$

$$numLambda := 300 \tag{1.7}$$

$data := seq\left(\left[\left(\frac{2 \cdot x}{numLambda} - 1\right), eigs\left(\frac{2 \cdot x}{numLambda} - 1\right)\right], x = 0 ..numLambda\right) :$

$data[1]; data[1][1]; data[1][2];$

$$\left[-1, \left[\begin{array}{c} \mathit{1\ ..\ 40\ Vector}_{column} \\ \mathit{Data\ Type:\ complex}_8 \\ \mathit{Storage:\ rectangular} \\ \mathit{Order:\ Fortran\_order} \end{array}\right]\right]$$

$$-1$$

$$\left[\begin{array}{c} \mathit{1\ ..\ 40\ Vector}_{column} \\ \mathit{Data\ Type:\ complex}_8 \\ \mathit{Storage:\ rectangular} \\ \mathit{Order:\ Fortran\_order} \end{array}\right] \tag{1.8}$$

$lambdaVals := j \rightarrow data[j][1];$

$$lambdaVals := j \rightarrow data_{j_1} \tag{1.9}$$

$lambdaVals(2);$

$$-\frac{149}{150} \tag{1.10}$$

$eigVals := j \rightarrow data[j][2];$

$$eigVals := j \rightarrow data_{j_2} \tag{1.11}$$

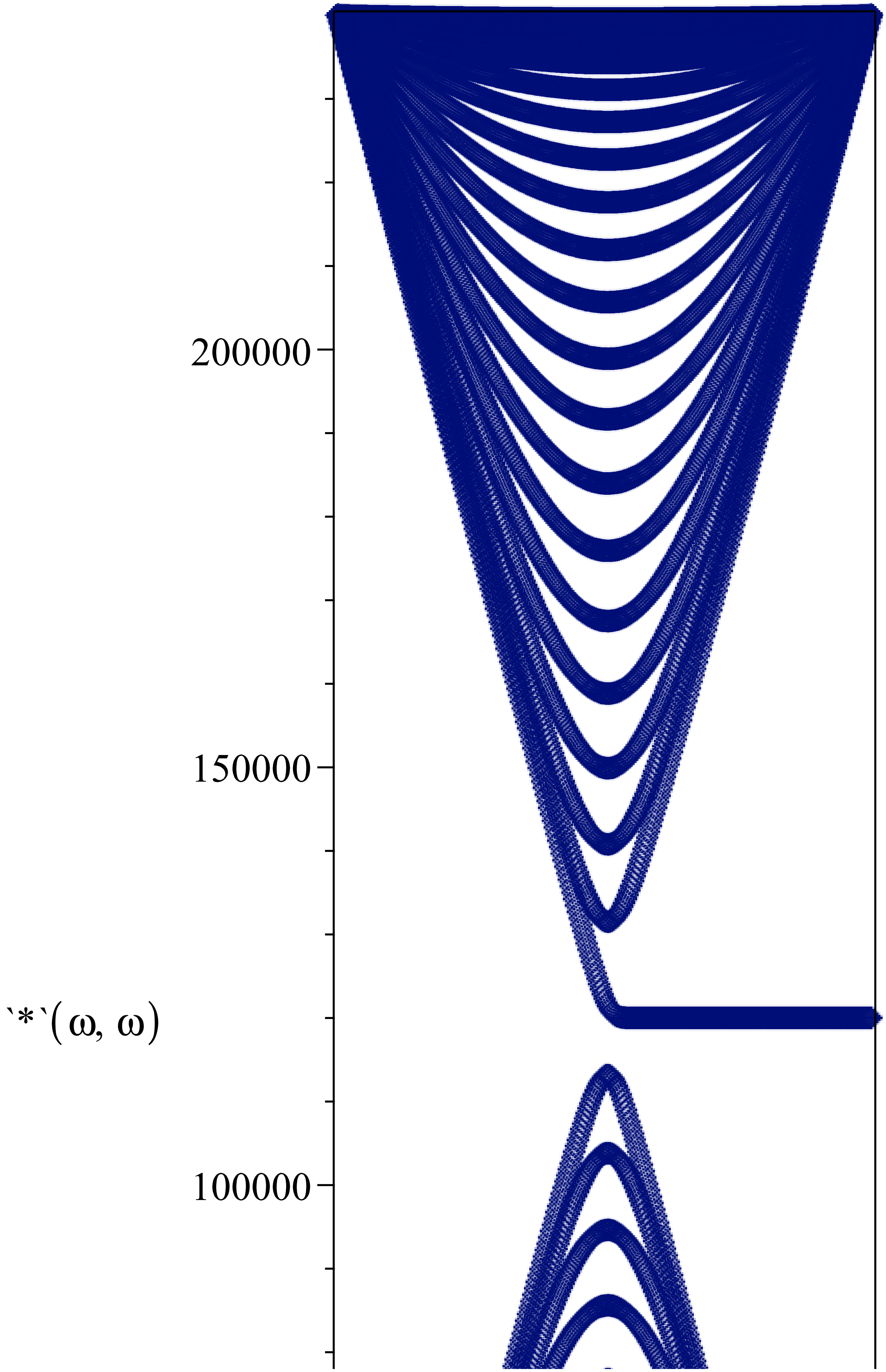
$eigVals(1);$

$$\left[\begin{array}{c} \text{1 .. 40 } Vector_{column} \\ \text{Data Type: } complex_8 \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{array}\right] \qquad \textbf{(1.12)}$$

I plot either the eigenvalue or the sqrt(eigenvalue). When sqrt, we recover omega.

$\#ScatterPlot(\langle seq(seq(lambdaVals(l), j = 1 ..2 \cdot N), l = 1 ..numLambda + 1)\rangle, \langle seq(\mathrm{sqrt}\sim(eigVals(j)), j = 1 ..numLambda + 1)\rangle, labels = [\text{'lambda','ω'}]);$

$ScatterPlot(\langle seq(seq(lambdaVals(l), j = 1 ..2 \cdot N), l = 1 ..numLambda + 1)\rangle, \langle seq(eigVals(j), j = 1 ..numLambda + 1)\rangle, labels = [\text{'lambda','ω ω'}]);$



200000
150000
100000
`*`(ω, ω)

Have to be careful to take Transpose at times since eigenvectors are columns (see further below)
$eigs := x \rightarrow Eigenvectors(subs(\text{lambda} = x, K), M);$

$$eigs := x \mapsto Eigenvectors\left(subs\left(\lambda = x, K\right), M\right) \quad \textbf{(1.13)}$$

$eigs(0);$

$$\left[\begin{array}{c} \textit{1 .. 40 Vector}_{column} \\ \textit{Data Type: complex}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right], \left[\begin{array}{c} \textit{40 x 40 Matrix} \\ \textit{Data Type: complex}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right] \quad \textbf{(1.14)}$$

$eigs(1.0)[1][3];$

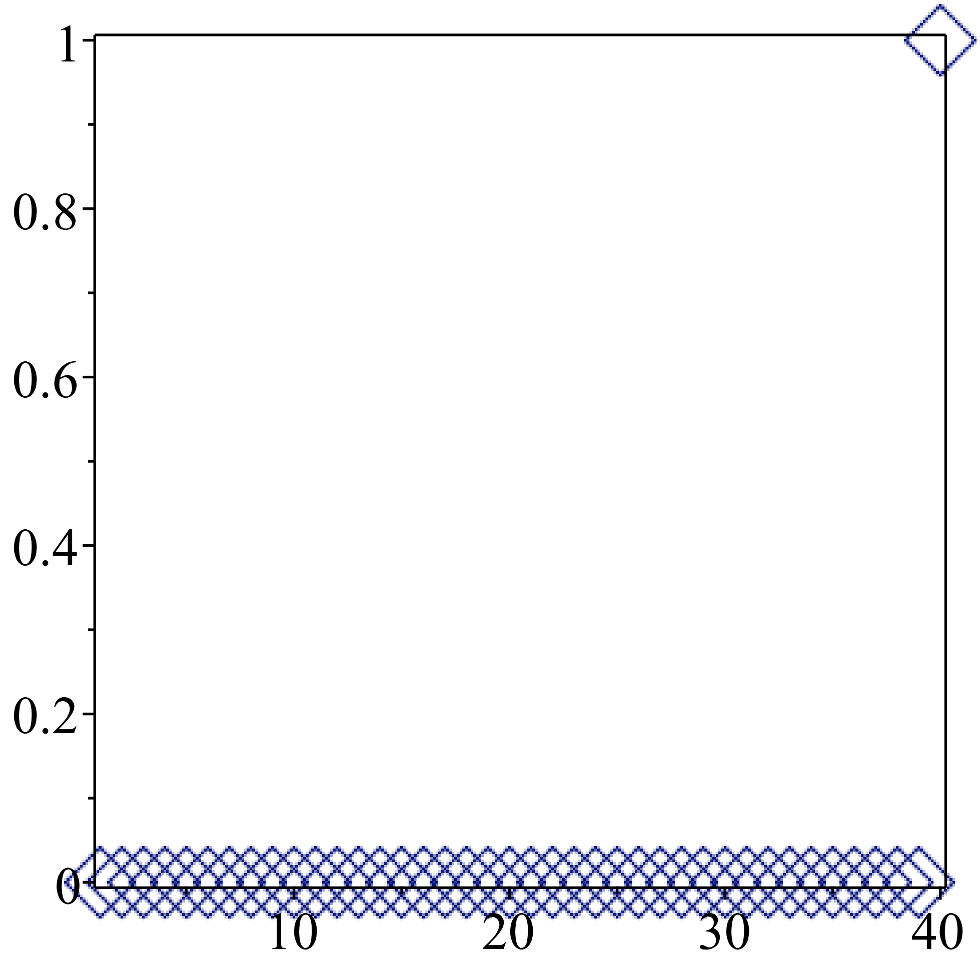
$$240000.000000000 + 0.\ \mathrm{I} \quad \textbf{(1.15)}$$

This is the mode on the right end of the chain
$eigs(1.0)[1][2 \cdot N]; Transpose(eigs(1.0)[2])[2 \cdot N];$

$$120000. + 0.\ \mathrm{I}$$

$$\left[\begin{array}{c} \textit{1 .. 40 Vector}_{row} \\ \textit{Data Type: complex}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right] \quad \textbf{(1.16)}$$

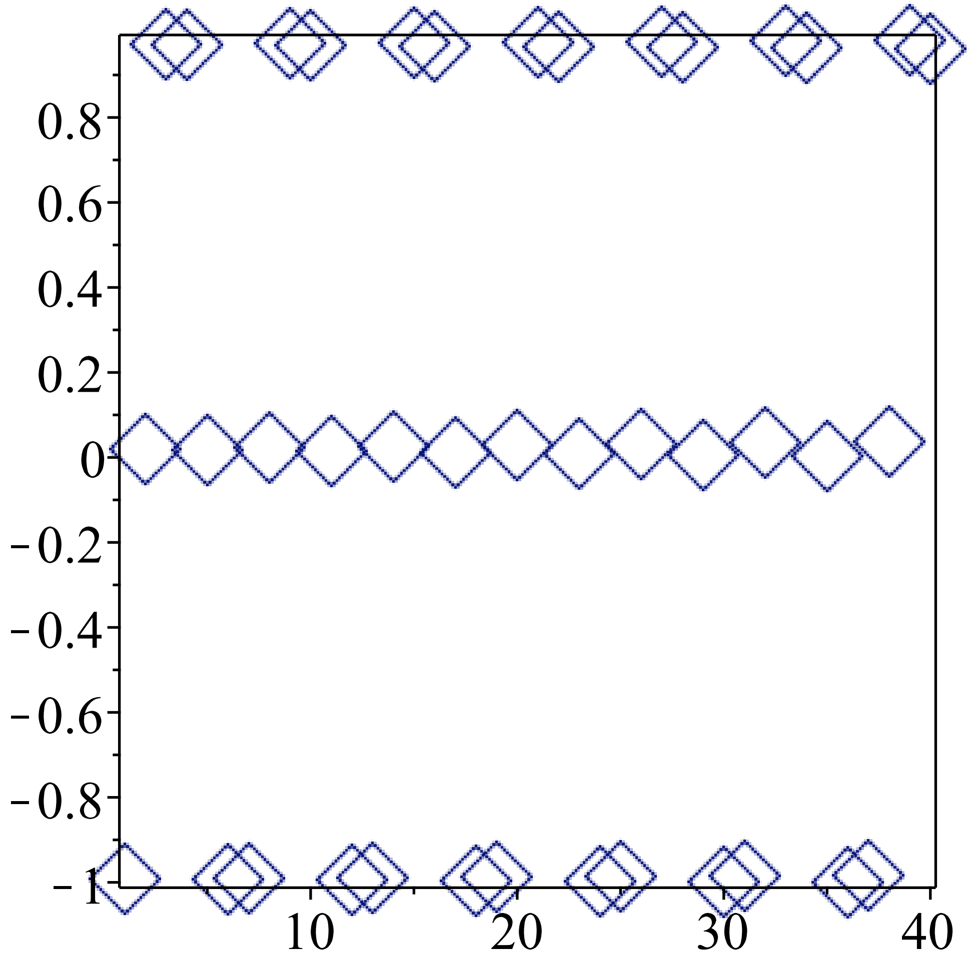
Note that the mode shape when lambda=1.0 has a 1 at mass 2N and the rest of the masses have very little response
$ScatterPlot(\langle seq(j, j = 1 ..2 \cdot N) \rangle, Transpose(eigs(1.0)[2])[2 \cdot N], symbolsize = 50);$



Note that the mode shape when lambda=0.02 the localized mode becomes distributed

$eigs(0.02)[1][2 \cdot N]$; $ScatterPlot(\langle seq(j, j = 1 ..2 \cdot N) \rangle, Transpose(eigs(0.02)[2])[2 \cdot N],$ $symbolsize = 50)$;

$$60008.4423987034 + 0.\ \mathrm{I}$$



Here is the topological mode at lambda = 0.375, 0.4, 0.5, 0.6, ..., 1.0 (turns out to be the last eigenvector returned)

$eigs\left(\frac{3}{8}\right)[1][2\cdot N]$; $ScatterPlot\left(\langle seq(j, j=1..2\cdot N)\rangle, Transpose\left(eigs\left(\frac{3}{8}\right)[2]\right)[2\cdot N], symbolsize=50, labels=['j','u']\right);$

$eigs(0.4)[1][2\cdot N]$; $ScatterPlot(\langle seq(j, j=1..2\cdot N)\rangle, Transpose(eigs(0.4)[2])[2\cdot N], symbolsize=50, labels=['j','u']);$

$eigs(0.5)[1][2\cdot N]$; $ScatterPlot(\langle seq(j, j=1..2\cdot N)\rangle, Transpose(eigs(0.5)[2])[2\cdot N], symbolsize=50, labels=['j','u']);$

$eigs(0.6)[1][2\cdot N]$; $ScatterPlot(\langle seq(j, j=1..2\cdot N)\rangle, Transpose(eigs(0.6)[2])[2\cdot N], symbolsize=50, labels=['j','u']);$

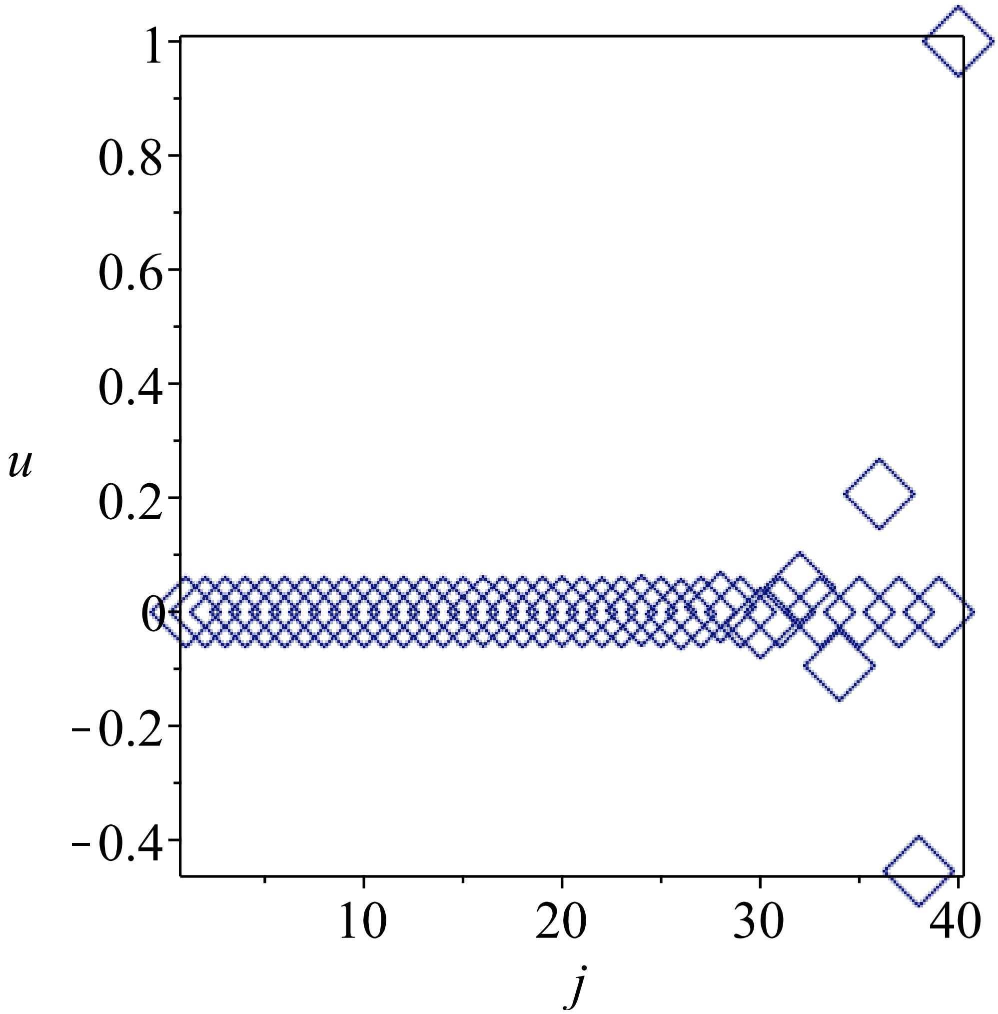
$eigs(0.7)[1][2\cdot N]$; $ScatterPlot(\langle seq(j, j=1..2\cdot N)\rangle, Transpose(eigs(0.7)[2])[2\cdot N], symbolsize=50, labels=['j','u']);$

$eigs(0.8)[1][2\cdot N]$; $ScatterPlot(\langle seq(j, j=1..2\cdot N)\rangle, Transpose(eigs(0.8)[2])[2\cdot N], symbolsize=50, labels=['j','u']);$

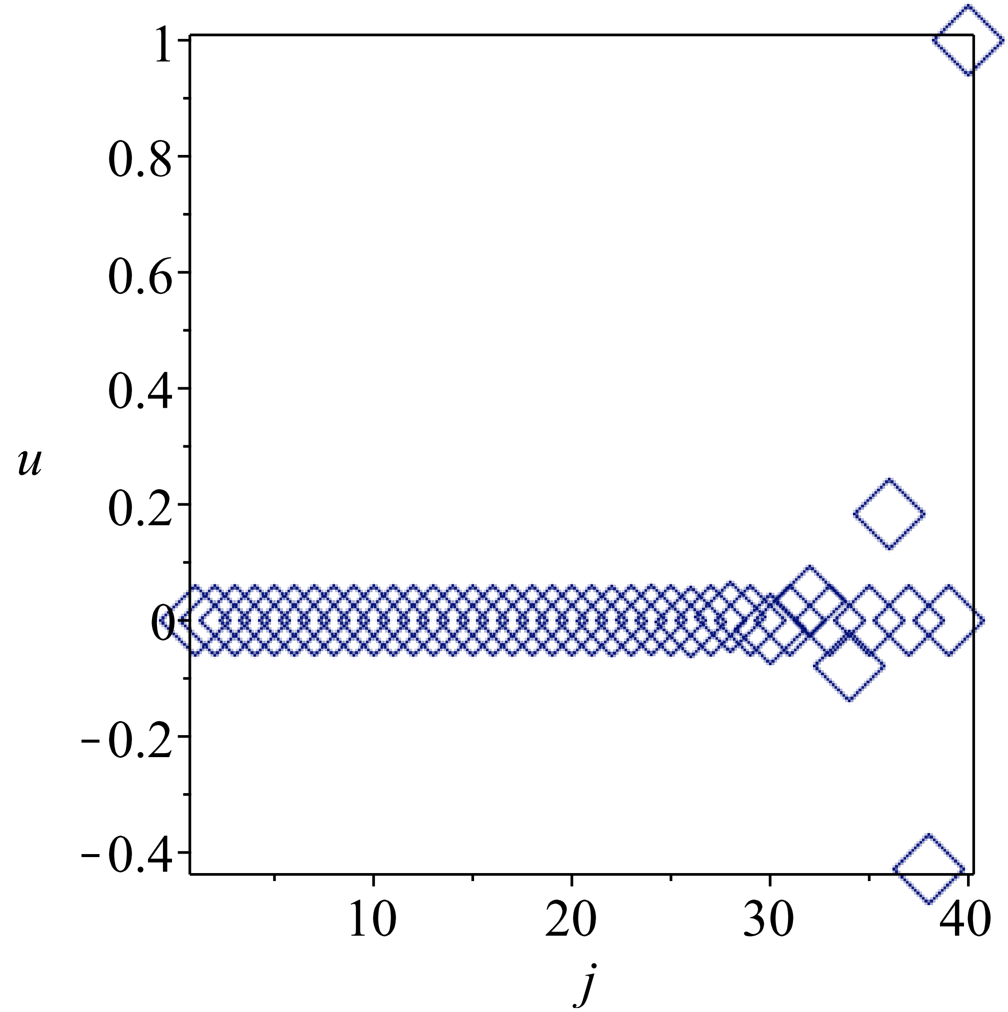
$eigs(0.9)[1][2\cdot N]$; $ScatterPlot(\langle seq(j, j=1..2\cdot N)\rangle, Transpose(eigs(0.9)[2])[2\cdot N], symbolsize=50, labels=['j','u']);$

$eigs(1.0)[1][2\cdot N]$; $ScatterPlot(\langle seq(j, j=1..2\cdot N)\rangle, Transpose(eigs(1.0)[2])[2\cdot N], symbolsize=50, labels=['j','u']);$

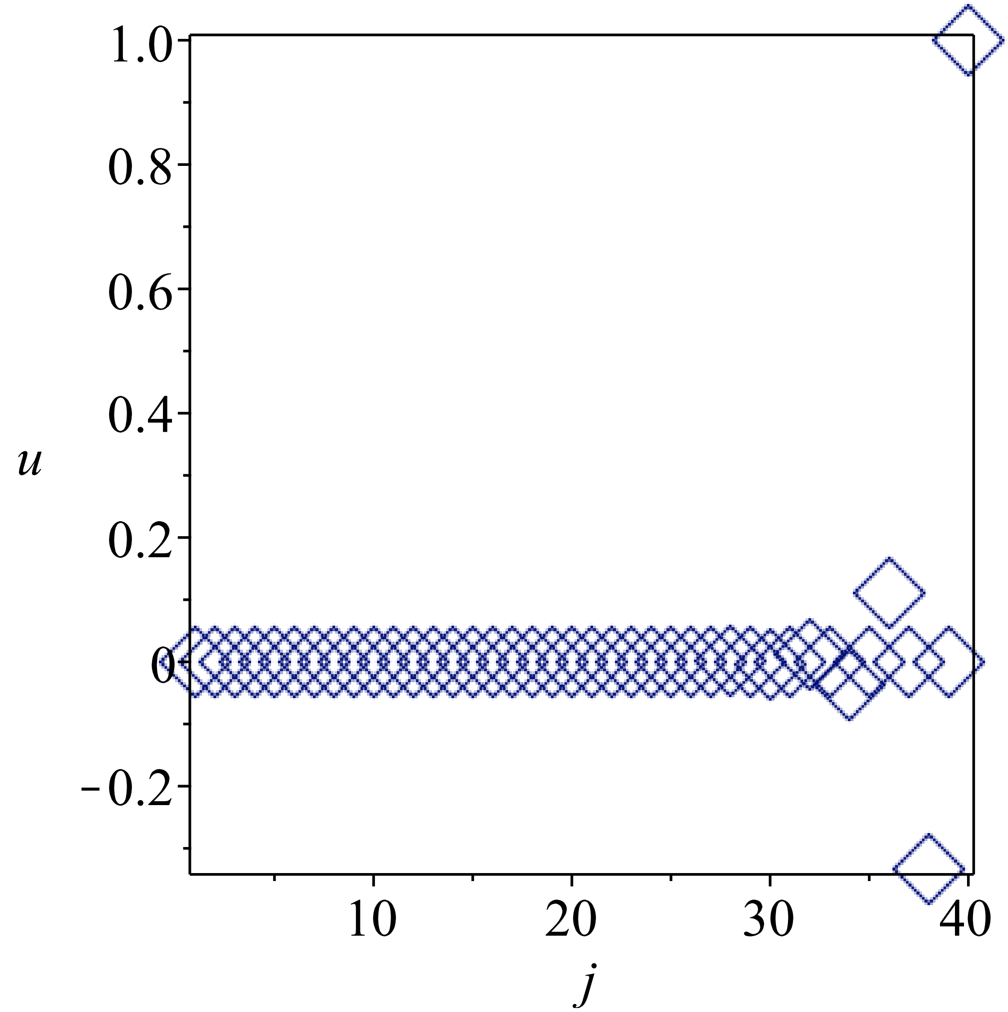
$$120000.000000001 + 0.\ I$$



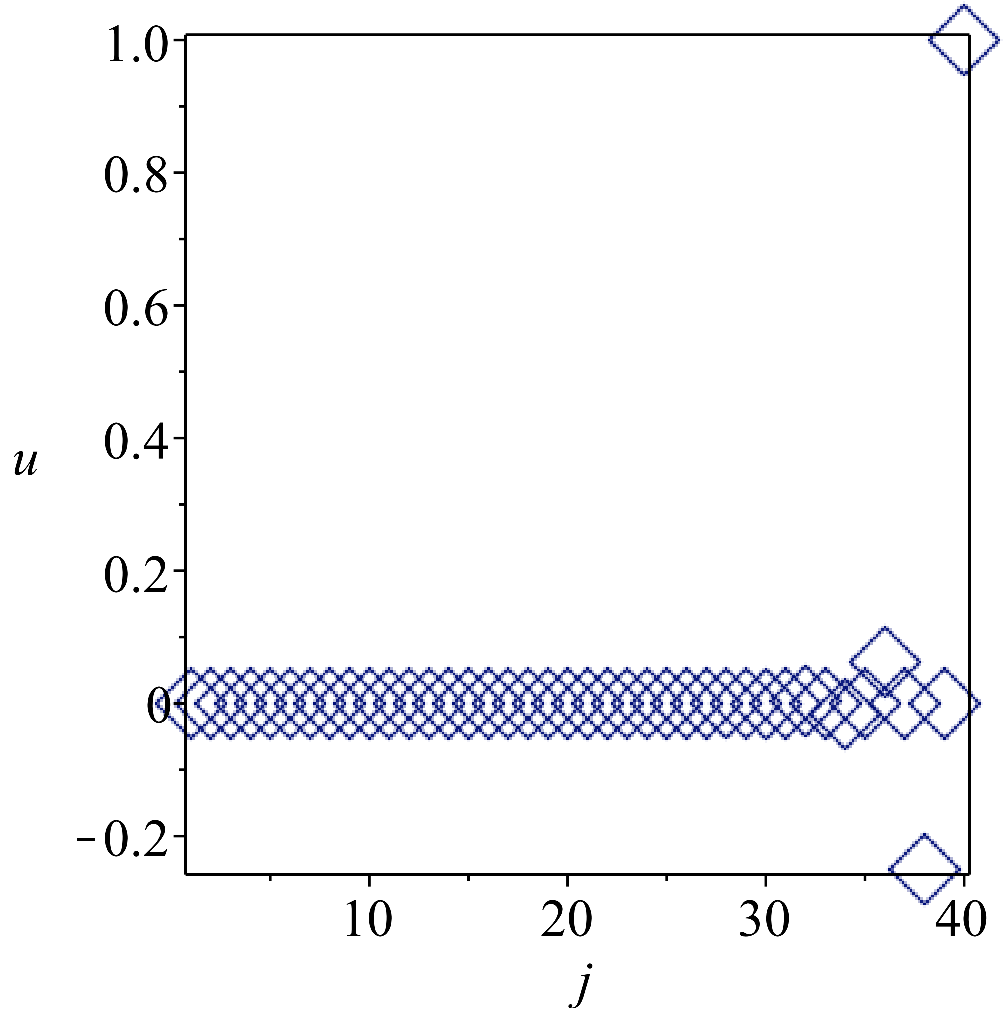
$$120000.000000000 + 0.\ I$$



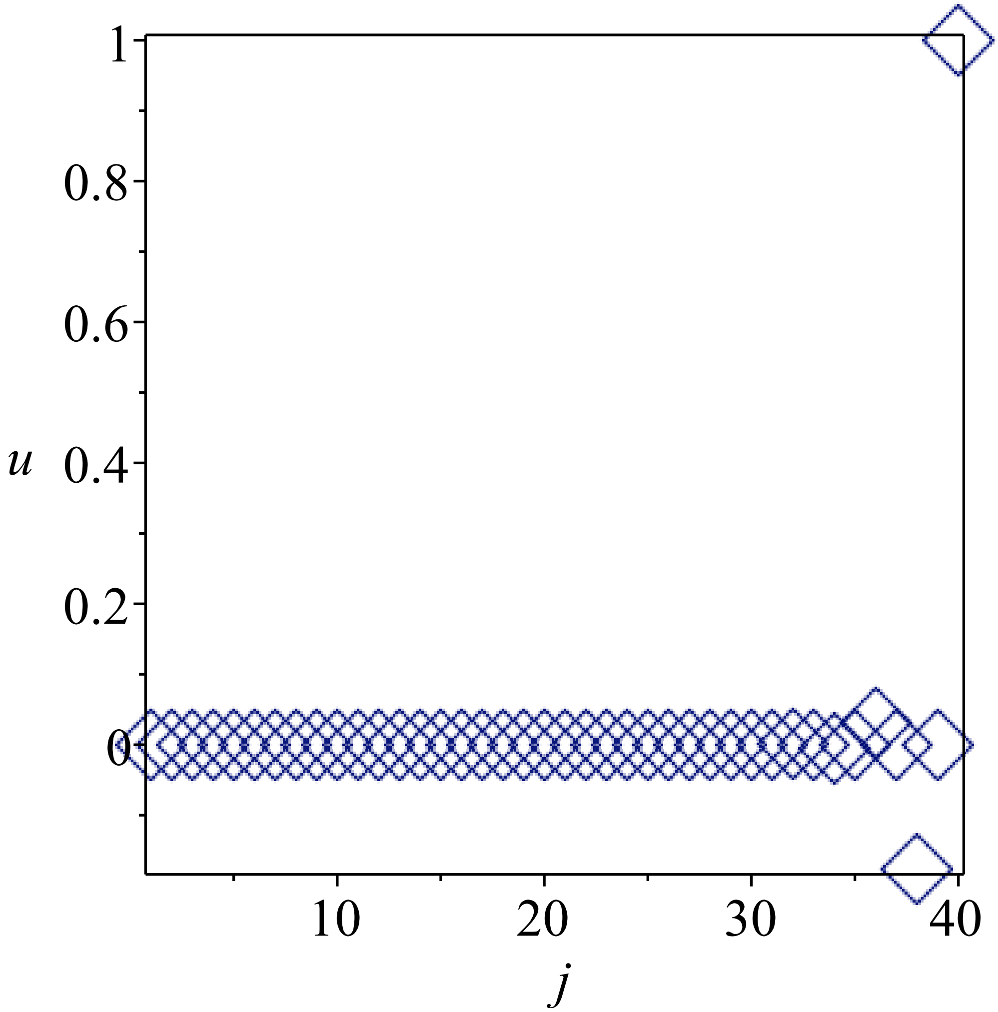
$120000. + 0.\ I$



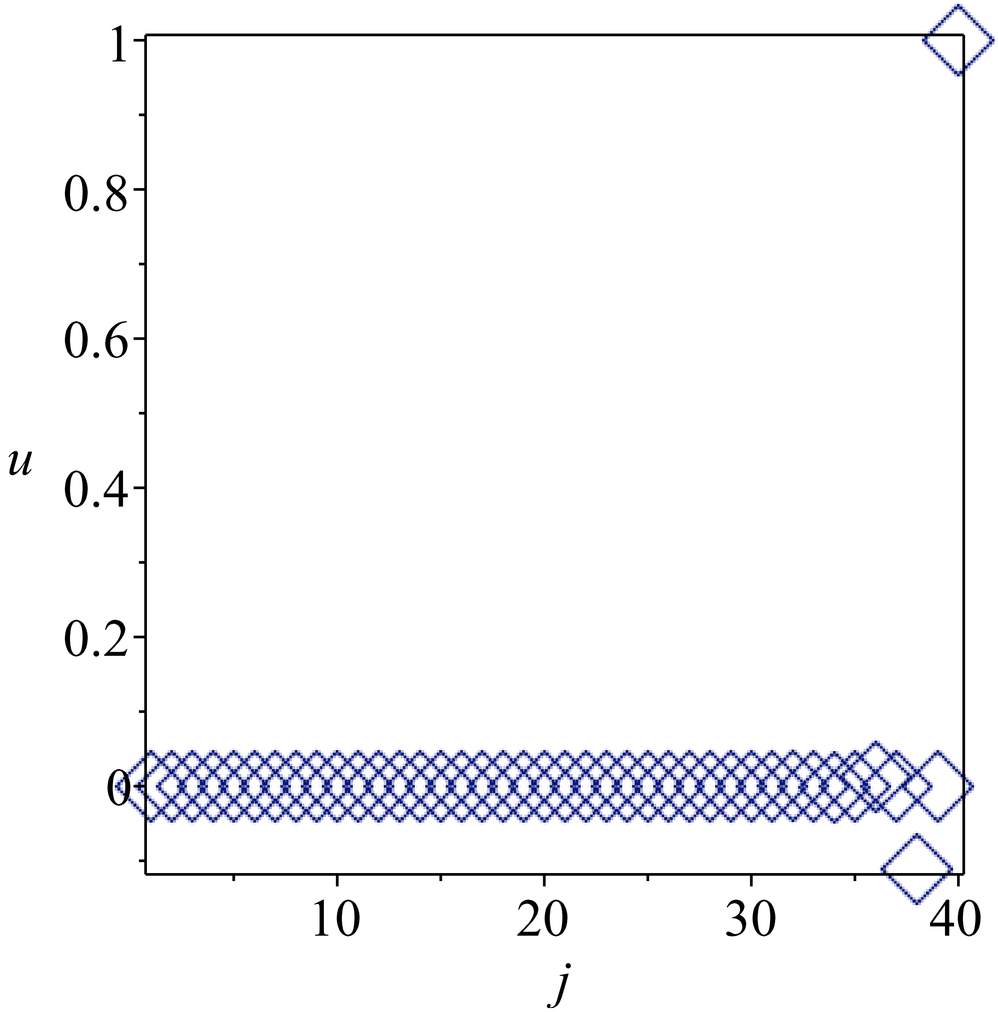
$120000.000000000 + 0.\ I$



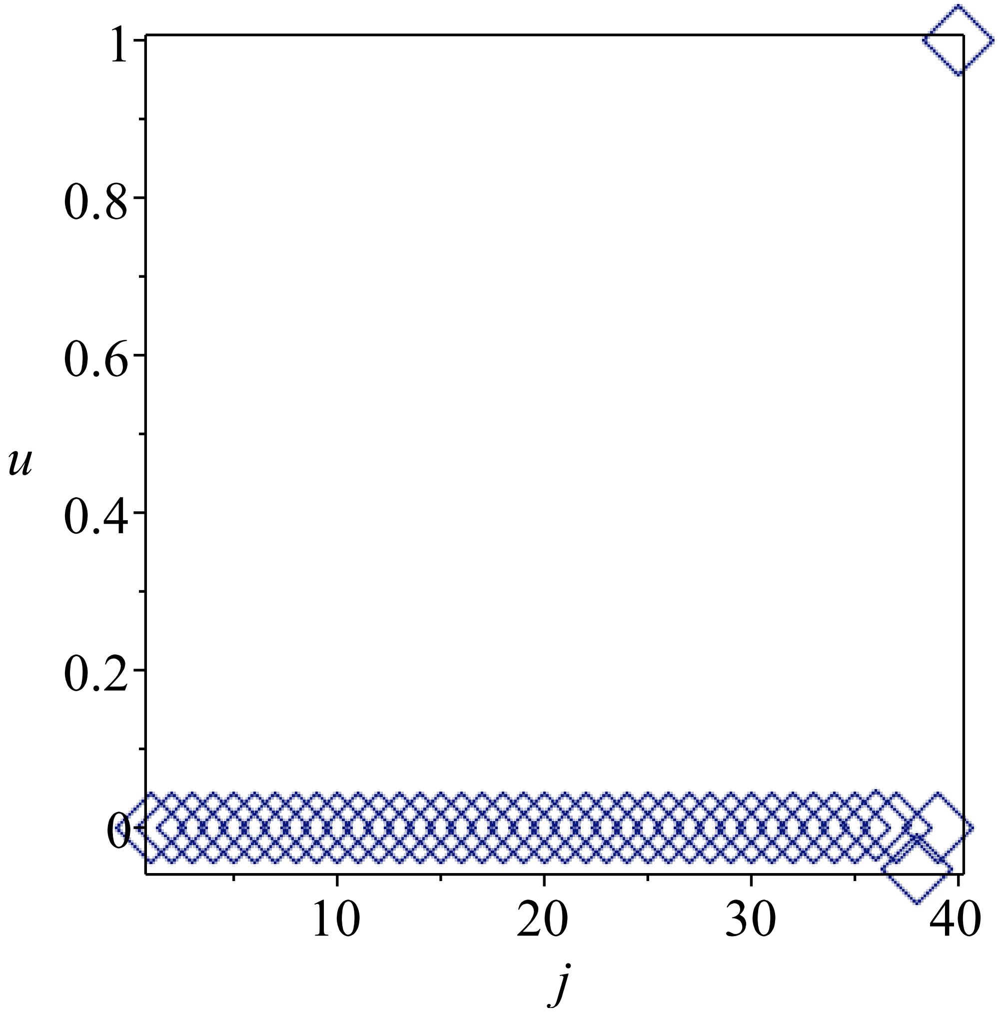
$120000.000000000 + 0.\ I$



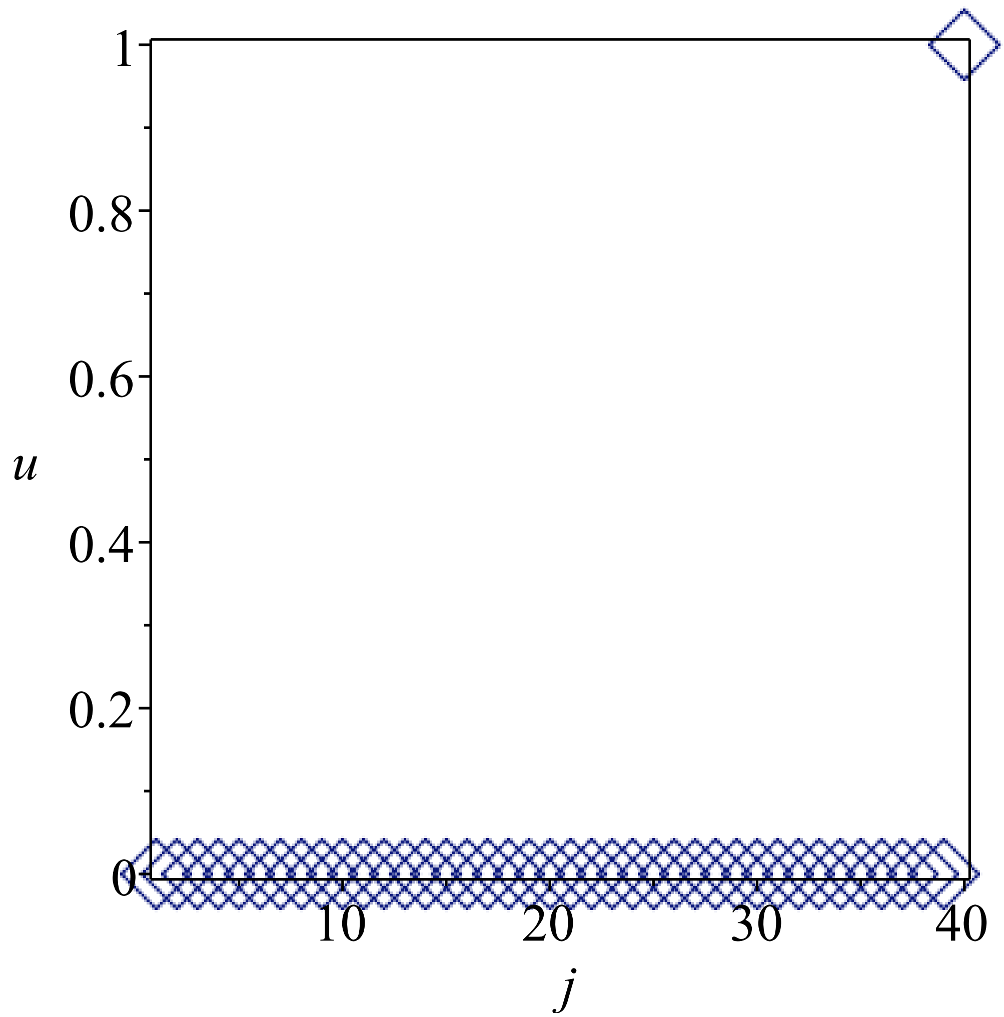
$120000.000000000 + 0.\ \mathrm{I}$



$120000. + 0.\ I$



$120000. + 0.\ I$



Springs needed in the experiment to see a localized edge mode

$subs\left(\text{lambda} = \frac{3}{8}, K1\right); subs\left(\text{lambda} = \frac{3}{8}, K2\right);$

$$3750$$
$$8250 \tag{1.17}$$

Expected edge mode frequency in Hz

$evalf\left(\frac{\text{sqrt}\left(eigs\left(\frac{3}{8}\right)[1][2 \cdot N]\right)}{2 \cdot \text{Pi}}\right);$

$$55.1328895276665 + 0.\ \text{I} \tag{1.18}$$